Steady equations describing flow of chemically reacting heat conducting compressible mixtures

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Steady system – formulation I

We consider the following system of PDEs:

$$
\begin{aligned}
\operatorname{div}(\varrho\mathbf{u}) &= 0,\\
\operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u}) - \operatorname{div}\mathbb{S} + \nabla\pi &= \varrho\mathbf{f},\\
\operatorname{div}(\varrho E\mathbf{u}) + \operatorname{div}(\pi\mathbf{u}) + \operatorname{div}\mathbf{Q} - \operatorname{div}(\mathbb{S}\mathbf{u}) &= \varrho\mathbf{f}\cdot\mathbf{u},\\
\operatorname{div}(\varrho Y_k\mathbf{u}) + \operatorname{div}\mathbf{F}_k &= m_k\omega_k, \quad k\in\{1,\dots,n\}.
\end{aligned}
\tag{1}
$$

Here, ϱ is the total density, $\mathbf{u}$ is the mean velocity field, $\mathbb{S}$ is the mean stress tensor, π is the pressure, E is the specific total energy, $E = \frac{1}{2}|\mathbf{u}|^2 + e$ with e the specific internal energy, $\mathbf{Q}$ is the heat flux, $Y_k = \varrho_k/\varrho$ are the mass fractions, ϱ_k is the density of the k-th constituent, $\mathbf{F}_k$, $k = 1,\dots,n$, are the multicomponent diffusion fluxes, ϑ is the temperature, m_k are the molar masses, and ω_k the chemical source terms, $k = 1, 2, \dots, n$.

Steady system – formulation II

We consider the following boundary conditions on $\partial\Omega$:

$$\mathbf{u} \cdot \mathbf{n} = 0, \qquad (\mathbb{S}\mathbf{n} + f\mathbf{u}) \times \mathbf{n} = \mathbf{0} \tag{2}$$

$(f \geq 0)$ or

$$\mathbf{u} = \mathbf{0}, \tag{3}$$

and for the temperature and diffusion flux

$$\begin{gathered} \mathbf{F}_k \cdot \mathbf{n} = 0, \\ -\mathbf{Q} \cdot \mathbf{n} + L(\vartheta - \vartheta_0) = 0. \end{gathered} \tag{4}$$

We prescribe:

$$\int_\Omega \varrho \,\mathrm{d}x = M > 0. \tag{5}$$

Main assumptions I

Pressure:

We have

$$\pi(\varrho,\vartheta) = \pi_c(\varrho) + \pi_m(\varrho,\vartheta) = \varrho^\gamma + \sum_{k=1}^{n} \frac{\varrho Y_k}{m_k}\vartheta.$$

Stress tensor:

$$\mathbb{S} = \mu(\vartheta)\Big[\nabla\mathbf{u} + (\nabla\mathbf{u})^t - \frac{2}{3}\operatorname{div}\mathbf{u}\,\mathbb{I}\Big] + \nu(\vartheta)\operatorname{div}\mathbf{u}\,\mathbb{I},$$

with the viscosity coefficients

$$\mu(\vartheta) \sim (1+\vartheta), \qquad 0 \le \nu(\vartheta) \le C(1+\vartheta).$$

Heat flux:

$$\mathbf{Q} = \sum_{k=1}^{n} h_k \mathbf{F}_k + \mathbf{q}$$

with

$$\mathbf{q} = -\kappa(\vartheta)\nabla\vartheta, \qquad \kappa(\vartheta) = (1+\vartheta^m)$$

and the partial enthalpies $h_k = c_{pk}\vartheta = (1/m_k + c_{vk})\vartheta$.

Main assumptions II

Internal energy:
We have

$$e(\varrho,\vartheta) = e_c(\varrho) + e_m(\vartheta,\vec{Y}) = \frac{\varrho^{\gamma-1}}{\gamma-1} + \vartheta\sum_{k=1}^{n} c_{vk} Y_k,$$

$\vec{Y} = (Y_1, Y_2, \dots, Y_n)$.

Chemical production rate:
We have

$$\omega_k(\vartheta,\vec{Y}) \geq -CY_k^r, \quad C, r > 0$$

ω_k bounded functions of ϑ, $\vec{Y}$. Moreover, $\sum_{k=1}^{n} m_k\omega_k = 0$.

