① V závislosti na parametru $a \in \mathbb{R}^+$ nalezněte rozvoj fce

9b

$$f(x) = \sin(ax)$$

do (sin.) Fourierovy řady na $(-\pi,\pi)$. Vyšetřete bodovou / stejn. / lok. stejn. konvergenci této řady.

Řešení

f lichá je lichá, proto $a_k = 0$ $k=0,1,\dots$ 1b

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(ax)\sin(kx)\,dx$$ 1b

a) Pokud $a \in \mathbb{N} \Rightarrow b_k = 0$ $k \in \mathbb{N}\setminus\{a\}$ 1b

a tedy $\sin ax = 1\cdot \sin ax$ $a \in \mathbb{N}$. 1b

b) Pokud $a \notin \mathbb{N}$, pak

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(ax)\sin(kx)\,dx = \frac{2}{\pi}\frac{1}{2}\int_0^{\pi}\left[\cos(k-a)x - \cos(k+a)x\right]$$

$$= \frac{1}{\pi}\left(\left[\frac{\sin(k-a)x}{k-a}\right]_0^{\pi} - \left[\frac{\sin(k+a)x}{k+a}\right]\right) = \frac{1}{\pi}\left(\frac{\sin(k-a)\pi}{k-a} - \frac{\sin(k+a)\pi}{k+a}\right)$$

$$= \frac{1}{\pi}\left(\frac{(-1)^{k+1}\sin(a\pi)}{k-a} + \frac{(-1)^{k+1}\sin(a\pi)}{k+a}\right) = \frac{(-1)^{k+1}}{\pi}\sin(a\pi)\,\frac{2k}{k^2-a^2}$$ 2b

$$\sin(ax) = \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}k\,\sin(a\pi)}{\pi(k^2-a^2)}\sin(kx) \quad x \in (-\pi,\pi)$$ 1b

1b

konvergence je bodová a lok. stejn. na $(-\pi,\pi)$ 1b (dle known vět]), v $x=\pi$ ale řada k ní nemá.

② Vhodně použitím Reziduové věty spočtěte (výsledek upravte, ať lze bez limity)

12b

$$\int_0^\infty \frac{\sin(ax)}{\sinh(4x)}\,dx \qquad a\in\mathbb{R}^+$$

Ověřte, že výsledný vzorec pro hodnotu integrálu splňuje $\lim_{a\to 0+}\int_0^\infty \frac{\sin(ax)}{\sinh 4x}\,dx = 0$ (tj. spočítejte příslušnou limitu).

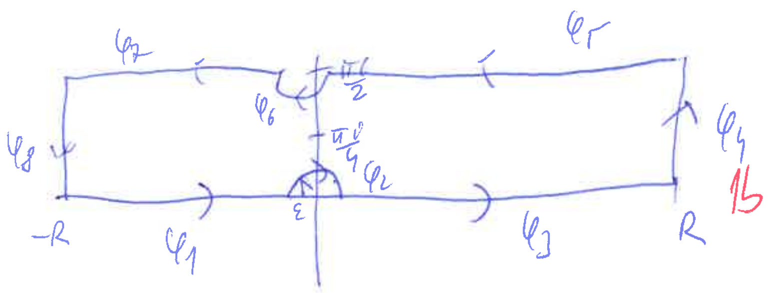
Řešení

$\forall a\in\mathbb{R}^+$ integrál konverguje: kolem 0 se chová jako $\frac{a}{4}$, v ∞ roste $\sinh(4x)$ exponenciálně, zatímco $\sin(ax)$ je omezený. 2b

Výsledná fce je sudá, proto 1b

$$I = \int_0^\infty \frac{\sin(ax)}{\sinh(4x)}\,dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\sin(ax)}{\sinh(4x)}\,dx = \frac{1}{2}\,\mathrm{Im}\;\underbrace{\mathrm{p.v.}\int_{-\infty}^{\infty}\frac{e^{iax}}{\sinh(4x)}\,dx}_{J}$$

1b 1b



1b

$$F(z) = \frac{e^{iaz}}{\sinh(4z)}$$

$$\int_{\varphi_1\oplus\cdots\oplus\varphi_8} F(z)\,dz = 2\pi i\,\mathrm{Res}_{\frac{\pi i}{4}}\frac{e^{iaz}}{\sinh(4z)} \quad \text{1b}$$

