Steady compressible Navier–Stokes–Fourier system and related results: Large data results

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results obtained in collaboration with M. Bulíček (Praha), E. Feireisl, V. Giovangigli (Paris), D. Jesslé (Toulon), A. Jüngel (Wien), O. Kreml, Y. Lu (Nanjing), P.B. Mucha (Warszawa), Š. Nečasová, A. Novotný (Toulon), T. Piasecki (Warszawa), N. Zamponi (Praha), E. Zatorska (London)

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Steady compressible Navier–Stokes–Fourier system I

$\Omega \subset \mathbb{R}^3$, bounded, smooth ($C^2$)

- Balance of mass

$$\operatorname{div}(\varrho \mathbf{u}) = 0 \tag{1}$$

$\varrho \colon \Omega \mapsto \mathbb{R}$ ... density of the fluid
$\mathbf{u} \colon \Omega \mapsto \mathbb{R}^3$ ... velocity field

- Balance of momentum

$$\operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - \operatorname{div} \mathbb{S} + \nabla p = \varrho \mathbf{f} \tag{2}$$

$\mathbb{S}$ ... viscous part of the stress tensor (symmetric tensor)
$\mathbf{f} \colon \Omega \mapsto \mathbb{R}^3$ ... specific volume force (given)
p ... pressure (scalar quantity)

- Balance of total energy

$$\operatorname{div}\left(\varrho E \mathbf{u}\right) + \operatorname{div}(\mathbf{q} + p\mathbf{u}) = \varrho \mathbf{f} \cdot \mathbf{u} + \operatorname{div}\left(\mathbb{S}\mathbf{u}\right) \tag{3}$$

$E = \frac{1}{2}|\mathbf{u}|^2 + e$ ... specific total energy
e ... specific internal energy (scalar quantity)
$\mathbf{q}$ ... heat flux (vector field)
(no energy sources assumed)

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Steady compressible Navier–Stokes–Fourier system II

- Boundary conditions at $\partial\Omega$: velocity

$$\begin{aligned} \mathbf{u}\cdot\mathbf{n} &= 0 \\ (\mathbb{I}-\mathbf{n}\otimes\mathbf{n})(\mathbb{S}\mathbf{n}+\lambda\mathbf{u}) &= \mathbf{0}, \end{aligned} \tag{4}$$

$\lambda \geq 0$

or

$$\mathbf{u} = \mathbf{0} \tag{5}$$

- Boundary conditions at $\partial\Omega$: temperature

$$\mathbf{q}\cdot\mathbf{n} - L(\vartheta - \Theta_0) = 0, \tag{6}$$

$L > 0$, $\Theta_0 > 0$

- Total mass

$$\int_\Omega \varrho \,\mathrm{d}x = M > 0 \tag{7}$$

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Thermodynamics I

We will work with basic quantities: density ϱ and temperature ϑ

We assume: $e = e(\varrho, \vartheta)$, $p = p(\varrho, \vartheta)$

Gibbs' relation

$$\frac{1}{\vartheta}\left(De(\varrho,\vartheta) + p(\varrho,\vartheta) D\left(\frac{1}{\varrho}\right)\right) = Ds(\varrho,\vartheta) \tag{8}$$

with $s(\varrho, \vartheta)$ the specific entropy.

The specific entropy fulfills formally the entropy balance

$$\operatorname{div}(\varrho s \mathbf{u}) + \operatorname{div}\left(\frac{\mathbf{q}}{\vartheta}\right) = \sigma = \frac{\mathbb{S} : \nabla \mathbf{u}}{\vartheta} - \frac{\mathbf{q} \cdot \nabla \vartheta}{\vartheta^2} \tag{9}$$

Second law of thermodynamics

$$\sigma = \frac{\mathbb{S} : \nabla \mathbf{u}}{\vartheta} - \frac{\mathbf{q} \cdot \nabla \vartheta}{\vartheta^2} \geq 0 \tag{10}$$

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Thermodynamics II

Another possibility is to work with internal energy balance (heat equation)

Balance of internal energy

$$\operatorname{div}\left(\varrho e \mathbf{u}\right) + \operatorname{div}\mathbf{q} + p\operatorname{div}\mathbf{u} = \mathbb{S} : \nabla\mathbf{u}$$

The troublemaker is the nonlinear term on the rhs. Anyway, this equation plays an important role in the construction of weak solutions.

Constitutive relations I

- Newtonian fluid

$$\mathbb{S} = \mathbb{S}(\vartheta, \nabla \mathbf{u}) = \mu(\vartheta)\Big[\nabla \mathbf{u} + (\nabla \mathbf{u})^T - \tfrac{2}{3}\operatorname{div}\mathbf{u}\mathbb{I}\Big] + \xi(\vartheta)\operatorname{div}\mathbf{u}\mathbb{I} \qquad (11)$$

$\mu(\cdot)$: $\mathbb{R}^+ \to \mathbb{R}^+$,
$\xi(\cdot)$: $\mathbb{R}^+ \to \mathbb{R}^+_0$: viscosity coefficients

- Fourier's law

$$\mathbf{q} = \mathbf{q}(\vartheta, \nabla\vartheta) = -\kappa(\vartheta)\nabla\vartheta \qquad (12)$$

$\kappa(\cdot)$: $\mathbb{R}^+ \to \mathbb{R}^+ \ldots$ heat conductivity

- Pressure law

$$\begin{aligned} p = p(\varrho, \vartheta) &= \varrho^\gamma + \varrho\vartheta \\ \text{or} \qquad &= (\gamma - 1)\varrho e(\varrho, \vartheta) \end{aligned} \qquad (13)$$

(we will not consider the latter, due to additional technicalities)

- Internal energy

$$e(\varrho, \vartheta) = c_v\vartheta + \frac{\varrho^{\gamma-1}}{\gamma - 1} \qquad (14)$$

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- Heat conductivity

$$\kappa(\vartheta) \sim (1+\vartheta)^m \tag{15}$$

$m \in \mathbb{R}^+$

- Viscosity coefficients

$$\begin{aligned} C_1(1+\vartheta)^\alpha \le \mu(\vartheta) \le C_2(1+\vartheta)^\alpha \\ 0 \le \xi(\vartheta) \le C_2(1+\vartheta)^\alpha \end{aligned} \tag{16}$$

$\mu(\cdot)$ globally Lipschitz continuous, $\xi(\cdot)$ continuous,
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Weak solution I

We consider the Navier boundary conditions for the velocity and the Newton boundary conditions for the temperature.

