Exercises PDEs II: Set 1 (February 27, 2026)

Sobolev spaces

1. Show the following claim:

   Let Ω be bounded, $1 \leq p < \infty$ and $u \in W^{1,p}(\Omega)$. Denote for $\delta > 0$

   $$\Omega_\delta := \{x \in \Omega \,|\, \mathrm{dist}\,(x, \partial\Omega) > \delta\}.$$

   The it holds for any $\delta > 0$ small

   $$\lim_{h \to 0} \left\| \Delta_i^h u - \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega_\delta)} = 0.$$

2. Show the following claim:

   Let Ω be an open set, $u \in W^{1,1}(\Omega)$. Let $D \subset \Omega$ denote

   $$D := \{x \in \Omega \,|\, \text{ there exists the classical partial derivative of } u \text{ with respect to } x_i\}.$$

   Then the weak and classical derivatives of u coincide almost everywhere in D.

3. Show that for $\Omega \in C^{1,1}$ there exists an extension operator $E\, W^{2,p}(\Omega) \to W^{2,p}(\mathbb{R}^d)$, $1 \leq p \leq \infty$. (Modify the proof for $k = 1$.) Try to discuss the general case $k \in \mathbb{N}$.

4. Show that the extension operator E: defined as

   $$EU(x) := \begin{cases} U(x) & x \in \overline{C^+} \\ U(x_1, \dots, x_{d-1}, -x_d) & x \in C^- \\ 0 & x \in \mathbb{R}^d \setminus C \end{cases}$$

   maps $C^1(C^+)$ functions with support in $C^+ \cup (-1,1)^{d-1} \times \{0\}$ to $W^{1,\infty}(C) \cap W_0^{1,1}(C)$.

5. Show that for domains $\Omega \in C^{0,\alpha}$, $\alpha \in (0,1)$ the optimal embedding theorem

   $$W^{1,p}(\Omega) \hookrightarrow L^{\frac{dp}{d-p}}(\Omega), \quad 1 \leq p < d$$

cannot hold. To this aim, consider domain $\Omega \subset \mathbb{R}^2$ with one part $\Omega_1 = \{(x,y)\,|\, x \in (0,1), y \in (-x^\mu, x^\mu)\}$ (the endpoints are connected by a smooth curve lying in the set $x > 1$), $\mu > 1$ and consider functions $u(x,y) = x^{-a}$.