

Elliptic PDEs

1. Consider the problem

$$\Delta^2 u + \alpha \Delta u + \beta u = f$$

in $\Omega \subset \mathbb{R}^d$ a bounded open set, where $\Delta^2 u = \Delta(\Delta u)$, α, β are real constants and $f \in L^2(\Omega)$. Consider the following boundary conditions

$$u = \frac{\partial u}{\partial \boldsymbol{\nu}} = 0$$

on $\partial\Omega$, where $\boldsymbol{\nu}$ is the (unit) exterior normal vector to $\partial\Omega$.

Formulate the problem weakly and discuss conditions under which there exists a unique weak solution to the problem above.

2. Let $\Omega \subset \mathbb{R}^d$ be an open bounded domain (but no assumption on the regularity!). Consider the problem

$$-\frac{\partial}{\partial x_i} \left(\sum_{i,j=1} a_{ij}(x) \frac{\partial u}{\partial x_j} \right) = f \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \partial\Omega$$

in the weak setting. Explain under the assumptions of ellipticity of the operator, boundedness of functions a_{ij} and $f \in L^2(\Omega)$ construction of a weak solution to this problem.

3. Let $\Omega \subset \mathbb{R}^d$ be Lipschitz, $f \in L^2(\Omega)$, $p \in [1, \infty]$ and the vector $\mathbf{c} \in L^p(\Omega; \mathbb{R}^d)$. Let moreover $\operatorname{div} \mathbf{c} \leq 0$ in the sense of distributions on Ω , i.e., for any non-negative $\varphi \in C_0^\infty(\Omega)$ it holds

$$\int_{\Omega} \mathbf{c} \cdot \nabla \varphi \geq 0.$$

Consider the weak formulation of the problem

$$-\frac{\partial}{\partial x_i} \left(\sum_{i,j=1} a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + \sum_{i=1}^d c_i \frac{\partial u}{\partial x_i} = f \quad \text{in } \Omega$$

$$u = u_0 \quad \text{on } \partial\Omega.$$

Show existence and uniqueness for a weak solution to this problem (under standard assumptions of a_{ij} and u_0) for the interval of p 's as large as possible (in dependence on the dimension).

4. Let Ω be an open Lipschitz domain in $\mathbb{R}^d$. Assume that $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$, where $\Omega_1 \cap \Omega_2 = \emptyset$, Ω_i are open, $i = 1, 2$ and let $|\Gamma|_{d-1} > 0$, finite, be the common part of the of the boundaries of Ω_1 and Ω_2 (i.e., $\overline{\Omega_1} \cap \overline{\Omega_2} = \overline{\Gamma}$). Under standard assumptions on a_{ij} , f consider the weak solution to

$$\begin{aligned} -\frac{\partial}{\partial x_i} \left(\sum_{j=1} a_{ij}(x) \frac{\partial u}{\partial x_j} \right) &= f && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Discuss, under suitable conditions on u , which weak problem satisfies the function u separately on Ω_1 and Ω_2 . This problem can be also viewed as follows.

Assume additionally that $u \in W^{2,2}(\Omega)$. What kind of problems satisfies u in Ω_1 and in Ω_2 . Or another point of view, assume that $u \in W^{1,2}(\Omega_i)$ satisfies in the weak setting

$$\begin{aligned} -\frac{\partial}{\partial x_i} \left(\sum_{j=1} a_{ij}(x) \frac{\partial u_i}{\partial x_j} \right) &= f && \text{in } \Omega_i \\ u_i &= u_i^0 && \text{on } \partial\Omega_i, \quad i = 1, 2. \end{aligned}$$

What are the necessary conditions on the common part of the boundary such that

$$u = u_1 1_{\Omega_1} + u_2 1_{\Omega_2}$$

is a weak solution to the original problem on Ω ?