Main assumptions III

Diffusion fluxes:

$$\mathbf{F}_k = -Y_k \sum_{l=1}^{n} D_{kl}\mathbf{d}_l$$

with

$$\mathbf{d}_k = \nabla\left(\frac{p_k}{\pi_m}\right) + \left(\frac{p_k}{\pi_m} - \frac{\varrho_k}{\varrho}\right)\nabla \log \pi_m = \frac{\nabla p_k}{\pi_m} - Y_k \frac{\nabla \pi_m}{\pi_m},$$

$$|Y_k D_{kl}| = |Y_k D_{kl}(\vartheta, \vec{Y}))| \leq C(\vec{Y})(1 + \vartheta^a).$$

The diffusion matrix $\mathbb{D}$ fulfills:

$$\begin{aligned} &\mathbb{D} = \mathbb{D}^t, \quad N(\mathbb{D}) = \mathbb{R}\vec{Y}, \quad R(\mathbb{D}) = \vec{Y}^\perp, \\ &\mathbb{D} \quad \text{is positive semidefinite over } \mathbb{R}^n \\ &\text{and positive definite over } (1, \dots, 1)^\perp. \end{aligned} \tag{6}$$

Therefore

$$\sum_{k=1}^{n} \mathbf{F}_k = \mathbf{0},$$

$$\delta\langle \mathbb{Y}^{-1}\vec{x}, \vec{x}\rangle \leq \langle \mathbb{D}\vec{x}, \vec{x}\rangle \quad \forall \vec{x} \in \vec{U}^\perp,$$

where $\mathbb{Y} = \mathrm{diag}\,(Y_1, \dots, Y_n)$.

Entropy I

The entropy is defined

$$s = \sum_{k=1}^{n} Y_k s_k$$

with

$$s_k = c_{vk} \log \vartheta - \frac{1}{m_k} \log \frac{\varrho Y_k}{m_k}.$$

Entropy balance:

$$\operatorname{div}(\varrho s \mathbf{u}) + \operatorname{div}\left(\frac{\mathbf{Q}}{\vartheta} - \sum_{k=1}^{n} \frac{g_k}{\vartheta} \mathbf{F}_k\right) = \sigma,$$

where $g_k = h_k - \vartheta s_k$, σ is the entropy production rate

$$\begin{aligned}
\sigma =& \frac{\mathbb{S} : \nabla \mathbf{u}}{\vartheta} - \frac{\mathbf{Q} \cdot \nabla \vartheta}{\vartheta^2} - \sum_{k=1}^{n} \mathbf{F}_k \cdot \nabla \left(\frac{g_k}{\vartheta}\right) - \frac{\sum_{k=1}^{n} m_k g_k \omega_k}{\vartheta} \\
=& \frac{\mathbb{S} : \nabla \mathbf{u}}{\vartheta} + \frac{\kappa |\nabla \vartheta|^2}{\vartheta^2} - \sum_{k=1}^{n} \frac{\mathbf{F}_k}{m_k} \cdot \nabla \log p_k - \frac{\sum_{k=1}^{n} m_k g_k \omega_k}{\vartheta}.
\end{aligned}$$

Entropy II

The Second Law of Thermodynamics dictates

$$\sigma \geq 0,$$

therefore we assume $-\sum_{k=1}^{n} m_k g_k \omega_k \geq 0$. Moreover,

$$-\sum_{k=1}^{n} \frac{\mathbf{F}_k}{m_k} \cdot \nabla \log p_k = \frac{\pi_m}{\varrho\vartheta} \sum_{k,l=1}^{n} D_{kl} \left[\frac{\nabla p_l}{\pi_m} - Y_l \frac{\nabla \pi_m}{\pi_m}\right]\left[\frac{\nabla p_k}{\pi_m} - Y_k \frac{\nabla \pi_m}{\pi_m}\right]$$
$$+ \frac{\pi_m}{\varrho\vartheta} \sum_{k,l=1}^{n} D_{kl} \left[\frac{\nabla p_l}{\pi_m} - Y_l \frac{\nabla \pi_m}{\pi_m}\right] Y_k \frac{\nabla \pi_m}{\pi_m} \geq 0,$$

as $\sum_{k=1}^{n} \mathbf{F}_k = \mathbf{0}$ implies

$$\sum_{k=1}^{n} Y_k D_{kl} = 0 \quad \forall l = 1, 2, \ldots, n.$$

V. Giovangigli: Multicomponent Flow Modelling 1999

Simplifications

We cannot read estimates of $\nabla \vec{Y}_k$ from the entropy inequality. One possibility was used in

P.B. Mucha, M.P., E. Zatorska: Heat-conducting, compressible mixtures with multicomponent diffusion: construction of a weak solution, SIMA 2015

It requires information on $\nabla \varrho$, the viscosity was density dependent with the BD entropy. Not available in the steady case.