- Weak formulation of the continuity equation

$$\int_\Omega \varrho \mathbf{u} \cdot \nabla \psi \, \mathrm{d}x = 0 \qquad \forall \psi \in C^1(\overline{\Omega}) \tag{17}$$

- Renormalized continuity equation
  ϱ extended by zero outside Ω, $\mathbf{u}$ extended outside Ω so that it remains in the $W^{1,p}$ space

$$\int_\Omega b(\varrho)\mathbf{u} \cdot \nabla \psi \, \mathrm{d}x + \int_\Omega \big(\varrho b'(\varrho) - b(\varrho)\big) \operatorname{div} \mathbf{u} \psi \, \mathrm{d}x = 0 \ \forall \psi \in C^1_0(\mathbb{R}^3) \tag{18}$$

  for all $b \in C^1([0,\infty))$ with $b'(z) = 0$ for $z \geq K > 0$.

- Weak formulation of the momentum equation

$$\begin{aligned} &\int_\Omega \big(-\varrho(\mathbf{u} \otimes \mathbf{u}) : \nabla \boldsymbol{\varphi} - p(\varrho,\vartheta) \operatorname{div} \boldsymbol{\varphi} + \mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \boldsymbol{\varphi} \big) \, \mathrm{d}x \\ &+ \lambda \int_{\partial\Omega} \mathbf{u} \cdot \boldsymbol{\varphi} \, \mathrm{d}\sigma = \int_\Omega \varrho \mathbf{f} \cdot \boldsymbol{\varphi} \, \mathrm{d}x \quad \forall \boldsymbol{\varphi} \in C^1_{\mathbf{n}}(\overline{\Omega};\mathbb{R}^3) \end{aligned} \tag{19}$$

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Weak solution II

Weak formulation of the total energy balance

$$\int_\Omega -\Big(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\Big)\mathbf{u}\cdot\nabla\psi\,\mathrm{d}x$$
$$= \int_\Omega \big(\varrho\mathbf{f}\cdot\mathbf{u}\psi + p(\varrho,\vartheta)\mathbf{u}\cdot\nabla\psi\big)\,\mathrm{d}x$$
$$- \int_\Omega \big((\mathbb{S}(\vartheta,\nabla\mathbf{u})\mathbf{u})\cdot\nabla\psi + \kappa(\vartheta)\nabla\vartheta\cdot\nabla\psi\big)\,\mathrm{d}x \tag{20}$$
$$- \int_{\partial\Omega} (L(\vartheta-\Theta_0) + \lambda|\mathbf{u}|^2)\psi\,\mathrm{d}\sigma$$
$$\forall\psi\in C^1(\overline{\Omega})$$

Definition

The triple $(\varrho,\mathbf{u},\vartheta)$ is called a renormalized weak solution to our system (1)–(7) if $\varrho\geq 0$, $\vartheta>0$ a.e. in Ω, $\mathbf{u}\cdot\mathbf{n}=0$ on $\partial\Omega$, $\int_\Omega \varrho\,\mathrm{d}x = M$, (17), (18), (19) and (20) hold true.

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Variational entropy solution I

- Weak formulation of the entropy inequality

$$\int_\Omega \left(\frac{\mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \mathbf{u}}{\vartheta} + \kappa(\vartheta) \frac{|\nabla \vartheta|^2}{\vartheta^2} \right) \psi \, \mathrm{d}x + \int_{\partial\Omega} \frac{L}{\vartheta} \Theta_0 \psi \, \mathrm{d}\sigma$$
$$\leq \int_{\partial\Omega} L\psi \, \mathrm{d}\sigma + \int_\Omega \left(\kappa(\vartheta) \frac{\nabla\vartheta \cdot \nabla\psi}{\vartheta} - \varrho s(\varrho, \vartheta) \mathbf{u} \cdot \nabla\psi \right) \mathrm{d}x \tag{21}$$
$$\forall \text{ nonnegative } \psi \in C^1(\overline{\Omega})$$

- Global total energy balance

$$\int_{\partial\Omega} \left(L(\vartheta - \Theta_0) + \lambda |\mathbf{u}|^2 \right) \mathrm{d}\sigma = \int_\Omega \varrho \mathbf{f} \cdot \mathbf{u} \, \mathrm{d}x \tag{22}$$

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Both definitions are reasonable in the sense that any smooth weak or entropy variational solution is actually a classical solution to (1)–(7) (weak-strong compatibility).

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$$\leq\int_{\partial\Omega}L\psi\,\mathrm{d}\sigma+\int_{\Omega}\Big(\kappa(\vartheta)\frac{\nabla\vartheta\cdot\nabla\psi}{\vartheta}-\varrho s(\varrho,\vartheta)\mathbf{u}\cdot\nabla\psi\Big)\,\mathrm{d}x \tag{21}$$
$$\forall\ \text{nonnegative}\ \psi\in C^1(\overline{\Omega})$$

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Both definitions are reasonable in the sense that any smooth weak or entropy variational solution is actually a classical solution to (1)–(7) (weak-strong compatibility).

