Introduction to the theory of weak solutions for PDE's

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May 13, 2026

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"There is no general theory known concerning the solvability of all partial differential equations. Such a theory is extremely unlikely to exist, given the rich variety of physical, geometric, and probabilistic phenomena which can be modeled by PDE. Instead, research focuses on various particular partial differential equations that are important for applications within and outside of mathematics, with the hope that insight from the origins of these PDE can give clues to their solutions."

"Our effort will be largely devoted to proving mathematically the existence of solutions to various sorts of partial differential equations, and not to so much to deriving formulas for these solutions. This may seem wasted or misguided effort, but in fact mathematicians are like theologians: we regard existence the prime attribute of what we study. But unlike most theologians, we need not always rely upon faith alone."

Lawrence C. Evans: Partial differential equations.

Chapter 1

Introduction

1.1 Why we introduce the weak solution

We expect that the reader has at least basic knowledge of the classical theory of partial differential equations. The aim of the introductory part is to explain that the notion of classical solutions is sometimes not sufficient or even not natural with respect to the studied problem and why it is natural to replace it by the notion of weak solutions. From many possible examples we chose the following ones.

- On the example of the Dirichlet problem for the Poisson equation we show in Subsection 1.2.1 that the classical solution requires certain *a priori* smoothness of the data. If the data do not fulfil these smoothness requirements, it is necessary to generalize the notion of the solution. We shall follow one possible path which will lead us to the notion of the weak solution and. We shall also present basic problems we need to consider in these Lecture Notes. It also indicates us requirements on the function spaces in which we look for solutions.
- In Subsections 1.2.2 and 1.2.3 we present two situations, where the weak solution appears naturally. In other words, the weak solution of a partial differential equation (PDE) is here the primary notion and the classical solution is the secondary notion which was used rather due to historical reasons.
  - The first example are the necessary conditions for the existence of critical points of functionals as the length of a curve, the measure of a surface, the total energy of some physical system etc. The critical points are in the classical mechanics solutions of Euler–Lagrange equations. These equations in the classical setting are, however, not the primary notion, they are deduced from the condition:

    the Gateaux derivative is zero at the critical point in any direction.

    This is in fact equivalent with the claim that the critical point is a weak solution to the Euler–Lagrange equations.
  - The balance laws of continuum mechanics are formulated on arbitrary sufficiently "nice" subsets of the given domain (so-called control volumes) which is filled by the body. This is another example of a problem, where the notion of the weak solution is primary. We shall show that from the balance laws (mass, linear momentum, total energy) it is possible to obtain directly the weak formulation and the classical formulation is secondary and can be obtained only under the extra assumption of sufficient regularity of the data and the solution itself.

1.2 Several examples

1.2.1 Homogeneous Dirichlet problem for the Poisson equation

Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with the boundary $\partial\Omega$. We consider the following problem[1]

$$\begin{aligned} -\Delta u(x) &= f(x) && \forall x \in \Omega \\ u(x) &= 0 && \forall x \in \partial\Omega, \end{aligned} \tag{1.2}$$

[1]Let us recall the notation. If $\mathbf{q} = \big(q_1, \dots, q_d\big)^T$ is a vector-valued quantity (for example the heat flux), then the equation

$$\operatorname{div} \mathbf{q} = f \quad \text{in } \Omega \tag{1.1}$$

balances the flux of the quantity $\mathbf{q}$ over the boundary $\partial\Omega$ and the source term f. If the flux $\mathbf{q}$ is proportional to ∇u (it means, it is linearly dependent on ∇u), where u is a scalar quantity $u\colon \Omega \to \mathbb{R}$ (for example the temperature), then we obtain from (1.1) equation $(1.2)_1$, since after using $\mathbf{q} = -\nabla u$ (in our example, the heat flows from warmer parts to colder ones) in the left-hand side of equation (1.1) we get

$$\operatorname{div} \mathbf{q} = \sum_{i=1}^{d} \frac{\partial q_i}{\partial x_i} = -\sum_{i=1}^{d} \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i}\right) = -\sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2} = -\Delta u.$$

where $f\colon \Omega \to \mathbb{R}$ is a given function. The boundary condition is for simplicity chosen to be constant (and zero is just our choice since we may simply add/subtract any constant value without changing the equation).

A classical solution to problem (1.2) is a function $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ which fulfils pointwisely (or equivalently in the sense of continuous functions) (1.2). If u is a classical solution to (1.2), then we necessarily require $f \in \mathcal{C}(\Omega)$. If the function f on the right-hand side of the equation is not continuous (which may easily be the case for many problems coming from physics), then the notion of classical solutions is not sufficient. Since such problems are interesting from application point of view, it is important to formulate them mathematically correctly. This requires to introduce a more general definition of a solution to the given problem to obtain a well defined object which we can work with in such examples.

Let us assume that

$$f \in L^2(\Omega), \tag{1.3}$$

we multiply equation $(1.2)_1$ on an arbitrary function $\varphi \in \mathcal{C}_0^\infty(\Omega)$ and integrate the obtained equality over Ω. We get

$$-\int_\Omega \Delta u \varphi \,\mathrm{d}x = \int_\Omega f\varphi \,\mathrm{d}x \quad \forall \varphi \in \mathcal{C}_0^\infty(\Omega).$$

We rewrite the left-hand side by virtue of the Gauss Theorem (more precisely, by its consequence called the Green formula which describes the integration by parts in higher space dimensions). We also recall that the boundary integral over $\partial\Omega$ is zero, since the function φ has compact support in Ω. It reads

$$\int_\Omega \nabla u \cdot \nabla \varphi \,\mathrm{d}x = \int_\Omega f\varphi \,\mathrm{d}x \quad \forall \varphi \in \mathcal{C}_0^\infty(\Omega), \tag{1.4}$$

or

$$(\nabla u, \nabla \varphi)_{L^2(\Omega)} = (f, \varphi)_{L^2(\Omega)} \quad \forall \varphi \in \mathcal{C}_0^\infty(\Omega). \tag{1.5}$$

We could go on and transfer one more derivative from u onto φ. We shall not follow this path (it leads to so-called very weak solutions), we namely require that the space in which we look for a solution is the same (or at least close to the) space, where we take the function φ from. Particularly, we raise the following questions:

Question 1.2.1. Is it possible to set in (1.5) $\varphi = u$?

If yes, then it follows from (1.4)

$$\|\nabla u\|_{L^2(\Omega)}^2 = \int_\Omega |\nabla u|^2 \,\mathrm{d}x = (f, u)_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)}, \tag{1.6}$$

where we used in the last step Hölder's (or Cauchy–Schwartz) inequality.

Let us assume that

$$\exists c > 0,\ \forall u\colon\ \|u\|_{L^2(\Omega)} \leq c\,\|\nabla u\|_{L^2(\Omega)}, \tag{1.7}$$

or the norm of the function is controlled by the norm of its gradient. In general, it cannot be true; it is enough to consider a constant function. However, in our case we have $u|_{\partial\Omega} = 0$ which excludes all nontrivial constant functions. A question remains:

Question 1.2.2. For which functions we may expect (1.7) to hold?

If (1.7) holds, then it follows from (1.6)

$$\|\nabla u\|_{L^2(\Omega)} \leq c\,\|f\|_{L^2(\Omega)}.$$

This inequality and (1.3) imply that $\nabla u \in L^2(\Omega)$ which leads to the natural definition of the space

$$W^{1,2}(\Omega) = \left\{ v \in L^2(\Omega) \,|\, \forall i = 1, \dots, d\colon \frac{\partial v}{\partial x_i} \in L^2(\Omega) \right\}.$$

This, however, raises another question:

Question 1.2.3. How can we define derivatives for functions which are not continuous?

We further introduce the space

$$W_0^{1,2}(\Omega) = \left\{ v\ \in W^{1,2}(\Omega) \,|\ v|_{\partial\Omega} = 0 \right\}.$$

Recall, however, that for functions $v \in L^p(\Omega)$, $p \in [1, \infty]$ it does not make sense to speak about values of the function on the boundary, as the boundary $\partial\Omega$ is in standard situations a set of zero d-dimensional Lebesgue measure. Functions from $W^{1,2}(\Omega)$ form a subset in $L^2(\Omega)$. We can therefore also talk about derivatives, at least in a certain sense. The question is if this information is sufficient to speak about values on the boundary. Hence, it will be necessary to solve the following problem:

Question 1.2.4. Is is possible to speak about boundary values for functions from $W^{1,2}(\Omega)$? If yes, in which sense?

Let us still return to Question 1.2.1. It follows from (1.5) that the answer will be positive, provided there is a positive answer to the question:

Question 1.2.5. Are the smooth functions with compact support dense in $W_0^{1,2}(\Omega)$?

If the answer is affirmative, we call $u \in W_0^{1,2}(\Omega)$ the weak solution to problem (1.2) if the equality

$$(\nabla u, \nabla \varphi)_{L^2(\Omega)} = (f, \varphi)_{L^2(\Omega)} \,.$$

holds for all $\varphi \in W_0^{1,2}(\Omega)$. We introduced a notion of solution which requires less knowledge about the regularity of the functions (both the given and the sought ones). The application of the new definition of a solution immediately raises the following questions.

1. What can be said about existence and uniqueness of a weak solution? Is it possible to show a continuous dependence on the data of the problem? If yes in which norm? Altogether, these questions deal with generalization of the well-posedness in the sense of J. Hadamard. They are denoted as problems of existence, uniqueness and continuous dependence on the data for weak solutions.
2. Is it possible to get an extra information about the smoothness of the solution in case when the data are smoother than required for the existence of a solution? Under which conditions on the data the solution will be a classical one? Altogether, these questions are denoted as regularity problem for weak solutions.

We shall study these fundamental problems not only for the motivational example (1.2), but mostly for linear and nonlinear elliptic and later also parabolic and hyperbolic problems in the relevant chapters of these Lecture Notes.

Let us note that the problem of existence and uniqueness of weak solutions is simpler than the regularity problem. The reason is that the definition of weak solutions is based on more complex structures, function spaces etc. For linear elliptic problems it is enough to use the Riesz representation Theorem and its generalizations.

1.2.2 Problems of calculus of variation — necessary conditions

The calculus of variations deals with the study of critical points of functionals, it means of mappings from a Banach space (complete normed vector space), usually infinite dimensional, to the space of real numbers. Typically $\varphi\colon X \to \mathbb{R}$, where X is a function space (in the classical setting X is typically $\mathcal{C}^1((a,b)) \cap C([a,b])$ with zero values at endpoints or $\mathcal{C}^1(\Omega) \cap \mathcal{C}_0(\overline{\Omega})$, where the latter denotes the space of continuous functions up to the boundary of Ω which are equal to zero on $\partial\Omega$).

Let us recall a few basic notions.

Definition 1.2.6 — Local maximum (minimum). We say that the functional $\varphi\colon X \to \mathbb{R}$ has at the point $x_0 \in X$ a local maximum (or minimum), if

$$\exists \delta > 0, \forall x \in U_\delta(x_0) : \varphi(x_0) \geq \varphi(x) \text{ (or } \varphi(x_0) \leq \varphi(x)),$$

where $U_\delta(x_0) = \{x \in X |\ \|x - x_0\|_X < \delta\}$.

Definition 1.2.7 — Gateaux derivative. We say that the functional $\varphi\colon X \to \mathbb{R}$ has at the point $x_0 \in X$ Gateaux derivative, if $\forall h \in X$, $\|h\|_X = 1$ there exists the limit

$$\delta\varphi(x_0; h) = \lim_{t \to 0} \frac{\varphi(x_0 + th) - \varphi(x_0)}{t}.$$

If x_0 is the point of the local minimum (or maximum) of the functional φ, or in other words, if x_0 is the extremal point of the functional φ and if the limit $\delta\varphi(x_0; h)$ exists for some $h \in X$, then necessarily $\delta\varphi(x_0; h) = 0$. In particular, if φ has the Gateaux derivative at the extremal point $x_0 \in X$, then necessarily

$$\forall h \in X : \delta\varphi(x_0; h) = 0. \tag{1.8}$$

The proof of the previous claim is simple. We denote $g_h(t) = \varphi(x_0 + th)$. Since $g_h(t)$ has at 0 a local extremum, the theory of real functions implies that if $g_h'(0)$ exists, then $g_h'(0) = 0$. However, according to the definition of the Gateaux derivative it holds $g_h'(0) = \delta\varphi(x_0; h)$.

Let us mention several classical problems of calculus of variations.

Example 1.2.8 (Brachistochrone problem)**.** Let

$$\gamma_y = \big\{(x, y) \in \mathbb{R}^2 \,|\, x \in [0, a],\ y = y(x),\ y(0) = 0,\ y(a) = b\big\}$$

be a curve connecting the origin of the cartesian coordinate system with the point $[a, b]$, given as a graph of a function $y(x)$. The Brachistochrone problem (Problem of the fastest descent curve) is a problem to find such a function v from the suitable class of functions (in the classical setting)

$$Y = \big\{y \in \mathcal{C}^1((0, a)) \cap \mathcal{C}([0, a]) \,|\, y(0) = 0, y(a) = b\big\}$$

which minimizes the functional

$$T[y] = \frac{1}{\sqrt{g}} \int_0^a \sqrt{\frac{1 + (y'(x))^2}{b - y(x)}}\, \mathrm{d}x.$$

This functional provides the time needed for a mass point to reach from the point $[a, b]$ the origin on the curve $y(x)$ in the homogeneous gravitational field g. Any friction is neglected here.

Example 1.2.9 (Curve of the minimal length)**.** Using the same notations as in the previous example, the functional

$$L[y] = \int_0^a \sqrt{1 + (y'(x))^2}\, \mathrm{d}x$$

provides the length of the curve described as the graph of the function $y(x)$. The problem is to minimize $L[y]$ in a certain class of curves Y (typically, with fixed endpoints and with possible limitation that the curve lies in a given manifold).

Example 1.2.10 (Minimal surface)**.** Let $\Omega \subset \mathbb{R}^2$ be a (reasonable) set and let the function $g\colon \partial\Omega \to \mathbb{R}$ be given. The problem of the minimal surface is to find such a function $\omega\colon \Omega \to \mathbb{R}$ that ω minimizes the functional

$$A[u] = \int_\Omega \sqrt{1 + |\nabla u|^2}\, \mathrm{d}x$$

which provides the measure of the surface described by the function u, where the set of surfaces over which we minimize the function is described as

$$Y = \left\{ u \in \mathcal{C}^1(\Omega) \cap \mathcal{C}(\overline{\Omega}) \,|\, u|_{\partial\Omega} = g \right\},$$

where g is a given function.

The spaces Y introduced in Examples 1.2.8–1.2.10 are, however, not linear. We therefore choose in Examples 1.2.8 and 1.2.9

$$X = \mathcal{C}^1((0, a)) \cap \mathcal{C}_0([0, a]),$$

$\mathcal{C}_0([0, a])$ denotes here the space of continuous functions in $[0, a]$ such that $u(0) = u(a) = 0$. We look for the critical point v in the form

$$v(x) = v_0(x) + \xi(x), \tag{1.9}$$

where $\xi \in X$ and $v_0\colon\ [0, a] \to \mathbb{R}$ is a smooth function fulfilling the boundary conditions $v_0(0) = 0$ and $v_0(a) = b$. Similarly, we choose in Example 1.2.10

$$X = \mathcal{C}^1(\Omega) \cap \mathcal{C}_0(\overline{\Omega})$$

and look for the critical point (the extremal) ω in the form

$$\omega(x, y) = \omega_0(x, y) + \xi(x, y),$$

where $\xi \in X$ and $\omega_0\colon \overline{\Omega} \to \mathbb{R}$ is a smooth function fulfilling the boundary condition $\omega_0|_{\partial\Omega} = g$.

For Example 1.2.9 we show that the requirement that the Gateaux derivative (1.8) is zero corresponds to the requirement to find a weak solution of the Euler–Lagrange equation; the Euler–Lagrange equation (in the classical formulation) is in this case a secondary notion, since it is deduced under the assumption of higher regularity of the considered functions.

It holds for arbitrary $\varphi \in X$

$$\begin{aligned}\delta L(v_0 + \xi; \varphi) &= \frac{\mathrm{d}}{\mathrm{dt}} \int_0^a \sqrt{1 + ((v_0(x) + \xi(x))' + t\varphi'(x))^2}\, \mathrm{d}x \bigg|_{t=0} \\ &= \int_0^a \frac{(v'(x) + t\varphi'(x))\, \varphi'(x)}{\sqrt{1 + (v'(x) + t\varphi'(x))^2}}\, \mathrm{d}x \bigg|_{t=0} = \int_0^a \frac{v'(x)\varphi'(x)}{\sqrt{1 + (v'(x))^2}}\, \mathrm{d}x.\end{aligned}$$

We rewrite in this case condition (1.8) as

$$\forall \varphi \in \mathcal{C}^1((0, a)) \cap \mathcal{C}_0([0, a])\colon \int_0^a \frac{v'(x)\varphi'(x)}{\sqrt{1 + (v'(x))^2}}\, \mathrm{d}x = 0. \tag{1.10}$$

Assuming higher regularity of the considered functions, for example if $\left(\frac{v'(x)}{\sqrt{1+(v'(x))^2}}\right)' \in \mathcal{C}((0, a))$, we get from the previous equation after integration by parts the relation

$$\forall \varphi \in \mathcal{C}_0([0, a])\colon\ -\int_0^a \left(\frac{v'(x)}{\sqrt{1 + (v'(x))^2}}\right)' \varphi(x)\, \mathrm{d}x = 0. \tag{1.11}$$

This relation already implies the pointwise validity of the Euler–Lagrange equation[2]

$$-\left(\frac{v'(x)}{\sqrt{1+(v'(x))^2}}\right)' = 0. \tag{1.12}$$

It is reasonable to ask, why we do not call the weak solution to equation (1.12) a function $v \in \mathcal{C}^1((0,a)) \cap \mathcal{C}_0([0,a])$ fulfilling equality (1.10), and why we rather work with more general functions[3]. A very serious objection against this possibility is the fact that $\mathcal{C}^1(\Omega)$ is not complete with respect to the integral norm $\|u\|_{\mathcal{C}^1(\Omega),\int} = \int_\Omega |u'(x)|\,\mathrm{d}x$, while the integral norm is for the given problem natural.

Assuming v in the form (1.9), then the function ξ from the aforementioned decomposition is a suitable test function in quality (1.10). Indeed, if we set $\xi = \varphi$, we get

$$\int_0^a \frac{v'(x)\varphi'(x)}{\sqrt{1+(v'(x))^2}}\,\mathrm{d}x = 0 \Longrightarrow \int_0^a \frac{(v'(x))^2}{\sqrt{1+(v'(x))^2}}\,\mathrm{d}x = \int_0^a \frac{v'(x)v_0'(x)}{\sqrt{1+(v'(x))^2}}\,\mathrm{d}x.$$

This equality and the fact that $\frac{|v'(x)|}{\sqrt{1+(v'(x))^2}} \le 1$ can be used to obtain the estimate

$$\begin{aligned}\int_0^a |v'|\,\mathrm{d}x &= \int_0^a \frac{|v'|}{(1+|v'|^2)^{\frac14}}(1+|v'|^2)^{\frac14}\,\mathrm{d}x \\ &\le \frac12\int_0^a \frac{|v'|^2}{\sqrt{1+|v'|^2}}\,\mathrm{d}x + \frac12\int_0^a \sqrt{1+|v'|^2}\,\mathrm{d}x \\ &\le \frac12\int_0^a \frac{|v'|}{\sqrt{1+|v'|^2}}|v_0'|\,\mathrm{d}x + \frac12\int_0^a \sqrt{1+|v'|^2}\,\mathrm{d}x \\ &\le \frac12\int_0^a |v_0'|\,\mathrm{d}x + \frac{a}{2} + \frac12\int_0^a |v'|\,\mathrm{d}x.\end{aligned}$$

Altogether, we get

$$\int_0^a |v'|\,\mathrm{d}x \le \int_0^a |v_0'|\,\mathrm{d}x + a \tag{1.13}$$

which shows that the L^1-norm of the derivative is a natural norm for the given problem.

At the end we present one claim connecting the present and the previous subsections.

Lemma 1.2.11 — Variational formulation of the Poisson equation. The function $u \in W_0^{1,2}(\Omega)$ is an extremal point (local minimum) of the functional

$$\phi[u] = \frac12\int_\Omega |\nabla u|^2\,\mathrm{d}x - \int_\Omega fu\,\mathrm{d}x,$$

if and only if it is a weak solution of problem (1.2), i.e., it fulfills equality (1.4) for all $\varphi \in W_0^{1,2}(\Omega)$.

Proof. **Step 1:** Proof of implication "⇒"
If $u \in W_0^{1,2}(\Omega)$ is an extremal point of the functional ϕ, then we have $\delta\phi(u;\varphi) = 0$ for all $\varphi \in W_0^{1,2}(\Omega)$. Since

$$\begin{aligned}\phi[u+t\varphi] = \frac12\int_\Omega |\nabla u|^2\,\mathrm{d}x - \int_\Omega fu\,\mathrm{d}x + t\left\{\int_\Omega \nabla u\cdot\nabla\varphi - f\varphi\,\mathrm{d}x\right\} \\ + \frac{t^2}{2}\int_\Omega |\nabla\varphi|^2\,\mathrm{d}x,\end{aligned} \tag{1.14}$$

we immediately get that

$$\delta\phi(u;\varphi) = (\nabla u, \nabla\varphi)_{L^2(\Omega)} - (f,\varphi)_{L^2(\Omega)}\,.$$

This proves the first implication.

Step 2: Proof of implication "⇐"
On the other hand, using (1.14) with $t = 1$ we easily see that (1.4) implies

$$\phi[u+\varphi] \ge \phi[u] \qquad \forall\varphi \in W_0^{1,2}(\Omega).$$

■

This example clearly shows that if we understand the elliptic problem as Euler–Lagrange equation of the corresponding variational problem, then the correct formulation is the weak one, while the classical formulation is a consequence of a further assumption, namely that the solution is sufficiently regular.

[2] Here we used the following claim: If $u \in \mathcal{C}(\Omega)$ fulfills for any $\varphi \in \mathcal{C}_0^\infty(\Omega)$ the equality $\int_\Omega u(x)\varphi(x)\,\mathrm{d}x = 0$, then $u = 0$ for all $x \in \Omega$.

[3] The first candidate for a suitable function space could be the Sobolev space $W^{1,1}((0,a)) = \{u \in L^1((0,a)) \,|\, u' \in L^1((0,a))\}$ or rather the space of functions with bounded variation.

1.2.3 Balance laws of continuum mechanics

Equations which describe the body movement in the framework of continuum mechanics and thermodynamics are based on the balance laws, whose validity is required for any open subset $\mathcal{B}$ of the domain Ω filled in by the body. The general formulation of these balance laws has the form ($\boldsymbol{\nu}$ is the external unit normal vector to $\partial\Omega$)

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_{\mathcal{B}} D(t,x)\,\mathrm{d}x + \int_{\partial\mathcal{B}} \mathbf{F}(t,x)\cdot\boldsymbol{\nu}\,\mathrm{d}S = \int_{\mathcal{B}} P(t,x)\,\mathrm{d}x, \tag{1.15}$$

where D denotes the density of the physical quantity (mass ρ, components of the linear momentum vector $\rho\mathbf{v}$, components of the angular momentum vector $\rho\mathbf{v}\times\mathbf{x}$, specific total energy $E=\frac{1}{2}|\mathbf{v}|^2+e$ with e the specific internal energy), $\mathbf{F}$ is the corresponding flux of this quantity through the boundary and P is the volume production of the given quantity.

To have better understanding of the functions D, $\mathbf{F}$ and P, let us present the balance of mass

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_{\mathcal{B}} \rho(t,x)\,\mathrm{d}x + \int_{\partial\mathcal{B}} \rho(t,x)\mathbf{v}(t,x)\cdot\boldsymbol{\nu}\,\mathrm{d}S = 0, \tag{1.16}$$

the balance of linear momentum ($\mathbb{T}$ is the stress tensor, $\mathbf{b}$ the volume external force)

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_{\mathcal{B}} \rho(t,x)\mathbf{v}(t,x)\,\mathrm{d}x + \int_{\partial\mathcal{B}} \left(\rho(t,x)\mathbf{v}(t,x)\otimes\mathbf{v}(t,x)-\mathbb{T}(t,x)\right)\boldsymbol{\nu}\,\mathrm{d}S = \int_{\Omega} \rho(t,x)\mathbf{b}(t,x)\,\mathrm{d}x, \tag{1.17}$$

and the balance of total energy ($\mathbf{q}$ is the heat flux, r is the density of the heat sources)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\int_{\mathcal{B}} \rho(t,x)\left(\frac{|\mathbf{v}(t,x)|^2}{2}+e(t,x)\right)\mathrm{d}x + \int_{\partial\mathcal{B}} \left(\rho(t,x)\left(\frac{|\mathbf{v}(t,x)|^2}{2}+e(t,x)\right)\mathbf{v}(t,x)+\mathbf{q}(t,x)-\mathbb{T}(t,x)\mathbf{v}(t,x)\right)\cdot\boldsymbol{\nu}\,\mathrm{d}S \\ = \int_{\Omega}\Big(\rho(t,x)\mathbf{b}(t,x)\cdot\mathbf{v}(t,x)+\rho(t,x)r(t,x)\Big)\,\mathrm{d}x.\end{aligned} \tag{1.18}$$

The standard procedure of deducing equations of continuum thermodynamics is based on rewriting (1.15) by virtue of the Green identity and the interchange of derivative and integral to the form

$$\int_{\mathcal{B}} \left(\partial_t D + \operatorname{div}\mathbf{F} - P\right)(t,x)\,\mathrm{d}x = 0,$$

where $\mathcal{B}\subset\overline{\mathcal{B}}\subset\Omega$ is an arbitrary control volume. Assuming the integrand continuous, the previous equality implies that it holds $\forall x\in\Omega$, $\forall t\in(0,T)$

$$\partial_t D(t,x) + \operatorname{div}\mathbf{F}(t,x) = P(t,x). \tag{1.19}$$

It is evident that (1.19) requires differentiability of functions which is in (1.15) not needed. The weak formulation is then "usually" deduced from (1.19) and the boundary conditions.

Our goal is to show that the general form of the balance law (1.15) implies directly the weak formulation. To this aim we shall need the following result; its proof can be found, e.g., in (Evans and Gariepy, 1992, Section 3.4.3 Theorem 2).

Theorem 1.2.12 — Coarea formula. Let $\eta\colon\mathbb{R}^d\to\mathbb{R}$ be a Lipschitz function and let $v\in L^1(\mathbb{R}^d)$. Then it holds

1. $v|_{\{x\in\mathbb{R}^d|\ \eta(x)=r\}}$ is integrable in the sense of the $(d-1)$ dimensional (Hausdorff) measure
2. $\int_{\mathbb{R}^d} v(x)\,|\nabla\eta(x)|\,\mathrm{d}x = \int_{\mathbb{R}}\left(\int_{\{x\in\mathbb{R}^d|\ \eta(x)=r\}} v(x)\,\mathrm{d}S\right)\mathrm{d}r.$

We now follow the procedure proposed in book Feireisl (2004). Let η be a smooth non-negative function with compact support, hence $\eta\in\mathcal{C}_0^\infty(\Omega)$. Then the set $\{x\in\Omega|\ \eta(x)>r\}$ has for almost every $r\in(0,+\infty)$ a smooth boundary $\{x\in\Omega|\ \eta(x)=r\}$ as follows by the Sard Theorem (see, e.g., (Lukeš and Malý, 1995, Theorem 34.17)). Let us now consider the general form of the balance law (1.15) with the control volume $\mathcal{B}=\{x\in\Omega|\ \eta(x)>r\}$ for $r\in(-\infty,+\infty)$ and integrate the balance law with such control volume from 0 to $+\infty$. We have (recall that the outer unit normal vector to the set $\{x\in\Omega|\ \eta(x)=r\}$ is $-\frac{\nabla\eta}{|\nabla\eta|}$).

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\int_0^{+\infty}\int_{\mathcal{B}=\{x\in\Omega\,|\,\eta(x)>r\}} D(t,x)\,\mathrm{d}x\,\mathrm{d}r - \int_0^{+\infty}\int_{\partial\mathcal{B}=\{x\in\Omega\,|\,\eta(x)=r\}} \mathbf{F}(t,x)\cdot\frac{\nabla\eta(x)}{|\nabla\eta(x)|}\,\mathrm{d}S\,\mathrm{d}r \\ = \int_0^{+\infty}\int_{\mathcal{B}=\{x\in\Omega\,|\,\eta(x)>r\}} P(t,x)\,\mathrm{d}x\,\mathrm{d}r.\end{aligned}$$

We now apply the Coarea formula from Theorem 1.2.12 to the second term and rewrite the first and the third ones based on the following calculation ($h(t,x) = D(t,x)$ or $h(t,x) = P(t,x)$)

$$\int_0^{+\infty} \int_{\mathcal{B}=\{x\in\Omega \,|\, \eta(x)>r\}} h(t,x)\,\mathrm{d}x\,\mathrm{d}r = \int_0^{+\infty} \int_\Omega \operatorname{sign}(\eta(x)-r)^+\, h(t,x)\,\mathrm{d}x\,\mathrm{d}r$$
$$= \int_\Omega h(t,x) \int_0^{+\infty} \operatorname{sign}(\eta(x)-r)^+\,\mathrm{d}r\,\mathrm{d}x = \int_\Omega h(t,x)\eta(x)\,\mathrm{d}x;$$

we get for arbitrary $\eta \in \mathcal{C}_0^\infty(\Omega)$ non-negative

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega D(t,x)\eta(x)\,\mathrm{d}x - \int_\Omega \mathbf{F}(t,x)\cdot\nabla\eta(x)\,\mathrm{d}x = \int_\Omega P(t,x)\eta(x)\,\mathrm{d}x.$$

It is not difficult to verify that this implies the validity of (1.19) in the sense of distributions (i.e., also for arbitrary $\eta \in C_0^\infty(\Omega)$).

Before we touch the questions of solvability of partial differential equations in the weak sense and qualitative behaviour of these solutions, we introduce the Sobolev spaces $W^{k,p}(\Omega)$ and we study their properties. We do it in two steps. First, we only list their most important properties and skip all technicalities needed in the proofs, then we apply them to the study of linear problems. We return to them later and give all details of the proofs of the main properties as the density of smooth functions $\mathcal{C}_0^\infty(\Omega)$, $\mathcal{C}^\infty(\Omega)$ or $\mathcal{C}^\infty(\overline{\Omega})$ in different types of Sobolev spaces, question of interpretation of the boundary values of Sobolev spaces (trace theorems), the connection of Sobolev spaces and other spaces (theorems on continuous and compact embeddings) and finally also to different equivalent definitions of the Sobolev spaces. Then we apply our knowledge to a few classes of nonlinear problems. Similarly we proceed for the evolutionary problems (parabolic and hyperbolic equations and the corresponding functions spaces, called Lebesgue–Bochner or Sobolev–Bochner spaces).

Exercise 1.2.13. Deduce both the weak and the strong formulations of the Euler–Lagrange equations for the function

$$L(x,u,\nabla u) = \frac{1}{p}|u|^p + \frac{1}{p}|\nabla u|^p,$$

where $|\nabla u|^p = \left(\sum_{i=1}^d |\frac{\partial u}{\partial x_i}|^2\right)^{\frac{p}{2}}$.

Exercise 1.2.14. Prove that

$$(u,v) = \int_\Omega (uv + \nabla u\cdot\nabla v)\,\mathrm{d}x$$

is a scalar product on the space

$$V = \{w \in C^1(\overline{\Omega}) \,|\, w = 0 \text{ on } \partial\Omega\},$$

but this space is not complete with respect to the norm associated to this scalar product.

Exercise 1.2.15. Based on computations below Theorem 1.2.12 deduce the weak and strong formulation of the balances of mass, linear momentum and total energy introduced in (1.16)–(1.18).

Chapter 2

An easy guide to Sobolev spaces

We saw in the previous chapter that sometimes the classical solution for different kinds of partial differential equations may not exist or it is not the appropriate type of solution with respect to the problem. On the other hand, it follows from the examples therein that in these situations it is possible to speak about a generalized solution which will be in what follows called a *weak solution*. To introduce this notion we must build the theory of the relevant function spaces, so called Sobolev spaces $W^{k,p}(\Omega)$ which play the main role in the modern theory of partial differential equations. In this chapter we only introduce them and explain their most important properties. Several finer results as well as all long and technical proofs will be presented later, in Chapter 6.

We assume in this chapter that the reader knows the elements of the theory of Lebesgue integral and Lebesgue spaces and knows the main properties of spaces of continuous, Hölder continuous and continuously differentiable functions. For completeness and for the reader's convenience we present a short overview of these results in Appendix A. We also assume that the reader has sufficient knowledge of functional analysis; a short overview of the main important results is presented in Appendix B.

2.1 Definitions, basic properties

Let us first recall the definition of the multiindex.

Notation 2.1.1 (Multiindex)**.** The ordered d–tuple $\alpha = (\alpha_1, \dots, \alpha_d)$, $\alpha_i \in \mathbb{N}_0$, is called the multiindex. The length of the multiindex is denoted $|\alpha|$ and is defined as $|\alpha| = \alpha_1 + \cdots + \alpha_d$.

In what follows we also use a shorten notation for partial derivatives.

Notation 2.1.2 (Partial derivative written by a multiindex)**.** The symbol $D^\alpha \phi$ denotes the partial derivative of a function ϕ

$$D^\alpha \phi(x) := \frac{\partial^{|\alpha|} \phi(x)}{\partial x_1^{\alpha_1} \, \dots \, \partial x_d^{\alpha_d}}.$$

We can now introduce the basic notion in the modern theory of partial differential equations[1].

Definition 2.1.3 — Weak derivative. Let $\Omega \subset \mathbb{R}^d$ be an open (possibly also unbounded) set and $\alpha = (\alpha_1, \dots, \alpha_d)$ be a multiindex. Let $u, v_\alpha \in L^1_{\mathrm{loc}}(\Omega)$. We say that v_α is a weak derivative of u with respect to x^α, if it holds for any $\phi \in \mathcal{C}_0^\infty(\Omega)$

$$\int_\Omega u D^\alpha \phi \,\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega v_\alpha \phi \,\mathrm{d}x.$$

The following properties of the weak derivative are more or less evident.

[1] The weak derivative is a special case of a more general notion — the distributional derivative. If $u \in L^1_{\mathrm{loc}}(\Omega)$, we may assign to the function u the regular distribution T_u defined as

$$\forall \varphi \in \mathcal{D}(\Omega): \ \langle T_u, \varphi \rangle := \int_\Omega u\varphi \,\mathrm{d}x,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $(\mathcal{D}(\Omega))^*$ and $\mathcal{D}(\Omega)$. Every distribution can be differentiated infinitely many times. A distribution G is the derivative of a distribution T with respect to x^α, if

$$\forall \varphi \in \mathcal{D}(\Omega): \ \langle T, D^\alpha \varphi \rangle = (-1)^{|\alpha|} \langle G, \varphi \rangle .$$

In particular, if $G = G_v$ and $T = T_u$ are regular distributions, it holds

$$\forall \varphi \in \mathcal{D}(\Omega): \ \int_\Omega u D^\alpha \varphi \,\mathrm{d}x = \langle T_u, D^\alpha \varphi \rangle = (-1)^{|\alpha|} \langle G_v, \varphi \rangle = (-1)^{|\alpha|} \int_\Omega v\varphi \,\mathrm{d}x;$$

it means that $v = D^\alpha u$ in the weak sense. While the derivative of a distribution exists for any order, it may not be the case for weak derivatives. On the other hand, if the weak derivative of the second (or higher) order exists and $\Omega \subset \mathbb{R}^d$ with $d \geq 2$, then the derivative is independent of the order of differentiation, see also Exercise 2.1.10 below.

Lemma 2.1.4 — Connection between weak and classical derivative I. The following claims hold.

1. Let $u \in \mathcal{C}^k(\Omega)$. Then for any $|\alpha| \leq k$ the classical and weak derivatives coincide.
2. The weak derivative is (in the sense of equality in $L^1_{\text{loc}}(\Omega)$, thus almost everywhere) given uniquely.

Proof. We leave the proof of these claims to a kind reader as a useful exercise. ■

Remark 2.1.5. If the classical derivative is continuous, it is necessarily equal to the weak derivative. Therefore, we shall use the same notation for both; if v_α is a weak derivative of u with respect to x^α, we shall write $D^\alpha u = v_\alpha$.

We are ready to present the most important definition of this chapter.