$\varphi_1(t) = t \quad t\in[-R,-\varepsilon]$

$\varphi_2(t) = \varepsilon e^{it} \quad t\in[\pi,0]$

$\varphi_3(t) = t \quad t\in[\varepsilon,R]$

$\varphi_4(t) = R+it \quad t\in[0,\frac{\pi}{2}]$

$\varphi_5(t) = t+\frac{\pi i}{2} \quad t\in[R,\varepsilon]$

$\varphi_6(t) = \frac{\pi i}{2}+\varepsilon e^{it} \quad t\in[0,-\pi]$

$\varphi_7(t) = t+\frac{\pi i}{2} \quad t\in[-\varepsilon,-R]$

$\varphi_8(t) = -R+it \quad t\in[\frac{\pi}{2},0]$

2b

$$\int_{\varphi_1\oplus\varphi_3} \xrightarrow[R\to+\infty]{\varepsilon\to0^+} J$$

$$\int_{\varphi_5\oplus\varphi_7} \xrightarrow[R\to\infty]{\varepsilon\to0^+} -e^{-\frac{a}{2}\pi} J$$

$$\int_{\varphi_2} \xrightarrow[\varepsilon\to0^+]{} -\pi i\,\mathrm{Res}_0\frac{e^{iaz}}{\sinh(4z)} = -\frac{\pi i}{4}$$

$$\int_{\varphi_6} \xrightarrow[\varepsilon\to0^+]{} -\pi i\,\mathrm{Res}_{\frac{\pi i}{2}}\frac{e^{iaz}}{\sinh(4z)} = \frac{-\pi i\cdot e^{-a\frac{\pi}{2}}}{4}$$

$$\int_{\varphi_4}+\int_{\varphi_8} \xrightarrow[R\to\infty]{} 0 \quad \left(\int_0^{\frac{\pi}{2}}\frac{e^{ia(R+it)}}{\sinh(4R+4it)}\,dt \xrightarrow[R\to\infty]{} 0\right)$$

3b

Celkem máme

$$\left(1-e^{-\frac{a}{2}\pi}\right)J = \frac{\pi i}{4}\left(1+e^{-\frac{a}{2}\pi}\right) + 2\pi i \frac{e^{-\frac{a\pi}{4}}}{-4}$$

$$J = \frac{\pi i}{4}\,\frac{1+e^{-\frac{a}{2}\pi}}{1-e^{-\frac{a}{2}\pi}} - \frac{\pi i}{2}\,\frac{e^{-\frac{a\pi}{4}}}{1-e^{-\frac{a}{2}\pi}}$$ 2b

$$I = \frac{\pi}{8}\,\frac{1+e^{-\frac{a}{2}\pi}}{1-e^{-\frac{a}{2}\pi}} - \frac{\pi}{4}\,\frac{e^{-\frac{a\pi}{4}}}{1-e^{-\frac{a}{2}\pi}}$$ 1b

Test $\lim_{a\to 0} \frac{\pi}{8}\left(\frac{1+e^{-\frac{a}{2}\pi}-2e^{-\frac{a\pi}{4}}}{1-e^{-\frac{a}{2}\pi}}\right)$

$$\overset{\text{l'H}}{=} \lim_{a\to 0} \frac{\pi}{8}\cdot\frac{\left(-\frac{\pi}{2}e^{-\frac{a}{2}\pi}+\frac{\pi}{2}e^{-\frac{a\pi}{4}}\right)}{\frac{\pi}{2}e^{-\frac{a}{2}\pi}} = 0.$$ 2b

(3) Uvažujte fci $f: \mathbb{R}^3 \to \mathbb{R}$

(15b)

$$f(x) = \frac{|x|^2}{1+|x|^4}$$

V jakých $L^p(\mathbb{R}^3)$-prostorech tato fce leží? Spočtěte její FT a vysvětlete výpočet.

Pozn. Připomeňme, že $F(T_f) = T_g$ kde $F(x) = f(r)$

$$g(|\xi|) = \lim_{R\to\infty}\int_0^R f(r)\,\frac{2r}{|\xi|}\,\sin(2\pi r|\xi|)\,dr,$$ (pokud limita $\exists$).

Odůvodněte, proč daná limita existuje!

