Stochastic Processes 2

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Literature

Basic Lecture Notes:

- Z. Prášková: Základy náhodných procesů II, Karolinum 2016, In Czech
- Z. Prášková: Stochastic Processes 2, on-line version
- J. Dvořák, M. Prokešová: Stochastic Processes 2, Collection of solved exercises, on-line

Supplementary texts :

- Brockwell, P. J., Davis, R. A. : Time Series: Theory and Methods. Springer-Verlag, New York 1991
- Anděl, J. : Statistická analýza časových řad. SNTL, Praha 1976 (In Czech)

Definitions and basic characteristics

Definition:

Let $(\Omega, \mathcal{A}, \mathrm{P})$ be a probability space, $(S, \mathcal{E})$ a measurable space, and $T \subset \mathbb{R}$. A family of (real-valued) random variables $\{X_t,\ t \in T\}$ defined on $(\Omega, \mathcal{A}, \mathrm{P})$ with values in S is called a stochastic (random) process.

$T = \mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$ or $T \subset \mathbb{Z}$ - $\{X_t,\ t \in T\}$ is discrete time stochastic process, time series

$T = [a, b],\ -\infty \leq a < b \leq \infty$ - $\{X_t,\ t \in T\}$ is *a continuous time stochastic process*.

For any $\omega \in \Omega$ fixed, $X_t(\omega)$ is a function on T with values in S which is called *a trajectory* of the process.

Definition:

A pair $(S, \mathcal{E})$, where S is a set of values of random variables X_t and $\mathcal{E}$ is a $\sigma-$algebra of subsets of S, is called the state space of the process $\{X_t,\ t \in T\}$.

Finite-dimensional distributions:
$\forall n \in \mathbb{N}$ and any finite subset $\{t_1, \ldots, t_n\} \subset T$ there is a system of random variables $X_{t_1}, \ldots, X_{t_n}$, with the joint distribution function

$$\mathrm{P}\left[X_{t_1} \leq x_1, \ldots, X_{t_n} \leq x_n\right] = F_{t_1,\ldots,t_n}(x_1, \ldots, x_n)$$

for all real $x_1, \ldots, x_n$.

A system of distribution functions is said to be consistent, if

- $F_{t_{i_1},\ldots,t_{i_n}}(x_{i_1}, \ldots, x_{i_n}) = F_{t_1,\ldots,t_n}(x_1, \ldots, x_n)$ for any permutation $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$ (symmetry)
- $\lim_{x_n \to \infty} F_{t_1,\ldots,t_n}(x_1, \ldots, x_n) = F_{t_1,\ldots,t_{n-1}}(x_1, \ldots, x_{n-1})$ (consistency)

A system of **characteristic functions**

The characteristic function of a random vector $\mathbf{X} = (X_1, \dots, X_n)$ is

$$\varphi_{\mathbf{X}}(\mathbf{u}) := \mathrm{E} e^{\mathrm{i}\mathbf{u}^\top \mathbf{X}} = \mathrm{E} e^{\mathrm{i} \sum_{j=1}^n u_j X_j}, \quad \mathbf{u} = (u_1, \dots, u_n)^\top \in \mathbb{R}_n$$

$$F_{t_1,\dots,t_n}(x_1, \dots, x_n) \leftrightarrow \varphi_{X_{t_1},\dots,X_{t_n}}(u_1, \dots, u_n) := \varphi(u_1, \dots, u_n)$$

Consistent system of characteristic functions:

- symmetry:

$$\varphi(u_{i_1}, \dots, u_{i_n}) = \varphi(u_1, \dots, u_n)$$

  for any permutation $(i_1, \dots, i_n)$ of $(1, \dots, n)$,

- consistency:

$$\lim_{u_n \to 0} \varphi_{X_{t_1},\dots,X_{t_n}}(u_1, \dots, u_n) = \varphi_{X_{t_1},\dots,X_{t_{n-1}}}(u_1, \dots, u_{n-1}).$$

Daniell-Kolmogorov theorem

For any stochastic process there exists a consistent system of distribution functions and,

Theorem 1:
Let $\{F_{t_1,\ldots,t_n}(x_1,\ldots,x_n)\}$ be a consistent system of distribution functions. Then there exists a stochastic process $\{X_t,\, t \in T\}$ such that for any $n \in \mathbb{N}$, any $t_1,\ldots,t_n \in T$ and any real $x_1,\ldots,x_n$ it holds

$$\mathrm{P}\left[X_{t_1} \le x_1,\ldots,X_{t_n} \le x_n\right] = F_{t_1,\ldots,t_n}(x_1,\ldots,x_n).$$

Proof: Štěpán (1987), Theorem I.10.3.