Definition 2.1.6 — Sobolev spaces. Let $\Omega \subset \mathbb{R}^d$ be an open set, $k \in \mathbb{N}$ and $p \in [1, \infty]$. The Sobolev space $W^{k,p}(\Omega)$ is defined as

$$W^{k,p}(\Omega) := \{u \in L^p(\Omega) \,|\, \forall |\alpha| \leq k \,|\, D^\alpha u \in L^p(\Omega)\}.$$

We endow this space with the norm

$$\|u\|_{k,p} = \|u\|_{W^{k,p}(\Omega)} := \begin{cases} \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}} & \text{for } p \in [1, \infty), \\ \max_{|\alpha| \leq k} \|D^\alpha u\|_{L^\infty(\Omega)} & \text{for } p = \infty. \end{cases}$$

If it is clear from the context on which set we work, we shall use the shorten notation $\|\cdot\|_{k,p}$. If there is a danger of ambiguity, we shall use the full notation $\|\cdot\|_{W^{k,p}(\Omega)}$. Similarly as in the case of the $L^p(\Omega)$ spaces, the elements of $W^{k,p}(\Omega)$ are in fact classes of functions which differ on a set of measure zero.

Remark 2.1.7. It is possible to define the Sobolev spaces for $k \in \mathbb{N}_0$ which means that the case $k = 0$ is included. For $k = 0$ we identify the Sobolev space with the Lebesgue space, i.e.,

$$W^{0,p}(\Omega) := L^p(\Omega).$$

Remark 2.1.8. We also often shorten the notation for partial derivatives of u. For $u \in W^{k,p}(\Omega)$, we define for $m = 1, \ldots, k$ the vector (the tensor of the m-th order) $\nabla^m u : \Omega \to \mathbb{R}^{d^m}$ as follows

$$[\nabla^m u]_{i_1 \cdots i_m} := \frac{\partial^m u}{\partial x_{i_1} \cdots \partial x_{i_m}}, \qquad \text{where } i_l = 1, \ldots, d.$$

If $m = 1$, we write shortly $\nabla u := \nabla^1 u$.

The correctness of Definition 2.1.6 is summarized in the following theorem.

Theorem 2.1.9 — Sobolev norm. The space $W^{k,p}(\Omega)$ is a normed linear space.

Proof. The space $W^{k,p}(\Omega)$ is clearly a linear space (cf. Exercise 2.1.10 below). It is therefore enough to verify that $\|\cdot\|_{k,p}$ is a norm. We consider only the case $p \in [1, \infty)$, the proof for $p = \infty$ is left for a kind reader as a useful exercise. We check step by step that $\|\cdot\|_{k,p}$ satisfies all the axioms of a norm.

Step 1: Property 1. of the norm
It evidently holds for any $u \in W^{k,p}(\Omega)$ that

$$0 \leq \|u\|_{k,p} < \infty.$$

Moreover, if $\|u\|_{k,p} = 0$, then also $\|u\|_p = 0$ and therefore (property of the $\|\cdot\|_{L^p(\Omega)}$-norm) we also have $u = 0$ almost everywhere in Ω , i.e., u is equivalent to a zero function. The opposite implication is straightforward. Whence it holds

$$u = 0 \quad \Longleftrightarrow \quad \|u\|_{k,p} = 0.$$

Step 2: Property 2. of the norm
The weak derivative satisfies $D^\alpha(\lambda u) = \lambda D^\alpha u$ (cf. Exercise 2.1.10), further also $\|\lambda D^\alpha u\|_p = |\lambda| \|D^\alpha u\|_p$. Altogether, we have

$$\begin{aligned} \|\lambda u\|_{k,p} &= \left(\sum_{|\alpha| \leq k} \|D^\alpha(\lambda u)\|^p_p \right)^{\frac{1}{p}} = \left(|\lambda|^p \sum_{|\alpha| \leq k} \|D^\alpha u\|^p_p \right)^{\frac{1}{p}} \\ &= |\lambda| \|u\|_{k,p}, \end{aligned}$$

and we verified that the proposed norm is positively 1-homogeneous.

Step 3: Property 3. of the norm
Weak derivative is clearly linear $D^\alpha(u+v) = D^\alpha u + D^\alpha v$ (cf. Exercise 2.1.10) and for the L^p-norm, the Minkowski (triangle) inequality holds (cf. Theorem A.3.10)

$$\|u+v\|_p \le \|u\|_p + \|v\|_p.$$

Moreover, also the "discrete" Minkowski inequality holds, i.e., we have for any non-negative $\{a_n, b_n\}_{n=0}^m$

$$\left(\sum_{n=0}^m (a_n+b_n)^p\right)^{\frac{1}{p}} \le \left(\sum_{n=0}^m a_n^p\right)^{\frac{1}{p}} + \left(\sum_{n=0}^m b_n^p\right)^{\frac{1}{p}}.$$

This inequality yields

$$\begin{aligned}\|u+v\|_{k,p} &= \left(\sum_{|\alpha|\le k} \|D^\alpha u + D^\alpha v\|_p^p\right)^{\frac{1}{p}} \le \left(\sum_{|\alpha|\le k} \left(\|D^\alpha u\|_p + \|D^\alpha v\|_p\right)^p\right)^{\frac{1}{p}} \\ &\le \left(\sum_{|\alpha|\le k} \|D^\alpha u\|_p^p\right)^{\frac{1}{p}} + \left(\sum_{|\alpha|\le k} \|D^\alpha u\|_p^p\right)^{\frac{1}{p}} = \|u\|_{k,p} + \|v\|_{k,p};\end{aligned}$$

we thus verified the triangle inequality. It follows from Steps 1–3 that $\|\cdot\|_{k,p}$ is a norm. ■

The following exercise contains elementary properties of the weak derivative. Their proofs are easy, however, we strongly recommend the reader to perform them in detail.

Exercise 2.1.10 (Properties of weak derivative)**.** Show that it holds for arbitrary two functions $u, v \in W^{k,p}(\Omega)$, where $k \in \mathbb{N}$, and an arbitrary multiindex α satisfying $|\alpha| \le k$:

1. $D^\alpha u \in W^{k-|\alpha|,p}(\Omega)$ and $D^\alpha(D^\beta u) = D^\beta(D^\alpha u) = D^{\alpha+\beta}u$, whenever $|\alpha| + |\beta| \le k$
2. $\lambda u + \mu v \in W^{k,p}(\Omega)$ and $D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v$ whenever $\lambda, \mu \in \mathbb{R}$
3. if $\widetilde{\Omega} \subset \Omega$ is open, then $u \in W^{k,p}(\widetilde{\Omega})$
4. if $\eta \in \mathcal{C}^\infty(\overline{\Omega})$, then $\eta u \in W^{k,p}(\Omega)$ and it holds that

$$D^\alpha(\eta u) = \sum_{\{\beta \mid \forall i=1,\dots,d,\, \beta_i \le \alpha_i\}} \binom{\alpha}{\beta} D^\beta \eta D^{\alpha-\beta} u,$$

where $\binom{\alpha}{\beta} := \prod_{i=1}^d \binom{\alpha_i}{\beta_i}$.

Let us now present several typical examples showing which functions belong or do not belong to the spaces $W^{k,p}(\Omega)$. The first example illustrates that Sobolev functions cannot have a jump across a $(d-1)$–dimensional manifold.

Example 2.1.11. The function

$$u(x) := \begin{cases} x & \text{in } (0,1) \\ 2 & \text{in } [1,2) \end{cases}$$

is not an element of $W^{1,p}((0,2))$, because the weak derivative, if it had existed, would have been equal to the classical one in the intervals $(0,1)$ and $(1,2)$; the classical derivative is the function

$$v(x) := \begin{cases} 1 & \text{in } (0,1) \\ 0 & \text{in } (1,2). \end{cases}$$

This function, however, is not a weak derivative of[2] u, but it is a weak derivative of a function

$$\tilde{u}(x) = \begin{cases} x & \text{in } (0,1) \\ 1 & \text{in } [1,2). \end{cases}$$

Generally, a function which has a jump discontinuity across a $(d-1)$–dimensional manifold in $\Omega \subset \mathbb{R}^d$ does not have a weak derivative in Ω.

[2]The function u indeed possesses a distributional derivative. It is equal to the distribution $T_{\chi_{(0,1)}} + \delta_1$, where χ_I is the characteristic function of the interval I and δ_s is the Dirac distribution with the support at the point s. But this distribution is not regular and the function u does not possess a weak derivative. The function u, however, belongs to the space $BV((0,2))$, i.e., to the space of functions with bounded variation. The space $BV(\Omega)$ is defined as a subspace of the function space $L^1(\Omega)$ for which all distributional partial derivatives of the first order are Radon measures, cf. (Lukeš and Malý, 1995, Section 21).

The second example shows a typical behaviour near a singularity.

Example 2.1.12. Let $\Omega = B_1(0) \subset \mathbb{R}^d$. Then $u(x) := \frac{1}{|x|^\alpha} \in W^{1,p}(\Omega) \Leftrightarrow \alpha < \frac{d-p}{p}$. We see that also unbounded functions belong to some $W^{1,p}(\Omega)$. Note that for $\alpha < \frac{d-p}{p}$ the function $u \in L^q(\Omega)$ for every $q \in [1, p^*)$, where $p^* := \frac{dp}{d-p}$ (compare with the Embedding Theorem 2.4.5).

Solution. Consider the function

$$u_i(x) := -\alpha \frac{x_i}{|x|^{\alpha+2}}$$

and show that $u_i(x) = \frac{\partial}{\partial x_i} \frac{1}{|x|^\alpha}$ (in the weak sense). To this aim, apply the definition of the weak derivative and consider any $\varphi \in \mathcal{C}_0^\infty(\Omega)$ (it is in fact enough to take $\varphi \in \mathcal{C}_0^1(\Omega)$)

$$-\int_\Omega u \frac{\partial \varphi}{\partial x_i} \, \mathrm{d}x = -\lim_{\varepsilon \to 0_+} \int_{\Omega \setminus B_\varepsilon(0)} u \frac{\partial \varphi}{\partial x_i} \, \mathrm{d}x$$

and apply the Green formula (integration by parts in higher dimensions) on the second integral; (it is possible, as both functions are sufficiently smooth on $\Omega \setminus B_\varepsilon(0)$). Then compute the limit $\varepsilon \to 0_+$. □

The last example illustrates the fact that the set of points, where the Sobolev function is discontinuous or unbounded, can be even dense in Ω.

Example 2.1.13. Let $\{r_k\}_{i=1}^\infty$ be a dense countable subset in $B_1(0)$. We define for $x \in B_1(0)$

$$u(x) = \sum_{i=1}^\infty \frac{1}{2^i} |x - r_i|^{-\alpha}.$$

If $p < d$ and $\alpha \in (0, \frac{d-p}{p})$, then $u \in W^{1,p}(B_1(0))$, but the function is not bounded on any open subset of $B_1(0)$.

The basic important properties of Sobolev spaces as completeness, separability and reflexivity are summarized in the following theorem.

Theorem 2.1.14 — On properties of Sobolev spaces. For every $k \in \mathbb{N}_0$ and $p \in [1, \infty]$ the space $W^{k,p}(\Omega)$ is a Banach space. For $p \in [1, \infty)$ the space $W^{k,p}(\Omega)$ is separable and for $p \in (1, \infty)$ the space is reflexive. For $p = 2$ the space $W^{k,2}(\Omega)$ is a Hilbert space with the scalar product

$$(u, v)_{W^{k,2}(\Omega)} = (u, v)_{k,2} := \sum_{|\alpha| \le k} \int_\Omega D^\alpha u D^\alpha v \, \mathrm{d}x. \tag{2.1}$$

Proof. **Step 1:** Completeness
The aim is to show that every Cauchy sequence in $W^{k,p}(\Omega)$ has a limit in $W^{k,p}(\Omega)$. Let $\{u_n\}_{n=1}^\infty \subset W^{k,p}(\Omega)$ be a Cauchy sequence, i.e.,

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n, m \ge n_0 : \ \|u_n - u_m\|_{k,p} < \varepsilon.$$

The definition of $\|\cdot\|_{k,p}$ implies that for any multiindex α such that $|\alpha| \le k$ it holds $\|D^\alpha u_n - D^\alpha u_m\|_p < \varepsilon$. Therefore all sequences $\{D^\alpha u_n\}_{n=1}^\infty \subset L^p(\Omega)$ are Cauchy sequences. The spaces $L^p(\Omega)$ are complete (cf. Theorem A.3.11), and therefore there exist limits

$$\begin{aligned} u_n &\to u && \text{in } L^p(\Omega) \\ D^\alpha u_n &\to u_\alpha && \text{in } L^p(\Omega),\ |\alpha| \le k. \end{aligned} \tag{2.2}$$

Since the limits of sequences $D^\alpha u_n$ were constructed separately, it is not clear whether we have $D^\alpha u = u_\alpha$. It remains to verify this claim. First, it holds $u_\alpha \in L^1_{\mathrm{loc}}(\Omega)$ (as $u_\alpha \in L^p(\Omega)$); we have verified the first property of the weak derivative. We take arbitrary α such that $|\alpha| \le k$. By virtue of the definition of the weak derivative it holds for every $\phi \in \mathcal{C}_0^\infty(\Omega)$

$$\int_\Omega u_n D^\alpha \phi \, \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega D^\alpha u_n \phi \, \mathrm{d}x.$$

We pass to the limit $n \to \infty$ on both sides of the equality. For the left-hand side we have due to $(2.2)_1$

$$\lim_{n \to \infty} \int_\Omega u_n D^\alpha \phi \, \mathrm{d}x = \int_\Omega u D^\alpha \phi \, \mathrm{d}x,$$

and for the right-hand side we obtain due to $(2.2)_2$

$$\lim_{n \to \infty} (-1)^{|\alpha|} \int_\Omega D^\alpha u_n \phi \, \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega u_\alpha \phi \, \mathrm{d}x.$$

Whence it must hold for any $\phi \in \mathcal{C}_0^\infty(\Omega)$ that $\int_\Omega u D^\alpha \phi \, \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega u_\alpha \phi \, \mathrm{d}x$. This implies $D^\alpha u = u_\alpha$.

Step 2: Reflexivity and separability
To show the reflexivity and separability we use the properties of the $L^p(\Omega)$ spaces, cf. Theorems A.3.34 and A.3.37. Denote $X = (L^p(\Omega))^\kappa$, where κ is the number of all multiindeces with the length equal or less than k. The space X is clearly reflexive (for $p \in (1,\infty)$) and separable (for $p \in [1,\infty)$).

We further define the mapping $I\colon W^{k,p}(\Omega) \to X$ as[3]

$$I(u) = [D^\alpha u]_{|\alpha|\le k} = \left[u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_d}, \dots, \frac{\partial^k u}{\partial x_d^k}\right].$$

Then I is an isomorphism between $W^{k,p}(\Omega)$ and $I(W^{k,p}(\Omega)) \subset X$. Due to the completeness of the space $W^{k,p}(\Omega)$, cf. Theorem 2.1.14, the set $I(W^{k,p}(\Omega))$ is a closed subset of X. Thus due to Theorem B.2.4 the space $W^{k,p}(\Omega)$ is separable, if $p \in [1,\infty)$, and reflexive, if $p \in (1,\infty)$.

Step 3: Case $p = 2$
We leave for a kind reader the verification that (2.1) is a scalar product. Since the associated norm is the standard norm in $W^{k,2}(\Omega)$, the space $W^{k,2}(\Omega)$ is a Hilbert space. ■

On the other hand, for the value $p = 1$ the Sobolev spaces (similarly as the Lebesgue ones) are not reflexive and for $p = \infty$ neither reflexive nor separable.

Theorem 2.1.15 — On non-reflexivity and non-separability. The Sobolev space $W^{k,\infty}(\Omega)$ is not separable and the Sobolev spaces $W^{k,1}(\Omega)$ and $W^{k,\infty}(\Omega)$ are not reflexive.

Proof. The proof of the first claim is left for the reader, cf. the following Exercise 2.1.16. The proof of the second claim can be found in (Kufner et al., 1977, Theorems 5.2.4 and 5.2.6). ■

Exercise 2.1.16 ($W^{k,\infty}(\Omega)$ is not separable)**.** Let $\Omega \subset \mathbb{R}^d$ and let $\delta > 0$ be such that $B_\delta(x_0) \subset \Omega$ for a certain x_0. Consider for $\xi = (\xi_1, \dots, \xi_d) \in B_\delta(x_0)$ functions $\varphi_\xi = \min(1, |x_1 - \xi_1|)$. Show that φ_ξ is an uncountable system of functions from $W^{1,\infty}(\Omega)$ such that $\|\varphi_\xi - \varphi_{\widetilde{\xi}}\|_{W^{1,\infty}(\Omega)} \ge 1$ for $\xi_1 \ne \widetilde{\xi}_1$.

In what follows we introduce certain subspaces of Sobolev spaces whose elements "are zero" on the boundary Ω. These subspaces play an important role when we introduce the solution to certain boundary value problems in the theory of PDEs as well as at the rigorous justification of integration by parts for Sobolev functions.

Definition 2.1.17 — The space $W_0^{k,p}(\Omega)$. Let $\Omega \subset \mathbb{R}^d$ be an open set, $p \in [1,\infty)$ and $k \in \mathbb{N}$. Denote

$$W_0^{k,p}(\Omega) := \overline{\mathcal{C}_0^\infty(\Omega)}^{\|\cdot\|_{k,p}}.$$

Remark 2.1.18. If we allow $p = \infty$ in the definition above, we would get

$$\overline{\mathcal{C}_0^\infty(\Omega)}^{\|\cdot\|_{k,\infty}} \subseteq \left\{u \in \mathcal{C}^k(\overline{\Omega}) \,|\, \forall |\alpha| \le k,\, \forall x \in \partial\Omega\colon D^\alpha u(x) = 0\right\}$$

which follows directly from the definition of the convergence in the norm $\|\cdot\|_{k,\infty}$.

The following relation between $W^{k,p}(\Omega)$ and $W_0^{k,p}(\Omega)$ is left as an exercise for a kind reader.

Exercise 2.1.19. Show that $W_0^{k,p}(\Omega)$ is a subspace of $W^{k,p}(\Omega)$. Show further that $W_0^{k,p}(\Omega) \subsetneq W^{k,p}(\Omega)$ for an arbitrary open $\Omega \subsetneq \mathbb{R}^d$.

The spaces $W_0^{k,p}(\Omega)$ share many properties with $W^{k,p}(\Omega)$ as it is formulated in the next theorem.

Theorem 2.1.20 — On properties of spaces $W_0^{k,p}(\Omega)$. For any $k \in \mathbb{N}$ and $p \in [1,\infty)$ the space $W_0^{k,p}(\Omega)$ is a Banach space. For $p \in [1,\infty)$ the space $W_0^{k,p}(\Omega)$ is separable and for $p \in (1,\infty)$ the space is reflexive. The space $W_0^{k,1}(\Omega)$ is not reflexive.

Proof. The proof is, similarly as the proof of Theorem 2.1.14, based on known properties of the Lebesgue spaces $L^p(\Omega)$. A more detailed proof can be found in (Kufner et al., 1977, Theorems 5.2.2, 5.2.4 and 5.2.6). ■

Remark 2.1.21. If we allow for $k = 0$ in the definition of the space $W_0^{k,p}(\Omega)$, then $W_0^{0,p}(\Omega) = L^p(\Omega)$ for $1 \le p < \infty$ and $\Omega \subset \mathbb{R}^d$ open, since the smooth compactly supported functions are dense in $L^p(\Omega)$ in this situation.

Finally, as a direct consequence, we deduce the formula for integration by parts for elements of Sobolev spaces $W_0^{1,p}(\Omega)$ and $W^{1,p'}(\Omega)$.

[3]The mapping I forms the vector of all possible (weak) partial derivatives of the order at most k.

Theorem 2.1.22 — On integration by parts I. Let $\Omega \subset \mathbb{R}^d$ be an open set, $k \in \mathbb{N}$ and $p \in [1, \infty)$. Then for any multiindex α such that $|\alpha| \le k$, every $u \in W_0^{k,p}(\Omega)$ and every[a] $v \in W^{k,p'}(\Omega)$ it holds

$$\int_\Omega D^\alpha u \, v \, \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega u \, D^\alpha v \, \mathrm{d}x. \tag{2.3}$$

[a]Recall that $p' := \frac{p}{p-1}$ with the convention that for $p = 1$ we have $p' = \infty$.

Proof. By virtue of Hölder's inequality A.3.12 it is not difficult to verify that both integrals in (2.3) are finite. Furthermore, from the definition of the space $W_0^{k,p}(\Omega)$ we know that there exists a sequence $\{u_n\}_{n=1}^\infty \subset \mathcal{C}_0^\infty(\Omega)$ such that for any multiindex α, $|\alpha| \le k$ we have

$$D^\alpha u_n \to D^\alpha u \qquad \text{in } L^p(\Omega).$$

As also $D^\alpha v \in L^{p'}(\Omega)$, we immediately obtain

$$\begin{aligned} \int_\Omega D^\alpha u \, v \, \mathrm{d}x &= \lim_{n\to\infty} \int_\Omega D^\alpha u_n \, v \, \mathrm{d}x \\ \int_\Omega u \, D^\alpha v \, \mathrm{d}x &= \lim_{n\to\infty} \int_\Omega u_n \, D^\alpha v \, \mathrm{d}x. \end{aligned} \tag{2.4}$$

Finally, directly from the definition of the weak derivative (recall that $u_n \in \mathcal{C}_0^\infty(\Omega)$) we deduce

$$\int_\Omega D^\alpha u_n \, v \, \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega u_n \, D^\alpha v \, \mathrm{d}x$$

and plugging this identity into (2.4) we get (2.3). ■

Inspired by Definition 2.1.17, we introduce at the end of this section yet other function spaces which we obtain as closure of smooth functions up to the boundary in the corresponding Sobolev norm.

Definition 2.1.23 — Sobolev spaces as closure. Let $\Omega \subset \mathbb{R}^d$ be an open set, $p \in [1, \infty)$ and $k \in \mathbb{N}$. The space $\widetilde{W}^{k,p}(\Omega)$ is defined as

$$\widetilde{W}^{k,p}(\Omega) := \overline{\mathcal{C}^\infty(\overline{\Omega})}^{\|\cdot\|_{k,p}}.$$

Remark 2.1.24. Analogously as in the case $W_0^{k,p}(\Omega)$ (cf. Remark 2.1.18) it does not make sense to define $\widetilde{W}^{k,\infty}(\Omega)$, because we would get due to the properties of $\|\cdot\|_{k,\infty}$ that $\widetilde{W}^{k,\infty}(\Omega) \subset \mathcal{C}^k(\overline{\Omega})$.

The following lemma summarizes the properties of the space defined as the closure of smooth functions up to the boundary in the Sobolev norm.

Lemma 2.1.25 — On properties of spaces $\widetilde{W}^{k,p}(\Omega)$. Let $k \in \mathbb{N}$ and $p \in [1, \infty)$. Then $\widetilde{W}^{k,p}(\Omega)$ is a closed subspace of $W^{k,p}(\Omega)$ (and thus a Banach space) which is separable and furthermore for $p \in (1, \infty)$ also reflexive. In particular, $\widetilde{W}^{k,p}(\Omega) \subset W^{k,p}(\Omega)$.

Proof. The fact that the space is closed follows directly from the definition. Other properties can by shown by virtue of Theorem 2.1.14. Their proof is left as a useful exercise for a kind reader. ■

The answer on the question when it holds $\widetilde{W}^{k,p}(\Omega) = W^{k,p}(\Omega)$ will be given in the next section; all proofs then later, in Chapter 6. The validity of such claim will require certain assumptions on the qualitative properties of the set Ω. We now only present a counterexample of this claim for a sufficiently "ugly" open set Ω.

Exercise 2.1.26 ($\widetilde{W}^{k,p}(\Omega) \neq W^{k,p}(\Omega)$). We define the set $\Omega \subset \mathbb{R}^2$ as

$$\Omega := B_1(0) \setminus \{(x, 0) : x \in [0, 1)\}, \qquad \text{see also Figure 2.2 from Example 2.2.11.}$$

Consider a function u defined as

$$u(x, y) := \begin{cases} 0 & \text{if } x \le 0, \\ 0 & \text{if } x > 0 \text{ a } y \ge 0, \\ x & \text{if } x > 0 \text{ a } y < 0. \end{cases}$$

Show that for every $p \in [1, \infty]$ it holds $u \in W^{1,p}(\Omega)$, but for $p < \infty$ we have $u \notin \widetilde{W}^{1,p}(\Omega)$.

Let us note that introducing the Sobolev spaces by Definitions 2.1.6 and 2.1.23 is not the only possibility. In the last section of Chapter 6 we present an alternative, but fully equivalent definition based on the so-called Beppo Levi spaces.

2.2 Density of smooth functions

Starting from this section, we shall rather present overview of properties of Sobolev spaces and skip the proofs. We first look at results concerning the density of different types of smooth functions. They are important since they allow the following proof strategy: we first show validity of certain integral identity for smooth functions and in the next step, based on the density argument, we show by suitable limit passage that the identity also holds for Sobolev functions. These results will be mostly based on mollification of Sobolev functions.

In what follows the notation η will be used exclusively for the mollification kernel, i.e., for smooth radially symmetric function supported in the unit ball with integral mean value equal to one, see also Definition A.3.28. The function $\eta_\varepsilon(x) := \varepsilon^{-d}\eta(\frac{x}{\varepsilon})$ stands then for its rescaling and finally for $u \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ we introduce the mollification of the function u, denoted u_ε, as (see also Definition A.3.30)

$$u_\varepsilon(x) := \eta_\varepsilon \star u(x) = \int_{\mathbb{R}^d} \eta_\varepsilon(x-y)u(y)\,\mathrm{d}y.$$

2.2.1 Local approximation of Sobolev functions

The first result concerns a direct application of the mollification of a Sobolev function.

Theorem 2.2.1 — On local approximation by smooth functions. Let $\Omega \subset \mathbb{R}^d$ be open, $p \in [1,\infty)$ and $u \in W^{k,p}(\Omega)$ be arbitrary. We define u as zero outside of Ω and let $u_\varepsilon := \eta_\varepsilon \star u$ denote the mollification of u. Then it holds:

1. $D^\alpha u_\varepsilon = (D^\alpha u)_\varepsilon$ almost everywhere in $\Omega_\varepsilon := \{x \in \Omega \,|\, \mathrm{dist}\,(x, \partial\Omega) > \varepsilon\}$
2. $u_\varepsilon \to u$ in $W^{k,p}(\Omega')$ for every open $\Omega' \subset \overline{\Omega'} \subset \Omega$.

This theorem has several corollaries. The first one is a precise characterization of $W^{k,p}(\mathbb{R}^d)$.

Lemma 2.2.2 — Connection of $W_0^{k,p}(\mathbb{R}^d)$ and $W^{k,p}(\mathbb{R}^d)$. Let $k \in \mathbb{N}$ a $p \in [1,\infty)$. Then $W^{k,p}(\mathbb{R}^d) = W_0^{k,p}(\mathbb{R}^d)$.

Another, "intuitively" straightforward corollary of the Theorem on local approximation by smooth functions 2.2.1 is the following claim dealing with Sobolev functions which have zero first order derivative almost everywhere (and, as follows from the theorem, the functions are constant).

Lemma 2.2.3 — On constant functions. Let $\Omega \subset \mathbb{R}^d$ be an open connected and $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$. Then the following two assertions are equivalent.

1. The function u is constant almost everywhere in Ω.
2. For any multiindex α of the length one it holds $D^\alpha u = 0$ almost everywhere in Ω.

We stated in the previous lemma that a Sobolev function has zero gradient on an open connected set, if and only if the function is constant there. We now formulate the result that if a function is constant on a *measurable* set, then all derivatives of the first order are there zero almost everywhere. This will allow us to present a claim about composition of Lipschitz and Sobolev functions. Recall that χ_B denotes a characteristic function of a set B.

Theorem 2.2.4 — On the derivative of a composite function. Let Ω be open and $u \in W^{1,p}(\Omega)$ for some $p \in [1,\infty]$. Denote for arbitrary $a \in \mathbb{R}$

$$\Omega_a := \{x \in \Omega \,|\, u(x) = a\}.$$

Then for any $i \in \{1,\dots,d\}$ it holds that $\frac{\partial u}{\partial x_i} = 0$ almost everywhere in Ω_a.

Further, let $f \in \mathcal{C}^{0,1}(\mathbb{R})$ (note that $f' \in L^\infty(\mathbb{R})$). Then $f \circ u - f(0) \in W^{1,p}(\Omega)$ and it holds

$$\frac{\partial f(u(x))}{\partial x_i} = f'(u(x))\frac{\partial u(x)}{\partial x_i}\chi_{\{x\in\Omega\,|\,u(x)\notin S_f\}} \quad \text{almost everywhere in } \Omega, \tag{2.5}$$

where $S_f := \{s \in \mathbb{R} \,|\, \text{ the classical derivative } f'(s) \text{ does not exist}\}$.

Note that the Rademacher Theorem A.2.16 ensures that the derivative f' exists almost everywhere in $\mathbb{R}$ and thus the set S_f is of zero measure. If moreover Ω has finite measure, then $f(0) \in W^{1,p}(\Omega)$ and thus $f(u) \in W^{1,p}(\Omega)$. Finally, as an easy corollary of Theorem 2.2.4 which is in fact a part of the proof we get a claim which is often used in the theory of partial differential equations.

Corollary 2.2.5. Let $u \in W^{1,1}(\Omega)$ be arbitrary and denote $u^+ := \max\{0,u\}$ and $u^- := -\min\{0,u\}$. Then we have for all $i \in \{1, \dots, d\}$ and almost everywhere in Ω that

$$\frac{\partial u^+}{\partial x_i} = \frac{\partial u}{\partial x_i} \chi_{\{x \in \Omega \,|\, u(x) > 0\}}$$
$$\frac{\partial u^-}{\partial x_i} = -\frac{\partial u}{\partial x_i} \chi_{\{x \in \Omega \,|\, u(x) < 0\}}$$
$$\frac{\partial |u|}{\partial x_i} = \operatorname{sign} u \frac{\partial u}{\partial x_i} \chi_{\{x \in \Omega \,|\, u(x) \neq 0\}}.$$

In particular, u^+, u^- and $|u| \in W^{1,1}(\Omega)$.

In general, a composition of two Sobolev functions may be not a Sobolev function.

Example 2.2.6. Consider the function $u(x) = x^3 \sin^3\left(\frac{1}{x}\right) \in W^{1,\infty}((-1,1))$ and further the function $f(z) = \sqrt[3]{z} \in W^{1,q}((-1,1))$ for $q < \frac{3}{2}$. Then the function $(f \circ u)(x) = x \sin\left(\frac{1}{x}\right)$ does not belong even to $W^{1,1}((-1,1))$.

2.2.2 Global approximation of Sobolev functions by functions from $\mathcal{C}^\infty(\Omega)$

We shall explain in this subsection that it is possible to strengthen Theorem 2.2.1 in the sense that the approximate sequence belongs to $\mathcal{C}^\infty(\Omega) \cap W^{k,p}(\Omega)$. Note that this "stronger" formulation does not require any extra assumptions on the set Ω, on the other hand, this result does not say anything about the global approximation by means of $\mathcal{C}^\infty(\overline{\Omega})$-funtions. This problem will be studied in the following subsection.

Theorem 2.2.7 — On approximation by smooth functions in Ω. Let $\Omega \subset \mathbb{R}^d$ be a bounded open set, $p \in [1, \infty)$ and $u \in W^{k,p}(\Omega)$. Then there exists a sequence $\{u_n\}_{n=1}^\infty \subset \mathcal{C}^\infty(\Omega) \cap W^{k,p}(\Omega)$ such that $u_n \to u$ in $W^{k,p}(\Omega)$.

2.2.3 Global approximation of Sobolev functions by functions from $\mathcal{C}^\infty(\overline{\Omega})$

We aim at obtaining global approximation of Sobolev functions by smooth functions up to the boundary. To this aim, we need to apply Theorem 2.2.1. Since the approximation must be done up to the boundary, it will already depend on the properties of the boundary. For general domains, this given result is not true (cf. Exercise 2.1.26), thus we must exclude such situations as in the above mentioned exercise. We shall present two results of this type. The first one is connected with a special type of domains, so-called star-shaped ones for which the proof is rather straightforward. The other case covers more general class of domains, but the definition of the domain is much more involved. Indeed, also the proof is more complex.

Definition 2.2.8 — Star-shaped domain. We say that an open set $\Omega \subset \mathbb{R}^d$ is star-shaped (with respect to a point x_0), if there exists a point $x_0 \in \Omega$ such that for any $x \in \Omega$, $x \neq x_0$ the half line starting at x_0 and going through x has exactly one common point with the boundary of Ω. It means that

$$\left\{y \in \mathbb{R}^d \,|\, \exists \tau \in \mathbb{R}_+, y = \tau(x - x_0) + x_0\right\} \cap \partial\Omega \text{ contains exactly one point.}$$

An example of a star-shaped domain is a ball or cube in $\mathbb{R}^d$ or, as indicated by the name of the domain — a symmetric star in the plane. Recall that a star-shaped set is necessarily connected.

Theorem 2.2.9 — On approximation up to the boundary for star-shaped domains. Let Ω be a star-shaped domain and $u \in W^{k,p}(\Omega)$ for $p \in [1, \infty)$. Then there exists a sequence $\{u_n\}_{n=1}^\infty \subset \mathcal{C}^\infty(\overline{\Omega})$ such that $u_n \to u$ in $W^{k,p}(\Omega)$.

The main idea of the proof is connected with the fact that we may "slide out" the function along the half lines starting at x_0 (the point with respect to which the domain is star-shaped). We can do it in such a way that the slid out function is close to the original one in the Sobolev norm. Then we molify this function (the mollification makes sense up to the boundary of Ω).

The star-shaped domains are too specific and do not include a big class of otherwise "nice" domains. Another class for which the approximation of Sobolev function by smooth functions holds true are so-called domains with continuous boundary. Since it is not more difficult to define domains with Hölder continuous (or even more general) boundary, we present the definition in full generality.

Definition 2.2.10 — Domain with $\mathcal{C}^{k,\mu}$-boundary. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, $k \in \mathbb{N}_0$ and $\mu \in [0,1]^a$. We say that Ω is a domain with the $\mathcal{C}^{k,\mu}$-boundary (shortly domain of the type $\mathcal{C}^{k,\mu}$ and denote $\Omega \in \mathcal{C}^{k,\mu}$), if there exist positive numbers α, β, and M cartesian coordinate systems, i.e., the coordinates of an arbitrary point $x \in \mathbb{R}^d$ in the r-th coordinate system are denoted as $x = (x_{r_1}, \dots, x_{r_d}) := (x'_r, x_{r_d})$, and M continuous functions

$a_r\colon \Delta_r \to \mathbb{R}$ of class $\mathcal{C}^{k,\mu}$, where we define for any $r \in \{1,\dots,M\}$

$$\Delta_r := \left\{x'_r \in \mathbb{R}^{d-1} \,|\, i = 1,\dots,d-1 : \; |x_{r_i}| < \alpha\right\}$$

such that the following holds.

1. If we denote by T_r the mapping (rotation and shift) which describes the change of coordinates from the r-th cartesian coordinate system (x'_r, x_{r_d}) to the global coordinate system (x', x_d), then for any $x \in \partial\Omega$ there exists a coordinate system (i.e., there exists $r \in \{1,\dots,M\}$), such that $x = T_r(x'_r, a_r(x'_r))$ for some $x'_r \in \Delta_r$.

2. If we define

$$\begin{aligned} V_r^+ &:= \left\{(x'_r, x_{r_d}) \in \mathbb{R}^d \,|\, x'_r \in \Delta_r,\; a_r(x'_r) < x_{r_d} < a_r(x'_r) + \beta\right\} \\ V_r^- &:= \left\{(x'_r, x_{r_d}) \in \mathbb{R}^d \,|\, x'_r \in \Delta_r,\; a_r(x'_r) - \beta < x_{r_d} < a_r(x'_r)\right\} \\ \Lambda_r &:= \left\{(x'_r, x_{r_d}) \in \mathbb{R}^d \,|\, x'_r \in \Delta_r,\; a_r(x'_r) = x_{r_d}\right\}, \end{aligned}$$

   then $T_r(V_r^+) \subset \Omega$, $T_r(V_r^-) \subset \mathbb{R}^d \setminus \overline{\Omega}$ and $T_r(\Lambda_r) \subset \partial\Omega$.