Mathematical results

Until 2009, in the literature there was no existence results except for small data results or one result by P.L. Lions, where, however, the fixed mass was replaced by the finite L^p norm of the density for p sufficiently large.

- P.B. Mucha, M.P.: *On the steady compressible Navier–Stokes–Fourier system*, Communications in Mathematical Physics **288** (2009), 349–377.
- P.B. Mucha, M.P.: *Weak solutions to equations of steady compressible heat conducting fluids*, Mathematical Models & Methods in Applied Sciences **20** (2010), 785–813.
- A. Novotný, M.P.: *Steady compressible Navier–Stokes–Fourier system for monoatomic gas and its generalizations*, Journal of Differential Equations **251** (2011), 270–315.
- A. Novotný, M.P.: *Weak and variational solutions to steady equations for compressible heat conducting fluids*, SIAM Journal on Mathematical Analysis **43** (2011), 1158–1188.
- D. Jesslé, A. Novotný, M.P.: *Steady Navier–Stokes–Fourier system with slip boundary conditions*, Mathematical Models & Methods in Applied Sciences **24** (2014), 751–781.
- P.B. Mucha, M.P., E. Zatorska: *Existence of Stationary Weak Solutions for the Heat Conducting Flows.* In: Giga, Yoshikazu, Novotný, Antonín (eds.): **Handbook of Mathematical Analysis in Mechanics of Viscous Fluids**, Springer Verlag, 2018, 2595–2662.

Approximate system I

We consider for simplicity Ω not axially symmetric and $\lambda = 0$. We have in this case Korn's inequalities of the form

$$\|\mathbf{u}\|_{1,p} \leq C\Big(\int_\Omega \frac{1}{\vartheta}\mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \mathbf{u} \, \mathrm{d}x \Big)^{\frac{1}{2}} \|\vartheta\|_{3m}^{\frac{1-\alpha}{2}},$$

where $p = \frac{6m}{3m+1-\alpha} < 2$ if $0 \leq \alpha < 1$, $p = 2$ if $\alpha = 1$. We first consider the easier case $\alpha = 1$.

We can prove existence of a solution to the following system for arbitrary $\delta > 0$ provided β, $B \gg 1$.

Continuity equation:

$$\int_\Omega \varrho_\delta \mathbf{u}_\delta \cdot \nabla \psi \, \mathrm{d}x = 0 \tag{23}$$

for all $\psi \in W^{1,\frac{30\beta}{25\beta-18}}(\Omega;\mathbb{R})$, as well as in the renormalized sense

Momentum equation:

$$\begin{aligned} &\int_\Omega \Big(-\varrho_\delta(\mathbf{u}_\delta \otimes \mathbf{u}_\delta) : \nabla \boldsymbol{\varphi} + \mathbb{S}(\vartheta_\delta, \nabla \mathbf{u}_\delta) : \nabla \boldsymbol{\varphi} \\ &-\big(p(\varrho_\delta, \vartheta_\delta) + \delta \varrho_\delta^\beta + \delta \varrho_\delta^2\big) \operatorname{div} \boldsymbol{\varphi}\Big) \, \mathrm{d}x = \int_\Omega \varrho_\delta \mathbf{f} \cdot \boldsymbol{\varphi} \, \mathrm{d}x \end{aligned} \tag{24}$$

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Approximate system II

Total energy balance:

$$\int_\Omega \Big(\Big(-\frac{1}{2}\varrho_\delta |\mathbf{u}_\delta|^2 - \varrho_\delta e(\varrho_\delta, \vartheta_\delta) \Big) \mathbf{u}_\delta \cdot \nabla\psi + \big(\kappa(\vartheta_\delta) + \delta\vartheta_\delta^B + \delta\vartheta_\delta^{-1} \big) \nabla\vartheta_\delta \cdot \nabla\psi \Big) \, \mathrm{d}x$$
$$+ \int_{\partial\Omega} (L + \delta\vartheta_\delta^{B-1})(\vartheta_\delta - \Theta_0)\psi \, \mathrm{d}\sigma = \int_\Omega \varrho_\delta \mathbf{f} \cdot \mathbf{u}_\delta \psi \, \mathrm{d}x + \int_\Omega \Big(\Big(- \mathbb{S}(\vartheta_\delta, \nabla\mathbf{u}_\delta)\mathbf{u}_\delta$$
$$+ \big(p(\varrho_\delta, \vartheta_\delta) + \delta\varrho_\delta^\beta + \delta\varrho_\delta^2 \big)\mathbf{u}_\delta \Big) \cdot \nabla\psi + \delta\vartheta_\delta^{-1}\psi \Big) \, \mathrm{d}x + \delta \int_\Omega \Big(\frac{1}{\beta - 1}\varrho_\delta^\beta + \varrho_\delta^2 \Big) \mathbf{u}_\delta \cdot \nabla\psi \, \mathrm{d}x \qquad (25)$$

for all $\psi \in C^1(\overline{\Omega}; \mathbb{R})$

Entropy inequality:

$$\int_\Omega \Big(\vartheta_\delta^{-1} \mathbb{S}(\vartheta_\delta, \mathbf{u}) : \nabla\mathbf{u}_\delta + \delta\vartheta_\delta^{-2} + \big(\kappa(\vartheta_\delta) + \delta\vartheta_\delta^B + \delta\vartheta_\delta^{-1} \big) \frac{|\nabla\vartheta_\delta|^2}{\vartheta_\delta^2} \Big) \psi \, \mathrm{d}x$$
$$\leq \int_\Omega \Big(\big(\kappa(\vartheta_\delta) + \delta\vartheta_\delta^B + \delta\vartheta_\delta^{-1} \big) \frac{\nabla\vartheta_\delta : \nabla\psi}{\vartheta_\delta} - \varrho s(\varrho_\delta, \vartheta_\delta)\mathbf{u}_\delta \cdot \nabla\psi \Big) \, \mathrm{d}x \qquad (26)$$
$$+ \int_{\partial\Omega} \frac{L + \delta\vartheta_\delta^{B-1}}{\vartheta_\delta} (\vartheta_\delta - \Theta_0)\psi \, \mathrm{d}\sigma,$$