Another possibility: take $m_1 = m_2 = \cdots = m_n = 1$. Then $\pi_m = \varrho\vartheta$, $\frac{\nabla p_l}{\pi_m} - Y_l \frac{\nabla \pi_m}{\pi_m} = \nabla Y_l$. Therefore

$$\mathbf{F}_k = -\sum_{l=1}^{n} Y_k D_{kl} \nabla Y_l$$

and

$$-\sum_{k=1}^{n} \frac{\mathbf{F}_k}{m_k} \cdot \nabla \log p_k = -\sum_{k=1}^{n} \mathbf{F}_k \cdot \Big(\frac{\nabla Y_k}{Y_k} + \frac{\nabla(\varrho\vartheta)}{\varrho\vartheta}\Big) = \sum_{k,l=1}^{n} D_{kl} \nabla Y_l \nabla Y_k \geq c|\nabla \vec{Y}|^2.$$

Definition of a weak solution I: $(\varrho, \mathbf{u}, \vartheta, \{\varrho_k\}_{k=1}^n)$

We look for functions $\varrho \geq 0$, $\int_\Omega \varrho \,\mathrm{d}x = M$, $\vartheta > 0$, $\varrho_k \geq 0$, $\varrho = \sum_{k=1}^n \varrho_k$, $\varrho_k = Y_k \varrho$ a.e. in Ω,

$$\begin{array}{c} \varrho \in L^\gamma(\Omega), \mathbf{u} \in W_0^{1,2}(\Omega) \\ \varrho|\mathbf{u}|^3 \in L^1(\Omega), \vec{Y} \in W^{1,2}(\Omega) \\ \vartheta^m \nabla\vartheta \in L^1(\Omega), \vartheta \in L^1(\partial\Omega) \end{array} \tag{7}$$

and the following identities hold

- the weak formulation of the continuity equation

$$\int_\Omega \varrho \mathbf{u} \cdot \nabla\psi \,\mathrm{d}x = 0 \tag{8}$$

holds for any test function $\psi \in C^\infty(\overline{\Omega})$;

- the weak formulation of the momentum equation

$$\int_\Omega \big(-\varrho\,(\mathbf{u} \otimes \mathbf{u}) : \nabla\boldsymbol{\varphi} + \mathbb{S} : \nabla\boldsymbol{\varphi}\big)\,\mathrm{d}x - \int_\Omega \pi \,\mathrm{div}\,\boldsymbol{\varphi}\,\mathrm{d}x = \int_\Omega \varrho\mathbf{f} \cdot \boldsymbol{\varphi}\,\mathrm{d}x \tag{9}$$

holds for any test function $\boldsymbol{\varphi} \in C_0^\infty(\Omega)$;

Definition of a weak solution II: $(\varrho, \mathbf{u}, \vartheta, \{\varrho_k\}_{k=1}^n)$

- the weak formulation of the species equations

$$-\int_\Omega Y_k \varrho \mathbf{u} \cdot \nabla \psi \, \mathrm{d}x - \int_\Omega \mathbf{F}_k \cdot \nabla \psi \, \mathrm{d}x = \int_\Omega \omega_k \psi \, \mathrm{d}x \tag{10}$$

holds for any test function $\psi \in C^\infty(\overline{\Omega})$ and for all $k = 1, \dots, n$;

- the weak formulation of the total energy balance

$$-\int_\Omega \left(\frac{1}{2}\varrho |\mathbf{u}|^2 + \varrho e\right) \mathbf{u} \cdot \nabla \psi \, \mathrm{d}x + \int_\Omega \kappa \nabla \vartheta \cdot \nabla \psi \, \mathrm{d}x - \int_\Omega \left(\sum_{k=1}^n h_k \mathbf{F}_k\right) \cdot \nabla \psi \, \mathrm{d}x$$
$$= \int_\Omega \varrho \mathbf{f} \cdot \mathbf{u} \psi \, \mathrm{d}x - \int_\Omega (\mathbb{S}\mathbf{u}) \cdot \nabla \psi \, \mathrm{d}x + \int_\Omega \pi \mathbf{u} \cdot \nabla \psi \, \mathrm{d}x - \int_\Omega L(\vartheta - \vartheta_0) \psi \, \mathrm{d}S \tag{11}$$

holds for any test function $\psi \in C^\infty(\overline{\Omega})$.