Property 1. implies furthermore that $\partial\Omega = \bigcup_{r=1}^M T_r(\Lambda_r)$. If we denote the open set $V_r := V_r^+ \cup V_r^- \cup \Lambda_r$, then $\partial\Omega \subset \bigcup_{r=1}^M T_r(V_r)$ and $\{T_r(V_r)\}_{r=1}^M$ is a finite open covering of a certain neighbourhood of $\partial\Omega$.

[a] For $\mu = 0$ we speak about domains with $\mathcal{C}^k$-boundary; if further $k = 0$, then by $\mathcal{C}^{0,0}$ we understand a continuous function and we speak about domains with $\mathcal{C}^0$- or $\mathcal{C}$-boundary.

Figure 2.1 displays the typical situation how the boundary should look like.

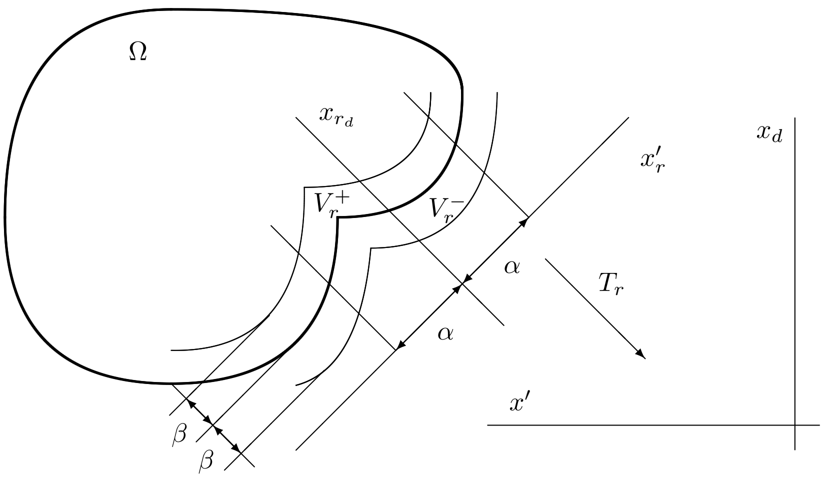

Figure 2.1: Domain Ω with the $\mathcal{C}^{0,1}$-boundary.

Example 2.2.11. Typical examples of different types of domains are as follows.

1. Domain defined as $(0,a)^d$, $a > 0$ (i.e., the d-dimensional cube) is a domain with the $\mathcal{C}^{0,1}$-boundary (Lipschitz boundary).
2. The ball in $\mathbb{R}^d$ is a domain with the $\mathcal{C}^\infty$-boundary.
3. The ball in $\mathbb{R}^d$ without one line (see Figure 2.2) is even not a domain with the $\mathcal{C}$-boundary.

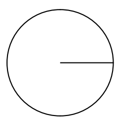

Figure 2.2: Domain which is not of the type $\mathcal{C}$.

Note that the third example is in fact a typical example of the domain assumed in Exercise 2.1.26 for which we do not have $\widetilde{W}^{k,p}(\Omega) = W^{k,p}(\Omega)$. On the other hand, as we shall see below, the equality holds for domains $\Omega \in \mathcal{C}$ and together with the continuity of the boundary, the key property is that we may speak about a well defined *interior* of the domain (represented by $T_r(V_r^+)$) and *exterior* of the domain (represented by $T_r(V_r^-)$) which allows us to modify the proof from the case of star-shaped domains.

Remark 2.2.12. Some of the domains have special names; the domain $\Omega \in \mathcal{C}$ is usually called the domain with continuous boundary, and $\Omega \in \mathcal{C}^{0,1}$ is called the domain with Lipschitz boundary.

Important tool in the proof of the theorem below which will also be very useful at several other situations, is the following lemma.

Lemma 2.2.13 — On partition of unity. Let $\{G_i\}_{i=1}^k$ be a finite number of open sets in $\mathbb{R}^d$ such that $\overline{\Omega} \subset \cup_{i=1}^k G_i$. Then there exist non-negative functions $\phi_i \in \mathcal{C}_0^\infty(G_i)$, $i = 1, \ldots, k$ such that for any i we have $\|\phi_i\|_{\mathcal{C}(\overline{\Omega})} \le 1$ and it holds for any $x \in \overline{\Omega}$

$$\sum_{i=1}^k \phi_i(x) = 1.$$

The main result reads as follows.

Theorem 2.2.14 — On the approximation up to the boundary for $\Omega \in \mathcal{C}$. Let $\Omega \in \mathcal{C}$, $p \in [1, \infty)$, $k \in \mathbb{N}$ and $u \in W^{k,p}(\Omega)$. Then there exists a sequence $\{u_n\}_{n=1}^\infty$ of functions from $\mathcal{C}^\infty(\overline{\Omega})$ such that $u_n \to u$ in $W^{k,p}(\Omega)$. In other words, for $p \in [1, \infty)$, $k \in \mathbb{N}$ and $\Omega \in \mathcal{C}$ we have $\widetilde{W}^{k,p}(\Omega) = W^{k,p}(\Omega)$.

The proof is based on the following. We consider each part of the boundary (the remaining part away from the boundary is easier and Theorem 2.2.1 can be applied) in the local coordinate system and apply the Lemma on partition of unity 2.2.13. We "slide the function out" in the direction of the last variable (i.e., out of Ω) and then mollify the resulted function. To have everything well defined, the continuity of the boundary is required.

2.3 Weak derivative and differences

We saw in the previous subsection that the Sobolev functions can approximated arbitrarily close in the Sobolev norm by smooth functions. We also know that if a function has continuous classical derivatives, then they coincide with the weak derivatives. We shall now consider the opposite relation, namely if the weak derivative can be considered as a certain approximation of the classical derivative.

We denote for arbitrary $i = 1, \ldots, d$ and $h \in \mathbb{R}$ the difference quotient

$$\Delta_i^h u(x) := \frac{u(x + h\mathbf{e}_i) - u(x)}{h}, \tag{2.6}$$

where $\mathbf{e}_i$ is the unit vector in the direction of the x_i axis. It is evident that if u has at the point x (classical) partial derivative with respect of the x_i variable, then

$$\lim_{h \to 0} \Delta_i^h u(x) = \frac{\partial u}{\partial x_i}.$$

We aim to clarify that something similar "in the mean" holds also for the weak derivative almost everwhere in Ω. The first result in this direction is the following.

Theorem 2.3.1 — On the connection between difference quotient and weak derivative I. Let Ω be open, $p \in [1, \infty]$ and $u \in L^p(\Omega)$. We denote for arbitrary $\delta > 0$

$$\Omega_\delta := \{x \in \Omega \,|\, \mathrm{dist}(x, \partial\Omega) > \delta\}.$$

Then we have the following.

1. If $u \in W^{1,p}(\Omega)$, then for any $i \in \{1, \ldots, d\}$, $\delta \in (0, 1)$ and $|h| \in (0, \frac{\delta}{2})$ it holds

$$\left\|\Delta_i^h u\right\|_{L^p(\Omega_\delta)} \le \left\|\frac{\partial u}{\partial x_i}\right\|_{L^p(\Omega)}.$$

2. If $p \in (1, \infty]$ and there exist constants $\{C_i\}_{i=1}^d$ such that for any $i \in \{1, \ldots, d\}$, $\delta \in (0, 1)$ and $|h| \in (0, \frac{\delta}{2})$ it holds

$$\left\|\Delta_i^h u\right\|_{L^p(\Omega_\delta)} \le C_i,$$

then $u \in W^{1,p}(\Omega)$. Moreover, we have for any $i \in \{1, \dots, d\}$ the estimate

$$\left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega)} \leq C_i.$$

Corollary 2.3.2. Note that (verify that the assumptions of the second part of Theorem 2.3.1 are satisfied) Lipschitz functions belong to $W^{1,\infty}(\Omega)$. We even have the embedding $\mathcal{C}^{0,1}(\overline{\Omega}) \hookrightarrow W^{1,\infty}(\Omega)$. The opposite inclusion requires certain regularity of the domain and will be discussed later.

Remark 2.3.3. The second part of Theorem 2.3.1 does not hold for $p = 1$, as a kind reader may verify for the function $u(x) := \operatorname{sign} x$ in the interval $(-1, 1)$. Nonetheless, the function fulfilling the assumptions of the second part of the theorem for $p = 1$ belongs to the space $BV(\Omega)$ — functions with bounded variation.

We now strengthen the previous theorem in the sense that we will replace weak convergence by the strong one.

Theorem 2.3.4 — On the connection between difference quotient and weak derivative II. Let Ω be bounded, $p \in [1, \infty)$ and $u \in W^{1,p}(\Omega)$. We denote for $\delta > 0$

$$\Omega_\delta := \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \delta\}.$$

Then it holds for any $\delta > 0$

$$\lim_{h \to 0} \left\| \Delta_i^h u - \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega_\delta)} = 0. \tag{2.7}$$

Remark 2.3.5. Since the strong convergence implies almost everywhere convergence, at least for a chosen subsequence, Theorem 2.3.4 guarantees that for any subsequence $h_n \to 0$ there exists a set of zero Lebesgue measure $N \subset \Omega$ such that (for a chosen subsequence)

$$\lim_{h_n \to 0} \left| \Delta_i^{h_n} u(x) - \frac{\partial u}{\partial x_i}(x) \right| = 0 \qquad \text{for all } x \in \Omega \setminus N.$$

This claim, however, does not imply existence of the classical derivative, since the function u can be changed on a set of measure zero which can be dense in Ω and thus after this change the function u is not continuous at any point and cannot have the classical derivative anywhere. A different situation is for $p > d$, where already a continuous representative of the function exists. We shall discuss this case later.

On the other hand, we may strengthen Lemma 2.1.4; we assumed there the continuity of the derivatives and we can relax the assumptions as follows.

Lemma 2.3.6 — Connection of weak and classical derivative II. Let Ω be an open set and $u \in W^{1,1}(\Omega)$. Let $D \subset \Omega$ be defined as

$$D := \{x \in \Omega \mid \text{ there exists the classical partial derivative of } u \text{ with respect to } x_i\}.$$

Then the weak and classical derivatives of u coincide almost everywhere in D.

Finally we present a small generalization of Theorem 2.3.1.

Lemma 2.3.7 — On the connection between difference quotient and weak derivative III. Let Ω be open, $p \in [1, \infty]$ and $u \in W^{1,p}(\Omega)$. We denote for $\delta > 0$, $h \in (-\frac{\delta}{2}, \frac{\delta}{2})$ and arbitrary unit vector $\mathbf{e} \in \mathbb{R}^d$

$$\Omega_\delta := \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \delta\}$$
$$\Delta_{\mathbf{e}}^h u(x) := \frac{u(x + h\mathbf{e}) - u(x)}{h}.$$

Then it holds

$$\left\| \Delta_{\mathbf{e}}^h u \right\|_{L^p(\Omega_{2|h|})} \leq \left\| \nabla u \cdot \mathbf{e} \right\|_{L^p(\Omega)}.$$

2.4 Theorems on continuous and compact embeddings

Before we present the results concerning the embeddings, we first mention two auxiliary results connected with extensions of Sobolev functions. They are used in the proofs of the further results, however, play also an important role in the theory of partial differential equations.

Theorem 2.4.1 — Extension operator. Let $\Omega \in \mathcal{C}^{0,1}$ and $p \in [1,\infty]$. Then there exists a continuous linear operator

$$E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^d)$$

such that:

1. $Eu = u$ in Ω
2. Eu has compact support in $\mathbb{R}^d$
3. there exists $C = C(d,\Omega) > 0$ such that

$$\|Eu\|_{W^{1,p}(\mathbb{R}^d)} \le C \|u\|_{W^{1,p}(\Omega)} .$$

Remark 2.4.2. In what follows we shall call Eu extension of $u \in W^{1,p}(\Omega)$ to $u \in W^{1,p}(\mathbb{R}^d)$ and the operator E will be called the extension operator. The extension can be defined so that it preserves the $\mathcal{C}^1$ regularity. It is possible to preserve also higher regularity, but more regularity of the boundary of Ω must be required. This is due to the fact that the construction can be easily done for flat boundaries, while for more general non-flat boundaries we first need to flatten the boundary.

Let us assume that the boundary is described (for simplicity, we do not consider the change of variables caused by different coordinate systems)

$$x_d = a(x'),\ x' \in \Delta = \left\{x' \in \mathbb{R}^{d-1} \,|\, \forall i \in \{1,\dots,d-1\}\ |x_i| < \alpha\right\}.$$

Recall the notation from Definition 2.2.10

$$\begin{aligned} V^+ &= \left\{(x',x_d) \in \mathbb{R}^d \,|\, x' \in \Delta,\ a(x') < x_d < a(x') + \beta\right\} \\ V^- &= \left\{(x',x_d) \in \mathbb{R}^d \,|\, x' \in \Delta,\ a(x') - \beta < x_d < a(x')\right\} \\ \Lambda &= \left\{(x',x_d) \in \mathbb{R}^d \,|\, x' \in \Delta,\ a(x') = x_d\right\} \\ V &= V^+ \cup V^- \cup \Lambda. \end{aligned}$$

We now look at the change of variables which "flattens" the boundary. We define new variables (y', y_d) and the mapping $\mathbf{F}$: $(-1,1)^d \to V$ by means of

$$\begin{aligned} x' &= \alpha y' \\ x_d &= a(\alpha y') + \beta y_d, \quad (\text{i.e., } x = \mathbf{F}(y)) \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} y' &= \frac{1}{\alpha} x' \\ y_d &= \frac{1}{\beta} x_d - \frac{1}{\beta} a(x'), \quad (\text{i.e., } y = \mathbf{F}^{-1}(x)), \end{aligned} \tag{2.9}$$

respectively. Denote further

$$\begin{aligned} C^+ &:= (-1,1)^{d-1} \times (0,1) \\ C^- &:= (-1,1)^{d-1} \times (-1,0). \end{aligned}$$

Then $\mathbf{F}$ maps C^+ onto V^+, C^- onto V^- and $(-1,1)^{d-1} \times \{0\}$ onto Λ. Furthermore, if a is at least Lipschitz, the following key lemma holds true.

Lemma 2.4.3 — Flattening of the boundary. Let $\mathbf{F}$ and $\mathbf{F}^{-1}$ be defined above by (2.8) and (2.9). Let a be a Lipschitz continuous function in $\overline{\Delta}$. Then the change of variables $\mathbf{F}$ and its inverse $\mathbf{F}^{-1}$ are Lipschitz continuous.

Moreover, there exists constants $C_1 = C_1(a,\alpha,\beta,d)$ and $C_2 = C_2(a,\alpha,\beta,d)$ such that for any $u \in W^{1,p}(V^+)$ with $p \in [1,\infty)$, the function $U := u \circ \mathbf{F} \in W^{1,p}(C^+)$ and it holds

$$C_1 \|u\|_{W^{1,p}(V^+)} \le \|U\|_{W^{1,p}(C^+)} \le C_2 \|u\|_{W^{1,p}(V^+)} . \tag{2.10}$$

Remark 2.4.4. There is also another method of extension of Sobolev functions, so called Calderón method (the method from Theorem 2.4.1 is called Nikolskii method) and to construct the extension for any k and $p \in (1,\infty)$ it is enough to have $\Omega \in \mathcal{C}^{0,1}$, while the Nikolskii method requires, due to the flattening of the boundary, at least $\mathcal{C}^{k-1,1}$ boundary. Disadvantage of the Calderón method is that it does not work for $p = 1$ and $p = \infty$.[4]

[4] The Calderón method is based on integral representation of functions. More details can be found (for $p = 2$) in Nečas (1967) or Ženíšek (2001), the generalization for $p \neq 2$ can the reader easily construct himself; it is based on the theory of Fourier multipliers. The restriction $p \neq 1$ and $p \neq \infty$ is connected with the fact that the $L^p - L^p$ estimates of the used singular integral operators generally do not hold true for $p = 1$ and $p = \infty$.

We turn now our attention to different embeddings (continuous or compact) of Sobolev spaces. From the definition of Sobolev spaces it directly follows that $u \in W^{k,p}(\Omega)$ belongs also to $L^p(\Omega)$. The main goal is to show that the function u belongs to a "much better" space, provided Ω has a Lipschitz boundary.

Recall Example 2.1.12. We studied for which $\alpha \in \mathbb{R}$ the function

$$f(x) := |x|^{-\alpha}$$

belongs to $W^{1,p}(B_1(0))$ and to $L^q(B_1(0))$, respectively. We showed that if $\alpha < \frac{d-p}{p}$, then $f \in W^{1,p}(B_1(0))$, and, simultaneously, $f \in L^q(B_1(0))$ pro any $q \in [1, \frac{dp}{d-p}]$. This on one hand indicates that if $u \in W^{1,p}(\Omega)$, for $p < d$, then $u \in L^q(\Omega)$ for $q \in [1, \frac{dp}{d-p}]$, too. On the other hand, this examples shows that the best we may expect for $p < d$ is that $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, where $q \leq \frac{dp}{d-p}$, higher exponent would contradict the above mentioned example.

Finally, for $p > d$, the assumption $u \in W^{1,p}(\Omega)$ implies that $-\alpha > 1 - \frac{d}{p} > 0$. Note that for such α we already have $f \in \mathcal{C}^{0,|\alpha|}(\overline{B_1(0)})$.

This simple example leads to the conjecture that for "reasonable" domains Ω it may hold that

$$W^{1,p}(\Omega) \hookrightarrow \begin{cases} L^{\frac{dp}{d-p}}(\Omega) & \text{for } p \in [1,d), \\ \mathcal{C}^{0,1-\frac{d}{p}}(\overline{\Omega}) & \text{for } p \in (d,\infty). \end{cases}$$

2.4.1 Theorems on continuous embedding

We denote for $p < d$

$$p^\star := \frac{dp}{d-p}. \tag{2.11}$$

As indicated above in the introduction to this section, the following results hold.

Theorem 2.4.5 — Embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $p < d$. Let $\Omega \in \mathcal{C}^{0,1}$ and $p \in [1,d)$. Then it holds for any $q \in [1, p^\star]$

$$W^{1,p}(\Omega) \hookrightarrow L^q(\Omega).$$

More precisely, for any $q \in [1, p^\star]$ there exists C which depends only on p, d, q and Ω such that for any $u \in W^{1,p}(\Omega)$ we have

$$\|u\|_{L^q(\Omega)} \leq C \|u\|_{W^{1,p}(\Omega)}. \tag{2.12}$$

The theorem above does not say anything about the case $p = d$, however, as a direct consequence of Theorem 2.4.5 we easily get the following.

Theorem 2.4.6 — Embedding $W^{1,d}(\Omega) \hookrightarrow L^q(\Omega)$. Let $\Omega \in \mathcal{C}^{0,1}$. Then it hold for any $q \in [1,\infty)$

$$W^{1,d}(\Omega) \hookrightarrow L^q(\Omega).$$

Let us now consider the case $p > d$. Define

$$\mu^\star := 1 - \frac{d}{p}, \tag{2.13}$$

where we set for $p = \infty$ the value $\mu^\star := 1$. The main result in this situation is the following.

Theorem 2.4.7 — Embedding $W^{1,p}(\Omega) \hookrightarrow \mathcal{C}^{0,\alpha}(\overline{\Omega})$ for $p > d$. Let $\Omega \in \mathcal{C}^{0,1}$ and $p \in (d,\infty]$. Then it holds for any $\alpha \in [0,\mu^\star]$

$$W^{1,p}(\Omega) \hookrightarrow \mathcal{C}^{0,\alpha}(\overline{\Omega}).$$

More precisely, for any $\alpha \in [0,\mu^\star]$ there exists a constant C which depends only on α, p, d and Ω such that for any $u \in W^{1,p}(\Omega)$ there exists a representative $u^\star \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$, i.e., $u^\star \in [u]$, satisfying

$$\|u^\star\|_{\mathcal{C}^{0,\alpha}(\overline{\Omega})} \leq C \|u\|_{W^{1,p}(\Omega)}.$$

It follows from the above stated theorems that the case $p = d$ is particular and the embedding from Theorem 2.4.6 is not "sharp". Nonetheless, as shown in the following exercise, better result on the scale of Lebesgue spaces cannot be expected.[5]

Exercise 2.4.8. Show that the function $f(x) := \ln\left(\ln\left(1 + \frac{1}{|x|}\right)\right)$ belongs for $d \geq 2$ to $W^{1,d}(B_1(0))$, but does not belong to $L^\infty(B_1(0))$. Whence $W^{1,d}(\Omega) \not\hookrightarrow L^\infty(\Omega)$.

[5] We show in Subsection 2.6 that if Ω is a bounded set, then

$$\left(\int_\Omega \left|u - \frac{1}{|\Omega|}\int_\Omega u \,\mathrm{d}y\right|^p \mathrm{d}x\right)^{\frac{1}{p}} \leq C \|\nabla u\|_{L^p(\Omega)}.$$

Recall the notation (cf. Remark 2.1.8)

$$\nabla u := \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_d}\right).$$

Then the Euclidean norm of ∇u is

$$|\nabla u| := \sqrt{\sum_{i=1}^{d} \left(\frac{\partial u}{\partial x_i}\right)^2}$$

and furthermore

$$\|\nabla u\|_{L^p(\Omega;\mathbb{R}^d)} := \||\nabla u|\|_{L^p(\Omega)}.$$

Then it is not difficult to see that

$$\|u\|_p + \|\nabla u\|_{L^p(\Omega;\mathbb{R}^d)}$$

is an equivalent norm in $W^{1,p}(\Omega)$. To simplify the notation, if no confusion may arise, we always work with ∇u instead of carefully writing all partial derivatives.

Example 2.4.9. Let us try to find a condition for q to satisfy: let $p \in [1, d)$, then there exists $C > 0$ such that for any $u \in W^{1,p}(\mathbb{R}^d)$ it holds

$$\|u\|_{L^q(\mathbb{R}^d)} \leq C \|\nabla u\|_{L^p(\mathbb{R}^d;\mathbb{R}^d)} . \tag{2.14}$$

We take $u \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ for which we assume (2.14) to hold, and define the functions

$$u_\lambda(x) := u(\lambda x), \qquad \text{with arbitrary } \lambda \in \mathbb{R}^+.$$

Inequality (2.14) should hold for all functions from $W^{1,p}(\mathbb{R}^d)$ with the constant C independent of u. In particular, it must hold for all functions u_λ. Using the change of variables, we easily get

$$\|u_\lambda\|_{L^q(\mathbb{R}^d)} = \lambda^{-\frac{d}{q}} \|u\|_{L^q(\mathbb{R}^d)}$$
$$\|\nabla u_\lambda\|_{L^p(\mathbb{R}^d;\mathbb{R}^d)} = \lambda^{1-\frac{d}{p}} \|\nabla u\|_{L^p(\mathbb{R}^d;\mathbb{R}^d)}$$

and from inequality (2.14) applied on u_λ it follows

$$\lambda^{-\frac{d}{q}} \|u\|_{L^q(\mathbb{R}^d)} \leq C\lambda^{1-\frac{d}{p}} \|\nabla u\|_{L^p(\mathbb{R}^d;\mathbb{R}^d)} .$$

Since the parameter λ is arbitrary, for the inequality above to hold it is necessary that

$$1 + \frac{d}{q} - \frac{d}{p} = 0, \text{ or } q = \frac{dp}{d-p}.$$

Indeed, inequality (2.14) really holds for $u \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, provided d, p and q satisfy $1 + \frac{d}{q} - \frac{d}{p} = 0$.

Theorem 2.4.10 — Gagliardo–Nirenberg. Let $p \in [1, d)$. Then it holds for any $u \in \mathcal{C}_0^1(\mathbb{R}^d)$

$$\|u\|_{L^{p^\star}(\mathbb{R}^d)} \leq \frac{p(d-1)}{d-p} \|\nabla u\|_{L^p(\mathbb{R}^d;\mathbb{R}^d)} . \tag{2.15}$$

Choosing $\Omega = B_1(0)$ and applying the change of variables $z = x + ry$, $y \in B_1(0)$ we get (perform carefully!)

$$\left(\int_{B_r(x)} \left|u - \frac{1}{|B_r(x)|}\int_{B_r(x)} u \,\mathrm{d}y\right|^p \mathrm{d}z\right)^{\frac{1}{p}} \leq Cr \|\nabla u\|_{L^p(B_r(x))} .$$

Particularly for $p = 1$ we have

$$\frac{1}{|B_r(x)|}\int_{B_r(x)} \left|u - \frac{1}{|B_r(x)|}\int_{B_r(x)} u \,\mathrm{d}y\right| \mathrm{d}z \leq Cr\frac{1}{|B_r(x)|}\int_{B_r(x)} |\nabla u| \,\mathrm{d}z$$
$$\leq Cr\frac{1}{|B_r(x)|}\left(\int_{B_r(x)} |\nabla u|^d \,\mathrm{d}z\right)^{\frac{1}{d}} |B_r(x)|^{1-\frac{1}{d}} \leq C \|\nabla u\|_{L^d(\mathbb{R}^d)} .$$

The left-hand side characterizes the space $BMO(\mathbb{R}^d)$ (bounded mean oscillations). This space is a Banach space and

$$[u]_{BMO(\mathbb{R}^d)} = \sup_{B_r(x)\subset\mathbb{R}^d} \frac{1}{|B_r(x)|}\int_{B_r(x)} \left|u - \frac{1}{|B_r(x)|}\int_{B_r(x)} u \,\mathrm{d}y\right| \mathrm{d}z$$

is the seminorm in this space. It plays an important role in the harmonic analysis, where it is often used as a replacement of the space $L^\infty(\Omega)$, see, e.g., Stein (1993).

Remark 2.4.11. Since the smooth functions with compact support are by definition dense in $W_0^{1,p}(\Omega)$ for arbitrary open sets Ω, inequality (2.15) holds also for them. More precisely, for the same constant as above and any $u \in W_0^{1,p}(\Omega)$,

$$\|u\|_{L^{p^\star}(\Omega)} \le \frac{p(d-1)}{d-p} \|\nabla u\|_{L^p(\Omega;\mathbb{R}^d)} .$$

If $u \in W^{1,p}(\Omega)$, then using the extension from Theorem 2.4.1 we get a similar inequality also in this case. However, we need to require that Ω is a bounded Lipschitz domain and on the right-hand side we need to have the full norm in $W^{1,p}(\Omega)$.

A natural question arises, whether the assumption $\Omega \in \mathcal{C}^{0,1}$ is really necessary for the optimal embedding exponent. The following example contains counterexamples, showing the necessity.

Example 2.4.12. We show that for $W^{1,p}(\Omega) \hookrightarrow L^{p^\star}(\Omega)$ to hold in the class of sets $\Omega \in \mathcal{C}^{0,\alpha}$ with $\alpha \in [0,1]$, the necessary condition is $\Omega \in \mathcal{C}^{0,1}$. Assume that $\Omega \subset \mathbb{R}^2$ has a part of the boundary in the neighbourhood of the point $(0,0)$ described by $|y| = x^\mu$, where $\mu > 1$ and $x \in (0,1)$, and the rest of the boundary is smooth, cf. Figure 2.3. Check carefully that then $\Omega \in \mathcal{C}^{0,\frac{1}{\mu}}$ (the description of the boundary $x = |y|^{\frac{1}{\mu}}$, $y \in [-1,1]$, must be considered).

Now, similarly as in Example 2.1.12, we consider functions of the type $u(x,y) := x^{-a}$, where we only deal with the part of the domain close to the origin, i.e. with the set $\Omega := \{(x,y)\,|\, x \in (0,1),\ y \in (-x^\mu, x^\mu)\}$, since outside the origin, the function is smooth. Then

$$\begin{aligned} \|u\|^q_{L^q(\Omega)} &= \int_0^1 \left(\int_{-x^\mu}^{x^\mu} x^{-aq}\,\mathrm{d}y \right) \mathrm{d}x = 2\int_0^1 x^{\mu-aq}\,\mathrm{d}x \\ \|\nabla u\|^p_{L^p(\Omega;\mathbb{R}^d)} &= |a|^p \int_0^1 \left(\int_{-x^\mu}^{x^\mu} x^{-(a+1)p}\,\mathrm{d}y \right) \mathrm{d}x = 2|a|^p \int_0^1 x^{\mu-(a+1)p}\,\mathrm{d}x. \end{aligned}$$

From here we immediately get that

$$\begin{aligned} u \in L^q(\Omega) &\iff q < \frac{1+\mu}{a} \\ u \in W^{1,p}(\Omega) &\iff a < \frac{1+\mu-p}{p}. \end{aligned}$$

This example confirms that we have for a two-dimensional domain $\Omega \in \mathcal{C}^{0,\frac{1}{\mu}}$ at most $W^{1,p}(\Omega) \hookrightarrow L^{q_\mu}(\Omega)$, where $q_\mu = \frac{(1+\mu)p}{1+\mu-p}$. This example can be easily transformed to d dimensions, where the correct choice is $q_\mu = \frac{(d-1+\mu)p}{d-1+\mu-p}$. For more details see (Adams, 1975, Theorem 5.35). It is possible to see that q_μ is a decreasing function of μ, having for $\mu = 1$, i.e., for a Lipschitz domain, optimal value $q_1 = p^\star$, and at infinity (domain with continuous boundary) only the value $q_\infty = p$.

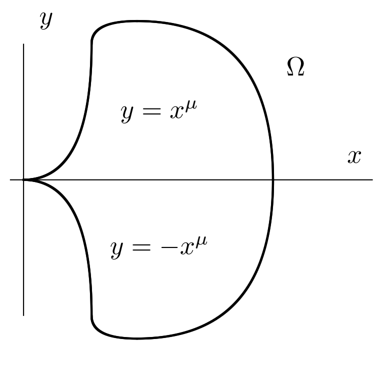

Figure 2.3: Domain Ω from Example 2.4.12.

We show at the end of this subsection that inequality (2.15) allows to prove interpolation inequalities of the type

$$\|u\|_{L^r(\mathbb{R}^d)} \le C \|\nabla u\|^\alpha_{L^p(\mathbb{R}^d;\mathbb{R}^d)} \|u\|^{1-\alpha}_{L^q(\mathbb{R}^d)} ,$$

for suitably chosen q, r, p and α.

Example 2.4.13. The following inequalities hold[6].

[6]The constants obtained above are not optimal and can be improved, see e.g., Temam (2001). These inequalities play an important role in the proof of existence, uniqueness and regularity of weak solutions to the incompressible Navier–Stokes equations and they are "responsible" for a significant difference between the results for two and three space dimensions.

1. We have for $d=2$ $\|v\|_{L^4(\mathbb{R}^2)} \leq 2^{\frac{1}{2}} \|\nabla v\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2;\mathbb{R}^2)} \|v\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}$.

2. We have for $d=3$ $\|v\|_{L^4(\mathbb{R}^3)} \leq \left(\frac{8}{3}\right)^{\frac{3}{4}} \|\nabla v\|^{\frac{3}{4}}_{L^2(\mathbb{R}^3;\mathbb{R}^3)} \|v\|^{\frac{1}{4}}_{L^2(\mathbb{R}^3)}$.

We start with inequality (2.15) for $p=1$. We take for $d=2$ the function $u=|v|^2$ and by virtue of Hölder's inequality we compute

$$\int_{\mathbb{R}^2} |v|^4 \,\mathrm{d}x \leq \left(\int_{\mathbb{R}^2} |\nabla |v|^2| \,\mathrm{d}x\right)^2 \leq 4\left(\int_{\mathbb{R}^2} |\nabla v||v| \,\mathrm{d}x\right)^2 \leq 4\|\nabla v\|^2_{L^2(\mathbb{R}^2)} \|v\|^2_{L^2(\mathbb{R}^2;\mathbb{R}^2)}.$$

Taking the fourth root we obtain the result.

For $d=3$ we choose $u=|v|^{\frac{8}{3}}$, then

$$\begin{aligned}\int_{\mathbb{R}^3} |v|^4 \,\mathrm{d}x &\leq \left(\int_{\mathbb{R}^3} |\nabla |v|^{\frac{8}{3}}| \,\mathrm{d}x\right)^{\frac{3}{2}} \leq \left(\frac{8}{3}\right)^{\frac{3}{2}} \left(\int_{\mathbb{R}^3} |\nabla v||v|^{\frac{5}{3}} \,\mathrm{d}x\right)^{\frac{3}{2}} \\ &\leq \left(\frac{8}{3}\right)^{\frac{3}{2}} \left(\int_{\mathbb{R}^3} |\nabla v||v|^{\frac{1}{3}}|v|^{\frac{4}{3}} \,\mathrm{d}x\right)^{\frac{3}{2}} \\ &\leq \left(\frac{8}{3}\right)^{\frac{3}{2}} \|\nabla v\|^{\frac{3}{2}}_{L^2(\mathbb{R}^3;\mathbb{R}^3)} \|v\|^{\frac{1}{2}}_{L^2(\mathbb{R}^3)} \|v\|^2_{L^4(\mathbb{R}^3)}.\end{aligned}$$

We now divide by $\|v\|^2_{L^4(\mathbb{R}^3)}$ and we conclude by taking the square root.

Based on similar considerations, it is possible to show a general claim which is left as a useful exercise for the kind reader.

Exercise 2.4.14 (General interpolation inequality)**.** Show that for any $\alpha \in [0,1)$ and any $r,p,q \in [1,\infty]$ satisfying

$$\frac{1}{r} = \alpha\left(\frac{1}{p} - \frac{1}{d}\right) + (1-\alpha)\frac{1}{q}$$

there exists a constant $C = C(p,q,r,d)$ such that it holds for any $u \in \mathcal{C}_0^\infty(\mathbb{R}^d)$

$$\|u\|_{L^r(\mathbb{R}^d)} \leq C \|\nabla u\|^{\alpha}_{L^p(\mathbb{R}^d;\mathbb{R}^d)} \|u\|^{1-\alpha}_{L^q(\mathbb{R}^d)}. \tag{2.16}$$

Moreover, if $p<d$, it is also possible to take $\alpha = 1$. Due to the density argument, these inequalities can be extended for functions $u \in L^r(\mathbb{R}^d)$ for which $\nabla u \in L^p(\mathbb{R}^d;\mathbb{R}^d)$.

As a corollary of the embedding theorem for $p>d$ we have

Corollary 2.4.15 (If $\Omega \in \mathcal{C}^{0,1}$, then $\mathcal{C}^{0,1}(\overline{\Omega}) = W^{1,\infty}(\Omega)$). Let $\Omega \in C^{0,1}$. Then $\mathcal{C}^{0,1}(\overline{\Omega}) = W^{1,\infty}(\Omega)$.

Proof. We already know (cf. Corollary 2.3.2), that for any open set $\mathcal{C}^{0,1}(\overline{\Omega}) \hookrightarrow W^{1,\infty}(\Omega)$. On the other hand, we know for Lipschitz domains that $W^{1,\infty}(\Omega) \hookrightarrow \mathcal{C}^{0,1}(\overline{\Omega})$ and thus also $\mathcal{C}^{0,1}(\overline{\Omega}) = W^{1,\infty}(\Omega)$. ■

The only "weakness" of this approach is the use of the Rademacher Theorem A.2.16 in the proof of the extension theorem. However, this can be removed, since it is possible to show a more general result which include the Rademacher Theorem as a special case.

Recall that a function u has at the point x total differential if there exists $a \in \mathbb{R}^d$ such that

$$\lim_{y\to x} \frac{|u(y)-u(x)-a\cdot(y-x)|}{|x-y|} = 0.$$

Recall also that a is equal to the gradient of u at the point x in the classical sense. In the theorem below we deal with the continuous representative $[u]$, since due to Theorem 2.4.7 it holds $W^{1,p}(\Omega) \hookrightarrow \mathcal{C}^0(\overline{\Omega})$.

Theorem 2.4.16 Let $u \in W^{1,p}(\Omega)$ and $p \in (d,\infty]$. Then the function u has total differential almost everywhere in Ω and the classical derivative is at almost all points, where the total differential exists, equal to the weak derivative.

2.4.2 Theorems on compact embedding

We now strengthen the results of the previous subsection and state that the corresponding compact embeddings for Sobolev spaces defined on Lipschitz domains hold true. Furthermore, we also claim that some results remain true even for less smooth domains, in particular for only domains with continuous boundary. We again consider separately the cases $p<d$, $p=d$ and $p>d$. Recall also the notation $p^\star = \frac{dp}{d-p}$ and $\mu^\star = 1 - \frac{d}{p}$. The main result for $p<d$ is the following.

Theorem 2.4.17 — On compact embedding of $W^{1,p}(\Omega)$ for $p < d$. Let $\Omega \in \mathcal{C}^{0,1}$ and $p \in [1, d)$. Then it holds for any $q \in [1, p^\star)$

$$W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega).$$

For $p \geq d$ we have the following result.