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Estimates independent of δ I

Use in the entropy inequality and in the total energy balance test functions $\psi \equiv 1$:

$$\int_\Omega \left(\kappa(\vartheta_\delta)+\delta\vartheta_\delta^B+\delta\vartheta_\delta^{-1}\right)\frac{|\nabla\vartheta_\delta|^2}{\vartheta_\delta^2}\,\mathrm{d}x + \int_\Omega\left(\frac{1}{\vartheta_\delta}\mathbb{S}(\vartheta_\delta,\mathbf{u}_\delta):\nabla\mathbf{u}_\delta+\delta\vartheta_\delta^{-2}\right)\mathrm{d}x$$

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Using suitable estimates of the Bogovskii-type we can get rid of the δ-dependent terms and we conclude:

$$\|\mathbf{u}_\delta\|_{1,2}+\|\nabla\vartheta_\delta^{\frac{m}{2}}\|_2+\|\nabla\ln\vartheta_\delta\|_2+\|\vartheta_\delta^{-1}\|_{1,\partial\Omega} +\delta\left(\|\nabla\vartheta_\delta^{\frac{B}{2}}\|_2^2+\|\nabla\vartheta_\delta^{-\frac{1}{2}}\|_2^2+\|\vartheta_\delta\|_{3B}^B+\|\vartheta_\delta^{-2}\|_1\right)\le C \tag{29}$$

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To estimate the density, we may use the Bogovskii-type estimates, but this leads to the bound $\gamma > \frac{3}{2}$. Therefore we apply another approach based on "potential"estimates of the pressure.

- Define for $1 \leq a \leq \gamma$, $0 < b < 1$

$$\mathcal{A} = \int_\Omega (\varrho_\delta^a |\mathbf{u}_\delta|^2 + \varrho_\delta^b |\mathbf{u}_\delta|^{2b+2})\, \mathrm{d}x \tag{31}$$

- Using the previous estimates we get, under some conditions on a and b

$$\begin{gathered} \|\mathbf{u}_\delta\|_{1,2} \leq C \\ \|\vartheta_\delta\|_{3m} \leq C\big(1 + \mathcal{A}^{\frac{a-b}{2(ab+a-2b)}}\big) \\ \int_\Omega \big(\varrho_\delta^{s\gamma} + \varrho_\delta^{(s-1)\gamma} p(\varrho_\delta, \vartheta_\delta) + (\varrho_\delta |\mathbf{u}_\delta|^2)^s + \delta \varrho_\delta^{\beta+(s-1)\gamma}\big)\, \mathrm{d}x \\ \leq C(1 + \mathcal{A}^{\frac{sa-b}{ab+a-b}}), \end{gathered} \tag{32}$$

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$$\varphi_i(x) \sim \frac{(x-y)_i}{|x-y|^A}.$$

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Estimates independent of δ III

Lemma

Let $y \in \Omega$, $R_0 < \frac{1}{3} dist(y, \partial\Omega)$. Then

$$\int_{B_{R_0}(y)} \Big(\frac{p(\varrho_\delta, \vartheta_\delta)}{|x-y|^A} + \frac{\varrho_\delta |\mathbf{u}_\delta|^2}{|x-y|^A} \Big) \mathrm{d}x \\ \leq C \big(1 + \|p(\varrho_\delta, \vartheta_\delta)\|_1 + \|\mathbf{u}_\delta\|_{1,2}(1 + \|\vartheta_\delta\|_{3m}) + \|\varrho_\delta |\mathbf{u}_\delta|^2\|_1\big), \tag{33}$$

provided $A < \min \Big\{ \frac{3m-2}{2m}, 1 \Big\}$.

Similar test functions can be used for y near and at the boundary. We obtain a similar result. More complex for the Dirichlet boundary conditions, leads to more restrictions.

- Let us consider

$$-\Delta h = \varrho_\delta^a + \varrho_\delta^b |\mathbf{u}_\delta|^{2b} - \frac{1}{|\Omega|} \int_\Omega (\varrho_\delta^a + \varrho_\delta^b |\mathbf{u}_\delta|^{2b}) \mathrm{d}x, \\ \frac{\partial h}{\partial \mathbf{n}}|_{\partial\Omega} = 0. \tag{34}$$

The unique strong solution can be written

$$h(y) = \int_\Omega G(x,y)(\varrho_\delta^a + \varrho_\delta^b |\mathbf{u}_\delta|^{2b}) \,\mathrm{d}x + l.o.t. \tag{35}$$

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Estimates independent of δ IV

As $G(x,y) \le C|x-y|^{-1}$, we get

$$\|h\|_\infty \le C(1+\mathcal{A}^\eta), \tag{36}$$

where $\eta = \eta(a,b,\gamma,m)$

- Next

$$\mathcal{A} \sim \int_\Omega -\Delta h |\mathbf{u}_\delta|^2 \, \mathrm{d}x = \int_\Omega \nabla h \cdot \nabla |\mathbf{u}_\delta|^2 \, \mathrm{d}x \le 2\|\nabla \mathbf{u}_\delta\|_2 B^{\frac{1}{2}}, \tag{37}$$

$$B = \int_\Omega |\nabla h \otimes \mathbf{u}_\delta|^2 \, \mathrm{d}x. \tag{38}$$