Definition:

A complex-valued random variable X is defined by $X = Y + \mathrm{i}Z$, where Y a Z are real random variables, $\mathrm{i} = \sqrt{-1}$.

The mean value of a complex-valued random variable $X = Y + \mathrm{i}Z$ is defined by

$$\mathrm{E}X = \mathrm{E}Y + \mathrm{i}\mathrm{E}Z$$

provided the mean values $\mathrm{E}Y$ and $\mathrm{E}Z$ exist.

The variance of a complex-valued random variable $X = Y + \mathrm{i}Z$ is defined by

$$\operatorname{var} X := \mathrm{E}\left[(X - \mathrm{E}X)(\overline{X} - \overline{\mathrm{E}X})\right] = \mathrm{E}|X - \mathrm{E}X|^2 \geq 0$$

provided the second moments of random variables Y and Z exist.

Definition:

A complex-valued stochastic process is a family of complex-valued random variables on $(\Omega, \mathcal{A}, \mathrm{P})$.

Definition:
Let $\{X_t,\ t \in T\}$ be a stochastic process such that $\mathrm{E}X_t$ exists for all $t \in T$. Then the function $\mu_t = \mathrm{E}X_t$ defined on T is called the mean value of the process $\{X_t,\ t \in T\}$. We say that the process is centred if its mean value is zero for all $t \in T$.

Definition:
Let $\{X_t,\ t \in T\}$ be a process with finite second moments, i.e., $\mathrm{E}|X_t|^2 < \infty$, $\forall t \in T$. Then a (complex-valued) function defined on $T \times T$ by

$$R(s,t) = \mathrm{E}\left[(X_s - \mu_s)(\overline{X}_t - \overline{\mu}_t)\right]$$

is called the autocovariance function of the process $\{X_t,\ t \in T\}$. The value $R(t,t)$ is the variance of the process at time t.

Definition:

Autocorrelation function of the process $\{X_t,\ t \in T\}$ with positive variances is defined by

$$r(s,t) = \frac{R(s,t)}{\sqrt{R(s,s)}\sqrt{R(t,t)}}, \quad s,t \in T.$$

Definition:

Stochastic process $\{X_t,\ t \in T\}$ is called Gaussian, if for any $n \in \mathbb{N}$ and $t_1, \dots, t_n \in T$, the vector $(X_{t_1}, \dots, X_{t_n})^\top$ is normally distributed $\mathcal{N}_n(\mathbf{m_t}, \mathbf{V_t})$, where $\mathbf{m_t} = (\mathrm{E}X_{t_1}, \dots, \mathrm{E}X_{t_n})^\top$ and

$$\mathbf{V_t} = \begin{pmatrix} \mathrm{var} X_{t_1} & \mathrm{cov}(X_{t_1}, X_{t_2}) & \dots & \mathrm{cov}(X_{t_1}, X_{t_n}) \\ \mathrm{cov}(X_{t_2}, X_{t_1}) & \mathrm{var} X_{t_2} & \dots & \mathrm{cov}(X_{t_2}, X_{t_n}) \\ \dots & \dots & \ddots & \dots \\ \mathrm{cov}(X_{t_n}, X_{t_1}) & \mathrm{cov}(X_{t_n}, X_{t_2}) & \dots & \mathrm{var} X_{t_n} \end{pmatrix}.$$

Definition:

Stochastic process $\{X_t,\ t \in T\}$ is said to be strictly stationary, if for any $n \in \mathbb{N}$, for any $x_1, \ldots, x_n$ real and for any $t_1, \ldots, t_n$ a h such that $t_k \in T, t_k + h \in T, 1 \leq k \leq n$,

$$F_{t_1,\ldots,t_n}(x_1, \ldots, x_n) = F_{t_1+h,\ldots,t_n+h}(x_1, \ldots, x_n).$$

Definition:

Stochastic process $\{X_t,\ t \in T\}$ with finite second moments is said to be weakly stationary or second order stationary, if its mean value is constant, $\mu_t = \mu$, $\forall t \in T$ and if its autocovariance function $R(s,t)$ is a function of $s - t$, only. If only the latter condition is satisfied, the process is called covariance stationary.