For the slip b.c. additional terms due to the b.c. appear (if $f \neq 0$), the velocity is required to have zero normal trace only and the test function in the momentum equation has zero normal trace only. (Important!)

Result I

Theorem (Giovangigli, Pokorný, Zatorska '15)

Let $\gamma > \frac{5}{3}$, $M > 0$, $m > 1$, $a < \frac{3m-2}{2}$. *Let* $\Omega \in C^2$. *Then there exists at least one weak solution to our problem above. Moreover,* $(\varrho, \mathbf{u})$ *is the renormalized solution to the continuity equation.*

V. Giovangigli, M. Pokorný M., E. Zatorska: On the steady flow of reactive gaseous mixture, Analysis, 2015.

Variational entropy solutions

We replace the the total energy balance by the entropy inequality and the global total energy balance (weak formulation of the total energy balance with test function $\psi \equiv 1$).

Entropy inequality

$$\int_\Omega \frac{\mathbb{S} : \nabla \mathbf{u}}{\vartheta} \psi \, \mathrm{d}x + \int_\Omega \kappa \frac{|\nabla \vartheta|^2}{\vartheta^2} \psi \, \mathrm{d}x - \int_\Omega \sum_{k=1}^n \frac{\omega_k \psi}{\vartheta} (c_{pk} - c_{vk} \log \vartheta + log Y_k) \, \mathrm{d}x$$
$$+ \int_\Omega \psi \sum_{k,l=1}^n D_{kl} \nabla Y_k \nabla Y_l \, \mathrm{d}x + \int_{\partial\Omega} \frac{L}{\vartheta} \vartheta_0 \psi \, \mathrm{d}S \leq \int \frac{\kappa \nabla \vartheta \cdot \nabla \psi}{\vartheta} \, \mathrm{d}x - \int_\Omega \varrho s \mathbf{u} \cdot \nabla \psi \, \mathrm{d}x$$
$$- \int_\Omega \sum_{k=1}^n c_{vk} \log \vartheta \mathbf{F}_k \cdot \nabla \psi \, \mathrm{d}x + \int_\Omega \sum_{k=1}^n \log Y_k \mathbf{F}_k \cdot \nabla \psi \, \mathrm{d}x + \int_{\partial\Omega} L \psi \, \mathrm{d}S$$

for any non-negative $\psi \in C^\infty(\overline{\Omega})$.

Global total energy balance

$$\int_{\partial\Omega} L(\vartheta - \vartheta_0) \psi \, \mathrm{d}S = \int_\Omega \varrho \mathbf{f} \cdot \mathbf{u} \, \mathrm{d}x$$

Changes for $f \neq 0$ as above in the global total energy balance.

Comments to the variational entropy solutions

The entropy balance has the form

$$\operatorname{div}(\varrho s\mathbf{u}) + \operatorname{div}\left(\frac{\mathbf{Q}}{\vartheta} - \sum_{k=1}^{n} \frac{g_k}{\vartheta}\mathbf{F}_k\right) = \sigma$$

with the entropy production rate

$$\sigma = \frac{1}{\vartheta}\mathbb{S} : \nabla\mathbf{u} - \frac{\mathbf{Q}\cdot\nabla\vartheta}{\vartheta^2} - \sum_{k=1}^{n} \mathbf{F}_k \cdot \nabla\left(\frac{g_k}{\vartheta}\right) - \frac{\sum_{k=1}^{n} g_k\omega_k}{\vartheta},$$

the Gibbs function

$$g_k = c_{pk}\vartheta - \vartheta s_k = c_{pk}\vartheta - \vartheta(c_{vk}\log\vartheta - \log\varrho - \log Y_k)$$

and the total entropy

$$s = \sum_{k=1}^{n} s_k Y_k.$$

We would have problems in the entropy inequality with terms containing $\log\varrho$ as the density may be equal to zero. But either the terms are of the form ϱs_k, or we may use in the definition that $\sum_{k=1}^{n}\omega_k = 0$, $\sum_{k=1}^{n}\mathbf{F}_k = \mathbf{0}$ and we do not consider these terms in the entropy inequality. The inequality instead of the equality is a consequence of the limit passages in the proof and the weak lower semicontinuity of some terms.