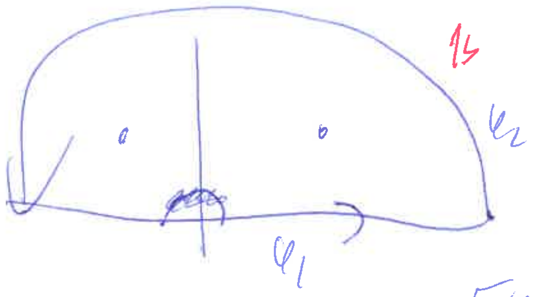
Řešení: $f \in L^2(\mathbb{R}^3)$ (navíc $f \in L^p(\mathbb{R}^3)$, $p > \frac{3}{2}$) 1b; proto $F(f) = \lim_{R\to\infty}\int_{B_R} f(x)\, e^{-2\pi i (x,\xi)}\,dx$. 1b

To, že existuje $\lim_{R\to\infty}\int_0^R f(r)\,\frac{2r}{|\xi|}\,\sin(2\pi r|\xi|)\,dr$ plyne z toho, že def. integrál $\left(\int_0^\infty \frac{P(r)}{Q(r)}\sin r\,dr\right)$ 2b existuje v konečném smyslu pro $\deg P = \deg Q - 1$ (popř. lze argumentovat důkazem Věty o FT radiálně symetrické funkce (pokud)).

Počítejme tedy (ve smyslu hlavní hodnoty limity)

$$\frac{1}{|\xi|}\int_0^\infty \frac{2r^3}{1+r^4}\sin(2\pi r|\xi|)\,dr = \frac{1}{|\xi|}\int_{-\infty}^{\infty}\frac{r^3}{1+r^4}\sin(2\pi r|\xi|)\,dr$$

$$= \frac{1}{|\xi|}\,\mathrm{Im}\;\underbrace{\mathrm{v.p.}\int_{-\infty}^{\infty}\frac{r^3 e^{i2\pi r|\xi|}}{1+r^4}\,dr}_{J}$$ 2b



$\varphi_1(t) = t \quad t \in [-R, R]$

$\varphi_2(t) \quad Re^{it} \quad t \in [0, \pi]$ 1b

$F(z) = \frac{z^3 e^{i 2\pi z |S|}}{1+z^4}$ 1b

$z^4 = -1$

$z = e^{-i\frac{\pi}{4} + j\frac{\pi}{2}} \quad j = 1,2$ (mej zgoraj)

$$\int_{\varphi_1 \oplus \varphi_2} F(z)\,dz = 2\pi i \left(\operatorname{Res}_{\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}} F(z) + \operatorname{Res}_{-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}} F(z) \right)$$ 1b

$$\int_{\varphi_1} F(z)\,dz \underset{R\to\infty}{\longrightarrow} J$$ 1b

$$\int_{\varphi_2} F(z)\,dz \underset{R\to\infty}{\longrightarrow} 0 \quad \text{(Jordanova lemma)}$$ 1b

$$\Rightarrow J = 2\pi i \left(\frac{1}{4} e^{i 2\pi |S| \left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)} + \frac{1}{4} e^{i 2\pi |S| \left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)} \right)$$

$$= \frac{1}{2}\pi i\, e^{-\sqrt{2}\pi |S|} \left(2 \cdot \cos(\sqrt{2}\pi |S|) \right)$$ 2b

$$\Rightarrow \mathcal{F}(F)(S) = \frac{\pi}{|S|} e^{-\sqrt{2}\pi |S|} \cos(\sqrt{2}\pi |S|) \quad (\in L^2(\mathbb{R}^3) \cap L^1(\mathbb{R}^3))$$

1b

(4) (18b)

Naj bo zaporedje distribucij

$T_n = n(\delta_{\frac{1}{n}} - \delta_0) \in \mathcal{D}'(\mathbb{R}), n \in \mathbb{N}.$

(i) Pokaži, da se gre za zaporedje temperiranih distribucij.

(ii) Poišči $T \in \mathcal{S}'(\mathbb{R})$ tako, da $T_n \to^* T$ v $\mathcal{S}'(\mathbb{R})$.