Employing once more integration by parts

$$B = -\int_\Omega h\Delta h |\mathbf{u}_\delta|^2 \, \mathrm{d}x - \int_\Omega h \nabla h \cdot \nabla \mathbf{u}_\delta \cdot \mathbf{u}_\delta \, \mathrm{d}x \le \|h\|_\infty (\mathcal{A} + \|\nabla \mathbf{u}_\delta\|_2 B^{\frac{1}{2}}),$$

i.e.,

$$B \le \|h\|_\infty \mathcal{A} + \frac{1}{2}\|\nabla \mathbf{u}_\delta\|_2^2 \|h\|_\infty^2. \tag{39}$$

Therefore

$$\mathcal{A} \le C\|\nabla \mathbf{u}_\delta\|_2^2 \|h\|_\infty. \tag{40}$$

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$$B = -\int_\Omega h\Delta h |\mathbf{u}_\delta|^2 \,\mathrm{d}x - \int_\Omega h \nabla h \cdot \nabla \mathbf{u}_\delta \cdot \mathbf{u}_\delta \,\mathrm{d}x \le \|h\|_\infty (\mathcal{A} + \|\nabla \mathbf{u}_\delta\|_2 B^{\frac{1}{2}}),$$

i.e.,

$$B \le \|h\|_\infty \mathcal{A} + \frac{1}{2}\|\nabla \mathbf{u}_\delta\|_2^2 \|h\|_\infty^2. \tag{39}$$

Therefore

$$\mathcal{A} \le C\|\nabla \mathbf{u}_\delta\|_2^2 \|h\|_\infty. \tag{40}$$

Estimates independent of δ IV

As $G(x,y) \le C|x-y|^{-1}$, we get

$$\|h\|_\infty \le C(1+\mathcal{A}^\eta), \tag{36}$$

where $\eta = \eta(a,b,\gamma,m)$

- Next

$$\mathcal{A} \sim \int_\Omega -\Delta h |\mathbf{u}_\delta|^2 \,\mathrm{d}x = \int_\Omega \nabla h \cdot \nabla |\mathbf{u}_\delta|^2 \,\mathrm{d}x \le 2\|\nabla \mathbf{u}_\delta\|_2 B^{\frac{1}{2}}, \tag{37}$$

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Estimates independent of δ V

- Then,

$$\mathcal{A} \leq C(1 + \mathcal{A}^{\tilde{\eta}})$$

  and we require $\tilde{\eta} < 1$. This leads to a set of conditions.
- Analyzing these conditions, we finally have

Lemma

Let $(\varrho_\delta, \mathbf{u}_\delta, \vartheta_\delta)$ solve our approximate problem. Let $\gamma > 1$ and $m > \frac{2}{4\gamma - 3}$. Then there exists $s > 1$ such that

$$\begin{aligned}
\sup_{\delta>0} \|\varrho_\delta\|_{\gamma s} &< +\infty \\
\sup_{\delta>0} \|\varrho_\delta \mathbf{u}_\delta\|_{s} &< +\infty \\
\sup_{\delta>0} \|\varrho_\delta |\mathbf{u}_\delta|^2\|_{s} &< +\infty \\
\sup_{\delta>0} \|\mathbf{u}_\delta\|_{1,2} &< +\infty \\
\sup_{\delta>0} \|\vartheta_\delta\|_{3m} &< +\infty \\
\sup_{\delta>0} \|\vartheta_\delta^{m/2}\|_{1,2} &< +\infty \\
\sup_{\delta>0} \delta \|\varrho_\delta^{\beta+(s-1)\gamma}\|_{1} &< +\infty.
\end{aligned} \tag{41}$$

Moreover, we can take $s > \frac{6}{5}$ provided $\gamma > \frac{5}{4}$, and $m > \max\{1, \frac{2\gamma+10}{17\gamma-15}\}$.

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Limit passage $\delta \to 0^+$ I

Continuity equation

$$\int_\Omega \rho \mathbf{u} \cdot \nabla \psi \, \mathrm{d}x = 0 \tag{42}$$

for all $\psi \in C^1(\overline{\Omega}; \mathbb{R})$

Momentum equation

$$\int_\Omega \Big(- \varrho(\mathbf{u} \otimes \mathbf{u}) : \nabla \varphi + \mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \varphi - \overline{p(\varrho, \vartheta)} \operatorname{div} \varphi \Big) \, \mathrm{d}x = \int_\Omega \varrho \mathbf{f} \cdot \varphi \, \mathrm{d}x \tag{43}$$

for all $\varphi \in C^1_{\mathbf{n}}(\overline{\Omega}; \mathbb{R}^3)$

Entropy inequality

$$\int_\Omega \Big(\vartheta^{-1} \mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \mathbf{u} + \kappa(\vartheta) \frac{|\nabla \vartheta|^2}{\vartheta^2} \Big) \psi \, \mathrm{d}x$$
$$\leq \int_\Omega \Big(\kappa(\vartheta) \frac{\nabla \vartheta : \nabla \psi}{\vartheta} - \overline{\varrho s(\varrho, \vartheta)} \mathbf{u} \cdot \nabla \psi \Big) \, \mathrm{d}x + \int_{\partial \Omega} \frac{L}{\vartheta} (\vartheta - \Theta_0) \psi \, \mathrm{d}\sigma, \tag{44}$$

for all $\psi \in C^1(\overline{\Omega}; \mathbb{R})$, nonnegative

Global total energy balance

$$\int_{\partial \Omega} L(\vartheta - \Theta_0) \, \mathrm{d}\sigma = \int_\Omega \varrho \mathbf{f} \cdot \mathbf{u} \, \mathrm{d}x \tag{45}$$

(total energy balance with test function $\psi \equiv 1$)

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Limit passage $\delta \to 0^+$ II