Result II (Dirichlet b.c.)

Theorem (Piasecki, Pokorný '16)

Let $\gamma > 1$, $M > 0$, $m > \max\{\frac{2}{3}, \frac{2}{3(\gamma-1)}\}$, $a < \frac{3m}{2}$. *Let* $\Omega \in C^2$. *Then there exists at least one variational entropy solution to our problem above. Moreover,* $(\varrho, \mathbf{u})$ *is the renormalized solution to the continuity equation.*
In addition, if $m > \max\{1, \frac{2\gamma}{3(3\gamma-4)}\}$, $\gamma > \frac{4}{3}$, $a < \frac{3m-2}{2}$, *then the solution is a weak solution in the sense above.*

T. Piasecki, M. Pokorný: Weak and variational entropy solutions to the system describing steady flow of a compressible reactive mixture, Nonlinear Analysis 2017.

Result III (Navier b.c.)

Theorem (Piasecki, Pokorný '17)

Let $\gamma > 1$, $M > 0$, $m > \max\{\frac{2}{3}, \frac{2}{3(\gamma-1)}\}$, $a < \frac{3m}{2}$, $\vartheta_0 \in L^1(\partial\Omega)$, $\vartheta_0 \geq K_0 > 0$ a.e. on $\partial\Omega$. Let $\Omega \in C^2$ be not axially symmetric. Then there exists at least one variational entropy solution to our problem. Moreover, $(\varrho, \mathbf{u})$ is the renormalized solution to the continuity equation.
In addition, if $m > 1$, $\gamma > \frac{5}{4}$, $a < \frac{3m-2}{2}$, then the solution is a weak solution in the sense above.
If Ω is axially symmetric, let $f > 0$. Then there exists at least one variational entropy solution to our problem. In addition, if $\gamma > \frac{5}{4}$, $m > 1$, $m > \frac{16\gamma}{15\gamma-16}$ (if $\gamma \in (\frac{5}{4}, \frac{4}{3}]$) or $m > \frac{18-6\gamma}{9\gamma-7}$ (if $\gamma \in (\frac{4}{3}, \frac{5}{3})$) then the solution is a weak solution in the sense above.

T. Piasecki, M. Pokorný: On steady solutions to a model of chemically reacting heat conducting compressible mixture with slip boundary conditions, Contemporary Mathematics 2018.

Approximation

Take $\delta > 0$ (regularization of pressure and temperature), $\varepsilon > 0$ (parabolic regularization of the continuity equation), $\lambda > 0$ (regularization of the fluxes), $\eta > 0$ (regularization of some constitutive relations), and $N \in \mathbb{N}$ (Galerkin approximation of the velocity).
We let subsequently $N \to \infty$, $\eta \to 0^+$, $\lambda \to 0^+$, and $\varepsilon \to 0^+$.

- We read the estimates either from the entropy inequality or from the energy equality, the limit passages are standard.
- For $\lambda > 0$ we read estimates on Y_k from additional terms.
- As soon as $\lambda = 0$, we know that $\sum_{k=1}^{n} Y_k = 1$ and we read estimates on Y_k from the diffusion matrix term.

Limit passage I ($\delta \to 0^+$)

For simplicity, we consider only weak compactness of the solutions. We have the following estimates independent of δ:
From the total energy balance with constant test function

$$f\|\mathbf{u}_\delta\|_{2,\partial\Omega}^2 + \|\vartheta_\delta\|_{1,\partial\Omega} \leq C\Big(1 + \Big|\int_\Omega \varrho_\delta \mathbf{u}_\delta \cdot \mathbf{f}\,\mathrm{d}x\Big|\Big). \tag{12}$$

From the entropy inequality with constant test function:

$$\|\nabla \vec{Y}_\delta\|_2^2 + \|\nabla \vartheta_\delta^{\frac{m}{2}}\|_2^2 + \|\nabla \mathbf{u}_\delta\|_2^2 + \|\vartheta_\delta^{-1}\|_{1,\partial\Omega} \leq C. \tag{13}$$

We have: $\sum_{k=1}^n Y_k = 1$ and $0 \leq (Y_k)_\delta \leq 1,\ k = 1, 2, \ldots, n.$

Estimates of the density I (Dirichlet b.c.)