Theorem 2.4.18 — On compact embedding of $W^{1,p}(\Omega)$ for $p \geq d$. Let $\Omega \in \mathcal{C}^{0,1}$ and $p \in [d, \infty]$. Then it holds for any $q \in [1, \infty)$

$$W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega).$$

Moreover, if $p > d$, then it holds for any $\alpha \in [0, \mu^\star)$

$$W^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathcal{C}^{0,\alpha}(\overline{\Omega}),$$

and thus also $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^\infty(\Omega)$.

As indicated in Example 2.4.12, the assumption on the Lipschitz continuity of the boundary is a key one, otherwise the embedding (continuous) up to the limit spaces cannot be true. On the other hand, the same example indicates that for domains with Hölder continuous boundary it is possible to expect some improvement in the integrability and we may hope for some compact embedding. The following theorem gives the claim which will be also fundamental for the proof of Theorem 2.4.17.

Theorem 2.4.19 — On compact embedding for domains with $\mathcal{C}$-boundary. Let $\Omega \in \mathcal{C}$ and $p \in [1, \infty)$. Then it holds

$$W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega).$$

As a trivial corollary we also have.

Corollary 2.4.20. Let $\Omega \in \mathcal{C}$ and $p \in [1, \infty)$. Then it holds for any $q \in [1, p]$

$$W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega).$$

2.4.3 General Sobolev embeddings

The previous subsections dealt with $W^{1,p}(\Omega)$. By induction, the results above can be generalized.

Theorem 2.4.21 — General Sobolev embeddings. Let $\Omega \in \mathcal{C}^{0,1}$, $k \in \mathbb{N}$ and $p \in [1, \infty]$. Let further $j \in \{0, 1, \ldots, k-1\}$ be arbitrary. Denote

$$m_0 := \frac{1}{p} - \frac{k-j}{d} \qquad \text{and if } m_0 \neq 0, \qquad m := \frac{1}{m_0}.$$

Then we have the following.

1. If $m_0 > 0$, then
   (a) $W^{k,p}(\Omega) \hookrightarrow W^{j,m}(\Omega)$
   (b) for any $m_1 \in [1, m)$ it holds $W^{k,p}(\Omega) \hookrightarrow\hookrightarrow W^{j,m_1}(\Omega)$.
2. If $m_0 = 0$, then it holds for any $q \in [1, \infty)$ that $W^{k,p}(\Omega) \hookrightarrow\hookrightarrow W^{j,q}(\Omega)$.
3. If $m_0 < 0$, we set $\mu := -dm_0$ and it holds
   (a) if $\mu \in (0, 1)$, then $W^{k,p}(\Omega) \hookrightarrow \mathcal{C}^{j,\mu}(\overline{\Omega})$ and for any $\alpha \in [0, \mu)$ we have $W^{k,p}(\Omega) \hookrightarrow\hookrightarrow \mathcal{C}^{j,\alpha}(\overline{\Omega})$
   (b) if $\mu = 1$, then
   $$\begin{cases} p \neq \infty : \ \forall \alpha \in [0, 1) : \ W^{k,p}(\Omega) \hookrightarrow\hookrightarrow \mathcal{C}^{j,\alpha}(\overline{\Omega}) \\ p = \infty : \ W^{k,\infty}(\Omega) \hookrightarrow \mathcal{C}^{k-1,1}(\overline{\Omega}) \end{cases}$$
   (c) if $\mu > 1$, then it holds for any $\alpha \in [0, 1]$ that $W^{k,p}(\Omega) \hookrightarrow\hookrightarrow \mathcal{C}^{j,\alpha}(\overline{\Omega})$.

The last result describes a special case which gives a slightly better result than expected.

Theorem 2.4.22 — On embedding of $W^{d,1}(\Omega)$. Let $\Omega \in \mathcal{C}^{0,1}$. Then $W^{d,1}(\Omega) \hookrightarrow \mathcal{C}^0_B(\Omega)$, where

$$\mathcal{C}^0_B(\Omega) = \{u \in \mathcal{C}(\Omega) \mid \sup_{x \in \Omega} |u(x)| =: \|u\|_{\mathcal{C}^0_B(\Omega)} < \infty\}.$$

Exercise 2.4.23. Prove the above result for $d = 1$!

2.5 Traces of Sobolev functions

The Sobolev functions are used to formulate the boundary value problems for partial differential equations. It is therefore necessary to have a possibility to speak about values of Sobolev functions on the boundary of the domain Ω. There are two approaches how to solve this problem. One is based on the notion of "capacity" of a set and can be found, e.g., in Mazja (1985). The other approach which is more commonly used is based on extension of a certain linear operator and will be presented here.

If $u \in W^{1,p}(\Omega)$, then for $p > d$ there exists a continuous representative (up to the boundary of Ω, provided the domain is sufficiently regular) which equals to the given function u almost everywhere in Ω. Whence in this case the boundary value is well defined (and is even Hölder continuous). Thus, below we discuss only the case $p \leq d$.

In this situation it is not any more clear what does the boundary value mean, since the d-dimensional measure of the boundary is zero. On the other hand, under suitable assumption on the regularity of the boundary of Ω (at least for $\Omega \in \mathcal{C}^0$) we know that smooth functions up to the boundary are dense in $W^{1,p}(\Omega)$, the restriction of functions $u \in W^{1,p}(\Omega)$ on $\partial\Omega$ is well defined for a dense subset of $W^{1,p}(\Omega)$. It therefore suffices to study if the restriction operator for functions $u \in \mathcal{C}^\infty(\overline{\Omega})$ on $\partial\Omega$ is well defined as an operator from $W^{1,p}(\Omega)$ into a certain function space defined on the boundary of Ω. Below we explain that it is possible. First, however, we need to define correctly the corresponding spaces.

2.5.1 Surface integral and spaces $L^p(\partial\Omega)$

In the whole subsection we assume that the boundary is of the class $\mathcal{C}^{0,1}$ and we use the notation from Definition 2.2.10. Further, let $\{\phi_r\}_{r=1}^M$ be the partition of unity on a certain small neighbourhood of $\partial\Omega$ covered by the local description of the boundary. We first define sets of zero measure.

Definition 2.5.1 — Set of zero measure on $\partial\Omega$. Let $A \subset \partial\Omega$. We say that A is a set of zero measure, if it holds for any $r \in \{1, \dots, M\}$

$$|\{x_r' \in \Delta_r \,|\, T_r(x_r', a_r(x_r')) \in A\}|_{d-1} = 0.$$

Similarly as in the case of the standard Lebesgue measure, in what follows, we use the terminology *almost everywhere on* $\partial\Omega$, if the claim holds for all $x \in \partial\Omega \setminus A$, where A is a set of zero measure.

We continue with the definition of measurable functions.

Definition 2.5.2 — Measurable functions on $\partial\Omega$. Let $f\colon \partial\Omega \to \mathbb{R}$. We say that f is measurable, if for any $r \in \{1, \dots, M\}$, $f \circ T_r$ is measurable on Δ_r with respect to the $(d-1)$-dimensional measure.

We are now ready to define the surface integral.

Definition 2.5.3 — Integral $\int_{\partial\Omega} \cdot \,\mathrm{d}S$. Let $u\colon \partial\Omega \to \mathbb{R}$ be measurable. Denote $u_r(x) := u(x)\phi_r(x)$, where $\{\phi_r\}_{r=1}^M$ is the partition of unity corresponding to the open covering of $\partial\Omega$ by $\cup_{r=1}^M V_r$. Then we define the integral of a function u over $\partial\Omega$ as

$$\int_{\partial\Omega} u \,\mathrm{d}S := \sum_{r=1}^M \int_{\Delta_r} u_r(T_r(x_r', a_r(x_r'))) \sqrt{1 + |\nabla a_r(x_r')|^2} \,\mathrm{d}x_r', \tag{2.17}$$

provided all the integrals on the right-hand side exist and are finite.

Remark 2.5.4. Note that in the definition above, each integrand on the right-hand side of (2.17) is a measurable function. It follows from the fact that u is measurable and thus, due to the smoothness of ϕ_r, also the function u_r is measurable. Furthermore, since functions a_r are Lipschitz, then the Rademacher Theorem A.2.16 ensures existence of ∇a_r almost everywhere in Δ_r and the gradient is also measurable.

The above defined surface integral seems to depend on the choice of parametrization (description) of the boundary $\partial\Omega$. The following key theorem justifies that, indeed, it is not the case.

Theorem 2.5.5 — Independence of surface integral of parametrization. Let $\Omega \in \mathcal{C}^{0,1}$ be arbitrary. Let $\{a_{r1}, \Delta_{r1}, T_{r1}\}_{r1=1}^{M_1}$ and $\{a_{r2}, \Delta_{r2}, T_{r2}\}_{r2=1}^{M_2}$ be two arbitrary descriptions of the boundary which are in agreement with Definition 2.2.10. Let further $\{\phi_{r1}\}_{r1=1}^{M_1}$ and $\{\phi_{r2}\}_{r2=1}^{M_2}$ be two partitions of unity corresponding to them. Then a function u is measurable with respect to the first parametrization, if and only if it is measurable with respect to the second parametrization. Furthermore, it holds for any measurable u

$$\begin{aligned} &\sum_{r1=1}^{M_1} \int_{\Delta_{r1}} u_{r1}(T_{r1}(x_{r1}', a_{r1}(x_{r1}'))) \sqrt{1 + |\nabla a_{r1}(x_{r1}')|^2} \,\mathrm{d}x_{r1}' \\ &= \sum_{r2=1}^{M_2} \int_{\Delta_{r2}} u_{r2}(T_{r2}(x_{r2}', a_{r2}(x_{r2}'))) \sqrt{1 + |\nabla a_{r2}(x_{r2}')|^2} \,\mathrm{d}x_{r2}'; \end{aligned} \tag{2.18}$$

whence the surface integral of u does not depend on the choice of parametrization of $\partial\Omega$.

The surface integral is thus well (and uniquely) defined and we can present the definition of the Lebesgue spaces on $\partial\Omega$. This definition is in fact a straightforward modification of the spaces $L^p(\Omega)$.

Definition 2.5.6 — Spaces $L^p(\partial\Omega)$. Let $\Omega \in \mathcal{C}^{0,1}$ and let u be a function defined almost everywhere in $\partial\Omega$. We say that the function u is an element of $L^p(\partial\Omega)$, $p \in [1,\infty)$, if

$$\int_{\partial\Omega} |u|^p \, \mathrm{d}S < \infty.$$

We further say that the function u is an element of $L^\infty(\partial\Omega)$, if (the essential supremum, more precisely the notion of a set with zero measure, is understood in the sense of Definition 2.5.1)

$$\operatorname*{ess\,sup}_{x\in\partial\Omega} |u(x)| < \infty.$$

We consider the following norms in the spaces $L^p(\partial\Omega)$

$$\|u\|_{L^p(\partial\Omega)} := \begin{cases} \left(\int_{\partial\Omega} |u|^p \, \mathrm{d}S\right)^{\frac{1}{p}} & p \in [1,\infty) \\ \operatorname*{ess\,sup}_{x\in\partial\Omega} |u(x)| & p = \infty. \end{cases} \tag{2.19}$$

Similarly as for the classical Lebesgue spaces we understand elements of $L^p(\partial\Omega)$ as classes of equivalent functions, i.e, we say that $u \sim v$, if $u = v$ almost everywhere in $\partial\Omega$. Based on this convention we have the following result.

Theorem 2.5.7 — Properties of spaces $L^p(\partial\Omega)$. Let $\Omega \in \mathcal{C}^{0,1}$. Then the space $L^p(\partial\Omega)$ with the norm defined in (2.19) is a Banach space which is separable for $p \in [1,\infty)$ and reflexive for $p \in (1,\infty)$.

Remark 2.5.8. Recall that if $\Omega \in \mathcal{C}^{0,1}$, then due to the Rademacher Theorem A.2.16 the functions a_r are differentiable in Δ_r and there exists a constant $C > 0$ such that $\left|\frac{\partial a_r}{\partial x'_{r_i}}\right| \le C < \infty$ almost everywhere in Δ_r for any $r \in \{1, 2, \dots, M\}$. Thus we may consider, for $p \in [1,\infty)$, instead of norm (2.19) an equivalent one,

$$\mathcal{N}(u)_{L^p(\partial\Omega)} := \left(\sum_{r=1}^{M} \int_{\Delta_r} |u_r(T_r(x'_r, a_r(x'_r)))|^p \, \mathrm{d}x'_r\right)^{\frac{1}{p}} \tag{2.20}$$

which is sometimes easier to work with.

The following lemma claims that yet another equivalent norm can be defined without the functions $\{\phi_r\}_{r=1}^M$ from the partition of unity.

Lemma 2.5.9 Let $\Omega \in \mathcal{C}^{0,1}$, a function u defined almost everywhere in $\partial\Omega$, be measurable such that it is non-zero only in $T_r(\Lambda_r)$ for some $r \in \{1, \dots, M\}$ fixed. Let it hold $\int_{\Delta_r} |u(T_r(x'_r, a_r(x'_r)))|^p \, \mathrm{d}x'_r < \infty$ for a certain $p \in [1,\infty)$. Then $u \in L^p(\partial\Omega)$ and there exists a constant $C = C(\partial\Omega)$ such that

$$\|u\|_{L^p(\partial\Omega)} \le C \left(\int_{\Delta_r} |u(T_r(x'_r, a_r(x'_r)))|^p \, \mathrm{d}x'_r\right)^{\frac{1}{p}}.$$

2.5.2 Theorems on traces for $W^{1,p}(\Omega)$

Before we introduce the main result of this subsection which allows to speak about boundary values for Sobolev functions, let us motivate the optimality of the result.

Example 2.5.10. Let $u(x) := |x|^{-\alpha}$. We already know from Example 2.1.12 that this function belongs to $W^{1,p}(B_1(0))$ provided $\alpha < \frac{d-p}{p}$. The same holds if we replace $B_1(0)$ by the half-ball $\{x \in \mathbb{R}^d \,|\, |x| < 1, x_d > 0\}$. Since this function is smooth outside the origin, we may expect that its trace will coincide with the function outside the origin. Computing its trace, the only interesting part is the set $x_d = 0$ (otherwise the function is smooth). We therefore have

$$\int_{\{B_1(0)\subset\mathbb{R}^{d-1}\,|\,x_d=0\}} |x|^{-\alpha q} \, \mathrm{d}S = C(d) \int_0^1 r^{-\alpha q + d - 2} \, \mathrm{d}r.$$

This integral is finite provided $-\alpha q + d - 2 > -1$, i.e., for $q < \frac{d-1}{\alpha}$. This indicates that the best result we may expect is that if $u \in W^{1,p}(\Omega)$, $p < d$, then $u \in L^q(\partial\Omega)$ for $q \in [1, \frac{(d-1)p}{d-p}]$.

We are now ready to formulate the promised main result.

Theorem 2.5.11 — On trace operator for $W^{1,p}(\Omega)$ with $p \in [1,d)$. Let $\Omega \in \mathcal{C}^{0,1}$. Define the linear continuous operator $T\colon \mathcal{C}^\infty(\overline{\Omega}) \to \mathcal{C}(\partial\Omega)$ by

$$Tu := u|_{\partial\Omega}.$$

Denote for arbitrary $p \in [1,d)$

$$p^\sharp := \frac{dp-p}{d-p}, \qquad \text{i.e.,} \qquad \frac{1}{p^\sharp} = \frac{1}{p} - \frac{p-1}{p(d-1)}.$$

Then there exists a uniquely defined extension of T such that the mapping is linear and

$$T\colon W^{1,p}(\Omega) \to L^q(\partial\Omega),$$

is bounded (thus continuous) for all $q \in [1,p^\sharp]$.

Remark 2.5.12. Since $C(\partial\Omega) \hookrightarrow L^q(\partial\Omega)$, the operator T defined as $Tu := u|_{\partial\Omega}$ is in fact defined as an operator from $\mathcal{C}^\infty(\overline{\Omega}) \to L^q(\partial\Omega)$ for any $q \in [1,\infty]$. However, it is possible to extend it to $W^{1,p}(\Omega)$ only for $q \le p^\sharp$.

The situation is much simpler for $p \ge d$ and the following claim holds.

Theorem 2.5.13 — On trace operator for $W^{1,p}(\Omega)$ with $p \ge d$. Let $\Omega \in \mathcal{C}^{0,1}$ and T be the trace operator defined in Theorem 2.5.11. Then the operator is continuous from $W^{1,d}(\Omega)$ to $L^q(\partial\Omega)$ for each $q \in [1,\infty)$. Moreover, the operator T is continuous from $W^{1,p}(\Omega)$ to $L^q(\partial\Omega)$ for each $q \in [1,\infty]$ if $p \in (d,\infty]$.

Next we consider further properties of the trace operator. In fact, except for $p=1$, the operator is compact as an operator from $W^{1,p}(\Omega)$ to certain Lebesgue spaces on $\partial\Omega$.

Theorem 2.5.14 — On compactness of trace operator. Let $\Omega \in \mathcal{C}^{0,1}$ and T be the operator defined in Theorem 2.5.11. Then it holds.

1. If $p \in (1,d)$, then the operator T is compact from $W^{1,p}(\Omega)$ to $L^q(\partial\Omega)$ for any $q \in [1,p^\sharp)$.
2. If $p > d$, then the operator T is compact from $W^{1,p}(\Omega)$ to $L^q(\partial\Omega)$ for each $q \in [1,\infty]$.
3. If $p = d$, then the operator T is compact from $W^{1,d}(\Omega)$ to $L^q(\partial\Omega)$ for each $q \in [1,\infty)$.

We saw in the section devoted to continuous and compact embeddings that we may hope for weakening the assumptions on the regularity of the boundary of Ω, from Lipschitz to (Hölder) continuous, for the price to get "non-optimal" results. The following example shows that any weakening of assumptions on the regularity of the boundary leads to problems with the definition of the traces.

Example 2.5.15. Let us consider a domain $\Omega \subset \mathbb{R}^2$ such that part of its boundary is formed by the curve (cf. Example 2.4.12)

$$|y| = x^\mu, \quad x \in [0,1], \ \mu > 1.$$

We showed in Example 2.4.12 that if the rest of the boundary is smooth, then the function $u(x,y) = x^{-a}$ belongs to $W^{1,2}(\Omega)$ if $a < \frac{1+\mu-2}{2} = \frac{\mu-1}{2}$.

If we compute the integral over the boundary, we get[7]

$$a < 1 \Longleftrightarrow \int_{\partial\Omega} |u| \, \mathrm{d}S < \infty.$$

Therefore, for $\mu > 3$, there exist functions from $W^{1,2}(\Omega)$ which do not belong to any $L^q(\partial\Omega)$ for arbitrary $q \ge 1$. It illustrates that the condition $\Omega \in \mathcal{C}^{0,1}$ cannot be weaken.

A question appears, whether $L^{p^\sharp}(\partial\Omega)$ is the range of the trace operator from $W^{1,p}(\Omega)$ (if $p \in [1,d)$). This question is important in connection to the boundary value problems for partial differential equations. The answer is negative. It only holds that the range of the corresponding trace operator is dense in $L^{p^\sharp}(\partial\Omega)$. The precise characterization requires to build the theory for spaces with fractional derivative. In fact, the range of the trace operator is the space $W^{1-\frac{1}{p},p}(\partial\Omega)$, $1 < p \le \infty$. For $p=1$ the range of the trace operator is $L^1(\partial\Omega)$, but the situation is rather complex here. More precise information about these spaces will be given in Chapter 6.

[7] If we consider the parametric description of the boundary $x=t$, $y=t^\mu$, $t \in [0,1]$, we thus compute

$$\int_0^1 t^{-a} \sqrt{1+(\mu t^{\mu-1})^2} \, \mathrm{d}t$$

which leads to the result presented above.

2.5.3 Characterization of $W_0^{1,p}(\Omega)$ and integration by parts

The theorems on traces have many important corollaries. It not only allows to speak about boundary values for Sobolev functions at the boundary $\partial\Omega$, but it also makes possible to generalize the classical theorem on integration by parts for Sobolev functions and to characterize precisely the space $W_0^{1,p}(\Omega)$. Let us start with the integration by parts. It is usually formulated for smooth Ω (piecewise smooth Ω) and functions with continuous derivatives up to the boundary. We saw in Theorem 2.1.22 that the assumption on smoothness up to the boundary can be weaken, provided we consider functions from $W_0^{1,p}(\Omega)$. We now generalize this result for Lipschitz domains and Sobolev functions (generally with nonzero trace on the boundary)

Theorem 2.5.16 — On integration by parts II. Let $\Omega \in \mathcal{C}^{0,1}$. Then the outer normal $\boldsymbol{\nu}$ exists almost everywhere on $\partial\Omega$. Let further $p, q \in [1,\infty)$ be such that one of the possibilities holds:

1. $p \in [1,d)$ and $q \in [1,d)$ such that $\frac{1}{p} + \frac{1}{q} \leq \frac{d+1}{d}$
2. $p = d$ and $q > 1$ (or $q = d$ and $p > 1$, respectively)
3. $p > d$ and $q \geq 1$ (or $q > d$ and $p \geq 1$, respectively).

Then it holds for any $u \in W^{1,p}(\Omega)$ and $v \in W^{1,q}(\Omega)$ ($\boldsymbol{\nu}$ is the unit outer normal vector to $\partial\Omega$)

$$\int_\Omega u \frac{\partial v}{\partial x_i} \,\mathrm{d}x = \int_{\partial\Omega} u v \nu_i \,\mathrm{d}S - \int_\Omega \frac{\partial u}{\partial x_i} v \,\mathrm{d}x. \tag{2.21}$$

Another important consequence of the theorem on traces is the following characterization of the space $W_0^{1,p}(\Omega)$.

Theorem 2.5.17 — Characterization of $W_0^{1,p}(\Omega)$. Let $\Omega \in \mathcal{C}^{0,1}$. Then

$$W_0^{1,p}(\Omega) = \left\{ u \in W^{1,p}(\Omega) \,|\, Tu = 0 \text{ almost everywhere on } \partial\Omega \right\}.$$

2.6 Poincaré inequalities and equivalent norms

We show in this section that in some cases it is possible to replace the standard norm on the space $W^{k,p}(\Omega)$ by different functionals which define there equivalent norms. These equivalent norms play an important role in the theory of partial differential equations, for example if we prescribe the boundary value on the full boundary or its parts, or if we consider functions with prescribed mean value. We start with one general lemma.

Lemma 2.6.1 Let $\Omega \in \mathcal{C}^0$, $k \in \mathbb{N}$ and $p \in [1,\infty)$. Denote by P_k polynomials of the degree at most k. Let $\{f_i\}_{i=1}^l$ be continuous bounded functionals (not necessarily linear) on $W^{k,p}(\Omega)$ which fulfil for any $u \in P_{k-1}$

$$\sum_{i=1}^l |f_i(u)| = 0 \quad \Longleftrightarrow \quad u = 0 \text{ almost everywhere in } \Omega.$$

Let it further hold for any $u \in W^{k,p}(\Omega)$, $\lambda \in \mathbb{R}$ and $i = 1, \dots, l$

$$|f_i(\lambda u)| \leq |\lambda| \, |f_i(u)| .$$

Then there exist positive constants c_1 and c_2 such that we have for any $u \in W^{k,p}(\Omega)$

$$c_1 \|u\|_{W^{k,p}(\Omega)} \leq \left(\sum_{|\alpha|=k} \|D^\alpha u\|_{L^p(\Omega)}^p + \sum_{i=1}^l |f_i(u)|^p \right)^{\frac{1}{p}} \leq c_2 \|u\|_{W^{k,p}(\Omega)} . \tag{2.22}$$

Proof. The second inequality in (2.22) is trivial, let us therefore consider only the first one.

For contradiction, let us assume that there exists a sequence of functions $\{\tilde{u}_n\}_{n=1}^\infty \subset W^{k,p}(\Omega)$ such that it holds

$$\left(\sum_{|\alpha|=k} \|D^\alpha \tilde{u}_n\|_{L^p(\Omega)}^p + \sum_{i=1}^l |f_i(\tilde{u}_n)|^p \right)^{\frac{1}{p}} < \frac{\|\tilde{u}_n\|_{W^{k,p}(\Omega)}}{n}.$$

Evidently $\tilde{u}_n \neq 0$, therefore $u_n := \tilde{u}_n / \|\tilde{u}_n\|_{W^{k,p}(\Omega)}$ is well defined. Dividing the above stated inequality by $\|\tilde{u}_n\|_{W^{k,p}(\Omega)}$

and using the assumptions on f_i we get

$$\left(\sum_{|\alpha|=k} \|D^\alpha u_n\|^p_{L^p(\Omega)} + \sum_{i=1}^{l} |f_i(u_n)|^p \right)^{\frac{1}{p}} < \frac{1}{n} \tag{2.23}$$

as well as $\|u_n\|_{W^{k,p}(\Omega)} = 1$. Since the sequence $\{u_n\}_{n=1}^\infty$ is bounded in $W^{k,p}(\Omega)$, due to the compact embedding $W^{k,p}(\Omega) \hookrightarrow\hookrightarrow W^{k-1,p}(\Omega)$ (Theorem 2.4.19 or 2.4.21) we may choose subsequence (relabeled) and find $u \in W^{k-1,p}(\Omega)$ such that $u_n \to u$ in $W^{k-1,p}(\Omega)$. Furthermore, it follows immediately from (2.23) that for $|\alpha| = k$ we have $D^\alpha u_n \to 0$ in $L^p(\Omega)$. This implies that $u_n \to u$ in $W^{k,p}(\Omega)$, too.

The strong convergence implies for the limit function u that $\|u\|_{W^{k,p}(\Omega)} = 1$ and $D^\alpha u = 0$ for any $|\alpha| = k$. Applying Lemma 2.2.3 and a simple induction it is not difficult to see that $u \in P_{k-1}$. Since the functionals f_i are continuous, we see that $\sum_{i=1}^l |f_i(u)|^p = 0$ and since $u \in P_{k-1}$, we also get $u = 0$ which contradicts to $\|u\|_{W^{k,p}(\Omega)} = 1$. ■

Remark 2.6.2. Functionals satisfying assumptions of Lemma 2.6.1 always exist, it is enough to take

$$f_\alpha(u) = \int_{\Omega^\star} x^\alpha u(x)\,\mathrm{d}x, \qquad \text{for all } |\alpha| \le k-1,$$

or

$$\tilde{f}_\alpha = \int_{\Omega^\star} D^\alpha u(x)\,\mathrm{d}x, \qquad \text{for all } |\alpha| \le k-1,$$

where $\Omega^\star$ is an arbitrary non-empty subdomain of Ω.

Lemma 2.6.1 has a number of applications. Let us present the most important ones.

Theorem 2.6.3 — On equivalent norms in $W^{1,p}(\Omega)$. Let $\Omega \in \mathcal{C}^{0,1}$. Let $\Omega^\star \subset \Omega$ be such that $|\Omega^\star|_d > 0$ and $\Gamma \subset \partial\Omega$ such that $|\Gamma|_{d-1} > 0$. Let further $p \in [1, \infty)$ and α_i, $i = 1, \dots, 4$, be non-negative numbers such that $\sum_{i=1}^4 \alpha_i > 0$. Then there exist positive constants c_1 a c_2 such that it holds for any $u \in W^{1,p}(\Omega)$

$$c_1 \|u\|_{W^{1,p}(\Omega)} \le \left(\|\nabla u\|^p_{L^p(\Omega)} + \alpha_1 \int_\Gamma |u|^p \,\mathrm{d}S + \alpha_2 \left| \int_\Gamma u \,\mathrm{d}S \right|^p + \alpha_3 \int_{\Omega^\star} |u|^p \,\mathrm{d}x + \alpha_4 \left| \int_{\Omega^\star} u \,\mathrm{d}x \right|^p \right)^{\frac{1}{p}} \le c_2 \|u\|_{W^{1,p}(\Omega)}.$$

Proof. Denote $f_1(u) = \left(\int_\Gamma |u|^p \,\mathrm{d}S\right)^{\frac{1}{p}}$, $f_2(u) = \left|\int_\Gamma u \,\mathrm{d}S\right|$, $f_3(u) = \left(\int_{\Omega^\star} |u|^p \,\mathrm{d}x\right)^{\frac{1}{p}}$ and $f_4(u) = \left|\int_{\Omega^\star} u \,\mathrm{d}x\right|$. All four functionals are clearly positive homogeneous (in the sense as presented in Lemma 2.6.1), bounded and continuous on $W^{1,p}(\Omega)$ (here we use the theorems on traces, i.e., Theorems 2.5.11 or 2.5.13 and the assumption $\Omega \in \mathcal{C}^{0,1}$). We finally apply Lemma 2.6.1 and it is enough to verify that if $u = const.$ almost everywhere in Ω, then for arbitrary $i \in \{1, \dots, 4\}$

$$f_i(u) = 0 \iff u = 0.$$

This equivalence is, however, evident. ■

Note that unlike in Lemma 2.6.1, we consider in Theorem 2.6.3 domains with Lipschitz boundary. This is connected with the fact that we speak about values of u on the boundary and use the theorems on traces (Theorems 2.5.11 and 2.5.13). If we do not consider integral over (a part of) the boundary, it would be enough to consider domains with continuous boundaries only.

Some inequalities (equivalent norms) have traditional names. Below we present their short list.

Remark 2.6.4 (Important inequalities). Let $\Omega \in \mathcal{C}^{0,1}$. Using Theorem 2.6.3 we can easily obtain the following inequalities.

1) Inequality $c_1 \|u\|_{W^{1,p}(\Omega)} \le \left(\|\nabla u\|^p_{L^p(\Omega)} + \int_\Gamma |u|^p \,\mathrm{d}S\right)^{\frac{1}{p}}$ is called Poincaré–Friedrichs inequality. Evidently, we can replace $\int_\Gamma |u|^p \,\mathrm{d}S$ by $\left(\int_\Gamma |u|^q \,\mathrm{d}S\right)^{\frac{p}{q}}$, where $q \in [1, \frac{dp-p}{d-p}]$ for $p \in [1, d)$ and $q \in [1, \infty)$ for $p \ge d$.

2) Inequality $c_1 \|u\|_{W^{1,p}(\Omega)} \le \left(\|\nabla u\|^p_{L^p(\Omega)} + \left|\int_\Omega u \,\mathrm{d}x\right|^p\right)^{\frac{1}{p}}$ is called Poincaré inequality. Its generalization for the space $W^{k,p}(\Omega)$ can be found below.

3) It is possible to replace $\int_{\Omega^\star} |u|^p \,\mathrm{d}x$ by $\left(\int_{\Omega^\star} |u|^q \,\mathrm{d}x\right)^{\frac{p}{q}}$, where $q \in [1, \frac{dp}{d-p}]$ for $p \in [1, d)$ and $q \in [1, \infty)$ for $p \ge d$.

Let us finally mention several important inequalities for Sobolev spaces of higher order.

Theorem 2.6.5 — On equivalent norms on $W^{k,p}(\Omega)$. Let $\Omega \in \mathcal{C}^0$, $\Omega^\star \subset \Omega$ be such that $|\Omega^\star|_d > 0$, $p \in [1, \infty)$ and $k \in \mathbb{N}$. Let α_1 and α_2 be non-negative numbers satisfying $\alpha_1 + \alpha_2 > 0$. Then there exist positive numbers c_1

and c_2 such that it holds for any $u \in W^{k,p}(\Omega)$

$$c_1 \|u\|_{W^{k,p}(\Omega)} \leq \left(\sum_{|\alpha|=k} \|D^\alpha u\|^p_{L^p(\Omega)} + \alpha_1 \left(\int_{\Omega^\star} |u| \,\mathrm{d}x \right)^p + \alpha_2 \sum_{|\alpha| \leq k-1} \left| \int_\Omega D^\alpha u \,\mathrm{d}x \right|^p \right)^{\frac{1}{p}} \leq c_2 \|u\|_{W^{k,p}(\Omega)} .$$

Proof. The proof is left for a kind reader as a useful exercise. ■

We did not consider integrals over boundary in the theorem above. There were two reasons for it. The above presented result holds for domains with only smooth boundary, since it is based on compact embedding and we do not need the theorems on traces and the Lipschitz boundary. Second reason is that unlike Theorem 2.6.3, it is not enough to consider for Sobolev spaces of higher order only integrals over the boundary; we need also certain qualitative assumptions of a part of the boundary. We shall illustrate it on the case of $W^{2,p}(\Omega)$ and leave the general situation for a kind reader.

Theorem 2.6.6 — On equivalent norms on $W^{2,p}(\Omega)$. Let $\Omega \in \mathcal{C}^{0,1}$ and $p \in [1, \infty)$. Let $\Gamma \subset \partial\Omega$ be such that Γ is not a hyperplane and satisfies $|\Gamma|_{d-1} > 0$. Then there exist positive numbers c_1 and c_2 such that it holds for any $u \in W^{2,p}(\Omega)$

$$c_1 \|u\|_{W^{2,p}(\Omega)} \leq \left(\sum_{|\alpha|=2} \|D^\alpha u\|^p_{L^p(\Omega)} + \int_\Gamma |u|^p \,\mathrm{d}S \right)^{\frac{1}{p}} \leq c_2 \|u\|_{W^{2,p}(\Omega)} .$$

Since $\partial\Omega$ cannot be a hyperplane, the above presented inequality holds always for $\Gamma = \partial\Omega$.

Proof. Due to Lemma 2.6.1 it is enough to check that if $u \in P_1$ and $\int_\Gamma |u|^p \,\mathrm{d}S = 0$, then $u = 0$. Let u be a linear function and $u = 0$ almost everywhere on Γ. This, however, may happen only in the case when Γ is a hyperplane. By our assumptions, this case is excluded. ■

We considered up to now only the questions concerning equivalent norms. We have seen that it is important to exclude the possibility that the function u is a non-zero polynomial of the $(k-1)$–th order. These functions cannot always be excluded, as for example in the weak formulation for the Poisson equation with the Neumann boundary condition. We therefore introduce subspaces of Sobolev functions which are equivalent up to polynomials of a certain order.

Definition 2.6.7 — Factor space $W^{k,p}(\Omega)/P$. Let $\Omega \subset \mathbb{R}^d$, $k \in \mathbb{N}$ and $p \in [1, \infty]$. Let $P \subset P_{k-1}$ be a subspace of polynomials of the $(k-1)$–th order. Denote by $W^{k,p}(\Omega)/P$ the factor space, i.e., we say that it holds $u_1 \sim u_2$ for $u_1, u_2 \in W^{k,p}(\Omega)$, if $u_1 - u_2 \in P$. We endow this space by the norm

$$\|u\|_{W^{k,p}(\Omega)/P} := \inf_{\tilde{u} \in W^{k,p}(\Omega) :\, \tilde{u} \sim u} \|\tilde{u}\|_{W^{k,p}(\Omega)} .$$

This space is evidently a Banach space which is for $p \in [1, \infty)$ separable and for $p \in (1, \infty)$ reflexive. The proof, as well as the proof of the following theorem, is left as an easy exercise for a kind reader.

Theorem 2.6.8 — Poincaré inequality for factor spaces. Let $\Omega \in \mathcal{C}^0$ and $k \in \mathbb{N}$. Then there exist positive constants c_1 and c_2 such that it holds for any $u \in W^{k,p}(\Omega)$

$$c_1 \|u\|_{W^{k,p}(\Omega)/P_{k-1}} \leq \left(\sum_{|\alpha|=k} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}} \leq c_2 \|u\|_{W^{k,p}(\Omega)/P_{k-1}} .$$

2.7 Dual spaces

Let $k \in \mathbb{N}$ and $p \in (1, \infty)$. Denote

$$\left(W_0^{k,p'}(\Omega) \right)^* = W^{-k,p}(\Omega),$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

The restriction $F \in W^{-k,p}(\Omega)$, $k \in \mathbb{N}$ onto $\mathcal{C}_0^\infty(\Omega)$ clearly defines a distribution. In what follows, we shall characterize these distributions more precisely.