Total energy balance

$$\int_\Omega \Big(\big(- \frac{1}{2} \varrho |\mathbf{u}|^2 - \overline{\varrho e(\varrho, \vartheta)} \big) \mathbf{u} \cdot \nabla \psi + \kappa(\vartheta) \nabla \vartheta : \nabla \psi \Big) \, \mathrm{d}x$$
$$+ \int_{\partial\Omega} \big(L(\vartheta - \Theta_0) \psi \, \mathrm{d}\sigma = \int_\Omega \varrho \mathbf{f} \cdot \mathbf{u} \psi \, \mathrm{d}x + \int_\Omega \big(- \mathbb{S}(\vartheta, \nabla \mathbf{u}) \mathbf{u} + \overline{p(\varrho, \vartheta)} \mathbf{u} \big) \cdot \nabla \psi \, \mathrm{d}x \tag{46}$$

for all $\psi \in C^1(\overline{\Omega}; \mathbb{R})$. We can pass only in certain situations, when we have better a priori estimates! We need $s > \frac{6}{5}$ and $m > 1$.

We need to show the strong convergence of the density!

Main ingredients:

- Effective viscous flux identity
- Oscillation defect measure estimate
- Renormalized continuity equation

Limit passage $\delta \to 0^+$ II

Total energy balance

$$\int_\Omega \Big(\big(-\frac{1}{2}\varrho|\mathbf{u}|^2 - \overline{\varrho e(\varrho,\vartheta)} \big)\mathbf{u}\cdot\nabla\psi + \kappa(\vartheta)\nabla\vartheta : \nabla\psi \Big)\,\mathrm{d}x$$

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Limit passage $\delta \to 0^+$ III

Item 1: Effective viscous flux

Using as test function $\zeta(x)\nabla\Delta^{-1}(1_\Omega T_k(\varrho_\delta))$ with $T_k(z) = kT(\frac{z}{k})$, $k \in N$ for

$$T(z) = \begin{cases} z \text{ for } 0 \le z \le 1, \\ \text{concave on } (0,\infty), \\ 2 \text{ for } z \ge 3, \end{cases}$$

in the approximative balance of momentum, and $\zeta(x)\nabla\Delta^{-1}(1_\Omega \overline{T_k(\varrho)})$ in its limit version we can deduce

$$\begin{aligned} &\overline{p(\varrho,\vartheta)T_k(\varrho)} - \left(\frac{4}{3}\mu(\vartheta) + \xi(\vartheta)\right)\overline{T_k(\varrho)\,\mathrm{div}\,\mathbf{u}} \\ &= \overline{p(\varrho,\vartheta)}\ \overline{T_k(\varrho)} - \left(\frac{4}{3}\mu(\vartheta) + \xi(\vartheta)\right)\overline{T_k(\varrho)}\,\mathrm{div}\,\mathbf{u} \end{aligned} \tag{47}$$

a.e. in Ω.

Item 2: Oscillation defect measure

We do not have L^2-bound on the density and thus we do not know whether the renormalized continuity equation for the limit holds. To show it, we introduce:

Oscillation defect measure

$$\mathbf{osc_q}[\varrho_\delta \to \varrho](Q) = \sup_{k>1} \left(\limsup_{\delta \to 0^+} \int_Q |T_k(\varrho_\delta) - T_k(\varrho)|^q \, \mathrm{d}x \right) \tag{48}$$

We have

$$\begin{aligned} \varrho_\delta &\rightharpoonup \varrho && \text{in } L^1(\Omega; \mathbb{R}), \\ \mathbf{u}_\delta &\rightharpoonup \mathbf{u} && \text{in } L^p(\Omega; \mathbb{R}^3), \\ \nabla \mathbf{u}_\delta &\rightharpoonup \nabla \mathbf{u} && \text{in } L^p(\Omega; \mathbb{R}^{3\times 3}) \end{aligned}$$

and

$$\mathbf{osc_q}[\varrho_\delta \to \varrho](\Omega) < \infty \tag{49}$$

for $q > p'$, then the limit density and velocity satisfy the renormalized continuity equation.

Assuming $m > \max\{\frac{2}{3(\gamma-1)}, \frac{2}{3}\}$, it can be verified that (49) holds true with some $2 < q < \gamma + 1$.

Limit passage $\delta \to 0^+$ IV

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Limit passage $\delta \to 0^+$ V

We also get

$$\limsup_{\delta\to 0^+} \int_\Omega |T_k(\varrho_\delta) - T_k(\varrho)|^{\gamma+1}\,\mathrm{d}x \le C \int_\Omega \left(\overline{p(\varrho,\vartheta)T_k(\varrho)} - \overline{p(\varrho,\vartheta)}\;\overline{T_k(\varrho)}\right)\mathrm{d}x, \tag{50}$$

$$\limsup_{\delta\to 0^+} \int_\Omega \frac{1}{1+\vartheta} |T_k(\varrho_\delta) - T_k(\varrho)|^{\gamma+1}\,\mathrm{d}x \le C \int_\Omega \frac{1}{1+\vartheta}\left(\overline{p(\varrho,\vartheta)T_k(\varrho)} - \overline{p(\varrho,\vartheta)}\;\overline{T_k(\varrho)}\right)\mathrm{d}x. \tag{51}$$

Item 3: Application of the renormalization

As $(\varrho_\delta, \mathbf{u}_\delta)$ and $(\varrho, \mathbf{u})$ verify the renormalized continuity equation, we have:

$$\int_\Omega T_k(\varrho)\,\mathrm{div}\,\mathbf{u}\,\mathrm{d}x = 0$$

and

$$\int_\Omega T_k(\varrho_\delta)\,\mathrm{div}\,\mathbf{u}_\delta\,\mathrm{d}x = 0, \qquad \text{i.e.} \int_\Omega \overline{T_k(\varrho)\,\mathrm{div}\,\mathbf{u}}\,\mathrm{d}x = 0$$