To get the missing density estimates we use the local estimates of the pressure based on the test function for the momentum equation

$$\frac{x - x_0}{|x - x_0|^\alpha}.$$

We may use it with suitable cut-off function if x_0 is far from the boundary to get

$$\int_{B_{R_0}(x_0)} \frac{\pi(\varrho_\delta, \vartheta_\delta)}{|x - x_0|^\alpha}\, \mathrm{d}x \tag{14}$$
$$\leq C\big(1 + \|\pi(\varrho_\delta, \vartheta_\delta)\|_1 + \|\mathbf{u}_\delta\|_{1,2}(1 + \|\vartheta_\delta\|_{3m}) + \|\varrho_\delta |\mathbf{u}_\delta|^2\|_1\big),$$

provided

$$\alpha < \min\left\{\frac{3m-2}{2m}, 1\right\}. \tag{15}$$

For $x_0 \in \partial\Omega$ we apply

$$\varphi^1(x) = d(x)\nabla d(x)(d(x) + |x - x_0|^a)^{-\alpha} \tag{16}$$

with $d(x)$ a regularized distance function. It yields the same estimate as above provided $\alpha < \frac{9m-6}{9m-2}$.

Estimates of the density II (Dirichlet b.c.)

For $x_0 \notin \partial\Omega$, but close to $\partial\Omega$ ($\mathrm{dist}\,\{x_0, \partial\Omega\} = 5\varepsilon$) we use

$$\varphi = K\varphi^1 + \varphi^2,$$

where K is a suitably chosen positive constant and

$$\varphi^2(x) = \begin{cases} \frac{x-x_0}{|x-x_0|^\alpha}\left(1 - \frac{1}{2^{\frac{\alpha}{2}}}\right), & |x-x_0| < \varepsilon, \\ (x-x_0)\left(\frac{1}{|x-x_0|^{\frac{\alpha}{2}}} - \frac{1}{(|x-x_0|+\varepsilon)^{\frac{\alpha}{2}}}\right), & |x-x_0| > \varepsilon, d(x) > \varepsilon, \\ (x-x_0)\left(\frac{1}{|x-x_0|^{\frac{\alpha}{2}}} - \frac{1}{(|x-x_0|+d(x))^{\frac{\alpha}{2}}}\right), & |x-x_0| > \varepsilon, d(x) \le \varepsilon. \end{cases} \tag{17}$$

It leads to the same result as above under the same restrictions. These estimates can be then transformed into the integrability estimate of the velocity and kinetic energy under some restrictions on the parameters γ and m.

Estimates of the density III (Navier b.c.)

For x_0 far from the boundary we proceed as above. To simplify the idea, let us assume that we deal with the part of boundary of Ω which is flat and is described by $x_3 = 0$, i.e. $\mathbf{n} = (0, 0, -1)$ and $\boldsymbol{\tau}_1 = (1, 0, 0)$, $\boldsymbol{\tau}_2 = (0, 1, 0)$ the tangent vectors. The general case can be studied using the standard technique of flattening the boundary. First let x_0 lies on the boundary of Ω, i.e. $(x_0)_3 = 0$. We use as the test function in the approximate momentum equation

$$\mathbf{w}(x) = \mathbf{v}(x - x_0),$$

where

$$\mathbf{v}(x) = \frac{1}{|x|^\alpha}(x_1, x_2, x_3) = (x \cdot \boldsymbol{\tau}_1)\boldsymbol{\tau}_1 + (x \cdot \boldsymbol{\tau}_2)\boldsymbol{\tau}_2 + ((0, 0, x_3 - z(x')) \cdot \mathbf{n})\mathbf{n}, \quad x_3 \geq 0.$$