Theorem 2.7.1 — Equivalent characterization of dual spaces to $W_0^{k,p'}(\Omega)$ I. Let $p \in (1, \infty)$, $k \in \mathbb{N}$. Then

$F \in W^{-k,p}(\Omega)$, if and only if there exist functions $\{f_\alpha\}_{|\alpha|\le k} \subset L^p(\Omega)$ such that

$$F = \sum_{|\alpha|\le k} (-1)^\alpha D^\alpha f_\alpha,$$

where $D^\alpha f_\alpha$ denotes the distributional derivatives, i.e., it holds for $u \in W_0^{k,p'}(\Omega)$

$$\langle F, u\rangle_{W_0^{k,p}(\Omega)} = \sum_{|\alpha|\le k} \int_\Omega f_\alpha D^\alpha u \,\mathrm{d}x. \tag{2.24}$$

Moreover,

$$\|F\|_{W^{-k,p}(\Omega)} = \inf \left(\sum_{|\alpha|\le k} \|f_\alpha\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}},$$

where the infimum is taken over all sets of functions $\{f_\alpha\}_{|\alpha|\le k}$ which fulfil (2.24).

Proof. For the general case, see, e.g., (Kufner et al., 1977, Theorem 5.9.2), the special case $k = 1$ and $p = 2$ can be found, e.g., in (Evans, 1998, Section 5.9 Theorem 1). ■

Another possible characterization is as follows.

Theorem 2.7.2 — Equivalent characterization of dual spaces to $W_0^{k,p'}(\Omega)$ II. Let $p \in (1,\infty)$, $k \in \mathbb{N}$ and $\Omega \in \mathcal{C}^{k,0}(\Omega)$. Then for any $g \in W_0^{k,p'}(\Omega)$ the formula

$$\langle \phi_g, f\rangle_{W_0^{k,p}(\Omega)} = \sum_{|\alpha|\le k} \int_\Omega D^\alpha g D^\alpha f \,\mathrm{d}x,\;\; f \in W_0^{k,p}(\Omega)$$

defines a continuous linear functional on $W_0^{k,p}(\Omega)$.

On the contrary, for any $\phi \in \left(W_0^{k,p}(\Omega)\right)^*$ there exists exactly one $g \in W_0^{k,p'}(\Omega)$ such that

$$\forall f \in W_0^{k,p}(\Omega):\;\; \langle \phi, f\rangle_{W_0^{k,p}(\Omega)} = \sum_{|\alpha|\le k} \int_\Omega D^\alpha g D^\alpha f \,\mathrm{d}x.$$

Furthermore, there exists a positive constant $K = K(d,k,p,\Omega)$ such that

$$K\,\|g\|_{W^{k,p'}(\Omega)} \le \|\phi\|_{\left(W_0^{k,p}(\Omega)\right)^*} \le \|g\|_{W^{k,p'}(\Omega)}.$$

Proof. The proof can be found in Simader (1972). ■

Chapter 3

Linear elliptic PDE's

Elliptic equations of second order belong to the most important types of partial differential equations. They are also the most studied ones and the best understood ones. The reason of such intensive study of this type of PDEs is the fact that they describe numerous real-world problems. We shall first show several simple problems and explain the main ideas connected with the notion of weak solution to these problems. Next we turn our attention to more complex (but still linear) equations with more complex boundary conditions. We concentrate our effort to questions of existence and uniqueness of solutions. We further ask the question whether more regular data imply also higher regularity of the weak solution and we shall also study the question under which conditions the weak solution becomes the classical one. Then we show that in some situations the weak solutions are minimizers of certain variational problems. Even though the reader may get impression that the theory is complete and in some situations rather simple, the general theory of these equations still contains many open problems and there are results showing that not everything is so simple as it might seem at the first glance. Nonlinear problems will be studied in Chapter 7.

3.1 A few simple problems

Let us recall the problem studied in Subsection 1.2.1 and consider several similar problems. At this moment we can dispose with a certain knowledge of Sobolev spaces, however, still quite limited.

First, let us consider

$$\begin{aligned} -\Delta u + u &= f \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{3.1}$$

Based on considerations in Subsection 1.2.1 we easily deduce that the corresponding weak formulation of Problem (3.1) is (for simplicity with $f \in L^2(\Omega)$) as follows. We look for $u \in W_0^{1,2}(\Omega)$ such that

$$\int_\Omega (\nabla u \cdot \nabla \varphi + u\varphi)\,\mathrm{d}x = \int_\Omega f\varphi\,\mathrm{d}x \qquad \forall \varphi \in W_0^{1,2}(\Omega). \tag{3.2}$$

The existence of a weak solution in the sense of (3.2) is based on direct application of the Riesz representation Theorem for linear bounded functionals in the Hilbert space (Theorem B.2.2). The left-hand side of (3.1) is clearly the scalar product in $W^{1,2}(\Omega)$ and we know that $W_0^{1,2}(\Omega)$ is a closed subspace of $W^{1,2}(\Omega)$. Thus we may consider the same scalar product also on $W_0^{1,2}(\Omega)$. We therefore only need to verify that the right-hand side of (3.2) can be understood as an application of a linear continuous functional on the function φ. This is, however, trivial, since

$$\langle F, \varphi \rangle_{W_0^{1,2}(\Omega)} := \int_\Omega f\varphi\,\mathrm{d}x$$

is clearly linear and

$$\Big| \int_\Omega f\varphi\,\mathrm{d}x \Big| \le \|f\|_{L^2(\Omega)}\|\varphi\|_{L^2(\Omega)} \le \|f\|_{L^2(\Omega)}\|\varphi\|_{W^{1,2}(\Omega)}.$$

Whence we know that for any $F \in W^{-1,2}(\Omega) = (W_0^{1,2}(\Omega))^*$ there exists uniquely defined $u \in W_0^{1,2}(\Omega)$ such that

$$\langle F, \varphi \rangle_{W_0^{1,2}(\Omega)} = (u, \varphi)_{W_0^{1,2}(\Omega)} \qquad \forall \varphi \in W_0^{1,2}(\Omega).$$

This function u is clearly a weak solution to our problem in the sense of (3.2). Moreover, it is easy to see that any classical solution to Problem (3.1) fulfils also (3.2). The other question, namely whether a more regular weak solution is also a classical solution (or an intermediate solution, called the strong solution) we skip for a moment and return to this at the end of this section. Note that in our considerations the regularity of the set Ω did not play any role. In fact, any open nontrivial set is sufficient for the definition of the weak solution as well as for the existence and uniqueness of the weak solution to Problem (3.2).

Next we return precisely to the problem considered in Subsection 1.2.1. We take

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.3}$$

The weak solution reads as follows. We look for $u \in W_0^{1,2}(\Omega)$ such that (again, we assume $f \in L^2(\Omega)$)

$$\int_\Omega \nabla u \cdot \nabla \varphi \, \mathrm{d}x = \int_\Omega f \varphi \, \mathrm{d}x \qquad \forall \varphi \in W_0^{1,2}(\Omega). \tag{3.4}$$

The main difficulty of this problem is connected with the fact that the left-hand side of (3.4) is not any more a scalar product in $W^{1,2}(\Omega)$. On the other hand, due to Poincaré–Friedrichs inequality, we have that there exists C independent of $u \in W_0^{1,2}(\Omega)$ such that

$$\|u\|_{L^2(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)} \tag{3.5}$$

for all $u \in W_0^{1,2}(\Omega)$; this implies (we recommend the reader to check it carefully!) that the left-hand side is indeed a scalar product in $W_0^{1,2}(\Omega)$. The norm associated to this scalar product is then equivalent on $W_0^{1,2}(\Omega)$ with the standard norm on $W^{1,2}(\Omega)$ (associated to the standard scalar product on $W^{1,2}(\Omega)$). We can now again employ the Riesz representation Theorem (the fact that the right-hand side of (3.4) defines a continuous linear functional on $W_0^{1,2}(\Omega)$ with respect to the norm associated to the scalar product defined by the left-hand side of (3.2) requires to use once more inequality (3.5)) to conclude that there exists unique weak solution to Problem (3.4). Again, we did not need any smoothness of the boundary.

Next, we still keep the Poisson equation, however, instead of the homogeneous Dirichlet boundary condition we consider the inhomogeneous one. The classical formulation reads as follows

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega \\ u &= u_0 && \text{on } \partial\Omega. \end{aligned} \tag{3.6}$$

The classical formulation requires that $u_0 \in C(\partial\Omega)$ at least. For the weak solution, the choice of the optimal function space is more complex and requires the use of fractional Sobolev spaces on the boundary of Ω. In fact, the range of the trace operator from $W^{1,2}(\Omega)$ can be for $\Omega \in C^{0,1}$ characterized as $W^{\frac{1}{2},2}(\Omega)$; this space will be introduced in more details in Chapter 6. To simplify, we assume that there exists $U_0 \in W^{1,2}(\Omega)$ such that $U_0 = u_0$ on $\partial\Omega$ in the sense of traces. Indeed, the choice of U_0 is non-unique.

The weak formulation of our problem (3.6) is the following. We look for $u \in W^{1,2}(\Omega)$ such that $u - U_0 \in W_0^{1,2}(\Omega)$ and

$$\int_\Omega \nabla u \cdot \nabla \varphi \, \mathrm{d}x = \int_\Omega f \varphi \, \mathrm{d}x \qquad \forall \varphi \in W_0^{1,2}(\Omega). \tag{3.7}$$

The application of the Riesz representation Theorem is less straightforward. We look for u in the form $u = v + U_0$, where $v \in W_0^{1,2}(\Omega)$ and satisfies

$$\int_\Omega \nabla v \cdot \nabla \varphi \, \mathrm{d}x = \int_\Omega f \varphi \, \mathrm{d}x - \int_\Omega \nabla U_0 \cdot \nabla \varphi \, \mathrm{d}x \qquad \forall \varphi \in W_0^{1,2}(\Omega). \tag{3.8}$$

Since evidently

$$\langle G, \varphi \rangle_{W_0^{1,2}(\Omega)} := \int_\Omega f \varphi \, \mathrm{d}x - \int_\Omega \nabla U_0 \cdot \nabla \varphi \, \mathrm{d}x \tag{3.9}$$

is a linear continuous functional on $W_0^{1,2}(\Omega)$ with respect to the norm associated to the scalar product defined by the left-hand side of (3.8), the existence of the unique $v = v(U_0)$ is a direct consequence of the Riesz representation Theorem, exactly as for Problem (3.4). However, it does not imply directly that also $u = v(U_0) + U_0$ is independent of the choice of the extension of the boundary data U_0.

Let U_0^1 and U_0^2 be two different functions from $W^{1,2}(\Omega)$ having the same trace, $U_0^1 = U_0^2 = u_0$ on $\partial\Omega$ in the sense of traces. Let v^1 and v^2 be the corresponding solutions to Problem (3.8). Then $w := (v^1 + U_0^1) - (v^2 + U_0^2)$ satisfies

$$\int_\Omega \nabla w \cdot \nabla \varphi \, \mathrm{d}x = 0 \qquad \forall \varphi \in W_0^{1,2}(\Omega).$$

Since $w \in W_0^{1,2}(\Omega)$ too, it is possible to use $\varphi := w$ which yields

$$\int_\Omega |\nabla w|^2 \, \mathrm{d}x = 0.$$

Whence $w = const$ almost everywhere in Ω and since the trace of w is zero, it implies that $w = 0$ almost everywhere in Ω. Thus, the solution u is independent of the choice of the boundary data extension U_0. Recall that $\Omega \in C^{0,1}$ is required in order to have the trace of a Sobolev function well defined.

The last problem we look at in this section is the Neumann problem for the Helmholtz equation

$$\begin{aligned} -\Delta u + u &= f \qquad \text{in } \Omega \\ \frac{\partial u}{\partial \boldsymbol{\nu}} &= g \qquad \text{on } \partial\Omega. \end{aligned} \tag{3.10}$$

Above, $\boldsymbol{\nu}$ is the unit exterior normal vector. Here, the weak formulation requires more attention. Formally, for φ smooth up to the boundary and assuming $\Omega \in C^{0,1}$, we get

$$\int_\Omega -\Delta u \varphi \,\mathrm{d}x = -\int_{\partial\Omega} \frac{\partial u}{\partial \boldsymbol{\nu}} \varphi \,\mathrm{d}S + \int_\Omega \nabla u \cdot \nabla \varphi \,\mathrm{d}x = -\int_{\partial\Omega} g\varphi \,\mathrm{d}S + \int_\Omega \nabla u \cdot \nabla \varphi \,\mathrm{d}x.$$

Therefore, we consider the following weak formulation. We look for $u \in W^{1,2}(\Omega)$ such that

$$\int_\Omega (\nabla u \cdot \nabla \varphi + u\varphi) \,\mathrm{d}x = \int_\Omega f\varphi \,\mathrm{d}x + \int_{\partial\Omega} g\varphi \,\mathrm{d}S \qquad \forall \varphi \in W^{1,2}(\Omega). \tag{3.11}$$

Assuming $g \in L^2(\partial\Omega)$ (we could reduce the assumption, e.g., to $g \in L^{\frac{2(d-1)}{d}}(\partial\Omega)$ if $d > 2$ or $L^q(\partial\Omega)$ for some $q > 1$ if $d = 2$) the Riesz representation Theorem provides us unique weak solution to our Problem (3.11).

Clearly, whenever we have existence of a classical problem to our Problems (3.1), (3.3), (3.6) or (3.10), it is also a weak solution in the corresponding sense. Let us now look at the opposite problem, namely whether a weak solution which is sufficiently regular is also a classical solution to our problem. This question (in both directions) is usually referred as consistency of weak solutions to the given problem. We start with Problem (3.2). Let $u \in W_0^{1,2}(\Omega)$ be a weak solution to Problem (3.2) and let furthermore, $u \in C^2(\Omega) \cap C(\overline{\Omega})$. Then clearly $u = 0$ at $\partial\Omega$ in the sense of continuous functions. Using the integration by parts formula (2.3) we have for any $\varphi \in C_0^\infty(\Omega)$

$$\int_\Omega (\nabla u \cdot \nabla \varphi + u\varphi) \,\mathrm{d}x = \int_\Omega (-\Delta u + u)\varphi \,\mathrm{d}x;$$

thus we have

$$\int_\Omega (-\Delta u + u - f)\varphi \,\mathrm{d}x = 0 \tag{3.12}$$

for any $\varphi \in C_0^\infty(\Omega)$. Therefore, it holds

$$-\Delta u + u = f$$

in Ω (in the sense of continuous functions; since $u \in C^2(\Omega)$, then also $f \in C(\Omega)$) and our function u is the classical solution to (3.1). Similarly we show the consistency of weak solutions for the weak formulations (3.4) and (3.7).

Problem (3.11) is slightly different. Assume that $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies (3.11) for all $\varphi \in W^{1,2}(\Omega)$. Then, using integration by parts (2.1.22) with $\varphi \in C_0^\infty(\Omega)$ we get as above that

$$-\Delta u + u = f$$

in Ω. Next, we take $\varphi \in C^\infty(\overline{\Omega})$. We get

$$\int_{\partial\Omega} g\varphi \,\mathrm{d}S = \int_\Omega (\nabla u \cdot \nabla \varphi + u\varphi - f\varphi) \,\mathrm{d}x = \int_\Omega (-\Delta u + u - f)\varphi \,\mathrm{d}x + \int_{\partial\Omega} \frac{\partial u}{\partial \boldsymbol{\nu}} \varphi \,\mathrm{d}S.$$

Since the volume integral is identically zero and the equality holds for any $\varphi \in C^\infty(\overline{\Omega})$, it is not difficult to see that

$$\frac{\partial u}{\partial \boldsymbol{\nu}} = g$$

on $\partial\Omega$, i.e., the function u is a classical solution to (3.10).

If we replace the assumption $u \in C^2(\Omega)$ by $u \in W^{2,2}(\Omega)$, we get then only

$$-\Delta u + u = f$$

a.e. in Ω. In this situation we speak about *strong solution.* This remark is relevant for all four problems considered above.

3.2 Linear elliptic partial differential equations of the second order

In this section we turn our attention to a general *second order differential operator* and we shall discuss the weak formulation for rather general boundary conditions. We define for $\Omega \subset \mathbb{R}^d$ and a function $u\colon \Omega \to \mathbb{R}$

$$Lu := -\sum_{i,j=1}^d \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^d c_i \frac{\partial u}{\partial x_i} + \sum_{i=1}^d \frac{\partial}{\partial x_i}(d_i u) + bu, \tag{3.13}$$

where $\mathbb{A} := (a_{ij})_{i,j=1}^d\colon \Omega \to \mathbb{R}^{d\times d}$, $\mathbf{c} := (c_i)_{i=1}^d\colon \Omega \to \mathbb{R}^d$, $\mathbf{d} := (d_i)_{i=1}^d\colon \Omega \to \mathbb{R}^d$ and $b\colon \Omega \to \mathbb{R}$ are the coefficients of L. To shorten the notation, we shall often use the tensor (vector) notation

$$Lu = -\mathrm{div}\,(\mathbb{A}\nabla u) + \mathbf{c} \cdot \nabla u + \mathrm{div}\,(\mathbf{d}u) + bu.$$

We now introduce the elliptic operator; it will be closely connected to our operator L.

Definition 3.2.1 — Elliptic operator I. Let $\Omega \subset \mathbb{R}^d$ be open. We say that the operator L defined in (3.13) is elliptic, if it holds for any $i, j \in \{1, \ldots, d\}$ that $a_{ij}, b, c_i, d_i \in L^\infty(\Omega)$ and there exists $C_1 > 0$ such that it holds for any $\mathbf{z} \in \mathbb{R}^d$ and almost every $x \in \Omega$

$$\mathbf{z}^T \mathbb{A}(x)\mathbf{z} = \sum_{i,j=1}^{d} a_{ij}(x) z_i z_j \geq C_1 |\mathbf{z}|^2. \tag{3.14}$$

We now focus on the elliptic problem we would like to solve. Below, we understand under the *data* the following.

- The domain $\Omega \subset \mathbb{R}^d$ will be Lipschitz and its boundary will be formed by
$$\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup M,$$
where Γ_i are open in $\partial\Omega$ for $i = 1, 2, 3$ and the $(d-1)$-dimensional Lebesgue measure of the set M is zero.
- The datum in Ω is
$$f \colon \Omega \to \mathbb{R}.$$
- The data on $\partial\Omega$ are
$$\begin{aligned} u_0 &\colon \Gamma_1 \to \mathbb{R} \\ g &\colon \Gamma_2 \cup \Gamma_3 \to \mathbb{R} \\ \sigma &\colon \Gamma_3 \to \mathbb{R}. \end{aligned}$$

In this chapter we shall mostly directly assume that there exists $U_0 \colon \Omega \to \mathbb{R}$ such that $U_0 = u_0$ on Γ_1 in the appropriate sense.

We aim at solving the following problem.

Definition 3.2.2 — Boundary value elliptic problem: scalar equation. For the given data Ω, f, u_0 (or U_0), g and σ and the operator L defined in (3.13) which is elliptic in the sense of Definition 3.2.1, we look for $u \colon \Omega \to \mathbb{R}$ such that ($\boldsymbol{\nu}$ is the exterior unit normal vector to $\partial\Omega$)

$$Lu = f \quad \text{in } \Omega \tag{3.15}$$
$$u = u_0 \quad \text{on } \Gamma_1 \text{ (Dirichlet boundary condition)} \tag{3.16}$$
$$(\mathbb{A}\nabla u - \mathbf{d}u) \cdot \boldsymbol{\nu} = g \quad \text{on } \Gamma_2 \text{ (Neumann boundary condition)} \tag{3.17}$$
$$(\mathbb{A}\nabla u - \mathbf{d}u) \cdot \boldsymbol{\nu} + \sigma u = g \quad \text{on } \Gamma_3 \text{ (Newton boundary condition).} \tag{3.18}$$

The above stated definition includes elliptic equation with three kinds of boundary conditions. Recall that equations (3.15)–(3.18) can be written as follows.

$$-\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{d} c_i \frac{\partial u}{\partial x_i} + \sum_{i=1}^{d} \frac{\partial}{\partial x_i}(d_i u) + bu = f \quad \text{in } \Omega \tag{3.19}$$
$$u = u_0 \quad \text{on } \Gamma_1 \tag{3.20}$$
$$\sum_{i=1}^{d}\left(\left(\sum_{j=1}^{d} a_{ij}\frac{\partial u}{\partial x_j}\right) - d_i u\right)\nu_i = g \quad \text{on } \Gamma_2 \tag{3.21}$$
$$\sum_{i=1}^{d}\left(\left(\sum_{j=1}^{d} a_{ij}\frac{\partial u}{\partial x_j}\right) - d_i u\right)\nu_i + \sigma u = g \quad \text{on } \Gamma_3. \tag{3.22}$$

The term $(\mathbb{A}\nabla u - \mathbf{d}u) \cdot \boldsymbol{\nu}$ is (for $\mathbf{d} = \mathbf{0}$) often replaced by $\nabla u \cdot \mathfrak{n}$, where $\mathfrak{n}$ is the so called conormal vector defined by $n_j := \sum_{i=1}^{d} a_{ij}\nu_i$.

Observation 3.2.3. Note that if $\mathbf{d} = \mathbf{0}$, the conormal vector $\mathfrak{n}(x) = (n_1, \ldots, n_d)$ points at every point $x \in \partial\Omega$ (where it exists) outside of Ω, since due to the assumption on $\mathbb{A}$ (see (3.14)) it holds $\mathfrak{n}(x) \cdot \boldsymbol{\nu}(x) = \sum_{i,j=1}^{d} a_{ij}(x)\nu_j(x)\nu_i(x) \geq C_1|\boldsymbol{\nu}(x)|^2 = C_1 > 0$.

Below we present two examples of (scalar) elliptic problems.

Example 3.2.4. Let $\mathbb{A}(x) = \mathbb{I}$ (thus $a_{ij}(x) = \delta_{ij}$ for each $x \in \Omega$), $b = 0$ a $\mathbf{c} = \mathbf{d} = \mathbf{0}$ in Ω. Then Problem (3.15)–(3.18) is reduced to

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega \\ u &= u_0 && \text{on } \Gamma_1 \\ \frac{\partial u}{\partial \boldsymbol{\nu}} = \frac{\partial u}{\partial \mathfrak{n}} &= g && \text{on } \Gamma_2 \\ \frac{\partial u}{\partial \boldsymbol{\nu}} + \sigma u &= g && \text{on } \Gamma_3. \end{aligned} \tag{3.23}$$

This problem describes thermal equilibrium in homogeneous and isotropic material under the conditions that on Γ_1 the temperature is prescribed as u_0, on Γ_2 the heat flux through the boundary is prescribed and on Γ_3 the heat flux is proportional to the difference of the outer (given) temperature and the interior (unknown) temperature.

Another example shows that elliptic problems can describe convection and diffusion.

Example 3.2.5. Let $b = 0$, $\mathbf{d} = \mathbf{0}$ and let $\mathbf{c}$ be such that $\operatorname{div} \mathbf{c} = 0$ in Ω. Then equation (3.15) is reduced to

$$-\operatorname{div} \mathbb{A}(\nabla u) + \mathbf{c} \cdot \nabla u \ = f \text{ in } \Omega. \tag{3.24}$$

This equation describes two phenomena: the term $\mathbb{A}\nabla u$ represents the diffusion (e.g., the Brownian motion) and the term $\mathbf{c} \cdot \nabla u$ represents the transport of the quantity u in an incompressible fluid moving with the velocity field $\mathbf{c}(x)$, $x \in \Omega$.

Even though we shall mostly be concerned with the "scalar" operator L defined in (3.13), most results can be shown for systems of equations. It means for problems, where the α-th component of the operator L, where $\alpha \in \{1, \dots, N\}$ and $N \in \mathbb{N}$ denotes the number of unknown functions, is defined for $\vec{u}\colon \Omega \to \mathbb{R}^N$ as

$$(L\vec{u})^\alpha := -\sum_{\beta=1}^{N}\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}^{\alpha\beta}\frac{\partial u^\beta}{\partial x_j}\right) + \sum_{\beta=1}^{N}\sum_{i=1}^{d} c_i^{\alpha\beta}\frac{\partial u^\beta}{\partial x_i} + \sum_{\beta=1}^{N}\sum_{i=1}^{d}\frac{\partial}{\partial x_i}(d_i^{\alpha\beta}u^\beta) + \sum_{\beta=1}^{N} b^{\alpha\beta}u^\beta. \tag{3.25}$$

The coefficients of the operator L should be understood in this sence and we define mappings $\vec{\mathbb{A}}\colon \Omega \to \mathbb{R}^{d\times d\times N\times N}$, $\vec{\mathfrak{c}}\colon \Omega \to \mathbb{R}^{d\times N\times N}$, $\vec{\mathfrak{d}}\colon \Omega \to \mathbb{R}^{d\times N\times N}$ and $\mathfrak{b}\colon \Omega \to \mathbb{R}^{N\times N}$ with the help of $(\vec{\mathbb{A}})_{ij}^{\alpha\beta} := a_{ij}^{\alpha\beta}$, $(\vec{\mathfrak{c}})_i^{\alpha\beta} := c_i^{\alpha\beta}$, $(\vec{\mathfrak{d}})_i^{\alpha\beta} := d_i^{\alpha\beta}$ and $(\mathfrak{b})^{\alpha\beta} := b^{\alpha\beta}$. To shorten the notation, we use again the vector (tensor) notation

$$L\vec{u} = -\operatorname{div}\left(\vec{\mathbb{A}}\nabla\vec{u}\right) + \vec{\mathfrak{c}}\cdot\nabla\vec{u} + \operatorname{div}(\vec{\mathfrak{d}}\vec{u}) + \mathfrak{b}\vec{u}$$

which should be now understood as equality in $\mathbb{R}^N$. The expression $\vec{\mathfrak{c}} \cdot \nabla\vec{u}$ is to be understood as $(\vec{\mathfrak{c}} \cdot \nabla\vec{u})^\alpha = \sum_{i=1}^d \sum_{\alpha=1}^N c_i^{\alpha\beta}\frac{\partial u^\beta}{\partial x_i}$, $\vec{\mathfrak{d}}\vec{u}$ as $(\vec{\mathfrak{d}}\vec{u})_i^\alpha = \sum_{\beta=1}^N d_i^{\alpha\beta}u^\beta$, $\mathfrak{b}\vec{u}$ as $(\mathfrak{b}\vec{u})^\alpha = \sum_{\beta=1}^N b^{\alpha\beta}u^\beta$.

Similarly as for the scalar problem ($N = 1$) we can also define the ellipticity for the operator defined in (3.25).

Definition 3.2.6 — Elliptic operator II. Let $\Omega \subset \mathbb{R}^d$ be open. We say that the operator L defined in (3.25) is elliptic, if it holds for every $i,j \in \{1,\dots,d\}$ and $\alpha,\beta \in \{1,\dots,N\}$ that $a_{ij}^{\alpha\beta}, d_i^{\alpha\beta}, c_i^{\alpha\beta}, b^{\alpha\beta} \in L^\infty(\Omega)$ and there exists $C_1 > 0$ such that for all $\vec{\mathbf{z}} \in \mathbb{R}^{d\times N}$ and almost every $x \in \Omega$ that

$$\vec{\mathbf{z}}^T\vec{\mathbb{A}}(x)\vec{\mathbf{z}} = \sum_{i,j=1}^{d}\sum_{\alpha\beta=1}^{N} a_{ij}^{\alpha\beta}(x)z_i^\alpha z_j^\beta \geq C_1|\vec{\mathbf{z}}|^2. \tag{3.26}$$

Similarly as in the scalar case, we consider the following *data* of our problem.

- The domain $\Omega \subset \mathbb{R}^d$ will be Lipschitz and its boundary will be formed by four parts

$$\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup M,$$

  where Γ_i are open in $\partial\Omega$ for $i = 1,2,3$ and the $(d-1)$-dimensional Lebesgue measure of the set M is zero.
- The datum in Ω is

$$\vec{f}\colon \Omega \to \mathbb{R}^N.$$

- The data on $\partial\Omega$ are

$$\begin{aligned}\vec{u}_0 &\colon \Gamma_1 \to \mathbb{R}^N\\ \vec{g} &\colon \Gamma_2 \cup \Gamma_3 \to \mathbb{R}^N\\ \sigma &\colon \Gamma_3 \to \mathbb{R}^{N\times N}.\end{aligned}$$

As for the scalar case, we mostly assume in this chapter that there exists $\vec{U}_0\colon \Omega \to \mathbb{R}^N$ such that $\vec{U}_0 = \vec{u}_0$ on Γ_1 in the appropriate sense.

We aim at solving the following problem:

Definition 3.2.7 — Boundary value elliptic problem: vector equation. For the given data $\Omega, \vec{f}, \vec{u}_0, \vec{g}$ and σ and the operator L defined in (3.25) which is elliptic in the sense of Definition 3.2.6, we look for $\vec{u}\colon \Omega \to \mathbb{R}^N$

satisfying ($\boldsymbol{\nu}$ is again the unit exterior normal vector to $\partial\Omega$)

$$L\vec{u} = \vec{f} \quad \text{in } \Omega, \tag{3.27}$$
$$\vec{u} = \vec{u}_0 \quad \text{on } \Gamma_1 \quad \text{(Dirichlet boundary conditions)} \tag{3.28}$$
$$(\vec{\mathbb{A}}\nabla\vec{u} - \vec{\mathfrak{d}}\vec{u})\cdot\boldsymbol{\nu} = \vec{g} \quad \text{on } \Gamma_2 \quad \text{(Neumann boundary condition} \tag{3.29}$$
$$(\vec{\mathbb{A}}\nabla\vec{u} - \vec{\mathfrak{d}}\vec{u})\cdot\boldsymbol{\nu} + \mathbb{\sigma}\vec{u} = \vec{g} \quad \text{on } \Gamma_3 \quad \text{(Newton boundary condition).} \tag{3.30}$$

The above stated definition includes system of elliptic equations with three kinds of boundary conditions. Recall that equations (3.27)–(3.30) can be written as follows. For every $\alpha \in \{1,\dots,N\}$ it holds:

$$-\sum_{\beta=1}^{N}\sum_{i,j=1}^{d}\frac{\partial}{\partial x_i}\left(a_{ij}^{\alpha\beta}\frac{\partial u^\beta}{\partial x_j}\right) + \sum_{\beta=1}^{N}\sum_{i=1}^{d}c_i^{\alpha\beta}\frac{\partial u^\beta}{\partial x_i} + \sum_{\beta=1}^{N}\sum_{i=1}^{d}\frac{\partial}{\partial x_i}(d_i^{\alpha\beta}u^\beta) + \sum_{\beta=1}^{N}b^{\alpha\beta}u^\beta = f^\alpha \quad \text{in } \Omega \tag{3.31}$$

$$u^\alpha = u_0^\alpha \quad \text{on } \Gamma_1 \tag{3.32}$$

$$\sum_{\beta=1}^{N}\sum_{i=1}^{d}\left(\left(\sum_{j=1}^{d}a_{ij}^{\alpha\beta}\frac{\partial u^\beta}{\partial x_j}\right) - (d_i^{\alpha\beta}u^\beta)\right)\nu_i = g^\alpha \quad \text{on } \Gamma_2 \tag{3.33}$$

$$\sum_{\beta=1}^{N}\sum_{i=1}^{d}\left(\left(\sum_{j=1}^{d}a_{ij}^{\alpha\beta}\frac{\partial u^\beta}{\partial x_j}\right) - (d_i^{\alpha\beta}u^\beta)\right)\nu_i + \sum_{\beta=1}^{N}\sigma^{\alpha\beta}u^\beta = g^\alpha \quad \text{on } \Gamma_3. \tag{3.34}$$

Often, if $\vec{\mathfrak{d}} = \vec{\mathfrak{o}}$, the term $\vec{\mathbb{A}}\nabla\vec{u}\cdot\boldsymbol{\nu}$ is replaced by $\nabla\vec{u}\cdot\vec{\mathfrak{n}}$, where $\vec{\mathfrak{n}}$ is the conormal vector defined as $n_j^{\alpha\beta} := \sum_{i=1}^{d} a_{ij}^{\alpha\beta}\nu_i$. We now present one example which includes also the "reactive" terms.

Example 3.2.8. Let $N = 2$, $\mathbb{A} = \mathbb{I}$ (i.e., $a_{ij}^{\alpha\beta} := \delta_{ij}\delta_{\alpha\beta}$), $\vec{\mathfrak{d}} = \vec{\mathfrak{c}} = \vec{\mathfrak{o}}$ and $b^{\alpha\beta} := (-1)^\alpha(1-\delta_{\alpha\beta})$. Then equations (3.27) are reduced to

$$\left.\begin{aligned} -\Delta u^1 - u^2 &= f^1 \\ -\Delta u^2 + u^1 &= f^2 \end{aligned}\right\} \text{in } \Omega. \tag{3.35}$$

The rest of this chapter contains the treatment of questions of solvability and further qualitative properties of solutions to problems stated in Definitions 3.2.2 and 3.2.7. The reader should, however, keep in mind that the most important issue is to understand different problems included in (3.15)–(3.18) and (3.27)–(3.30) rather than building the general theory. This is also the main motivation of the definition of weak solution, although in some cases, the definition could be different.

3.3 Definition of weak solution

The procedure how to deduce the weak formulation was partially shown (at least for a simple elliptic operator) in Subsection 3.1. We shall follow this idea here and deduce the formulation for our rather complex problem. We shall first proceed for the scalar case, then we also explain the idea for the vector one. Assume that u is a classical solution to (3.15)–(3.18) and assume that all coefficients of the elliptic operator are smooth. Let $\varphi \in \mathcal{C}^\infty(\overline{\Omega})$ be arbitrary such that $\varphi = 0$ na Γ_1. We multiply (3.15) on φ, and integrate over Ω. It yields (see also (3.19))

$$\int_\Omega \Big(-\sum_{i,j=1}^{d}\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{d}c_i\frac{\partial u}{\partial x_i} + \sum_{i=1}^{d}\frac{\partial}{\partial x_i}(d_i u) + bu\Big)\varphi\,\mathrm{d}x = \int_\Omega f\varphi\,\mathrm{d}x.$$

We now look at the terms with the divergence operator and perform the integration by parts.

$$\int_\Omega\Big(\sum_{i,j=1}^{d}a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial\varphi}{\partial x_i} + \sum_{i=1}^{d}c_i\frac{\partial u}{\partial x_i}\varphi - \sum_{i=1}^{d}d_i u\frac{\partial\varphi}{\partial x_i} + bu\varphi\Big)\,\mathrm{d}x + \int_{\partial\Omega}\Big(-\sum_{i,j=1}^{d}a_{ij}\frac{\partial u}{\partial x_j}\nu_i + \sum_{i=1}^{d}d_i u\nu_i\Big)\varphi\,\mathrm{d}S = \int_\Omega f\varphi\,\mathrm{d}x.$$

To treat the boundary integrals we employ the boundary conditions (3.21)–(3.22) together with the fact that $\varphi = 0$ on Γ_1 and rewrite the above identity to the final form

$$\int_\Omega\Big(\sum_{i,j=1}^{d}a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial\varphi}{\partial x_i} + \sum_{i=1}^{d}c_i\frac{\partial u}{\partial x_i}\varphi - \sum_{i=1}^{d}d_i u\frac{\partial\varphi}{\partial x_i} + bu\varphi\Big)\,\mathrm{d}x + \int_{\Gamma_3}\sigma u\varphi\,\mathrm{d}S = \int_\Omega f\varphi\,\mathrm{d}x + \int_{\Gamma_2\cup\Gamma_3}g\varphi\,\mathrm{d}S.$$

Using the (vector-) matrix-calculus we can rewrite this identity into

$$\int_\Omega\Big(\mathbb{A}\nabla u\cdot\nabla\varphi + \mathbf{c}\cdot\nabla u\varphi - \mathbf{d}\cdot\nabla\varphi u + bu\varphi\Big)\,\mathrm{d}x + \int_{\Gamma_3}\sigma u\varphi\,\mathrm{d}S = \int_\Omega f\varphi\,\mathrm{d}x + \int_{\Gamma_2\cup\Gamma_3}g\varphi\,\mathrm{d}S. \tag{3.36}$$

For fixed operator L and fixed σ we introduce the following bilinear form which represents the left-hand side of (3.36)

$$B_{L,\sigma}(u,\varphi) := \int_\Omega \Big(\mathbb{A}\nabla u\cdot\nabla\varphi + \mathbf{c}\cdot\nabla u\varphi - \mathbf{d}\cdot\nabla\varphi u + bu\varphi\Big)\,\mathrm{d}x + \int_{\Gamma_3}\sigma u\varphi\,\mathrm{d}S, \tag{3.37}$$

where we assume that all integrals exist and are finite. We have already seen in Subsection 3.1 that the correct space for our problem is the Sobolev space $W^{1,2}(\Omega)$. To recall the motivation we use the special case $\mathbb{A}=\mathbb{I}$, $b=1$ and f arbitrary square integrable function and all other parameters zero. If we use $\varphi := u$ in (3.36), we get

$$\|u\|_{1,2}^2 = \int_\Omega fu\,\mathrm{d}x \le \frac{1}{2}\|f\|_2^2 + \frac{1}{2}\|u\|_{1,2}^2, \tag{3.38}$$

where we used on the right-hand side the Young inequality. We see that the second term can be subtracted from both sides of the inequality and the norm which we naturally have under control (at least formally) is the $W^{1,2}$-norm. Following Subsection 3.1 we shall assume in the rest of this chapter that $U_0 \in W^{1,2}(\Omega)$ is such a function that the trace of U_0 is equal to u_0 on Γ_1. This leads us to the following definition of the weak solution to (3.15)–(3.18).