Autocovariance function of weakly stationary process:

$$R(t) := R(t,0),\ t \in T,$$

(function of one variable).

Autocorrelation function in such case:

$$r(t) = \frac{R(t)}{R(0)}.$$

Theorem 2:

Strictly stationary stochastic process $\{X_t,\ t \in T\}$ with finite second moments is also weakly stationary.

Proof:

$\{X_t,\ t \in T\}$ strictly stationary $\Rightarrow$ X_t are equally distributed for all $t \in T$ and thus with the mean value

$$\mathrm{E}X_t = \mathrm{E}X_{t+h},\ \forall t \in T,\ \forall h:\ t+h \in T$$

especially, for $h = -t$: $\mathrm{E}X_t = \mathrm{E}X_0 = \text{const}$

Proof of Theorem 2, continued

Similarly, (X_t, X_s) are equally distributed and

$$\mathrm{E}\,[X_t\, X_s] = \mathrm{E}\,[X_{t+h}\, X_{s+h}] \;\forall s, t \in T, \forall h: \; s+h \in T, \; t+h \in T$$

especially, for $h = -t : \mathrm{E}\,[X_t X_s] = \mathrm{E}\,[X_0 X_{s-t}]$
is a function of $s - t$. $\square$

Example:

$\{X_t,\ t \in T\}$ - a sequence of iid random variables with a distribution function F

$$\begin{aligned}
F_{t_1,\dots,t_n}(x_1,\dots,x_n) &= \mathsf{P}\left[X_{t_1} \le x_1,\dots,X_{t_n} \le x_n\right] = \\
&= \prod_{i=1}^{n} \mathsf{P}\left[X_{t_i} \le x_i\right] = \prod_{i=1}^{n} F(x_i), \\
F_{t_1+h,\dots,t_n+h}(x_1,\dots,x_n) &= \mathsf{P}\left[X_{t_1+h} \le x_1,\dots,X_{t_n+h} \le x_n\right] = \\
&= \prod_{i=1}^{n} \mathsf{P}\left[X_{t_i+h} \le x_i\right] = \prod_{i=1}^{n} F(x_i),
\end{aligned}$$

$\Rightarrow \{X_t,\ t \in T\}$ is strictly stationary.

Example:

$\{X_t,\ t \in \mathbb{Z}\}$ - a sequence defined by

$$X_t = (-1)^t X,$$

where X is a random variable:

$$X = \begin{cases} -\frac{1}{4} & \text{with probability } \frac{3}{4}, \\ \frac{3}{4} & \text{with probability } \frac{1}{4}. \end{cases}.$$

Then $\{X_t,\ t \in \mathbb{Z}\}$ is weakly stationary, since

$$\begin{aligned} \mathrm{E} X_t &= 0, \\ \operatorname{var} X_t &= \sigma^2 = \frac{3}{16}, \\ R(s,t) &= \sigma^2(-1)^{s+t} = \sigma^2(-1)^{s-t}, \end{aligned}$$

but it is not strictly stationary (variables X a $-X$ are not equally distributed).

Theorem 3:
A weakly stationary Gaussian process $\{X_t,\ t \in T\}$ is also strictly stationary.

Proof:
Weak stationarity of the process $\{X_t, t \in T\}$ implies
$\mathrm{E}X_t = \mu,\ \mathrm{cov}\,(X_t, X_s) = R(t-s) = \mathrm{cov}\,(X_{t+h}, X_{s+h}), t, s \in T$,
thus

$$\mathrm{E}(X_{t_1}, \ldots, X_{t_n}) = \mathrm{E}(X_{t_1+h}, \ldots, X_{t_n+h}) = (\mu, \ldots, \mu) := \boldsymbol{\mu}$$

$$\mathrm{var}\,(X_{t_1}, \ldots, X_{t_n}) = \mathrm{var}\,(X_{t_1+h}, \ldots, X_{t_n+h}) := \boldsymbol{\Sigma}$$

$$\boldsymbol{\Sigma} = \begin{pmatrix} R(0) & R(t_2 - t_1) & \dots & R(t_n - t_1) \\ R(t_2 - t_1) & R(0) & \dots & R(t_n - t_2) \\ \vdots & \vdots & \ddots & \vdots \\ & & \dots & R(0) \end{pmatrix}.$$