If x_0 is close to the boundary but not on the boundary, i.e. $(x_0)_3 > 0$, but small, we lose control of some terms for $0 < x_3 < (x_0)_3$. In this case, we must modify the test functions. We first consider

$$\mathbf{v}^1(x) = \begin{cases} \frac{1}{|x - x_0|^\alpha}\big((x - x_0)_1, (x - x_0)_2, (x - x_0)_3\big), & x_3 \geq \frac{(x_0)_3}{2}, \\ \frac{1}{|x - x_0|^\alpha}\Big((x - x_0)_1, (x - x_0)_2, 4(x - x_0)_3 \frac{x_3^2}{|(x - x_0)_3|^2}\Big), & 0 < x_3 < \frac{(x_0)_3}{2}. \end{cases}$$

Estimates of the density IV (Navier b.c.)

Nonetheless, we still miss control of some terms from the convective term, more precisely of those, which contain at least one velocity component u_3, however, only close to the boundary, i.e. for $x_3 < (x_0)_3/2$. Hence we further consider

$$\mathbf{v}^2(x) = \begin{cases} \dfrac{(0,0,x_3)}{(x_3 + |x-x_0||\ln|x-x_0||^{-1})^\alpha}, & |x-x_0| \le 1/K, \\ \dfrac{(0,0,x_3)}{(x_3 + 1/K|\ln K|^{-1})^\alpha}, & |x-x_0| > 1/K \end{cases}$$

for K sufficiently large (but fixed, independently of the distance of x_0 from $\partial\Omega$). Both functions have zero normal trace, belong to $W^{1,q}(\Omega)$ and their norms are bounded uniformly provided $1 \le q < \frac{3}{\alpha}$. We finally use as the test function in the approximate momentum balance

$$\varphi = \mathbf{v}^1(x) + K_1\mathbf{v}^2(x) \tag{18}$$

with K_1 suitably chosen (large).
More details on both cases can be found in

P.B. Mucha, M. Pokorný M., E. Zatorska: Existence of stationary weak solutions for the heat conducting fluids, Handbook of Mathematical Analysis in Mechanics of Viscous Fluids (eds. Y. Giga, A. Novotný), Springer, 2017.

Limit passage II ($\delta \to 0^+$)

We finish the proof of the strong convergence of the density and thus of the main result using the standard machinery known for the steady compressible Navier–Stokes–Fourier system, see e.g. the reference above. We either pass to the limit only in the entropy inequality and get the variational entropy solutions, or, if we have enough information, i.e. more restrictions on the parameters, we may pass to the limit also in the weak formulation of the total energy balance. The main difference consists in the fact whether or not we can estimate the kinetic energy in $L^s(\Omega)$ for some $s > \frac{6}{5}$. This can be verified for $m > \max\{1, \frac{2\gamma}{3(3\gamma-4)}\}$ and $\gamma > \frac{4}{3}$, while we can only verify $s > 1$ for $\gamma > 1$ and $m > \max\{\frac{2}{3}, \frac{2\gamma}{9(\gamma-1)}\}$.

The standard machinery to get the strong convergence of the density based on the effective viscous flux identity and control of oscillation defect measure leads to further restriction $m > \frac{2}{3(\gamma-1)}$ and the limit passage in the total energy balance requires $a < \frac{3m-2}{2}$ while in the entropy inequality only $a < \frac{3m}{2}$. Similarly we proceed for the Navier boundary conditions.

Limit passage III ($\delta \to 0^+$)

The use of the Bogovskii-type estimates only

$$\|\varrho_\delta\|_{\gamma+\alpha} \leq C, \qquad 0 < \alpha \leq \min\{2\gamma - 3, \frac{3m-2}{3m+2}\gamma\},$$

leads to immediate restriction $\gamma > \frac{3}{2}$. Further restriction due to the total energy balance

$$\int_\Omega \varrho_\delta |\mathbf{u}_\delta|^3 \,\mathrm{d}x \leq C$$

leads to the result in the paper by V. Giovangigli, M.P. and E. Zatorska.

Model DDGG I

In order to avoid the restriction of the same molar masses, we consider another model which is due to

W. Dreyer, P.-E. Druet, P. Gajewski, C. Guhlke: Existence of weak solutions for improved Nernst–Planck–Poisson models of compressible reacting electrolytes, Preprint WIAS 2016.