Definition 3.3.1 — Weak solution of boundary value problem for elliptic equation. Let $\Omega \in \mathcal{C}^{0,1}$, $\{\Gamma_i\}_{i=1}^3$ be the corresponding parts of the boundary and the operator L be elliptic in the sense of Definition 3.2.6. We define

$$V := \{\varphi \mid \varphi \in W^{1,2}(\Omega),\ T\varphi|_{\Gamma_1} = 0\}$$

with the standard norm $\|\varphi\|_V := \|\varphi\|_{W^{1,2}(\Omega)}$. Let further the function $u \in W^{1,2}(\Omega)$ and the data $U_0 \in W^{1,2}(\Omega)$ [a], $\sigma \in L^\infty(\Gamma_3)$, $g \in (W^{\frac{1}{2},2}(\Gamma_2\cup\Gamma_3))^*$ [b] and $f \in V^*$. We say that u is a weak solution to (3.19)–(3.22), if

$$\begin{aligned} &u - U_0 \in V\\ &B_{L,\sigma}(u,\varphi) = \langle f,\varphi\rangle_V + \langle g,\varphi\rangle_{W^{\frac{1}{2},2}(\Gamma_2\cup\Gamma_3)} \quad \text{for any } \varphi \in V. \end{aligned} \tag{3.39}$$

[a] $U_0 = u_0$ on Γ_1 in the sense of traces.
[b] Dual space to the range of the trace operator from $W^{1,2}(\Omega)$.

Due to the assumptions on the data and on the coefficients of the operator L, all terms in (3.39) are well defined and finite. This can be easily seen from Hölder's inequality and from the fact (not yet proved) that the range of the trace operator can be well characterized and it is a Banach space. Note that we replaced the integrability assumption on f and g by the assumption that they belong to certain dual spaces. Nevertheless, if we assume $f \in L^2(\Omega)$ and $g \in L^2(\Omega)$, we may simply define

$$\langle f,\varphi\rangle_V := \int_\Omega f\varphi\,\mathrm{d}x, \qquad \langle g,\varphi\rangle_{W^{\frac{1}{2},2}(\partial\Omega)} := \int_{\partial\Omega} g\varphi\,\mathrm{d}S,$$

and solve problem (3.39). On the other hand, the general form of (3.39) allows us to cover also the case when the right-hand side is defined as derivative of a non-smooth function. If, e.g., $\Gamma_1 = \partial\Omega$ and $f := \operatorname{div}\mathbf{F}$, where $F_i \in L^2(\Omega)$, we may easily set

$$\langle f,\varphi\rangle_V := -\int_\Omega \mathbf{F}\cdot\nabla\varphi\,\mathrm{d}x.$$

This definition is, moreover, consistent with the prefered integral formulation, as if $F_i \in W_0^{1,2}(\Omega)$ and thus $f \in L^2(\Omega)$, we may due to integration by parts (Theorem 2.1.22) identify

$$\langle f,\varphi\rangle_V := -\int_\Omega \mathbf{F}\cdot\nabla\varphi\,\mathrm{d}x = \int_\Omega \operatorname{div}\mathbf{F}\varphi\,\mathrm{d}x = \int_\Omega f\varphi\,\mathrm{d}x.$$

We showed at the beginning of this section that if u is a classical solution to our Problem (3.15)–(3.18), then it satisfies also the weak formulation. We will now look at the opposite implication, i.e., if u is a weak solution which is additionally sufficiently smooth, then it is also the classical solution. This implication is the key property to justify the weak formulation as it says that whenever we may show sufficient regularity of the weak solution, then the weak solution becomes the classical one. If we can furthermore show also uniqueness of weak solutions, then the weak formulation is the only correct choice. Everything is summarized in the following theorem.

Theorem 3.3.2 — On consistency of weak solutions I. Let all assumptions of Definition 3.3.1 be satisfied. Let for any $i,j \in \{1,\dots,d\}$ it hold $a_{ij}, d_i \in \mathcal{C}^1(\overline{\Omega})$, $c_i, b, f \in \mathcal{C}(\overline{\Omega})$, $\sigma \in \mathcal{C}(\overline{\Gamma_3})$, $g \in \mathcal{C}(\overline{\Gamma_2\cup\Gamma_3})$ and $U_0, u \in \mathcal{C}^1(\overline{\Omega})\cap\mathcal{C}^2(\Omega)$. Then u is a classical solution to (3.19)–(3.22), if and only if u is a weak solution.

Proof. The first part of the proof that any classical solution is a weak solution was explained at the beginning of this section and we shall not repeat it. Let us look at the proof of the second implication. Let $u \in \mathcal{C}^1(\overline{\Omega})\cap\mathcal{C}^2(\Omega)$. We want to show that if u satisfies (3.39), then it also satisfies (3.19)–(3.22). Let the function $\varphi \in \mathcal{C}_0^\infty(\Omega)$ be arbitrary.

Using this test function in (3.39) and due to assumptions on f we get (note that all boundary terms disappear due to the fact that φ has compact support)

$$\int_\Omega \Big(\mathbb{A}\nabla u\cdot\nabla\varphi + \mathbf{c}\cdot\nabla u\varphi - \mathbf{d}\cdot\nabla\varphi u + bu\varphi\Big)\,\mathrm{d}x = \int_\Omega f\varphi\,\mathrm{d}x.$$

We apply in the first and in the third terms integration by parts (it is possible due to strong assumptions on all functions) and using the fact that the boundary terms disappear we get

$$\int_\Omega Lu\varphi\,\mathrm{d}x = \int_\Omega f\varphi\,\mathrm{d}x.$$

Since φ is an arbitrary smooth compactly supported function, we immediately obtain the validity of (3.15). The validity of (3.16) follows from the definition of the function spaces; we therefore look at the validity of (3.17)–(3.18). We repeat the procedure with the difference that $\varphi \in \mathcal{C}^\infty(\overline{\Omega})$ such that $\varphi = 0$ on Γ_1. These functions satisfy $\varphi \in V$ and thus can be used in (3.39). We get

$$\int_\Omega \Big(\mathbb{A}\nabla u\cdot\nabla\varphi + \mathbf{c}\cdot\nabla u\varphi - \mathbf{d}\cdot\nabla\varphi u + bu\varphi\Big)\,\mathrm{d}x + \int_{\Gamma_3}\sigma u\varphi\,\mathrm{d}S = \int_\Omega f\varphi\,\mathrm{d}x + \int_{\Gamma_2\cup\Gamma_3} g\varphi\,\mathrm{d}S.$$

We use again in the first and in the third terms integration by parts, but we must include now also the boundary terms (note that the integral over Γ_1 again disappears as $\varphi = 0$ on Γ_1)

$$\int_\Omega Lu\varphi\,\mathrm{d}x + \int_{\Gamma_2\cup\Gamma_3}\Big(\mathbb{A}\nabla u\cdot\boldsymbol{\nu}\varphi - u\mathbf{d}\cdot\boldsymbol{\nu}\Big)\varphi\,\mathrm{d}S + \int_{\Gamma_3}\sigma u\varphi\,\mathrm{d}S = \int_\Omega f\varphi\,\mathrm{d}x + \int_{\Gamma_2\cup\Gamma_3} g\varphi\,\mathrm{d}S.$$

Due to the previous step we know that the first terms on both sides are equal. Since φ can be an arbitrary smooth function on $\Gamma_2\cup\Gamma_3$, we immediately get (3.17)–(3.18). ■

Before coming to the vector case, let us define another notion of the solution which lies between the weak and the classical one.

Definition 3.3.3 — Strong solution I. Let all assumptions of Definition 3.3.1 be satisfied. Let, moreover, for any $i,j\in\{1,\dots,d\}$ it hold $a_{ij}, d_i \in W^{1,\infty}(\Omega)$, $f\in L^2(\Omega)$, $g\in L^2(\partial\Omega)$ and $U_0\in W^{2,2}(\Omega)$. We say that $u\in W^{2,2}(\Omega)$ is a strong solution to our problem (3.15)–(3.18), if (3.15) holds almost everywhere in Ω and (3.16)–(3.18) hold almost everywhere on $\partial\Omega$.

It is again not very difficult to see that if $u\in W^{2,2}(\Omega)$ and u is a weak solution, then it is also a strong solution and vice versa. The proof can be performed exactly as the proof of Theorem 3.3.2.

We now repeat the same procedure as above for the vector problem (system of elliptic equations). We multiply the α-th equation on the function φ^α from $C^\infty(\overline{\Omega})$, where $\varphi^\alpha = 0$ on Γ_1. We sum the resulted equalities over $\alpha = 1,2,\dots,N$ and integrate over Ω. It yields

$$\sum_{\alpha,\beta=1}^N \int_\Omega \Big(-\sum_{i,j=1}^d \frac{\partial}{\partial x_i}\Big(a_{ij}^{\alpha\beta}\frac{\partial u^\beta}{\partial x_j}\Big) + \sum_{i=1}^d c_i^{\alpha\beta}\frac{\partial u^\beta}{\partial x_i} + \sum_{i=1}^d \frac{\partial}{\partial x_i}(d_i^{\alpha\beta}u^\beta) + b^{\alpha\beta}u^\beta\Big)\varphi^\alpha\,\mathrm{d}x = \int_\Omega \sum_{\alpha=1}^N f^\alpha\varphi^\alpha\,\mathrm{d}x.$$

As in the scalar case, we perform integration by parts at several term and get the identity

$$\begin{aligned}\sum_{\alpha,\beta=1}^N \int_\Omega \Big(\sum_{i,j=1}^d a_{ij}^{\alpha\beta}\frac{\partial u^\beta}{\partial x_j}\frac{\partial\varphi^\alpha}{\partial x_i} + \sum_{i=1}^d c_i^{\alpha\beta}\frac{\partial u^\beta}{\partial x_i}\varphi^\alpha - \sum_{i=1}^d d_i^{\alpha\beta}u^\beta\frac{\partial\varphi^\alpha}{\partial x_i} + b^{\alpha\beta}u^\beta\varphi^\alpha\Big)\,\mathrm{d}x\\ + \sum_{\alpha,\beta=1}^N\int_{\partial\Omega}\Big(-\sum_{i,j=1}^d a_{ij}^{\alpha\beta}\frac{\partial u^\beta}{\partial x_j}\nu_i + \sum_{i=1}^d d_i^{\alpha\beta}u^\beta\nu_i\Big)\varphi^\alpha\,\mathrm{d}S = \int_\Omega\sum_{\alpha=1}^N f^\alpha\varphi^\alpha\,\mathrm{d}x.\end{aligned}$$

We replace the boundary terms by (3.33)–(3.34) and use the fact that $\varphi^\alpha = 0$ on Γ_1. The resulted identity has the form (as in the scalar case, we define $\vec{g}$ to be zero on Γ_1)

$$\begin{aligned}\sum_{\alpha,\beta=1}^N \int_\Omega \Big(\sum_{i,j=1}^d a_{ij}^{\alpha\beta}\frac{\partial u^\beta}{\partial x_j}\frac{\partial\varphi^\alpha}{\partial x_i} + \sum_{i=1}^d c_i^{\alpha\beta}\frac{\partial u^\beta}{\partial x_i}\varphi^\alpha - \sum_{i=1}^d d_i^{\alpha\beta}u^\beta\frac{\partial\varphi^\alpha}{\partial x_i} + b^{\alpha\beta}u^\beta\varphi^\alpha\Big)\,\mathrm{d}x\\ + \sum_{\alpha,\beta=1}^N\int_{\Gamma_3}\sigma^{\alpha\beta}u^\beta\varphi^\alpha\,\mathrm{d}S = \int_\Omega\sum_{\alpha=1}^N f^\alpha\varphi^\alpha\,\mathrm{d}x + \sum_{\alpha=1}^N\int_{\Gamma_2\cap\Gamma_3} g^\alpha\varphi^\alpha\,\mathrm{d}S.\end{aligned}$$

Using the (vector-) matrix-calculus we can rewrite the identity into shorter and more understandable form (we use as in the previous section $\vec{u} = (u^1,\dots,u^N)$ and similarly for other terms)

$$\int_\Omega \Big(\vec{\mathbb{A}}\nabla\vec{u} : \nabla\vec{\varphi} + (\mathfrak{c}\nabla\vec{u})\cdot\vec{\varphi} - (\vec{\mathfrak{d}}\nabla\vec{\varphi})\cdot\vec{u} + \mathfrak{b}\vec{u}\cdot\vec{\varphi}\Big)\,\mathrm{d}x + \int_{\Gamma_3}(\sigma\vec{u})\cdot\vec{\varphi}\,\mathrm{d}S = \int_\Omega \vec{f}\cdot\vec{\varphi}\,\mathrm{d}x + \int_{\Gamma_2\cap\Gamma_3}\vec{g}\cdot\vec{\varphi}\,\mathrm{d}S. \tag{3.40}$$

Above, the denoted $\mathbb{A}\nabla\vec{u} : \nabla\vec{\varphi} := \sum_{\alpha=1}^{N}\sum_{i=1}^{d}(\sum_{\beta=1}^{N}\sum_{j=1}^{d} a_{ij}^{\alpha\beta}\frac{\partial u^\beta}{\partial x_j})\frac{\partial\varphi^\alpha}{\partial x_i}$. We again introduce the following bilinear form (with L and $\mathfrak{o}$ fixed)

$$B_{L,\mathfrak{o}}(\vec{u},\vec{\varphi}) := \int_\Omega \left(\mathbb{A}\nabla\vec{u} : \nabla\vec{\varphi} + (\mathfrak{c}\nabla\vec{u})\cdot\vec{\varphi} - (\mathfrak{d}\nabla\vec{\varphi})\cdot\vec{u} + (\mathfrak{b}\vec{u})\cdot\vec{\varphi}\right)\mathrm{d}x + \int_{\Gamma_3}(\mathfrak{o}\vec{u})\cdot\vec{\varphi}\,\mathrm{d}S, \tag{3.41}$$

where we assume that all integrals exist and are finite. This allows us to introduce the definition of the weak solution to (3.27)–(3.30).

Definition 3.3.4 — Weak solution of boundary value problem for system of elliptic equations. Let $\Omega \in \mathcal{C}^{0,1}$ with the corresponding parts of the boundary $\{\Gamma_i\}_{i=1}^3$ and let L be elliptic in the sense of Definition 3.2.6. We define

$$V := \{\vec{\varphi} = (\varphi^1,\ldots,\varphi^N)\,|\,\forall\alpha = 1,\ldots,N,\ \varphi^\alpha \in W^{1,2}(\Omega),\ T\varphi^\alpha|_{\Gamma_1} = 0\}$$

with the standard norm $\|\varphi\|_V^2 := \sum_{\alpha=1}^N \|\varphi^\alpha\|^2_{W^{1,2}(\Omega)}$. Further, let for any $\alpha,\beta = 1,\ldots,N$ the functions $u^\alpha \in W^{1,2}(\Omega)$ and the data $U_0^\alpha \in W^{1,2}(\Omega)$, $\sigma^{\alpha\beta} \in L^\infty(\Gamma_3)$, $g^\alpha \in (W^{\frac{1}{2},2}(\partial\Omega))^*$ and $f^\alpha \in V^*$. We say that $\vec{u}$ is a weak solution to (3.31)–(3.34), if

$$\begin{aligned} &\vec{u} - \vec{U}_0 \in V \\ &B_{L,\mathfrak{o}}(\vec{u},\vec{\varphi}) = \langle \vec{f},\vec{\varphi}\rangle_V + \sum_{\alpha=1}^N \langle g^\alpha,\varphi^\alpha\rangle_{W^{\frac{1}{2},2}(\Gamma_2\cap\Gamma_3)} \quad \text{for any } \vec{\varphi}\in V. \end{aligned} \tag{3.42}$$

Similarly as below Definition 3.3.1 it is possible to verify that under our assumptions all integrals (dualities) are well defined and finite.

Furthermore, the same claim as in the scalar case can be shown concerning the consistency of weak solutions.

Theorem 3.3.5 — On consistency of weak solutions II. Let all assumptions of Definition 3.3.4 be fulfilled. Let for any $i,j \in \{1,\ldots,d\}$ and any $\alpha,\beta\in\{1,\ldots,N\}$ it hold $a_{ij}^{\alpha\beta}, d_i^{\alpha\beta} \in \mathcal{C}^1(\overline{\Omega})$, $c_i^{\alpha\beta}, b^{\alpha\beta}, f^\alpha \in \mathcal{C}(\overline{\Omega})$, $\sigma^{\alpha\beta}\in\mathcal{C}(\overline{\Gamma_3})$, $g^\alpha \in \mathcal{C}(\overline{\Gamma_2\cup\Gamma_3})$ and $U_0^\alpha, u^\alpha \in \mathcal{C}^1(\overline{\Omega})\cap\mathcal{C}^2(\Omega)$. Then $\vec{u}$ is a classical solution of (3.31)–(3.34), if and only if $\vec{u}$ is a weak solution.

Proof. The proof is similar to the scalar case. ∎

Exercise 3.3.6. Prove Theorem 3.3.5.

Finally we also define the strong solution in the vector case.

Definition 3.3.7 — Strong solution II. Let all assumptions of Definition 3.3.4 be satisfied. Let for any $i,j \in \{1,\ldots,d\}$ and any $\alpha,\beta\in\{1,\ldots,N\}$ it holds $a_{ij}^{\alpha\beta}, d_i^{\alpha\beta} \in W^{1,\infty}(\Omega)$, $f^\alpha \in L^2(\Omega)$, $g^\alpha \in L^2(\partial\Omega)$ and $U_0^\alpha \in W^{2,2}(\Omega)$. We say that $\vec{u}$ such that $u^\alpha \in W^{2,2}(\Omega)$ for any $\alpha \in \{1,2,\ldots,N\}$ is a strong solution, if (3.31) holds almost everywhere in Ω and (3.32)–(3.34) hold almost everywhere on $\partial\Omega$.

It is again not difficult to show that $u^\alpha \in W^{2,2}(\Omega)$ for any $\alpha\in\{1,\ldots,N\}$ and $\vec{u} = (u^1,\ldots,u^N)$ is a weak solution, then it is a strong solution and vice versa.

Exercise 3.3.8. Prove the claim about the consistency for strong solutions.

3.4 Existence of weak solutions, uniqueness and continuous dependence on data for coercive operators

We have already seen in Subsection 3.1 that operators of the type $-\Delta u + u$ are elliptic and provide good estimates for the solution of corresponding elliptic problems: the bilinear form

$$B_L(u,v) = \int_\Omega (\nabla u\cdot\nabla v + uv)\,\mathrm{d}x$$

has the property that

$$B_L(u,u) = \int_\Omega (|\nabla u|^2 + u^2)\,\mathrm{d}x = \|u\|_{1,2}^2.$$

We shall consider in this subsection similar operators (even for the vector case) which will however be as general as possible. In Subsection 3.1 the main tool to get existence of a weak solution was the Riesz representation Theorem. This, however, requires that the bilinear form can be viewed as scalar product which means that the main part of the elliptic operator is symmetric; more precisely, the matrix $\mathbb{A}$ from the differential operator of the second order $-\sum_{i,j=1}^d \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial u}{\partial x_j})$ must be symmetric. We shall first relax this condition.

Theorem 3.4.1 — Lax–Milgram. Let V be a real Hilbert space with the scalar product $(\cdot,\cdot)_V$ and the norm $\|\cdot\|_V = (\cdot,\cdot)^{1/2}$. Let $B\colon V \times V \to \mathbb{R}$ be a bilinear form on V which is:

a) V-elliptic, i.e., there exists $m > 0$ such that for any $u \in V$ it holds

$$B(u,u) \geq m\|u\|_V^2 \tag{3.43}$$

b) V-bounded, i.e., there exists $M > 0$ such that for any $u, v \in V$ it holds

$$|B(u,v)| \leq M\|u\|_V\|v\|_V. \tag{3.44}$$

Then for any $F \in V^*$ there exists unique $u \in V$ such that for any $v \in V$ it holds

$$B(u,v) = \langle F, v\rangle_V. \tag{3.45}$$

Moreover, u satisfies the estimate

$$\|u\|_V \leq \frac{1}{m}\|F\|_{V^*}. \tag{3.46}$$

Proof. Let us first look at the simpler part of the proof. If u satisfies (3.45) for any $v \in V$, then choosing $v := u$ in (3.45) and using (3.43) we get

$$m\|u\|_V^2 \leq B(u,u) = \langle F, u\rangle_V \leq \|F\|_{V^*}\|u\|_V$$

which directly implies (3.46). Furthermore, let u_1, u_2 be two solutions of (3.45). Then the bilinearity of B and the linearity of the duality immediately yield

$$B(u_1 - u_2, v) = 0 \quad \text{for any } v \in V.$$

Choosing $v := u_1 - u_2$ and using the V–ellipticity we see that $u_1 = u_2$ which implies the uniqueness of solutions to our problem.

It remains to show existence of u which solves (3.45). We first use the Riesz representation Theorem B.2.2 and find for the given $F \in V^*$ unique $w \in V$ such that for any $v \in V$ it holds $(w,v)_V = \langle F, v\rangle_V$. Problem (3.45) transforms into the problem to find $u \in V$ such that for any $v \in V$ it holds

$$B(u,v) = (w,v)_V. \tag{3.47}$$

We now rewrite the left-hand side to the form of the scalar product. More precisely, due to the bilinearity and the V-boundedness of B we see that for a fixed $u \in V$ it holds $B(u,\cdot) \in V^*$. Due to the Riesz representation Theorem there exists unique element $A(u) \in V$ such that for any $v \in V$ it holds

$$B(u,v) = (A(u), v)_V. \tag{3.48}$$

Equation (3.47) then rewrites into

$$(A(u), v)_V = (w,v)_V. \tag{3.49}$$

To finish the proof of existence of $u \in V$ which fulfils for all $v \in V$ equation (3.49), it is enough to show that the mapping $A\colon V \to V$ is onto.

Let us first summarize properties of A which directly follow from the properties of the form B.

(i) A is linear. Indeed, since B is bilinear, we have for any $u_1, u_2, v \in V$

$$\begin{aligned}(A(u_1 + u_2), v)_V &\overset{(3.48)}{=} B(u_1 + u_2, v) = B(u_1, v) + B(u_2, v)\\ &\overset{(3.48)}{=} (A(u_1), v)_V + (A(u_2), v)_V\end{aligned}$$

and thus A is linear.

(ii) A is injective and $A(V)$ is a closed subspace of V. The property that $A(V)$ is a subspace follows from the linearity of A. Further, due to the V-ellipticity of B we have for any $u \in V$

$$m\|u\|_V^2 \leq B(u,u) = (A(u), u)_V \leq \|A(u)\|_V\|u\|_V;$$

thus

$$\|u\|_V \leq \frac{1}{m}\|A(u)\|_V$$

which due to the linearity implies the injectivity. Furthermore, if $\{A(u^n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence, then due to the linearity of A and the above presented estimate the sequence $\{u^n\}_{n\in\mathbb{N}}$ is Cauchy and thus we see that $A(V)$ is a closed subspace.

(iii) A is bounded. Indeed, due to the V-boundedness of B we have

$$\|A(u)\|_V^2 = (A(u), A(u))_V \overset{(3.48)}{=} B(u, A(u)) \le M\|u\|_V\|A(u)\|_V$$

and thus $\|A(u)\|_V \le M\|u\|_V$.

From the properties above it follows that A: $V \to V$ is linear, bounded (thus continuous) and injective operator with the closed range $A(V)$ which additionally forms a subspace of V. Assume now that $A(V) \neq V$. Then there exists $v \in V \setminus A(V)$ which can be chosen so that $\|v\|_V = 1$, and for any $w \in A(V)$ it holds $(v, w)_V = 0$. Choosing $w := A(v) \in A(V)$ and using the V-ellipticity of the form B we get

$$0 = (v, A(v))_V = (A(v), v)_V = B(v, v) \ge m\|v\|_V^2 = m > 0$$

which leads to contradiction. The operator A is thus onto and the proof is finished. ■

The goal of this section is to formulate assumptions on the operator L and on the data of Problem (3.27)–(3.30) so that we may apply the Lax–Milgram Theorem. For a moment we consider only the vector problem since the scalar problem in the setting of the following theorem is on the same difficulty level as the vector problem. Before we do so, let us look at the special problem of the Neumann problem. Let us set $\vec{\mathfrak{c}} = \vec{\mathfrak{d}} = \vec{\mathfrak{o}}$ and $\mathfrak{b} = \mathfrak{o}$ in the definition of the operator L and assume that $|\Gamma_2| = |\partial\Omega|$. In this situation the space V in Definition 3.3.4 reduces to

$$V := \{\vec{\varphi} = (\varphi^1, \dots, \varphi^N);\ \text{for any } \alpha \in \{1, \dots, N\} \text{ it holds that } \varphi^\alpha \in W^{1,2}(\Omega)\}$$

and the weak formulation has the form

$$\int_\Omega \vec{\mathbb{A}}\nabla\vec{u} : \nabla\vec{\varphi}\,\mathrm{d}x = \langle \vec{f}, \vec{\varphi}\rangle_V + \langle \vec{g}, \vec{\varphi}\rangle_{W^{\frac{1}{2},2}(\partial\Omega;\mathbb{R}^N)} \qquad \text{for all } \vec{\varphi} \in V. \tag{3.50}$$

If we now choose in this identity $\vec{\varphi} := (0, \dots, 0, 1, 0, \dots, 0)$, we get that for any $\alpha \in \{1, \dots, N\}$ it must hold

$$\langle f^\alpha, 1\rangle_{W^{1,2}(\Omega)} + \langle g^\alpha, 1\rangle_{W^{\frac{1}{2},2}(\partial\Omega)} = 0 \tag{3.51}$$

which for $\vec{f}$ and $\vec{g}$ integrable does not mean anything else than

$$\int_\Omega f^\alpha\,\mathrm{d}x + \int_{\partial\Omega} g^\alpha\,\mathrm{d}S = 0 \text{ for any } \alpha \in \{1, \dots, N\}. \tag{3.52}$$

We see that condition (3.51), or (3.52), respectively, are necessary conditions for the existence of a solution. In what follows we show that these conditions are also sufficient. We now show one abstract result telling us that the V-ellipticity of the bilinear form $B_{L,\sigma}$ defined in (3.41) is the sufficient condition for existence of the unique weak solution.

Theorem 3.4.2 — On existence and uniqueness for coercive operators. Let all assumptions of Definition 3.3.4 be satisfied.

a) If $B_{L,\sigma}$ is V-elliptic, then there exists unique solution to Problem (3.27)–(3.30).

b) If $\vec{\mathfrak{c}} = \vec{\mathfrak{d}} = \vec{\mathfrak{o}}$, $\mathfrak{b} = \mathfrak{o}$, $|\Gamma_2| = |\partial\Omega|$ and conditions (3.51) are fulfilled, then there exists a weak solution to Problem (3.50) which is unique up to an additive constant.

Proof. Let us first look at the simpler case b). We showed in (3.51) the necessary condition for the existence of a solution. Since in this case the condition is reduced to

$$\int_\Omega \vec{\mathbb{A}}\nabla\vec{u} : \nabla\vec{\varphi}\,\mathrm{d}x = \langle \vec{f}, \vec{\varphi}\rangle_V + \sum_{\alpha=1}^N \langle g^\alpha, \varphi^\alpha\rangle_{W^{\frac{1}{2},2}(\partial\Omega)},$$

we see that replacing $\widetilde{\vec{u}} := \vec{u} + \vec{k}$, where $\vec{k} \in \mathbb{R}^N$ is arbitrary, is again a solution if $\vec{u}$ were so. This leads to the idea that we should consider instead of the space V rather the factor space V/P_0^N; we say that $\vec{u}_1 \sim \vec{u}_2$, if $\vec{u}_1 - \vec{u}_2 = \vec{k} \in \mathbb{R}^N$ almost everywhere in Ω. The space V/P_0^N is again a Hilbert space and due to Theorem 2.6.8 can be endowed by the norm

$$\|\vec{u}\|_{V/P_0^N}^2 := \|\nabla\vec{u}\|_2^2 = \sum_{\alpha=1}^N \|\nabla u^\alpha\|_2^2;$$

furthermore, clearly $(\vec{u}, \vec{v})_{V/P_0^N} := \int_\Omega \nabla\vec{u} \cdot \nabla\vec{v}\,\mathrm{d}x$ is a scalar product. Due to the assumption on the matrix $\vec{\mathbb{A}}$ the expression

$$B(\vec{u}, \vec{v}) := \int_\Omega \vec{\mathbb{A}}\nabla\vec{u} : \nabla\vec{v}\,\mathrm{d}x$$

is a bilinear form. Assumption (3.26) yields

$$B(\vec{u},\vec{u}) := \int_\Omega \mathbb{A}\nabla\vec{u} : \nabla\vec{u}\,\mathrm{d}x \geq C_1\|\nabla\vec{u}\|_2^2 = C_1\|\vec{u}\|^2_{V/P_0^N}$$

and B is also elliptic. To be able to apply the Lax–Milgram Theorem 3.4.1 we have to find $\vec{F} \in (V/P_0^N)^*$ so that

$$\langle \vec{F},\vec{\varphi}\rangle_{V/P_0^N} = \langle \vec{f},\vec{\varphi}\rangle_V + \sum_{\alpha=1}^N \langle \vec{g},\vec{\varphi}\rangle_{W^{\frac12,2}(\partial\Omega)}.$$

Due to the trace theorem (recall that the space $W^{\frac12,2}(\Omega;\mathbb{R}^N)$ is the range of the trace operator) we can define $\vec{F} \in V^*$ by means of

$$\langle \vec{F},\vec{\varphi}\rangle_V = \langle \vec{f},\vec{\varphi}\rangle_V + \sum_{\alpha=1}^N \langle g^\alpha,\varphi^\alpha\rangle_{W^{\frac12,2}(\partial\Omega)} =: \langle \vec{f},\vec{\varphi}\rangle_V + \langle \vec{g},\vec{\varphi}\rangle_{W^{\frac12,2}(\partial\Omega;\mathbb{R}^N)}.$$

It remains to show that this $\vec{F}$ belongs also to $(V/P_0^N)^*$; in other words we need to show that for any $\vec{k} \in \mathbb{R}^N$ and any $\vec{\varphi} \in V$ it holds that

$$\langle \vec{F},\vec{\varphi}+\vec{k}\rangle_V = \langle \vec{F},\vec{\varphi}\rangle_V.$$

This is, however, a simple consequence of (3.51). We can therefore apply the Lax–Milgram Theorem (Theorem 3.4.1) and we get existence of a unique $\vec{u} \in V/P_0^N$ solving our problem. As $\vec{u}$ is unique in the factor space V/P_0^N, we get uniqueness of the weak solution up to an additive constant which finishes the proof for the Neumann problem.

Let us now deal with the case a). Similarly as above the space V is Hilbert space a due to the properties of the traces of Sobolev functions and assumptions on $\vec{f}$ and $\vec{g}$ we may define $F \in V^*$ by means of

$$\langle \vec{F},\vec{\varphi}\rangle_V := \langle \vec{f},\vec{\varphi}\rangle_V + \langle \vec{g},\vec{\varphi}\rangle_{W^{\frac12,2}(\Gamma_2\cup\Gamma_3;\mathbb{R}^N)}.$$

Consider now the form $B_{L,\sigma}$ defined in (3.41); due to the linearity of the integral and assumptions on the data (the reader can easily check it using Hölder's inequality), that the form is a bilinear bounded form on $W^{1,2}(\Omega;\mathbb{R}^N)$. Since V is a closed subspaces of this space (an easy corollary of the trace theorem), then $B_{L,\sigma}$ is a bilinear bounded form on V. Hence assume according to a), that $B_{L,\sigma}$ is V-elliptic.

Let us first consider uniqueness of solution to problem

$$B_{L,\sigma}(\vec{u},\vec{\varphi}) = \langle \vec{F},\vec{\varphi}\rangle_V \quad \text{for any } \vec{\varphi} \in V. \tag{3.53}$$

Let $\vec{u_1}$ and $\vec{u}_2$ be two solutions such that $\vec{u}_1 - \vec{U}_0^1 \in V$ and $\vec{u}_2 - \vec{U}_0^2 \in V$ (recall that $\vec{u}_0$ is the prescribed value on Γ_1 and $\vec{U}_0^1$ and $\vec{U}_0^2$ are two functions having on Γ_1 the same trace equal to $\vec{u}_0$). Then also $\vec{U}_0^1 - \vec{U}_0^2 \in V$ and we can choose in (3.53) the test function $\vec{\varphi} := \vec{u}_1 - \vec{u}_2 \in V$. Subtracting the corresponding equations for $\vec{u}_1$ and $\vec{u}_2$ we obtain

$$B_{L,\sigma}(\vec{u}_1 - \vec{u}_2, \vec{u}_1 - \vec{u}_2) = 0$$

which due to the V-ellipticity (it is possible to use it as $\vec{u}_1 - \vec{u}_2 \in V$) leads to $\vec{u}_1 = \vec{u}_2$ (almost everywhere in Ω) giving us the uniqueness of the solution.

Let us now deal with existence of a solution. We look $\vec{u}$ in the form $\vec{u} = \vec{v} + \vec{U}_0$, where $\vec{v} \in V$. Plugging this into (3.53) and due to the linearity of $B_{L,\sigma}$ we easily see that to find the function $\vec{u}$ it is enough to show existence of $\vec{v} \in V$ which satisfies

$$B_{L,\sigma}(\vec{v},\vec{\varphi}) = \langle \vec{F},\vec{\varphi}\rangle_V - B_{L,\sigma}(\vec{U}_0,\vec{\varphi}) \quad \text{for any } \vec{\varphi} \in V. \tag{3.54}$$

First, due to the bilinearity of $B_{L,\sigma}$ on $W^{1,2}(\Omega;\mathbb{R}^N)$ we can find $\vec{G} \in V^*$ such that

$$\langle \vec{G},\vec{\varphi}\rangle_V = \langle \vec{F},\vec{\varphi}\rangle_V - B_{L,\sigma}(\vec{U}_0,\vec{\varphi}) \quad \text{for any } \vec{\varphi} \in V.$$

Equation (3.54) transforms therefore to

$$B_{L,\sigma}(\vec{v},\vec{\varphi}) = \langle \vec{G},\vec{\varphi}\rangle_V. \tag{3.55}$$

We can now apply the Lax–Milgram Theorem 3.4.1, since we verified that $B_{L,\sigma}$ is a V-bounded bilinear form and due to our assumptions it is also V-elliptic. Therefore there exists unique $\vec{v} \in V$ satisfying (3.55) and thus also $\vec{u}$, a weak solution to problem (3.27)–(3.30). ∎

We realized that to the proof of existence and uniqueness of solutions to problem (3.27)–(3.30) it is enough to show that $B_{L,\sigma}$ is a V-elliptic form. Furthermore, for the Neumann problem we showed that conditions (3.51) are necessary and sufficient for the existence of a solution. We further concentrate on the identification of assumptions on the parameters of the operator L which allow us to obtain the desired ellipticity. Note that the ellipticity condition is fundamental as we shall see in the following sections and violation of this condition can lead to non-existence or non-uniqueness of solutions. Later, in the nonlinear case, this condition will be replaced by the so-called coercivity which will be equivalent to the ellipticity in general Banach spaces.

The results which we formulate and show use "brute force" to show the ellipticity of $B_{L,\sigma}$. In many cases, as will be shown below in several examples, they cannot be applied, although it can be verified by finer techniques that the form $B_{L,\sigma}$ is elliptic.

For the next part, we first define two auxiliary functions

$$
\begin{aligned}
\widetilde{b}(x) &:= \inf_{\{\vec{z}\in\mathbb{R}^N;\, |\vec{z}|=1\}} \sum_{\alpha,\beta=1}^{N} b^{\alpha\beta}(x) z^\alpha z^\beta \\
\widetilde{\sigma}(x) &:= \inf_{\{\vec{z}\in\mathbb{R}^N;\, |\vec{z}|=1\}} \sum_{\alpha,\beta=1}^{N} \sigma^{\alpha\beta}(x) z^\alpha z^\beta.
\end{aligned}
\tag{3.56}
$$

Note that in the scalar case, i.e., $N = 1$, it trivially holds $\widetilde{b} = b$ and $\widetilde{\sigma} = \sigma$.