Since the normal distribution is uniquely defined by the mean value vector and the variance matrix,
$(X_{t_1}, \dots, X_{t_n}) \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, and
$(X_{t_1+h}, \dots, X_{t_n+h}) \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \Rightarrow \{X_t,\ t \in T\}$ is strictly stationary. $\square$

Properties of autocovariance function

Theorem 4:
Let $\{X_t,\ t \in T\}$ be a process with finite second moments. Then its autocovariance function satisfies

$$R(t,t) \geq 0,$$
$$|R(s,t)| \leq \sqrt{R(s,s)}\sqrt{R(t,t)}.$$

Proof:
The first assertion follows from the definition of the variance. The second one follows from the Schwarz inequality, since

$$|R(s,t)| = |\mathrm{E}(X_s - \mathrm{E}X_s)(\overline{X_t} - \overline{\mathrm{E}X_t})| \leq \mathrm{E}|(X_s - \mathrm{E}X_s)(\overline{X_t} - \overline{\mathrm{E}X_t})|$$
$$\leq (\mathrm{E}|X_s - \mathrm{E}X_s|^2)^{\frac{1}{2}}(\mathrm{E}|X_t - \mathrm{E}X_t|^2)^{\frac{1}{2}} = \sqrt{R(s,s)}\sqrt{R(t,t)}$$

□

Thus, for weakly stationary process $R(0) \geq 0$ a $|R(t)| \leq R(0)$.

Definition:
Let $f(s,t)$ be a complex-valued function defined on $T \times T$, $T \subset \mathbb{R}$. We say that f is positive semidefinite, sometimes: non-negative definite, if $\forall n \in \mathbb{N}$, any complex numbers $c_1, \dots, c_n$ and any $t_1, \dots, t_n \in T$ it holds

$$\sum_{j=1}^{n} \sum_{k=1}^{n} c_j \overline{c_k} f(t_j, t_k) \geq 0.$$

We say that a complex-valued function g on T is positive semidefinite, if $\forall n \in \mathbb{N}$, any complex numbers $c_1, \dots, c_n$ and any $t_1, \dots, t_n \in T$, such that $t_j - t_k \in T$, it holds

$$\sum_{j=1}^{n} \sum_{k=1}^{n} c_j \overline{c_k} g(t_j - t_k) \geq 0.$$

Definition:

We say that a complex-valued function f on $T \times T$ is Hermitian, if $f(s,t) = \overline{f(t,s)}$ $\forall s,t \in T$. A complex-valued function g of one variable is called Hermitian, if $g(-t) = \overline{g(t)}$ $\forall t \in T$.

Theorem 5:

Any positive semidefinite function is also Hermitian.

Proof:

Use the definition of positive semidefiniteness and for $n = 1$ choose $c_1 = 1$; for $n = 2$ choose $c_1 = 1, c_2 = 1$ and $c_1 = 1, c_2 = i (= \sqrt{-1})$. □

Remark:

A positive semidefinite real-valued function f on $T \times T$, is symmetric, i.e., $f(s,t) = f(t,s)$ for all $s,t \in T$. A positive semidefinite real-valued function g on T is symmetric, i.e, $g(t) = g(-t)$ for all $t \in T$.

Theorem 6:
Let $\{X_t,\ t \in T\}$ be a process with finite second moments. Then its autocovariance function is positive semidefinite on $T \times T$.

Proof:
Suppose wlog that the process is centred. Then for any $n \in \mathbb{N}$, complex constants $c_1, \dots, c_n$ and $t_1, \dots, t_n \in T$

$$0 \le \mathrm{E}\left|\sum_{j=1}^{n} c_j X_{t_j}\right|^2 = \mathrm{E}\left[\sum_{j=1}^{n} c_j X_{t_j} \overline{\sum_{k=1}^{n} c_k X_{t_k}}\right]$$

$$= \sum_{j=1}^{n}\sum_{k=1}^{n} c_j \overline{c_k} \mathrm{E}(X_{t_j}\overline{X_{t_k}}) = \sum_{j=1}^{n}\sum_{k=1}^{n} c_j \overline{c_k} R(t_j, t_k).$$

□

Theorem 7:
To any positive semidefinite function R on $T \times T$ there exists a stochastic process $\{X_t,\ t \in T\}$ with finite second moments such that its autocovariance function is R.