We add the thermal effects, for simplicity neglect electrostatic field and chemical reactions on the boundary. The model reads

$$
\begin{aligned}
\operatorname{div}(\varrho_i \mathbf{v} + \mathbf{J}_i) &= r_i, \qquad i = 1, 2, \dots, N \\
\operatorname{div}(\sigma \mathbf{v} \otimes \mathbf{v} - \mathbb{S}) + \nabla p &= \sigma \mathbf{b} \\
\operatorname{div}(\sigma E \mathbf{v} + p\mathbf{v} + \mathbf{Q} - \mathbb{S}\mathbf{v}) &= \sigma \mathbf{b} \cdot \mathbf{v}
\end{aligned}
\tag{19}
$$

with the same boundary conditions as above

$$
\begin{aligned}
(\mathbb{S}\mathbf{n}) \cdot \boldsymbol{\tau} + \alpha_1 \mathbf{v} \cdot \boldsymbol{\tau} &= 0, \qquad \mathbf{v} \cdot \mathbf{n} = 0 \\
\mathbf{J}_i \cdot \mathbf{n} &= 0, \qquad i = 1, 2, \dots, N \\
-\mathbf{Q} \cdot \mathbf{n} + \alpha_2(\theta - \Theta_0) &= 0.
\end{aligned}
\tag{20}
$$

Above, $\sigma = \varrho_1 + \varrho_2 + \cdots + \varrho_N$ the rest is similar. We use slightly different thermodynamic concept:

Model DDGG II

We assume that all thermodynamic quantities as the chemical potentials $\{\mu_i\}_{i=1}^N$, specific internal energy e, specific entropy s and the pressure p are derived from the Helmholtz free energy

$$h = \sigma\psi(\vec{\varrho}, \theta)$$

as follows

$$\begin{aligned}
\mu_i &= \frac{\partial(\sigma\psi)}{\partial\varrho_i}, \qquad i = 1, 2, \dots, N \dots \text{ chemical potential} \\
\sigma e &= \sigma\psi - \theta\frac{\partial}{\partial\theta}(\sigma\psi) \dots \text{ internal energy} \\
\sigma s &= -\frac{\partial}{\partial\theta}(\sigma\psi) \dots \text{ entropy} \\
p &= \sigma\psi + \sum_{i=1}^{N} \varrho_i \frac{\partial}{\partial\varrho_i}(\sigma\psi) \dots \text{ pressure.}
\end{aligned} \tag{21}$$

The partial fluxes are given as

$$\mathbf{J}_i = -\sum_{j=1}^{N} M_{ij}\nabla\frac{\mu_j}{\theta} - M_i\nabla\left(\frac{1}{\theta}\right), \tag{22}$$

Model DDGG III

where the matrix $\mathbb{M}$ and the vector $\vec{M}$ fulfil

$$\begin{aligned} \sum_{i=1}^{N} M_{ij} &= 0, \qquad j = 1, 2, \ldots, N \\ \sum_{i=1}^{n} M_i &= 0. \end{aligned} \tag{23}$$

Moreover, we assume that

$$\sum_{i,j=1}^{N} M_{ij} w_i w_j \geq C |\Pi \vec{w}|^2 \qquad \forall \vec{w} \in \mathbb{R}^N, \tag{24}$$

where $\Pi = \mathbb{I} - \frac{1}{N}\vec{1} \otimes \vec{1}$. We also require $\sum_{i=1}^{N} r_i = 0$. Then, summing up $(19)_1$, we get the continuity equation.

Model DDGG IV

Instead of the vector of partial densities $\vec{\varrho}$ we consider the quantities

$$q_i = \frac{\mu_i - \mu_N}{N}, \qquad i = 1, 2, \ldots, N-1, \qquad \text{and } \sigma = \varrho_1 + \varrho_2 + \cdots + \varrho_N.$$

Due to the properties of the fluxes we get compactness of q_i, $i = 1, 2, \ldots, N-1$ and using standard technique from the compressible fluid mechanics we get compactness of σ. Then we may transform this back to the compactness of ϱ_i, $i = 1, 2, \ldots, N$. For $\gamma \geq \frac{5}{3}$ we can get the existence of a weak solution even for different molar masses, for lower γ's we need again to assume that they are equal. The details will be available soon.

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THANK YOU FOR

YOUR ATTENTION