We first formulate the existence theorem for the situation when at least on a part of the boundary (nontrivial one) the Dirichlet condition is given.

Theorem 3.4.3 — On existence for $|\Gamma_1| > 0$. Let all assumptions of Definition 3.3.4 be satisfied and $|\Gamma_1| > 0$. Assume that there exists $\varepsilon > 0$ dependent only on Ω and Γ_1 such that if data and coefficients of L satisfy

$$
\begin{aligned}
C_1\widetilde{b}(x) - |\vec{\mathfrak{c}}(x)|^2 - |\vec{\mathfrak{d}}(x)|^2 &\geq -\varepsilon C_1^2 \quad \text{almost everywhere in } \Omega, \\
\widetilde{\sigma}(x) &\geq -\varepsilon C_1 \quad \text{almost everywhere on } \Gamma_3,
\end{aligned}
\tag{3.57}
$$

where $\widetilde{b}$ and $\widetilde{\sigma}$ are defined in (3.56) and C_1 is the constant from the ellipticity of the operator L. Then there exists unique weak solution to (3.27)–(3.30). Moreover, there exists $C > 0$ dependent only on Ω, Γ_1, C_1 and ε such that the solution $\vec{u}$ satisfies

$$
\|\vec{u}\|_{1,2} \leq C\left(\|\vec{f}\|_{V^*} + \|\vec{U}_0\|_{W^{1,2}(\Omega;\mathbb{R}^N)} + \|\vec{g}\|_{(W^{\frac{1}{2},2}(\Gamma_2\cup\Gamma_3;\mathbb{R}^N))^*}\right). \tag{3.58}
$$

This theorem claims that if the Dirichlet condition is prescribed on a part of the boundary, then existence and uniqueness of weak solutions is ensured if either all terms are sufficiently small or the terms $\mathfrak{b}$ and σ are good (non-negative) and they control the potentially bad terms $\vec{\mathfrak{c}}$ a $\vec{\mathfrak{d}}$. Moreover, it is evident that the stronger the ellipticity of $\vec{\mathbb{A}}$ is (hence, the higher C_1 is) the worse situations can be covered. We leave as an easy exercise the proof of existence and uniqueness of weak solutions in the following situations: (i) problem (3.23) for $\sigma \geq 0$ and $|\Gamma_1| > 0$ (ii) problem (3.23) with $(3.23)_1$ replaced by (3.24) for $\vec{\mathfrak{c}} = 0$ and $|\Gamma_1| > 0$ (iii) vector problem with the equations (3.35) and such boundary conditions of the type discussed above with $|\Gamma_1| > 0$.

We now formulate the theorem for the case when we prescribe only Neumann or Newton boundary condition. The main difference with respect to the previous theorem is the fact that not only need that the terms $\vec{\mathfrak{b}}$ and σ are not bad, but one of them must be even sufficiently good.

Theorem 3.4.4 — On existence for $|\Gamma_1| = 0$. Let all assumptions of Definition 3.3.4 be satisfied and let $|\Gamma_1| = 0$. Let C_1 be the constant from the ellipticity of the operator L. Then the following holds.

1. Let

$$
\begin{aligned}
C_1\widetilde{b}(x) - |\vec{\mathfrak{c}}(x)|^2 - |\vec{\mathfrak{d}}(x)|^2 \geq 0 \quad &\text{almost everywhere in } \Omega, \\
\int_\Omega \left(C_1\widetilde{b}(x) - |\vec{\mathfrak{c}}(x)|^2 - |\vec{\mathfrak{d}}(x)|^2\right) \mathrm{d}x > 0.&
\end{aligned}
\tag{3.59}
$$

   Then there exists $\varepsilon > 0$ (however, now dependent on (3.59), C_1 and Ω) such that if σ fulfils

$$
\widetilde{\sigma}(x) \geq -\varepsilon \quad \text{almost everywhere on } \Gamma_3,
$$

   then there exists unique weak solution to (3.27)–(3.30).

2. Let

$$
\begin{aligned}
\widetilde{\sigma}(x) \geq 0 \quad &\text{almost everywhere on } \Gamma_3, \\
\int_{\Gamma_3} \widetilde{\sigma}\, \mathrm{d}S > 0.&
\end{aligned}
\tag{3.60}
$$

   Then there exists $\varepsilon > 0$ (dependent on (3.60), C_1 and Ω) such that if $\mathfrak{b}$, $\vec{\mathfrak{c}}$ and $\vec{\mathfrak{d}}$ fulfil

$$
C_1\widetilde{b}(x) - |\vec{\mathfrak{c}}(x)|^2 - |\vec{\mathfrak{d}}(x)|^2 \geq -\varepsilon \quad \text{almost everywhere on } \Gamma_3,
$$

   then there exists unique weak solution to (3.27)–(3.30).

Moreover, in both cases, there exists a constant $C > 0$ dependent only on Ω, Γ_1, C_1 and ε such that the solution $\vec{u}$ satisfies

$$\|\vec{u}\|_{1,2} \leq C \left(\|\vec{f}\|_{V^*} + \|\vec{U}_0\|_{W^{1,2}(\Omega;\mathbb{R}^N)} + \|\vec{g}\|_{(W^{\frac{1}{2},2}(\Gamma_2\cup\Gamma_3;\mathbb{R}^N))^*} \right). \tag{3.61}$$

Corollary 3.4.5. Since the problem is linear, then the mapping $\mathcal{F}$: $[\vec{f}, \vec{g}, \vec{U}_0] \longmapsto \vec{u}$ as a mapping from $V^* \times (W^{\frac{1}{2},2}(\Gamma_2 \cup \Gamma_3; \mathbb{R}^N))^* \times W^{1,2}(\Omega; \mathbb{R}^N)$ to V is bounded and continuous.

Proof of Theorem 3.4.3 and Theorem 3.4.4. Due to Theorem 3.4.2 it is enough to show that the bilinear form $B_{L,\sigma}$ is V-elliptic. Let us take arbitrary $\vec{u} \in V$ and plug it in formula (3.41). Using the ellipticity of the matrix $\vec{\mathbb{A}}$ (see (3.26)) and definitions of $\widetilde{b}$ and $\widetilde{\sigma}$ (see (3.56)) we get the inequality

$$\begin{aligned} B_{L,\sigma}(\vec{u},\vec{u}) &= \int_\Omega \left(\vec{\mathbb{A}}\nabla\vec{u} : \nabla\vec{u} + \mathfrak{c}\nabla\vec{u}\cdot\vec{u} - \vec{\mathfrak{d}}\nabla\vec{u}\cdot\vec{u} + \mathfrak{b}\vec{u}\cdot\vec{u} \right) \mathrm{d}x + \int_{\Gamma_3} (\sigma\vec{u})\cdot\vec{u}\,\mathrm{d}S \\ &\geq \int_\Omega \left(C_1|\nabla\vec{u}|^2 - |\mathfrak{c}||\nabla\vec{u}||\vec{u}| - |\vec{\mathfrak{d}}||\nabla\vec{u}||\vec{u}| + \widetilde{b}|\vec{u}|^2 \right) \mathrm{d}x + \int_{\Gamma_3} \widetilde{\sigma}|\vec{u}|^2\,\mathrm{d}S. \end{aligned}$$

Applying the Young inequality we may further estimate

$$|\mathfrak{c}||\nabla\vec{u}||\vec{u}| + |\vec{\mathfrak{d}}||\nabla\vec{u}||\vec{u}| \leq \frac{C_1}{2}|\nabla\vec{u}|^2 + \frac{|\mathfrak{c}|^2 + |\vec{\mathfrak{d}}|^2}{C_1}|\vec{u}|^2$$

and the above mentioned inequality reduces to the form

$$B_{L,\sigma}(\vec{u},\vec{u}) \geq \int_\Omega \left(\frac{C_1}{2}|\nabla\vec{u}|^2 + |\vec{u}|^2 \frac{C_1\widetilde{b} - |\mathfrak{c}|^2 - |\vec{\mathfrak{d}}|^2}{C_1} \right) \mathrm{d}x + \int_{\Gamma_3} \widetilde{\sigma}|\vec{u}|^2\,\mathrm{d}S. \tag{3.62}$$

From now on we proceed separately for $|\Gamma_1| > 0$ and $|\Gamma_1| = 0$.

In the former we use the Poincaré inequality and the Theorem on traces to obtain the required inequality. More precisely, using Theorem 2.6.3 we get existence of c_1 dependent only on Ω and Γ_1 such that

$$c_1^2\|\vec{u}\|_{1,2}^2 \leq \int_\Omega |\nabla\vec{u}|^2\,\mathrm{d}x + \int_{\Gamma_1} |\vec{u}|^2\,\mathrm{d}S.$$

Similarly due to the Theorem on trace operator 2.5.11 we get existence of c_2 dependent only on Ω such that

$$c_2^2 \int_{\Gamma_3} |\vec{u}|^2\,\mathrm{d}S \leq \|\vec{u}\|_{1,2}^2.$$

We apply these inequalities in (3.62) to get

$$\begin{aligned} B_{L,\sigma}(\vec{u},\vec{u}) &\geq c_1^2\frac{C_1}{2}\|\vec{u}\|_{1,2}^2 + \int_\Omega |\vec{u}|^2 \frac{C_1\widetilde{b} - |\mathfrak{c}|^2 - |\vec{\mathfrak{d}}|^2}{C_1}\,\mathrm{d}x + \int_{\Gamma_3} \widetilde{\sigma}|\vec{u}|^2\,\mathrm{d}S \\ &\geq c_1^2\frac{C_1}{4}\|\vec{u}\|_{1,2}^2 + c_1^2c_2^2\frac{C_1}{4}\int_{\Gamma_3} |\vec{u}|^2\,\mathrm{d}S \\ &\quad + \int_\Omega |\vec{u}|^2 \frac{C_1\widetilde{b} - |\mathfrak{c}|^2 - |\vec{\mathfrak{d}}|^2}{C_1}\,\mathrm{d}x + \int_{\Gamma_3} \widetilde{\sigma}|\vec{u}|^2\,\mathrm{d}S \\ &= c_1^2\frac{C_1}{4}\|\nabla\vec{u}\|_2^2 + C_1^{-1}\int_\Omega |\vec{u}|^2(c_1^2 4^{-1}C_1^2 + C_1\widetilde{b} - |\mathfrak{c}|^2 - |\vec{\mathfrak{d}}|^2)\,\mathrm{d}x \\ &\quad + \int_{\Gamma_3} (c_1^2c_2^2 4^{-1}C_1 + \widetilde{\sigma})|\vec{u}|^2\,\mathrm{d}S \\ &\geq c_1^4\frac{C_1}{4}\|\vec{u}\|_{1,2}^2 + C_1^{-1}\int_\Omega |\vec{u}|^2(c_1^2 4^{-1}C_1^2 + C_1\widetilde{b} - |\mathfrak{c}|^2 - |\vec{\mathfrak{d}}|^2)\,\mathrm{d}x \\ &\quad + \int_{\Gamma_3} (c_1^2c_2^2 4^{-1}C_1 + \widetilde{\sigma})|\vec{u}|^2\,\mathrm{d}S, \end{aligned} \tag{3.63}$$

where in the last inequality we used again Theorem 2.6.3. If we choose

$$\varepsilon := c_1^2 4^{-1}\min(1, c_2^2),$$

hence $\varepsilon > 0$ depends only on Ω and Γ_1, then it holds for any data satisfying (3.57)

$$B_{L,\sigma}(\vec{u},\vec{u}) \geq c_1^4\frac{C_1}{4}\|\vec{u}\|_{1,2}^2 = c_1^4\frac{C_1}{4}\|\vec{u}\|_V^2.$$

The bilinear form $B_{L,\sigma}$ is therefore V-elliptic and the proof is in this case finished.

Let us now turn our attention to the latter, i.e., $|\Gamma_1| = 0$. We cannot dispose any more with the Poincaré inequality which works with the values of the function on the boundary. However, we may replace this by requirement that at least one of the terms in (3.62) is good. Let us consider now case 1. from Theorem 3.4.4. Due to (3.59) there exists $\Omega^\star \subset \Omega$ and $\alpha_3 > 0$ such that $|\Omega^\star|_d > 0$ and

$$C_1\widetilde{b}(x) - |\vec{\mathfrak{c}}(x)|^2 - |\vec{\mathfrak{d}}(x)|^2 \geq \frac{C_1^2\alpha_3}{2} \quad \text{almost everywhere in } \Omega^\star.$$

Using this inequality in (3.62) and using the fact that expression in (3.59) is non-negative in the whole Ω, we have

$$B_{L,\sigma}(\vec{u},\vec{u}) \geq \frac{C_1}{2}\left(\int_\Omega |\nabla\vec{u}|^2 \,\mathrm{d}x + \alpha_3\int_{\Omega^\star} |\vec{u}|^2 \,\mathrm{d}x\right) + \int_{\Gamma_3} \widetilde{\sigma}|\vec{u}|^2 \,\mathrm{d}S. \tag{3.64}$$

We can now repeat the procedure from the previous part;, we apply Theorem 2.6.3 to get from the first term the whole norm and Theorem on trace operator 2.5.11 to estimate the last term. Altogether we have

$$B_{L,\sigma}(\vec{u},\vec{u}) \geq \frac{c_1^2C_1}{2}\|\vec{u}\|_{1,2}^2$$
$$+\int_{\Gamma_3} \widetilde{\sigma}|\vec{u}|^2 \,\mathrm{d}S \geq \frac{c_1^2C_1}{4}\|\vec{u}\|_{1,2}^2 + \int_{\Gamma_3}\left(\frac{c_2^2c_1^2C_1}{2} + \widetilde{\sigma}\right)|\vec{u}|^2 \,\mathrm{d}S.$$

Choosing

$$\varepsilon := \frac{c_2^2c_1^2C_1}{2}$$

we get the desired estimate, provided $\widetilde{\sigma}$ satisfies assumptions of point 1. Point 2. can be shown similarly and is left as an exercise to a kind reader. ■

This part contained basic techniques and procedures how to show that our operator is elliptic (hence the form $B_{L,\sigma}$ is V-elliptic). We saw that if we prescribe Dirichlet boundary values on at least a part of the boundary, we can handle also the unpleasant terms, provided they are not too big. If the Dirichlet conditions are not prescribed, at least one of the other terms must be sufficiently good. The whole proof did not use any structure of the unpleasant terms and the following exercises contain several cases which do not fall into our theory, developed up to now.

Exercise 3.4.6. Let us consider $\Omega \subset \mathbb{R}^d$ open bounded, not necessarily Lipschitz. For a given $f \in L^2(\Omega)$ we want to find a weak solution to

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Solution. We already discussed this case, however, the main issue now will be the properties of the boundary. In this situation, we cannot speak about the trace of u, nonetheless, the natural candidate for the suitable function space is $V := W_0^{1,2}(\Omega)$. Let us recall that this space is defined as a closure of smooth compactly supported functions in Ω. The weak solution is called a function $u \in V$ such that

$$B(u,\varphi) := \int_\Omega \nabla u \cdot \nabla\varphi \,\mathrm{d}x = \int_\Omega f\varphi \,\mathrm{d}x =: \langle F,\varphi\rangle_V \quad \text{for any } \varphi \in V.$$

The form B is evidently bilinear and V-bounded, it is enough to verify its V-ellipticity in order to be able to get from the Lax–Milgram Theorem 3.4.1 existence of exactly one weak solution. Since Ω is bounded, we may find a sufficiently large ball $B_R(0) \subset \mathbb{R}^d$ such that $\Omega \subset B_R(0)$. If we define u to be zero outside of Ω, we get $u \in W_0^{1,2}(B_R(0))$. (Verify carefully this step!) Since the ball has Lipschitz boundary (even a C^∞ one), we may use Theorem 2.6.3 to get

$$B(u,u) = \int_\Omega |\nabla u|^2 \,dx = \int_{B_R(0)} |\nabla u|^2 \,\mathrm{d}x \geq c_1^2\|u\|^2_{W^{1,2}(B_R(0))} = c_1^2\|u\|^2_{W^{1,2}(\Omega)}.$$

The form B is V-elliptic and the existence of a unique weak solution is a direct consequence of the theory developed above. □

This exercise shows that if we prescribe the Dirichlet boundary condition u_0 on the whole boundary, we are not obliged to deal with Lipschitz domains, but it is enough to to assume that there exists $U_0 \in W^{1,2}(\Omega)$ such that the trace of U_0 is equal to u_0 on $\partial\Omega$. We then look for our solution u in the form $u = U_0 + v$, where $v \in V$, or rather $(u - U_0) \in W_0^{1,2}(\Omega)$. Note that if Ω is Lipschitz and u_0 is sufficiently regular, the existence of such U_0 is guaranteed, but even some situation with less regular domain Ω can be considered.

Exercise 3.4.7. Let $\Omega = (-1,1)^2$. For a given $f \in L^2(\Omega)$ and $a \in L^\infty(-1,1)$ such that $\|a\|_\infty < 2$, let us show existence and uniqueness of weak solution to the problem

$$\begin{aligned} Lu(x) := -\frac{\partial^2 u}{\partial x_1^2}(x) - \frac{\partial^2 u}{\partial x_2^2}(x) + a(x_1)\frac{\partial^2 u}{\partial x_1\partial x_2}(x) &= f(x) && \text{in } \Omega \\ u(x) &= 0 && \text{on } \Omega. \end{aligned}$$

Solution. The expression on the left-hand side is not in the divergence form and thus we cannot use the theory developed above. However, we aim to rewrite in particular the third term into the divergence form. *A big mistake* would be the following

$$a\frac{\partial^2 u}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_1}\left(a\frac{\partial u}{\partial x_2}\right) - \frac{\partial a}{\partial x_1}\frac{\partial u}{\partial x_2}.$$

Formally, we achieved the form we need, however, by our assumptions we have no information about any derivative of a (even not about a weak one) and thus the expression on the right-hand side is not meaningful. On the other hand, since a is independent of x_2, we may proceed as follows:

$$a\frac{\partial^2 u}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_2}\left(a\frac{\partial u}{\partial x_1}\right).$$

If we define the matrix $\mathbb{A}(x)$ as

$$\mathbb{A}(x) := \begin{pmatrix} 1 & 0 \\ -a(x_1) & 1 \end{pmatrix},$$

we have

$$-\operatorname{div}\big(\mathbb{A}(x)\nabla u(x)\big) = -\sum_{i,j=1}^{2}\frac{\partial}{\partial x_i}\left((\mathbb{A}(x))_{ij}\frac{\partial u}{\partial x_j}\right) = Lu(x).$$

The operator L is now rewritten into the correct form and we need to verify the ellipticity of the matrix $\mathbb{A}$, i.e., to verify condition (3.26). A simple computation shows that it holds for any $\vec{z} \in \mathbb{R}^2$

$$\begin{aligned}\sum_{i,j=1}^{2}\big(\mathbb{A}(x)\big)_{ij} z_i z_j &= z_1^2 + z_2^2 - a(x_1)z_1 z_2 \ge z_1^2 - z_1^2(1-\varepsilon) - \frac{z_2^2(a(x_1))^2}{4(1-\varepsilon)} + z_2^2 \\ &\ge \varepsilon z_1^2 + \left(1 - \frac{(a(x_1))^2}{4(1-\varepsilon)}\right) z_2^2,\end{aligned}$$

where we used the Young inequality. We see that if $\|a\|_\infty < 2$, choosing $\varepsilon := (2 - \|a\|_\infty)/2 > 0$ we ensure

$$\left(1 - \frac{(a(x_1))^2}{4(1-\varepsilon)}\right) \ge \varepsilon \quad \text{almost everywhere in } \Omega.$$

Thus

$$\sum_{i,j=1}^{d}\big(\mathbb{A}(x)\big)_{ij} z_i z_j \ge \varepsilon|\mathbf{z}|^2.$$

Hence, the operator L is elliptic and we may use the theory developed above to prove existence and uniqueness of a weak solution to our problem. □

The following exercise deals with equation (3.24). We already know that for $|\mathbf{c}| \ll 1$ (the smallness depends now on the ellipticity constant C_1 of the matrix $\mathbb{A}$) there exists unique weak solution. We now show how we can generalize this result for particular vectors $\mathbf{c}$ which often appear in applications.

Exercise 3.4.8. Let $\Omega \subset \mathbb{R}^d$ be Lipschitz, $f \in L^2(\Omega)$, $p \in (2,\infty]$ and the vector $\mathbf{c} = (c_1,\dots,c_d)$ be such that $c_i \in L^p(\Omega)$, $1 = 1,2,\dots,d$. Let moreover $\operatorname{div}\mathbf{c} \le 0$ in Ω in the sense of distributions in Ω, i.e., for any non-negative $\varphi \in \mathcal{C}_0^\infty(\Omega)$ it holds

$$\int_\Omega \mathbf{c}\cdot\nabla\varphi \,\mathrm{d}x = \int_\Omega \sum_{i=1}^{d} c_i \frac{\partial\varphi}{\partial x_i}\,\mathrm{d}x \ge 0. \tag{3.65}$$

For p's from an interval as large as possible show the existence and uniqueness of a weak solution for equation (3.24) with homogeneous Dirichlet boundary condition without any restriction on the size of $\|\mathbf{c}\|_p$.

Solution. First, we have to formulate correctly the problem. The choice of the function space is evident, $V := W_0^{1,2}(\Omega)$, and $u \in V$ is called a weak solution to our problem, provided is satisfies for any $\varphi \in V$

$$\int_\Omega \Big(\mathbb{A}\nabla u\cdot\nabla\varphi + \mathbf{c}\cdot\nabla u\varphi\Big)\,\mathrm{d}x =: B(u,\varphi) = \langle F,\varphi\rangle_V := \int_\Omega f\varphi\,\mathrm{d}x.$$

Since we have no smallness of a norm of $\mathbf{c}$, our above proved results cannot be applied. We therefore try to apply directly the Lax–Milgram Theorem (Theorem 3.4.1) on our problem. Clearly $F \in V^*$ and the form B is bilinear. It remains to verify its V–boundedness and V–ellipticity. We start with the boundedness. The Hölder inequality give us the following estimate

$$\begin{aligned}|B(u,\varphi)| &\le \int_\Omega \Big(|\mathbb{A}||\nabla u||\nabla\varphi| + |\mathbf{c}||\nabla u||\varphi|\Big)\,\mathrm{d}x \\ &\le \|\mathbb{A}\|_\infty\|\nabla u\|_2\|\nabla\varphi\|_2 + \|\mathbf{c}\|_p\|\nabla u\|_2\|\varphi\|_{\frac{2p}{p-2}} \\ &\le C(\mathbb{A},\mathbf{c})\|u\|_V(\|\varphi\|_V + \|\varphi\|_{\frac{2p}{p-2}}).\end{aligned}$$

The last term contains a certain norm; to bound it by the norm in V we need to apply the continuous embedding $W^{1,2}(\Omega) \hookrightarrow L^{\frac{2p}{p-2}}(\Omega)$. Using Theorem 2.4.5 we obtain

$$W^{1,2}(\Omega) \hookrightarrow L^{\frac{2p}{p-2}}(\Omega) \Longleftrightarrow \begin{cases} p > 2 & \text{for } d = 2, \\ p \geq d & \text{for } d > 2. \end{cases}$$

In what follows we consider only such p's a and we know that our form B is V–bounded.

Let us now concentrate on the V–ellipticity. Using our assumptions on $\mathbb{A}$ (see (3.26)) together with the Poincaré inequality (see Theorem 2.6.3) we get the estimate

$$\begin{aligned} B(u,u) &= \int_\Omega \Big(\mathbb{A}\nabla u \cdot \nabla u + \mathbf{c} \cdot \nabla u u \Big) \,\mathrm{d}x \geq C_1 \|\nabla u\|_2^2 + \int_\Omega \mathbf{c} \cdot \nabla u u \,\mathrm{d}x \\ &\geq c_1^2 C_1 \|u\|_V^2 + \int_\Omega \mathbf{c} \cdot \nabla u u \,\mathrm{d}x. \end{aligned}$$

Let us now look at the last integral. Using the definition of $W_0^{1,2}(\Omega)$ we may find a sequence $\{u^n\}_{n\in\mathbb{N}} \in \mathcal{C}_0^\infty(\Omega)$ such that $u^n \to u$ in $W_0^{1,2}(\Omega)$. Moreover, due to the continuous embedding and our choice of p's we also have that $u^n \to u$ in $L^{\frac{2p}{p-2}}(\Omega)$. The last integral can be rewritten to the following form:

$$\begin{aligned} \int_\Omega \mathbf{c} \cdot \nabla u u \,\mathrm{d}x &= \lim_{n\to\infty} \int_\Omega \mathbf{c} \cdot \nabla u^n u^n \,\mathrm{d}x = \lim_{n\to\infty} \int_\Omega \sum_{i=1}^d c_i \frac{\partial u^n}{\partial x_i} u^n \,\mathrm{d}x \\ &= \frac{1}{2} \lim_{n\to\infty} \int_\Omega \sum_{i=1}^d c_i \frac{\partial |u^n|^2}{\partial x_i} \,\mathrm{d}x \overset{(3.65)}{\geq} 0. \end{aligned}$$

In the last inequality, we used (3.65) with $\varphi := (u^n)^2$ which is clearly non-negative and due to the properties of u^n also smooth and compactly supported. The form B is thus V–elliptic; the existence and uniqueness of a weak solution for $p \geq d$ (or $p > 2$ for $d = 2$, respectively) is a direct consequence of Lax–Milgram Theorem 3.4.1. □

Exercise 3.4.9. Consider for $\mathbb{A} \in L^\infty(\Omega; \mathbb{R}^{d\times d})$, $f \in L^p(\Omega)$, $b \in L^q(\Omega)$ and $\mathbf{c} \in L^r(\Omega; \mathbb{R}^d)$ the following problem: find $u \in W_0^{1,2}(\Omega)$ such that

$$\int_\Omega \Big(\sum_{i,j=1}^d a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + \sum_{i=1}^d c_i \frac{\partial u}{\partial x_i} v + buv \Big) \,\mathrm{d}x = \int_\Omega fv \,\mathrm{d}x$$

for all $v \in W_0^{1,2}(\Omega)$. Assume that the bilinear form on the left-hand side is $W_0^{1,2}$-elliptic. In dependence on d find minimal requirements on p,q and r such that there exists a solution to the problem above (i.e., show under which conditions the bilinear form is bounded and the right-hand side forms a bounded linear functional).

3.5 Existence of a weak solution by means of the Fredholm alternative

The previous section dealt with existence (and uniqueness) of a weak solution for a large class of data and we showed that the existence of a weak solution can be shown, provided the corresponding operator is coercive; in other words, provided we are able to show a priori estimates. This was typically possible in the cases, when b or σ was sufficiently good (positive). In this section, we rather concentrate ourselves on the characterization of the data for which a solution exists (and is possibly unique). For simplicity[1] we shall not deal with inhomogeneous Dirichlet boundary condition and in case we prescribe values $\vec{u}_0$ on the boundary, then we always take $\vec{u}_0 = \vec{0}$. Recall first the operator L

$$(L\vec{u})^\alpha := -\sum_{\beta=1}^N \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left(a_{ij}^{\alpha\beta} \frac{\partial u^\beta}{\partial x_j} \right) + \sum_{\beta=1}^N \sum_{i=1}^d c_i^{\alpha\beta} \frac{\partial u^\beta}{\partial x_i} + \sum_{\beta=1}^N \sum_{i=1}^d \frac{\partial}{\partial x_i} (d_i^{\alpha\beta} u^\beta) + \sum_{\beta=1}^N b^{\alpha\beta} u^\beta \tag{3.66}$$

and its shorter notation

$$L\vec{u} = -\mathrm{div}\, \left(\vec{\mathbb{A}} \nabla \vec{u} \right) + \vec{\mathfrak{c}} \cdot \nabla \vec{u} + \mathrm{div}\, (\vec{\mathfrak{d}} \vec{u}) + \mathfrak{b} \vec{u}$$

which will be elliptic, i.e., its coefficients satisfy Definition 3.2.6. Recall also that we aim to study solvability (weak) of the following problem

$$\begin{aligned} L\vec{u} &= \vec{f} && \text{in } \Omega, \\ \vec{u} &= \vec{0} && \text{on } \Gamma_1 \\ (\vec{\mathbb{A}}\nabla\vec{u} - \vec{\mathfrak{d}}\vec{u}) \cdot \vec{\nu} &= \vec{g} && \text{on } \Gamma_2 \\ (\vec{\mathbb{A}}\nabla\vec{u} - \vec{\mathfrak{d}}\vec{u}) \cdot \vec{\nu} + \sigma u &= \vec{g} && \text{on } \Gamma_3. \end{aligned} \tag{3.67}$$

We now define the adjoint operator L^*.

[1] The whole theory can be developed also for inhomogeneous Dirichlet conditions which have, however, to be more regular.

Definition 3.5.1 — Adjoint operator. Let L be an elliptic operator defined by (3.66). The operator L^* defined as

$$(L^*\vec{\varphi})^\alpha := -\sum_{\beta=1}^N \sum_{i,j=1}^d \frac{\partial}{\partial x_i}\left(a_{ji}^{\beta\alpha}\frac{\partial \varphi^\beta}{\partial x_j}\right) - \sum_{\beta=1}^N \sum_{i=1}^d \frac{\partial}{\partial x_i}(c_i^{\beta\alpha}\varphi^\beta) - \sum_{\beta=1}^N \sum_{i=1}^d d_i^{\beta\alpha}\frac{\partial \varphi^\beta}{\partial x_i} + \sum_{\beta=1}^N b^{\beta\alpha}\varphi^\beta \tag{3.68}$$

is called the adjoint operator to L.

In the short notation, the adjoint operator is written as

$$L^*\vec{\varphi} = -\mathrm{div}\left(\vec{\mathbb{A}}^*\nabla\vec{\varphi}\right) - \mathrm{div}\,(\vec{\mathfrak{c}}^*\vec{\varphi}) - \vec{\mathfrak{d}}^*\cdot\nabla\vec{\varphi} + \mathfrak{b}^*\vec{\varphi},$$

where we denoted

$$(\vec{\mathbb{A}}^*)_{ij}^{\alpha\beta} := (\vec{\mathbb{A}})_{ji}^{\beta\alpha}, \quad (\vec{\mathfrak{c}}^*)_i^{\alpha\beta} := (\vec{\mathfrak{c}})_i^{\beta\alpha}, \quad (\vec{\mathfrak{d}}^*)_i^{\alpha\beta} := (\vec{\mathfrak{d}})_i^{\beta\alpha}, \quad (\mathfrak{b}^*)^{\alpha\beta} := \mathfrak{b}^{\beta\alpha}.$$

Note further that in the scalar case (i.e., for $N = 1$) we have $\mathbb{A}^* = \mathbb{A}^T$, $b^* = b$, $\mathbf{c}^* = \mathbf{c}$ and $\mathbf{d}^* = \mathbf{d}$. We now define the adjoint problem.

Definition 3.5.2 — Adjoint problem. We say that the problem

$$\begin{aligned} L^*\vec{\varphi} &= \vec{f} && \text{in } \Omega\\ \vec{\varphi} &= \vec{0} && \text{on } \Gamma_1\\ (\vec{\mathbb{A}}^*\nabla\vec{\varphi} + \vec{\mathfrak{c}}^*\vec{\varphi})\cdot\vec{\nu} &= \vec{g} && \text{on } \Gamma_2\\ (\vec{\mathbb{A}}^*\nabla\vec{\varphi} + \vec{\mathfrak{c}}^*\vec{\varphi})\cdot\vec{\nu} + \sigma^*\vec{\varphi} &= \vec{g} && \text{on } \Gamma_3 \end{aligned} \tag{3.69}$$

is adjoint problem to (3.67). In (3.69), we denoted $(\sigma^*)^{\alpha\beta} := (\sigma)^{\beta\alpha}$.

Since this section deals with question for which data the weak solution exists (while in the previous one the main question was not only existence of weak solution, but also its uniqueness), we introduce the following notation.

Definition 3.5.3 We say that $(\vec{f},\vec{g}) \in \mathbb{L}^2$, if $\vec{f} = (f^1,\dots,f^N)$ and $\vec{g} = (g^1,\dots,g^N)$ and for any $\alpha \in \{1,\dots,N\}$ it holds $f^\alpha \in L^2(\Omega)$ and $g^\alpha \in L^2(\Gamma_2\cup\Gamma_3)$.

Remark 3.5.4. The space $\mathbb{L}^2$ is a Hilbert space endowed with the scalar product

$$\left((\vec{f},\vec{g}),(\vec{u},\vec{v})\right)_{\mathbb{L}^2} := \int_\Omega \vec{f}\cdot\vec{u}\,\mathrm{d}x + \int_{\Gamma_2\cup\Gamma_3}\vec{g}\cdot\vec{v}\,\mathrm{d}S.$$

We can now clarify why we speak about the adjoint operator and the adjoint problem. If we define the following bilinear forms

$$\begin{aligned} B_{L,\sigma}(\vec{u},\vec{\varphi}) :=& \sum_{\alpha,\beta=1}^N \int_\Omega \sum_{i,j=1}^d \left(a_{ij}^{\alpha\beta}\frac{\partial u^\beta}{\partial x_j}\frac{\partial \varphi^\alpha}{\partial x_i} + \sum_{i=1}^d c_i^{\alpha\beta}\frac{\partial u^\beta}{\partial x_i}\varphi^\alpha\right)\mathrm{d}x - \sum_{\alpha,\beta=1}^N \int_\Omega \sum_{i=1}^d d_i^{\alpha\beta}\frac{\partial\varphi^\alpha}{\partial x_i}u^\beta\Big)\,\mathrm{d}x\\ &+ \sum_{\alpha,\beta=1}^N \int_\Omega b^{\alpha\beta}u^\beta\varphi^\alpha\,\mathrm{d}x + \sum_{\alpha,\beta=1}^N\int_{\Gamma_3}\sigma^{\alpha\beta}u^\beta\varphi^\alpha\,\mathrm{d}S\\ B_{L^*,\sigma^*}(\vec{\varphi},\vec{u}) :=& \sum_{\alpha,\beta=1}^N \int_\Omega \Big(\sum_{i,j=1}^d (a^*)_{ij}^{\alpha\beta}\frac{\partial\varphi^\beta}{\partial x_j}\frac{\partial u^\alpha}{\partial x_i} + \sum_{i=1}^d (c^*)_i^{\alpha\beta}\frac{\partial u^\alpha}{\partial x_i}\varphi^\beta\Big)\,\mathrm{d}x - \sum_{\alpha,\beta=1}^N\int_\Omega\sum_{i=1}^d (d^*)_i^{\alpha\beta}\frac{\partial\varphi^\beta}{\partial x_i}u^\alpha\,\mathrm{d}x\\ &+ \sum_{\alpha,\beta=1}^N\int_\Omega (b^*)^{\alpha\beta}\varphi^\beta u^\alpha\,\mathrm{d}x + \sum_{\alpha,\beta=1}^N\int_{\Gamma_3}(\sigma^*)^{\alpha\beta}\varphi^\beta u^\alpha\,\mathrm{d}S, \end{aligned}$$

we see that $\vec{u} \in V$ is a weak solution to the problem (3.67), if and only if it holds for any $\vec{\varphi} \in V$ (see Definition 3.3.4)

$$B_{L,\sigma}(\vec{u},\vec{\varphi}) = \int_\Omega \vec{f}\cdot\vec{\varphi}\,\mathrm{d}x + \int_{\Gamma_2\cup\Gamma_3}\vec{g}\cdot\vec{\varphi}\,\mathrm{d}S. \tag{3.70}$$

Similarly (again from Definition 3.3.4) $\vec{\varphi} \in V$ is a weak solution to (3.69), if and only if it holds for any $\vec{u} \in V$

$$B_{L^*,\sigma^*}(\vec{\varphi},\vec{u}) = \int_\Omega \vec{f}\cdot\vec{u}\,\mathrm{d}x + \int_{\Gamma_2\cup\Gamma_3}\vec{g}\cdot\vec{u}\,\mathrm{d}S. \tag{3.71}$$

Moreover, from the definition of $\vec{\mathbb{A}}^*$, $\vec{\mathfrak{c}}^*$, $\vec{\mathfrak{d}}^*$, $\mathfrak{b}^*$ and σ^* we get that for any $\vec{u},\vec{\varphi} \in V$

$$B_{L,\sigma}(\vec{u},\vec{\varphi}) = B_{L^*,\sigma^*}(\vec{\varphi},\vec{u}). \tag{3.72}$$

Relation (3.72) can be also rewritten as

$$\langle L\vec{u}, \vec{\varphi}\rangle_V = \langle L^*\vec{\varphi}, \vec{u}\rangle_V,$$

and therefore we speak about the adjoint operator and the adjoint problem. We should however keep in mind that the equality above is also related to a certain Newton boundary condition represented by $\mathfrak{o}$ or $\mathfrak{o}^*$, respectively.