Proof:
The proof will be given for real-valued function R, only. For the proof in a complex case see, e.g., Loève (1955), Chap. X, Par. 34.
Since R is positive semidefinite, then for any $n \in \mathbb{N}$ and any real $t_1, \dots, t_n \in T$ the matrix

$$\mathbf{V_t} = \begin{pmatrix} R(t_1,t_1) & R(t_1,t_2) & \dots & R(t_1,t_n) \\ R(t_2,t_1) & R(t_2,t_2) & \dots & R(t_2,t_n) \\ \dots & \dots & \dots & \dots \\ R(t_n,t_1) & R(t_n,t_2) & \dots & R(t_n,t_n) \end{pmatrix}$$

is positive semidefinite.

Proof of Theorem 7, continued

Function

$$\varphi(\mathbf{u}) = \exp\left\{-\frac{1}{2}\mathbf{u}^\top \mathbf{V_t}\mathbf{u}\right\}, \quad \mathbf{u} \in \mathbb{R}^n$$

is the characteristic function of the normal distribution $\mathcal{N}_n(\mathbf{0}, \mathbf{V_t})$. In this way, $\forall n \in \mathbb{N}$ and any real $t_1, \ldots, t_n \in T$ we get a consistent system of characteristic functions. The corresponding system of the distribution functions is also consistent. Thus according to Daniell-Kolmogorov theorem (Theorem 1) there exists a Gaussian stochastic process, covariances of which are given the values of the function $R(s,t)$; hence, function R is the autocovariance function of this process.

$\square$

Example:

Decide whether function $\cos t$, $t \in T = (-\infty, \infty)$ is an autocovariance function of a stochastic process.

Solution:

It suffices to show, that $\cos t$ is a positive semidefinite function. Consider $n \in \mathbb{N}$, $c_1, \ldots, c_n \in \mathbb{C}$ a $t_1, \ldots, t_n \in \mathbb{R}$. Then we have

$$\sum_{j=1}^{n} \sum_{k=1}^{n} c_j \bar{c}_k \cos(t_j - t_k) = \sum_{j=1}^{n} \sum_{k=1}^{n} c_j \bar{c}_k (\cos t_j \cos t_k + \sin t_j \sin t_k)$$

$$= \left| \sum_{j=1}^{n} c_j \cos t_j \right|^2 + \left| \sum_{k=1}^{n} c_k \sin t_k \right|^2 \geq 0.$$

Function $\cos t$ is positive semidefinite, and according to Theorem 6 there exists a (Gaussian) stochastic process $\{X_t,\ t \in T\}$, autocovariance function of which is $R(s,t) = \cos(s-t)$.

Theorem 8:
The sum of two positive semidefinite functions is a positive semidefinite function.

Proof:
It follows from the definition of the positive semidefinite function. If f and g are positive semidefinite and $h = f + g$, then for any $n \in \mathbb{N}$, complex $c_1, \ldots, c_n$ and $t_1, \ldots, t_n \in T$

$$\sum_{j=1}^{n} \sum_{k=1}^{n} c_j \overline{c}_k h(t_j, t_k) = \sum_{j=1}^{n} \sum_{k=1}^{n} c_j \overline{c}_k [f(t_j, t_k) + g(t_j, t_k)]$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{n} c_j \overline{c}_k f(t_j, t_k) + \sum_{j=1}^{n} \sum_{k=1}^{n} c_j \overline{c}_k g(t_j, t_k) \geq 0.$$

□

Corollary:

Sum of two autocovariance functions is an autocovariance function of a stochastic process with finite second moments.

Proof:

It follows from Theorems 6 - 8. □

Theorem 9:

The real part of an autocovariance function is an autocovariance function. The imaginary part is an autocovariance function (ACF)if and only if it is zero.

Proof:

Wlog, we prove the assertion for centred processes only. If $X_t = Y_t + \mathrm{i}Z_t$ is complex with zero mean, then $\mathrm{E}Y_t = \mathrm{E}Z_t = 0$ and $R(s,t) = \mathrm{E}X_s\overline{X}_t = \mathrm{E}\left[(Y_s + \mathrm{i}Z_s)(Y_t - \mathrm{i}Z_t)\right] = \mathrm{E}Y_sY_t + \mathrm{E}Z_sZ_t + \mathrm{i}(\mathrm{E}Z_sY_t - \mathrm{E}Y_sZ_t)$. The real part is an autocovariance function according to the previous Corollary. If imaginary part is zero, it is an ACF. If the imaginary part is ACF, then, since for $s = t$ (the variance in s) it is zero, the imaginary part must be zero. □