To this aim, use

$$\mathrm{div}(b(\varrho)\mathbf{u}) + \big(\varrho b'(\varrho) - b(\varrho)\big)\,\mathrm{div}\,\mathbf{u} = 0 \text{ in } \mathcal{D}'(\mathbb{R}^3)$$

with

$$b(\varrho) = \varrho \int_1^\varrho \frac{T_k(z)}{z^2}\,\mathrm{d}z.$$

Limit passage $\delta \to 0^+$ V

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We also get

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Item 3: Application of the renormalization

As $(\varrho_\delta, \mathbf{u}_\delta)$ and $(\varrho, \mathbf{u})$ verify the renormalized continuity equation, we have:

$$\int_\Omega T_k(\varrho) \operatorname{div} \mathbf{u} \,\mathrm{d}x = 0$$

and

$$\int_\Omega T_k(\varrho_\delta) \operatorname{div} \mathbf{u}_\delta \,\mathrm{d}x = 0, \qquad \text{i.e.} \int_\Omega \overline{T_k(\varrho) \operatorname{div} \mathbf{u}} \,\mathrm{d}x = 0$$

To this aim, use

$$\operatorname{div}(b(\varrho)\mathbf{u}) + \left(\varrho b'(\varrho) - b(\varrho)\right) \operatorname{div} \mathbf{u} = 0 \text{ in } \mathcal{D}'(\mathbb{R}^3)$$

with

$$b(\varrho) = \varrho \int_1^\varrho \frac{T_k(z)}{z^2} \,\mathrm{d}z.$$

Limit passage $\delta \to 0^+$ V

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$$\int_\Omega \frac{1}{\frac{4}{3}\mu(\vartheta)+\xi(\vartheta)}\left(\overline{p(\varrho,\vartheta)T_k(\varrho)} - \overline{p(\varrho,\vartheta)}\ \overline{T_k(\varrho)}\right)\,\mathrm{d}x = \int_\Omega \left(T_k(\varrho) - \overline{T_k(\varrho)}\right)\operatorname{div}\mathbf{u}\,\mathrm{d}x. \tag{52}$$

As $\lim_{k\to\infty}\|T_k(\varrho)-\varrho\|_1 = \lim_{k\to\infty}\|\overline{T_k(\varrho)}-\varrho\|_1 = 0$, the definition of the oscillation defect measure together with (49)

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Results I (Navier b.c.)

We proved:

Theorem

Let $\Omega \in C^2$ *be a bounded domain in* $\mathbb{R}^3$, $\mathbf{f} \in L^\infty(\Omega;\mathbb{R}^3)$, $\Theta_0 \geq K_0 > 0$ *a.e. at* $\partial\Omega$, $\Theta_0 \in L^1(\partial\Omega)$. *Let* $\gamma > 1$, $m > \max\left\{\frac{2}{3}, \frac{2}{3(\gamma-1)}\right\}$.
Let Ω *be not axially symmetric. Then there exists a variational entropy solution to our problem. Moreover,* $(\varrho, \mathbf{u})$ *is a renormalized solution to the continuity equation.*
Additionally, if $m > 1$ *and* $\gamma > \frac{5}{4}$, *then the solution is a weak solution, i.e. also the weak formulation of the total energy balance is fulfilled.*

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The case $\alpha = 0$

The situation when the viscosity is independent of the temperature was studied in

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- The a priori estimates for the velocity were obtained from momentum equation, not from the entropy inequality, therefore it was possible have the velocity gradient in $L^2(\Omega)$
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Changes for $0 < \alpha < 1$

Recall that

$$\|\mathbf{u}\|_{1,p} \le C\Big(\int_\Omega \frac{1}{\vartheta}\mathbb{S}(\vartheta,\nabla \mathbf{u}) : \nabla \mathbf{u}\, \mathrm{d}x\Big)^{\frac{1}{2}} \|\vartheta\|_{3m}^{\frac{1-\alpha}{2}},$$

i.e., for $\alpha < 1$ we control only $W^{1,p}$-norm of the velocity, $p < 2$.

For $\gamma > \frac{3}{2}$ it is possible to estimate the density by the Bogovskii-type estimates and in dependence on γ and m it is possible to obtain either the weak or the variational entropy solutions as was shown in

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Therein, the steady flow of compressible, heat-conducting, radiative gas was studied.

For small γ and/or m it is possible to repeat the estimates of the pressure and momentum from the previous part. However, it is not possible to get from them the estimates of the velocity and density as above. Moreover, the integration-by-parts argument does not work! We can replace it with certain properties of Bessel kernels and Bessel potential spaces. The proof itself is similar, but more technical and the results are more messy, we have three parameters: α, γ and m. This is a recent project with O. Kreml.

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Chemically reacting mixtures I

$$\begin{gathered}\operatorname{div}(\varrho\mathbf{u}) = 0,\\ \operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u}) - \operatorname{div}\mathbb{S} + \nabla\pi = \varrho\mathbf{f},\\ \operatorname{div}(\varrho E\mathbf{u}) + \operatorname{div}(\pi\mathbf{u}) + \operatorname{div}\mathbb{Q} - \operatorname{div}(\mathbb{S}\mathbf{u}) = \varrho\mathbf{f}\cdot\mathbf{u},\\ \operatorname{div}(\varrho Y_k\mathbf{u}) + \operatorname{div}\mathbf{F}_k = m_k\omega_k, \quad k\in\{1,\dots,n\}\end{gathered} \tag{53}$$

with the boundary conditions

$$\mathbf{u} = \mathbf{0}, \tag{54}$$

$$\mathbf{F}_k\cdot\mathbf{n} = 0, \tag{55}$$

$$-\mathbf{Q}\cdot\mathbf{n} + L(\vartheta - \Theta_0) = 0, \tag{56}$$

and the given total mass

$$\int_\Omega \varrho\,\mathrm{d}x = M > 0. \tag{57}$$

Indeed, $\sum_{k=1}^n \mathbf{F}_k = \mathbf{0}$, $\sum_{k=1}^n m_k\omega_k = 0$ and we must construct solutions such that $\sum_{k=1}^N Y_k = 1$.