(iii) ~~Na podlagi neke ... sledi ...~~

Direktno pokaži, da $\mathcal{F}(T_n) \to^* \mathcal{F}(T)$

(iv) Na zadnji dveh po T_n ponovi (i)–(iii) za zaporedje

$\Theta_n := n^2(\delta_{\frac{1}{n}} - 2\delta_0 + \delta_{-\frac{1}{n}}).$

Rešitev

(i) $\delta_{\frac{1}{n}}$ in δ_0 sta temperirani distribuciji (distribuciji s kompaktnim nosilcem). 1b

(ii) Poglejmo za $\varphi \in \mathcal{S}(\mathbb{R}^N)$

$\langle T_n, \varphi\rangle = \langle n(\delta_{\frac{1}{n}} - \delta_0), \varphi\rangle = \frac{\varphi(\frac{1}{n}) - \varphi(0)}{\frac{1}{n}} \xrightarrow[n\to\infty]{} \varphi'(0) = \langle -\delta_0', \varphi\rangle.$ 1b 1b

Torej $T_n \to^* -\delta_0'$. 1b

(iii) Poglejmo

$\mathcal{F}(T_n) = n(\mathcal{F}(\delta_{\frac{1}{n}}) - \mathcal{F}(\delta_0))$ 1b, kjer $\mathcal{F}(\delta_{\frac{1}{n}}) = T_{e^{-2\pi i \frac{\xi}{n}}}$ in $\mathcal{F}(\delta_0) = T_1$.

$\langle \mathcal{F}(\delta_{\frac{1}{n}}), \varphi\rangle = \langle \delta_{\frac{1}{n}}, \mathcal{F}(\varphi)\rangle = \mathcal{F}(\varphi)(\tfrac{1}{n}) = \int_{\mathbb{R}} e^{-2\pi i \frac{\xi}{n}} \varphi(\xi)\, d\xi = \langle T_{e^{-2\pi i \frac{\xi}{n}}}, \varphi\rangle$ 2b

$\lim_{n\to\infty} \langle \mathcal{F}(T_n), \varphi\rangle = \lim_{n\to\infty} \int_{\mathbb{R}} n(e^{-2\pi i \frac{\xi}{n}} - 1)\varphi(\xi)\, d\xi$ 1b

$= \lim_{n\to\infty} \int_{\mathbb{R}} \left(\frac{e^{-2\pi i \frac{\xi}{n}} - 1}{\frac{1}{n}}\right)\varphi(\xi)\, d\xi \overset{\text{Izrek o dom. konv.}}{=} \int_{\mathbb{R}} (-2\pi i \xi)\varphi(\xi)\, d\xi = \langle T_{-2\pi i \xi}, \varphi\rangle.$ 1b 1b

(omejeno z const (ξ)) 1b

Na drugi strani $\langle \mathcal{F}(-\delta_0'), \varphi\rangle = \mathcal{F}(\varphi)'(0) = \int_{\mathbb{R}} (-2\pi i x)\varphi(x)\, dx = \langle T_{-2\pi i x}, \varphi\rangle.$

2b

(iv) Analog $G_n \in \mathcal{S}'(\mathbb{R}^n)$

a $\forall \varphi \in \mathcal{S}(\mathbb{R}^n)$ dolů 1b

$$\lim_{n\to\infty} \langle G_n, \varphi \rangle = \lim_{n\to\infty} \frac{\varphi(\frac{1}{n}) - 2\varphi(0) + \varphi(-\frac{1}{n})}{\frac{1}{n^2}}$$ 1b

$$= \lim_{n\to\infty} \left(\frac{\varphi(\frac{1}{n}) - \varphi(0)}{\frac{1}{n}} - \frac{\varphi(0) - \varphi(-\frac{1}{n})}{\frac{1}{n}} \right) \frac{1}{n}$$

$$= \lim_{n\to\infty} \frac{\varphi'(\xi_{n_1}) - \varphi'(\xi_{n_2})}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{\varphi(0) + \varphi'(0)\frac{1}{n} + \frac{1}{2}\varphi''(0)(\frac{1}{n})^2 + o(\frac{1}{n}) + (\varphi(0) + \varphi'(0)(-\frac{1}{n}) + \frac{1}{2}\varphi''(0)(\frac{1}{n})^2 + o(\frac{1}{n^2})) - 2\varphi(0)}{\frac{1}{n^2}}$$ 1b

$$= \varphi''(0) = \langle \delta_0'', \varphi \rangle .$$

Tedy $G_n \xrightarrow{*} \delta_0''$ 1b