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Based on similar ideas presented for the N-S-F system the existence of weak and variational entropy solutions can be established in the case of the same molar masses (closely connected with information from the entropy inequality which plays a central role here).

V. Giovangigli, M.P., E. Zatorska: *On the steady flow of reactive gaseous mixture*, Analysis (Berlin) **35** (2015), no. 4, 319–341.

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T. Piasecki, M.P.: *On steady solutions to a model of chemically reacting heat conducting compressible mixture with slip boundary conditions.* Mathematical analysis in fluid mechanics–selected recent results, 223–242, Contemp. Math., **710**, Amer. Math. Soc., Providence, RI, 2018.

The case of different molar masses for a slightly different thermodynamic concept is ongoing project with M. Bulíček, A. Jüngel and N. Zamponi.

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T. Piasecki, M.P.: *On steady solutions to a model of chemically reacting heat conducting compressible mixture with slip boundary conditions.* Mathematical analysis in fluid mechanics–selected recent results, 223–242, Contemp. Math., **710**, Amer. Math. Soc., Providence, RI, 2018.

The case of different molar masses for a slightly different thermodynamic concept is ongoing project with M. Bulíček, A. Jüngel and N. Zamponi.

Time periodic solutions I

Time periodic solutions to the compressible Navier–Stokes–Fourier system were constructed in

- E. Feireisl, Eduard, Piotr B. Mucha, Antonín Novotný, MP: Time-periodic solutions to the full Navier–Stokes–Fourier system, Arch. Ration. Mech. Anal. **204** (2012), 745–786.

- The proof combines the evolutionary with estimates similar to the steady problem
- Due to the lack of time-compactness of the temperature the variational entropy formulation must be considered
- The difficult part is also the construction of the approximate solutions

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Let $\varepsilon > 0$ be a small number, measures the mutual distance between the holes

$$\Omega_\varepsilon = \Omega \setminus \bigcup_{n=1}^{N(\varepsilon)} \overline{T}_{n,\varepsilon}, \tag{58}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded C^2-domain and $\{T_{n,\varepsilon}\}_{n=1}^{N(\varepsilon)}$ are C^2-domains of the diameter comparable to ε^α for some $\alpha \geq 1$ such that there exist δ_0, δ_1 and δ_2 positive for which

$$T_{n,\varepsilon} = x_{\varepsilon,n} + \varepsilon^\alpha T^0_{n,1} \subset B_{\delta_0 \varepsilon^\alpha}(x_{n,\varepsilon}) \subset B_{2\delta_0 \varepsilon^\alpha}(x_{n,\varepsilon}) \subset B_{\delta_1 \varepsilon}(x_{n,\varepsilon}) \subset B_{\delta_2 \varepsilon}(x_{n,\varepsilon}) \subset \Omega. \tag{59}$$

The balls $B_{\delta_2 \varepsilon}(x_{n,\varepsilon})$ centred at $x_{\varepsilon,n}$ with diameter $\delta_2 \varepsilon$ are pairwise disjoint and we assume that the domains $\{T^0_{n,1}\}_{n=1}^{N(\varepsilon)}$ are uniformly C^2-domains.

Homogenization for the steady NSF problem II

Theorem

Let $\mathbf{f} \in L^\infty(\Omega;\mathbb{R}^3)$, $M_\varepsilon > 0$ *with* $\sup_\varepsilon M_\varepsilon = M_1 < \infty$, $\inf_\varepsilon M_\varepsilon = M_0 > 0$, $L > 0$ *and let* $\vartheta_0 \geq T_0 > 0$ *in* Ω *be defined so that it has finite* L^q*-norm over arbitrary smooth two-dimensional surface with finite surface area contained in* Ω *for some* $q > 1$. *Let* $(\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon)$ *denote the corresponding renormalized weak entropy solution to our problem for fixed* $\varepsilon > 0$, *extended suitably to the whole* Ω, *for which in particular the extensions preserve their values in* Ω_ε. *Let* $\alpha > 3$, $m > 2$ *and* $\gamma > 2$ *fulfil* $\alpha > \max\{\frac{2\gamma-3}{\gamma-2}, \frac{3m-2}{m-2}\}$. *Then, for* $\varepsilon \in (0,1]$ *the solutions are uniformly bounded*

$$\|\varrho_\varepsilon\|_{L^{\gamma+\Theta}(\Omega)} + \|\mathbf{u}_\varepsilon\|_{W_0^{1,2}(\Omega)} + \|\vartheta_\varepsilon\|_{W^{1,2}\cap L^{3m}(\Omega)} \leq C, \tag{60}$$

where $\Theta := \min\left\{2\gamma - 3, \gamma\frac{3m-2}{3m+2}\right\}$ *and* C *is independent of* ε. *Moreover, the corresponding weak limit of the sequence for* $\varepsilon \to 0^+$ *is a renormalized weak solution to our problem in* Ω, *i.e., it fulfils the continuity equation in the weak and renormalized sense, the mass balance and the total energy balance in the weak sense in* Ω, *and* $\varrho \geq 0$ *and* $\vartheta > 0$ *a.e. in* Ω.

The details are contained in

Y. Lu, M.P.: Homogenization of stationary Navier–Stokes–Fourier system in domains with tiny holes, accepted to J. Differ. Equations (2021).

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THANK YOU VERY MUCH

FOR YOUR ATTENTION!