In what follows we shall study properties of Problems (3.67) and (3.69). Let us first mention consequence of Theorems 3.4.3–3.4.4.

Theorem 3.5.5 — Generalized existence theorem I. Let $\Omega \subset \mathbb{R}^d$ be Lipschitz, $\{\Gamma_i\}_{i=1}^3$ be the corresponding parts of the boundary and L be an elliptic operator. Then there exists $\gamma_0 \geq 0$ such that for any $\gamma \geq \gamma_0$ and any $(\vec{f}, \vec{g}) \in \mathbb{L}^2$ there exists unique weak solution to the problem

$$\begin{aligned} L_\gamma \vec{u} := L\vec{u} + \gamma\vec{u} &= \vec{f} && \text{in } \Omega \\ \vec{u} &= \vec{0} && \text{on } \Gamma_1 \\ (\vec{\mathbb{A}}\nabla\vec{u} - \vec{\mathfrak{d}}\vec{u})\cdot\vec{\nu} + \gamma\vec{u} &= \vec{g} && \text{on } \Gamma_2 \\ (\vec{\mathbb{A}}\nabla\vec{u} - \vec{\mathfrak{d}}\vec{u})\cdot\vec{\nu} + \mathfrak{o}\vec{u} + \gamma\vec{u} &= \vec{g} && \text{on } \Gamma_3. \end{aligned} \tag{3.73}$$

Moreover, there exists C dependent only on L, $\mathfrak{o}$ and Ω such that for any $\gamma \geq \gamma_0$ the solution $\vec{u}$ satisfies

$$\|\vec{u}\|_V \leq C\|(\vec{f}, \vec{g})\|_{\mathbb{L}^2}. \tag{3.74}$$

Proof. If $|\Gamma_1| > 0$, we use Theorem 3.4.3. The sufficient condition (3.57) has in this case the following form

$$\begin{aligned} C_1\big(\gamma + \widetilde{b}(x)\big) - |\vec{\mathfrak{c}}(x)|^2 - |\vec{\mathfrak{d}}(x)|^2 &\geq -\varepsilon C_1^2 && \text{almost everywhere in } \Omega, \\ \gamma + \widetilde{\sigma}(x) &\geq -\varepsilon C_1 && \text{almost everywhere on } \Gamma_3, \\ \gamma &\geq -\varepsilon C_1 && \text{almost everywhere on } \Gamma_2. \end{aligned} \tag{3.75}$$

If we choose $\gamma_0 := C_1^{-1}(\|\vec{\mathfrak{c}}\|_\infty^2 + \|\vec{\mathfrak{d}}\|_\infty^2 + \|\mathfrak{b}\|_\infty + \|\mathfrak{o}\|_\infty)$, then for any $\gamma \geq \gamma_0$ the condition (3.75) is fulfiled, therefore Theorem 3.4.3 finishes the proof. If $|\Gamma_1| = 0$, we similarly apply Theorem 3.4.4. ■

We present without proof a similar result for the adjoint problem. Its proof is, however, identical to the proof of the previous theorem.

Theorem 3.5.6 — Generalized existence theorem II. Let $\Omega \subset \mathbb{R}^d$ be Lipschitz, $\{\Gamma_i\}_{i=1}^3$ be the corresponding parts of the boundary and L the elliptic operator. Then there exists $\gamma_0 \geq 0$ such that for any $\gamma \geq \gamma_0$ and any $(\vec{f}, \vec{g}) \in \mathbb{L}^2$ there exists unique weak solution to the problem

$$\begin{aligned} L_\gamma^*\vec{\varphi} := L^*\vec{\varphi} + \gamma\vec{\varphi} &= \vec{f} && \text{in } \Omega \\ \vec{\varphi} &= \vec{0} && \text{on } \Gamma_1 \\ (\vec{\mathbb{A}}^*\nabla\vec{\varphi} + \vec{\mathfrak{c}}^*\vec{\varphi})\cdot\vec{\nu} + \gamma\vec{\varphi} &= \vec{g} && \text{on } \Gamma_2 \\ (\vec{\mathbb{A}}^*\nabla\vec{\varphi} + \vec{\mathfrak{c}}^*\vec{\varphi})\cdot\vec{\nu} + \mathfrak{o}^*\vec{\varphi} + \gamma\vec{\varphi} &= \vec{g} && \text{on } \Gamma_3. \end{aligned} \tag{3.76}$$

Moreover, there exists a constant C dependent only on L, $\mathfrak{o}$ and Ω such that for any $\gamma \geq \gamma_0$ the solution $\vec{\varphi}$ satisfies

$$\|\vec{\varphi}\|_V \leq C\|(\vec{f}, \vec{g})\|_{\mathbb{L}^2}. \tag{3.77}$$

The following results will be based on the Fredholm alternative and its consequences (see Theorem B.3.6). For the reader's convenience, we repeat it here.

Theorem 3.5.7 — Fredholm alternative. Let H be a Hilbert space and let $K\colon H \to H$ be a compact linear operator. Then it holds:

1. $N(I - K)$ is finite dimensional
2. $R(I - K)$ is closed
3. $R(I - K) = N(I - K^*)^\perp$
4. $N(I - K) = \{0\} \Leftrightarrow R(I - K) = H$
5. $\dim N(I - K) = \dim N(I - K^*)$
6. The spectrum of K is at most countable and contains 0. If the spectrum is infinite, then 0 is its only accumulation point.

Let us recall the notation used above: N denotes the kernel of the operator, R is the range of the operator, K^* is the adjoint operator, i.e., the operator satisfying for any $u, v \in H$ that $(Ku, v)_H = (u, K^*v)_H$. The spectrum is defined in Appendix, in Definition B.3.1. Finally, quite often, only Property 4. from Theorem 3.5.7 is called the Fredholm alternative. This claim says that either for any $f \in H$ the problem $u - Ku = f$ has (unique) solution or there exists a nontrivial solution to problem $u - Ku = 0$.

We can now formulate the most important result of this section.

Theorem 3.5.8 — Fredholm alternative for elliptic operators. Let $\Omega \subset \mathbb{R}^d$ be Lipschitz, $\{\Gamma_i\}_{i=1}^3$ be the corresponding parts of the boundary and let L be the elliptic operator.

1) Either for any $(\vec{f}, \vec{g}) \in \mathbb{L}^2$ problem (3.67) possesses exactly one weak solution or for $(\vec{f}, \vec{g}) := (\vec{0}, \vec{0})$ there exists a nontrivial solution to problem (3.67).

2) Denote

$$N_L := \{\vec{u} \in V; \mid \vec{u} \text{ is a weak solution to problem (3.67) for } (\vec{f}, \vec{g}) := (\vec{0}, \vec{0})\},$$
$$N_{L^*} := \{\vec{\varphi} \in V; \mid \varphi \text{ is a weak solution to problem (3.69) for } (\vec{f}, \vec{g}) := (\vec{0}, \vec{0})\}.$$

Then N_L and N_{L^*} are closed subspaces of V and it holds $\dim N_L = \dim N_{L^*} < \infty$.

3) Problem (3.67) has a weak solution for a given $(\vec{f}, \vec{g}) \in \mathbb{L}^2$, if and only if it holds for any $\vec{\varphi} \in N_{L^*}$

$$0 = \big((\vec{f}, \vec{g}), (\vec{\varphi}, \vec{\varphi}|_{\Gamma_2 \cup \Gamma_3})\big)_{\mathbb{L}^2} = \int_\Omega \vec{f} \cdot \vec{\varphi} \, \mathrm{d}x + \int_{\Gamma_2 \cup \Gamma_3} \vec{g} \cdot \vec{\varphi} \, \mathrm{d}S.$$

Proof. For an arbitrary $\gamma > 0$ we define bilinear forms

$$\begin{aligned} B^\gamma_{L,\sigma}(\vec{u}, \vec{\varphi}) &:= B_{L,\sigma}(\vec{u}, \vec{\varphi}) + \gamma \int_\Omega \vec{u} \cdot \vec{\varphi} \, \mathrm{d}x + \gamma \int_{\Gamma_2 \cup \Gamma_3} \vec{u} \cdot \vec{\varphi} \, \mathrm{d}S \\ B^\gamma_{L^*,\sigma^*}(\vec{\varphi}, \vec{u}) &:= B_{L^*,\sigma^*}(\vec{\varphi}, \vec{u}) + \gamma \int_\Omega \vec{\varphi} \cdot \vec{u} \, \mathrm{d}x + \gamma \int_{\Gamma_2 \cup \Gamma_3} \vec{\varphi} \cdot \vec{u} \, \mathrm{d}S. \end{aligned} \tag{3.78}$$

These bilinear forms correspond to problems (3.73) and (3.76). In other words, for $(\vec{f}, \vec{g}) \in \mathbb{L}^2$ the function $\vec{u} \in V$ is called a weak solution to problem (3.73) if it holds for any $\vec{\varphi} \in V$

$$B^\gamma_{L,\sigma}(\vec{u}, \vec{\varphi}) = \int_\Omega \vec{f} \cdot \vec{\varphi} \, \mathrm{d}x + \int_{\Gamma_2 \cup \Gamma_3} \vec{g} \cdot \vec{\varphi} \, \mathrm{d}S = \big((\vec{f}, \vec{g}), (\vec{\varphi}, \vec{\varphi}|_{\Gamma_2 \cup \Gamma_3})\big)_{\mathbb{L}^2}. \tag{3.79}$$

Similarly, $\vec{\varphi} \in V$ is a weak solution to problem (3.76) for $(\vec{f}, \vec{g}) \in \mathbb{L}^2$, if it holds for any $\vec{u} \in V$

$$B^\gamma_{L^*,\sigma^*}(\vec{\varphi}, \vec{u}) = \int_\Omega \vec{f} \cdot \vec{g} \, \mathrm{d}x + \int_{\Gamma_2 \cup \Gamma_3} \vec{g} \cdot \vec{u} \, \mathrm{d}S = \big((\vec{f}, \vec{g}), (\vec{u}, \vec{u}|_{\Gamma_2 \cup \Gamma_3})\big)_{\mathbb{L}^2}. \tag{3.80}$$

Recall that due to the trace operator and Lipschitz continuity of the boundary of Ω it makes sense to speak about values of functions from V on the boundary and in this sense we also understand $\vec{u}|_{\Gamma_2 \cup \Gamma_3}$ and $\vec{\varphi}|_{\Gamma_2 \cup \Gamma_3}$.

Using Theorems 3.5.5–3.5.6 we fix $\gamma > 0$ such that problems (3.73) and (3.76) have unique weak solution for any $(\vec{f}, \vec{g}) \in \mathbb{L}^2$. We define the operator $L_\gamma^{-1} : \mathbb{L}^2 \to \mathbb{L}^2$ by

$$L_\gamma^{-1} : (\vec{f}, \vec{g}) \mapsto (\vec{u}, \vec{u}|_{\Gamma_2 \cup \Gamma_3}),$$

it means that we assign to $(\vec{f}, \vec{g})$ unique weak solution to problem (3.73), i.e., the function $\vec{u}$ satisfies (3.79). Similarly we define the operator $(L_\gamma^*)^{-1}$ as

$$(L_\gamma^*)^{-1} : (\vec{f}, \vec{g}) \mapsto (\vec{\varphi}, \vec{\varphi}|_{\Gamma_2 \cup \Gamma_3}),$$

where $\vec{\varphi} \in V$ is the unique weak solution to (3.76), i.e., the function $\vec{\varphi}$ satisfies (3.80). Both operators are clearly linear and due to (3.74) and (3.77) also bounded (we apply here the Theorem on trace operator 2.5.11). Furthermore, due to the compact embedding $W^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$ (Theorem 2.4.17) and due to the compactness of the trace operator (Theorem 2.5.14) both operators L_γ^{-1} and $(L_\gamma^*)^{-1}$ are linear continuous compact operators from $\mathbb{L}^2$ to $\mathbb{L}^2$.

We now define the operator $K \colon \mathbb{L}^2 \to \mathbb{L}^2$ as $K := \gamma L_\gamma^{-1}$. Evidently, also this operator is linear continuous compact.

We now rewrite problem (3.67) using the operator K. Due to the weak formulation of (3.70), weak formulation for the operator L_γ (i.e., identity (3.79)), definition of the operator L_γ^{-1} and definition and linearity of the operator

K we subsequently obtain

$$
\begin{aligned}
&\vec{u} \in V \text{ solves (3.67)} \\
&\overset{\text{def}}{\Longleftrightarrow} B_{L,\sigma}(\vec{u},\vec{\varphi}) = \int_\Omega \vec{f}\cdot\vec{\varphi}\,\mathrm{d}x + \int_{\Gamma_2\cup\Gamma_3} \vec{g}\cdot\vec{\varphi}\,\mathrm{d}S \quad \forall\vec{\varphi}\in V, \\
&\Longleftrightarrow B^{\gamma}_{L,\sigma}(\vec{u},\vec{\varphi}) = \int_\Omega (\gamma\vec{u}+\vec{f})\cdot\vec{\varphi}\,\mathrm{d}x + \int_{\Gamma_2\cup\Gamma_3} (\gamma\vec{u}+\vec{g})\cdot\vec{\varphi}\,\mathrm{d}S \quad \forall\vec{\varphi}\in V, \\
&\Longleftrightarrow (\vec{u},\vec{u}|_{\Gamma_2\cup\Gamma_3}) = L_\gamma^{-1}\big((\gamma\vec{u}+\vec{f}),(\gamma\vec{u}|_{\Gamma_2\cup\Gamma_3}+\vec{g})\big) \\
&\Longleftrightarrow (\vec{u},\vec{u}|_{\Gamma_2\cup\Gamma_3}) - K(\vec{u},\vec{u}|_{\Gamma_2\cup\Gamma_3}) = \vec{h}, \quad \text{where } \vec{h} := \frac{1}{\gamma}K(\vec{f},\vec{g}).
\end{aligned}
\tag{3.81}
$$

Let us now show Claim 1. from the theorem. Since K is compact, we may use the Fredholm alternative 3.5.7, where Property 4. claims that either there exists a nontrivial solution $(\vec{a},\vec{b})\in\mathbb{L}^2$ to problem

$$(\vec{a},\vec{b}) - K(\vec{a},\vec{b}) = (\vec{0},\vec{0}) \tag{3.82}$$

or for any $(\vec{a},\vec{b})\in\mathbb{L}^2$ there exists unique solution $(\vec{v},\vec{w})\in\mathbb{L}^2$ to problem

$$(\vec{v},\vec{w}) - K(\vec{v},\vec{w}) = (\vec{a},\vec{b}). \tag{3.83}$$

Let us first verify that $K(\vec{f},\vec{g}) = (\vec{0},\vec{0})$, if and only if $(\vec{f},\vec{g}) = (\vec{0},\vec{0})$. One impliction is, due to the linearity of K, evident. Let us therefore show that if $K(\vec{f},\vec{g}) = (\vec{0},\vec{0})$, then $(\vec{f},\vec{g}) = (\vec{0},\vec{0})$. The definition of the operator K implies that $\vec{u}\equiv\vec{0}$ solves (3.73). This means that we have for any $\vec{\varphi}\in V$

$$\int_\Omega \vec{f}\cdot\vec{\varphi}\,\mathrm{d}x + \int_{\Gamma_2\cup\Gamma_3}\vec{g}\cdot\vec{\varphi}\,\mathrm{d}S = 0.$$

Since $[\mathcal{C}_0^\infty(\Omega)]^N \subset V$, we immediately see that $\vec{f}\equiv\vec{0}$. Similarly we also get $\vec{g}=\vec{0}$. We further show that (3.82) is equivalent to the existence of a nontrivial solution to (3.67) with $\vec{f}=\vec{0}$ and $\vec{g}=\vec{0}$. The definition of K implies that any nontrivial solution $(\vec{a},\vec{b})$ of (3.82) can be written as $(\vec{a},\vec{b}) = (\vec{u},\vec{u}|_{\Gamma_2\cup\Gamma_3})$, where $\vec{u}\in V$. Thus a nontrivial solution to (3.82) exists, if and only if there exists a nontrivial $\vec{u}\in V$ which satisfies

$$(\vec{u},\vec{u}|_{\Gamma_2\cup\Gamma_3}) - K(\vec{u},\vec{u}|_{\Gamma_2\cup\Gamma_3}) = (\vec{0},\vec{0}) = K(\vec{0},\vec{0}). \tag{3.84}$$

Comparing this equality with (3.81) we see that this property is equivalent with existence of a nontrivial solution to (3.67).

We thus have that if no nontrivial solution to (3.67) with zero data exists, then there does not exist any nontrivial solution to (3.82). Hence for any $(\vec{a},\vec{b})\in\mathbb{L}^2$ there exists unique solution to problem (3.83) and in particular for any $(\vec{f},\vec{g})\in\mathbb{L}^2$ there exists unique solution $(\vec{v},\vec{w})\in\mathbb{L}^2$ to problem (we use the fact that $K(\vec{f},\vec{g})\in\mathbb{L}^2$)

$$(\vec{v},\vec{w}) - K(\vec{v},\vec{w}) = \frac{1}{\gamma}K(\vec{f},\vec{g}). \tag{3.85}$$

Due to the definition of operator K, there exist $\vec{u}^1,\vec{u}^2\in V$ such that

$$(\vec{u}^1,\vec{u}^1|_{\Gamma_2\cup\Gamma_3}) = K(\vec{v},\vec{w}), \qquad (\vec{u}^2,\vec{u}^2|_{\Gamma_2\cup\Gamma_3}) = \frac{1}{\gamma}K(\vec{f},\vec{g}).$$

Therefore

$$(\vec{v},\vec{w}) = K(\vec{v},\vec{w}) + \frac{1}{\gamma}K(\vec{f},\vec{g}) = (\vec{u}^1,\vec{u}^1|_{\Gamma_2\cup\Gamma_3}) + (\vec{u}^2,\vec{u}^2|_{\Gamma_2\cup\Gamma_3}) = (\vec{u},\vec{u}|_{\Gamma_2\cup\Gamma_3}),$$

where $\vec{u} := \vec{u}^1+\vec{u}^2\in V$. Thus we get from (3.85) that for any $(\vec{f},\vec{g})\in\mathbb{L}^2$ there exists unique $\vec{u}\in V$ such that

$$(\vec{u},\vec{u}|_{\Gamma_2\cup\Gamma_3}) - K(\vec{u},\vec{u}|_{\Gamma_2\cup\Gamma_3}) = \frac{1}{\gamma}K(\vec{f},\vec{g}).$$

We may now use (3.81) and we obtain that for any $(\vec{f},\vec{g})$ there exists a weak solution to problem (3.67).

Let us now assume that there exists a nontrivial solution $\vec{w}\in V$ for problem (3.67) with $(\vec{f},\vec{g})\equiv(\vec{0},\vec{0})$. Then if arbitrary $\vec{u}\in V$ solves (3.67), due to the linearity $\vec{u}+\vec{w}$ also solves the same problem and we showed the nonuniqueness of solutions to (3.67).

Let us now show Claim 2. We showed in the preceding part that $N(I-K) = \{(\vec{u},\vec{u}|_{\Gamma_2\cup\Gamma_3})\,|\,\vec{u}\in N_L\}$. Therefore the finite dimension of N_L follows from Property 1. of the Fredholm alternative applied on the operator K. We now need to speak about the adjoint operator to K; to this aim, we define the operator (but this operator *may not* be the adjoint operator to K) $K^* := \gamma(L^*_\gamma)^{-1}$. Repeating the previous step we may show that $N(I-K^*) = \{(\vec\varphi, \vec\varphi|_{\Gamma_2\cup\Gamma_3}) \,|\, \vec\varphi \in N_{L^*}\}$. To finish the proof we need to verify that K^* is indeed the adjoint operator to K, i.e., we need to verify that

$$\big(K(\vec v,\vec w),(\vec\psi,\vec\theta)\big)_{\mathbb{L}^2} = \big((\vec v,\vec w),K^*(\vec\psi,\vec\theta)\big)_{\mathbb{L}^2} \quad \text{for any } (\vec v,\vec w),(\vec\psi,\vec\theta)\in\mathbb{L}^2. \tag{3.86}$$

Formula (3.86) is due to the definitions of both operators and their linearity equivalent to

$$\big(L_\gamma^{-1}(\vec v,\vec w),(\vec\psi,\vec\theta)\big)_{\mathbb{L}^2} = \big((\vec v,\vec w),(L^*_\gamma)^{-1}(\vec\psi,\vec\theta)\big)_{\mathbb{L}^2} \quad \text{for any } (\vec v,\vec w),(\vec\psi,\vec\theta)\in\mathbb{L}^2. \tag{3.87}$$

The definitions of the operators and Theorems 3.5.5 and 3.5.6 provide us $\vec u,\vec\varphi\in V$ satisfying $(\vec u,\vec u|_{\Gamma_2\cup\Gamma_3}) = L_\gamma^{-1}(\vec v,\vec w)$ and similarly $(\vec\varphi,\vec\varphi|_{\Gamma_2\cup\Gamma_3}) = (L^*_\gamma)^{-1}(\vec\psi,\vec\theta)$. Thus $\vec u$ a $\vec\varphi$ are unique solutions to

$$B^\gamma_{L,\sigma}(\vec u,\vec h) = \int_\Omega \vec v\cdot\vec h\,\mathrm{d}x + \int_{\Gamma_2\cup\Gamma_3}\vec w\cdot\vec h\,\mathrm{d}S \qquad \text{for any } \vec h\in V, \tag{3.88}$$

$$B^\gamma_{L^*,\sigma^*}(\vec\varphi,\vec z) = \int_\Omega \vec\psi\cdot\vec z\,\mathrm{d}x + \int_{\Gamma_2\cup\Gamma_3}\vec\theta\cdot\vec z\,\mathrm{d}S \qquad \text{for any } \vec z\in V. \tag{3.89}$$

Identity (3.87) can thus be rewritten to the form

$$\int_\Omega \vec u\cdot\vec\psi\,\mathrm{d}x + \int_{\Gamma_2\cup\Gamma_3}\vec u\cdot\vec\theta\,\mathrm{d}S = \int_\Omega \vec v\cdot\vec\varphi\,\mathrm{d}x + \int_{\Gamma_2\cup\Gamma_3}\vec w\cdot\vec\varphi\,\mathrm{d}S. \tag{3.90}$$

Choosing $\vec h := \vec\varphi\in V$ in equation (3.88) and $\vec z := \vec u\in V$ in equation (3.89) we see that (3.90) is the same as

$$B^\gamma_{L^*,\sigma^*}(\vec\varphi,\vec u) = B^\gamma_{L,\sigma}(\vec u,\vec\varphi).$$

By virtue of definition (3.78) it is not difficult to see that this equality is equivalent to

$$B_{L^*,\sigma^*}(\vec\varphi,\vec u) = B_{L,\sigma}(\vec u,\vec\varphi).$$

This equality, however, holds always since the problems are adjoint, see (3.72). Thus K^* is indeed the adjoint operator to K and the proof of Claim 2. is finished.

Let us show Claim 3. Due to (3.81) we know that $\vec u\in V$, solution to problem (3.67), exists, if and only if $\vec u\in V$ satisfies

$$(\vec u,\vec u|_{\Gamma_2\cup\Gamma_3}) - K(\vec u,\vec u|_{\Gamma_2\cup\Gamma_3}) = \frac{1}{\gamma}K(\vec f,\vec g). \tag{3.91}$$

Exactly as in (3.81) we may show that $\vec\varphi\in V$ solves (3.69) with $(\vec f,\vec g) := (\vec 0,\vec 0)$ (and thus $\vec\varphi\in N_{L^*}$) is equivalent to

$$(\vec\varphi,\vec\varphi|_{\Gamma_2\cup\Gamma_3}) - K^*(\vec\varphi,\vec\varphi|_{\Gamma_2\cup\Gamma_3}) = \frac{1}{\gamma}K^*(\vec 0,\vec 0) = (\vec 0,\vec 0). \tag{3.92}$$

Due to Property 3. of the Fredholm alternative (Theorem 3.5.7) a solution to (3.91) exists, if and only if it holds for any $\vec\varphi\in N_{L^*}$

$$\begin{aligned}
0 &= \Big(\frac{1}{\gamma}K(\vec f,\vec g),(\vec\varphi,\vec\varphi|_{\Gamma_2\cup\Gamma_3})\Big)_{\mathbb{L}^2} = \Big(K(\vec f,\vec g),(\vec\varphi,\vec\varphi|_{\Gamma_2\cup\Gamma_3})\Big)_{\mathbb{L}^2}\\
&= \Big((\vec f,\vec g),K^*(\vec\varphi,\vec\varphi|_{\Gamma_2\cup\Gamma_3})\Big)_{\mathbb{L}^2} = \Big((\vec f,\vec g),(\vec\varphi,\vec\varphi|_{\Gamma_2\cup\Gamma_3})\Big)_{\mathbb{L}^2}\\
&= \int_\Omega \vec f\cdot\vec\varphi\,\mathrm{d}x + \int_{\Gamma_2\cup\Gamma_3}\vec g\cdot\vec\varphi\,\mathrm{d}S,
\end{aligned} \tag{3.93}$$

where we subsequently used the fact that K^* is the adjoint operator to K as well as identity (3.92). The proof of Claim 3. is finished. ∎

The Fredholm alternative characterizes equivalently solvability of problem (3.67). We now show that due to the Fredholm alternative we may characterize the solvability of elliptic problems even more precisely. We show that it is closely connected with the spectrum of a certain operator. Let us now consider the following generalization of problem (3.67)

$$\begin{aligned}
L\vec u &= \lambda\vec u + \vec f \quad \text{in } \Omega,\\
\vec u &= \vec 0 \quad \text{on } \Gamma_1,\\
(\vec{\mathbb{A}}\nabla\vec u - \vec{\mathfrak{d}}\vec u)\vec\nu &= \lambda\vec u + \vec g \quad \text{on } \Gamma_2,\\
(\vec{\mathbb{A}}\nabla\vec u - \vec{\mathfrak{d}}\vec u)\vec\nu + \sigma\vec u &= \lambda\vec u + \vec g \quad \text{on } \Gamma_3.
\end{aligned} \tag{3.94}$$

The solvability of this problem is connected with λ. We have already shown (see Theorems 3.5.5 and 3.5.6), that if $\lambda < 0$ is sufficiently small, then the above stated problem has always the unique solution. We shall now study this problem in more details. Let us first present the following natural definition.

Definition 3.5.9 — Spectrum of the elliptic operator. Let $\Omega \subset \mathbb{R}^d$ be Lispchitz and $\{\Gamma_i\}_{i=1}^3$ be the corresponding parts of the boundary. Let L be the elliptic operator as above and $\sigma^{\alpha\beta} \in L^\infty(\Gamma_3)$ for $\alpha,\ \beta = 1, 2, \dots, N$. We say that $\lambda \in \mathbb{R}$ belongs to the real spectrum of the operator L, if there exists a nontrivial solution to problem (3.94) with $(\vec{f}, \vec{g}) = (\vec{0}, \vec{0})$. The set of such λ, called *the real spectrum*, is denoted by Σ.

It is important to realize that the definition of the spectrum is connected not only to the operator L, but takes into account also the boundary conditions. Therefore we have in general different spectrum for, e.g., homogeneous Dirichlet and homogeneous Neumann boundary conditions.

We finally show, how is the spectrum connected with the solvability of the original problem and we also show that the spectrum is at most countable.

Theorem 3.5.10 — On properties of the spectrum. Let $\Omega \subset \mathbb{R}^d$ be Lipschitz and $\{\Gamma_i\}_{i=1}^3$ be the corresponding parts of the boundary. Let L be an elliptic operator and $\{\sigma^{\alpha\beta}\}_{\alpha,\beta=1}^N \in L^\infty(\Gamma_3)$. Let Σ be the real spectrum of the operator L. Then it holds.

1. Σ is at most countable. Moreover, if it is infinite, $\Sigma = \{\lambda_k\}_{k=1}^\infty$, where $\lambda_k \to \infty$.
2. The following two claims are equivalent:
   - $\lambda \notin \Sigma$
   - for any $(\vec{f}, \vec{g}) \in \mathbb{L}^2$ there exists a unique weak solution to (3.94).
3. For any $\lambda \notin \Sigma$ there exists $C > 0$ such that for any $(\vec{f}, \vec{g}) \in \mathbb{L}^2$ and the corresponding unique weak solutions $\vec{u} \in V$ to problem (3.94) it holds
$$\|\vec{u}\|_V \le C\|(\vec{f}, \vec{g})\|_{\mathbb{L}^2}. \tag{3.95}$$

Proof. Let us start with the proof of Claim 1. Due to Theorem 3.5.5 we already know that it holds $\Sigma \cap (-\infty, -\gamma_0] = \emptyset$. Moreover, if $\lambda \le -\gamma_0$, all claims of the theorem hold. We may therefore assume $\lambda > -\gamma_0$. Using (3.81) with $\gamma > \gamma_0$ we may again show that $\vec{u} \in V$ solves problem (3.94), if and only if

$$\begin{aligned} (\vec{u}, \vec{u}|_{\Gamma_2\cup\Gamma_3}) - K(\vec{u}, \vec{u}|_{\Gamma_2\cup\Gamma_3}) &= \frac{1}{\gamma} K(\vec{f} + \lambda\vec{u}, \vec{g} + \lambda\vec{u}|_{\Gamma_2\cup\Gamma_3}) \\ \iff \gamma(\vec{u}, \vec{u}|_{\Gamma_2\cup\Gamma_3}) - (\gamma+\lambda)K(\vec{u}, \vec{u}|_{\Gamma_2\cup\Gamma_3}) &= K(\vec{f}, \vec{g}). \end{aligned} \tag{3.96}$$

Let $(\vec{f}, \vec{g}) = (\vec{0}, \vec{0})$ and $\vec{u} \in V$ be a nontrivial solution to (3.96). Then (recall that $\gamma + \lambda \ge \gamma - \gamma_0 > 0$)

$$0 < \frac{\gamma}{\gamma + \lambda} =: \mu$$

is the eigenvalue of operator K. Due to Claim 6. of Theorem 3.5.7 the spectrum of K is at most countable. Thus the set of such λ in Σ must be at most countable. Moreover, it is is infinite, i.e. $\Sigma = \{\lambda_k\}_{k=1}^\infty$, then there must exist eigenvalues of operator K such that

$$0 < \frac{\gamma}{\gamma + \lambda_k} = \mu_k \implies \lambda_k = \frac{\gamma(1-\mu_k)}{\mu_k}.$$

Since the only accumulation point of the spectrum of K is 0, we see that $\lambda_k \to \infty$ and the proof of Claim 1. is finished. Claim 2. follows directly from Claim 1. of Theorem 3.5.8.

We show Claim 3. by contradiction. Let $\lambda \notin \Sigma$ be fixed. Assume (3.95) does not hold. Then there exist $\{(\vec{f}_k, \vec{g}_k)\}_{k=1}^\infty \subset \mathbb{L}^2$ and the corresponding $\{\vec{u}_k\}_{k=1}^\infty \subset V$ (due to Claim 2. solutions $\vec{u}_k$ exist and are unique) such that $\|\vec{u}_k\|_V > k(\|\vec{f}_k\|_{L^2(\Omega)} + \|\vec{g}_k\|_{L^2(\Gamma_2\cup\Gamma_3)})$ for any $k \in \mathbb{N}$. Moreover, due to the linearity we may assume without lost of generality that $\|\vec{u}_k\|_V = 1$; we rescale $\{(\vec{f}_k, \vec{g}_k)\}$, if necessary. These solutions satisfy for any $\vec{\varphi} \in V$

$$B_{L,\sigma}(\vec{u}_k, \vec{\varphi}) = \int_\Omega (\vec{f}_k + \lambda\vec{u}_k) \cdot \vec{\varphi} \,\mathrm{d}x + \int_{\Gamma_2\cup\Gamma_3} (\vec{g}_k + \lambda\vec{u}_k) \cdot \vec{\varphi} \,\mathrm{d}S. \tag{3.97}$$

Since the space V is reflexive and $\|\vec{u}_k\|_V = 1$, we may choose a subsequence (we may relabel it) such that

$$\vec{u}_k \rightharpoonup \vec{u} \qquad \text{weakly in } V. \tag{3.98}$$

Theorem on compact embedding, compactness of the trace operator (see Theorem 2.4.17 and Theorem 2.5.14) and the definition of the sequence $\{(\vec{f}_k, \vec{g}_k)\}$ imply that there exists a subsequence (we again relabel it) such that for any $\alpha \in \{1, \dots, N\}$

$$u_k^\alpha \to u^\alpha \qquad \text{strongly in } L^2(\Omega), \tag{3.99}$$
$$u_k^\alpha \to u^\alpha \qquad \text{strongly in } L^2(\Gamma_2 \cup \Gamma_3), \tag{3.100}$$
$$f_k^\alpha \to 0 \qquad \text{strongly in } L^2(\Omega), \tag{3.101}$$
$$g_k^\alpha \to 0 \qquad \text{strongly in } L^2(\Gamma_2 \cup \Gamma_3). \tag{3.102}$$

We now pass to the limit $k \to \infty$ in (3.97). It is evident from (3.99)–(3.102) that

$$\begin{aligned}\lim_{k\to\infty}\left(\int_\Omega (\vec{f}_k + \lambda \vec{u}_k)\cdot\vec{\varphi}\,\mathrm{d}x + \int_{\Gamma_2\cup\Gamma_3}(\vec{g}_k + \lambda\vec{u}_k)\cdot\vec{\varphi}\,\mathrm{d}S\right)\\ = \int_\Omega \lambda\vec{u}\cdot\vec{\varphi}\,\mathrm{d}x + \int_{\Gamma_2\cup\Gamma_3}\lambda\vec{u}\cdot\vec{\varphi}\,\mathrm{d}S.\end{aligned} \tag{3.103}$$

Let us concentrate now on the left-hand side of (3.97). Relation (3.98) implies that it holds for any $w \in L^2(\Omega)$, $\beta \in \{1,\dots,N\}$ and $j \in \{1,\dots,d\}$

$$\lim_{k\to\infty}\int_\Omega \frac{\partial u_k^\beta}{\partial x_j} w\,\mathrm{d}x = \int_\Omega \frac{\partial u^\beta}{\partial x_j} w\,\mathrm{d}x. \tag{3.104}$$

Choosing $w := a_{ij}^{\alpha\beta}\frac{\partial v^\alpha}{\partial x_i}$ and $w := c_j^{\alpha\beta}\varphi^\alpha$ this identity yields

$$\lim_{k\to\infty}\int_\Omega \sum_{i,j=1}^d a_{ij}^{\alpha\beta}\frac{\partial u_k^\beta}{\partial x_j}\frac{\partial\varphi^\alpha}{\partial x_i}\,\mathrm{d}x = \int_\Omega \sum_{i,j=1}^d a_{ij}^{\alpha\beta}\frac{\partial u^\beta}{\partial x_j}\frac{\partial\varphi^\alpha}{\partial x_i}\,\mathrm{d}x,$$

$$\lim_{k\to\infty}\int_\Omega c_i^{\alpha\beta}\frac{\partial u_k^\beta}{\partial x_i}\varphi^\alpha\,\mathrm{d}x = \int_\Omega c_i^{\alpha\beta}\frac{\partial u^\beta}{\partial x_i}\varphi^\alpha\,\mathrm{d}x.$$

Using these convergences and (3.99)–(3.100) we obtain

$$\lim_{k\to\infty} B_{L,\sigma}(\vec{u}_k,\vec{\varphi}) = B_{L,\sigma}(\vec{u},\vec{\varphi}).$$

We received, altogether with (3.103) that for any $\vec{\varphi}\in V$ it holds

$$B_{L,\sigma}(\vec{u},\vec{\varphi}) = \int_\Omega \lambda\vec{u}\cdot\vec{\varphi}\,\mathrm{d}x + \int_{\Gamma_2\cup\Gamma_3}\lambda\vec{u}\cdot\vec{\varphi}\,\mathrm{d}S. \tag{3.105}$$

It means that $\vec{u}\in V$ is a weak solution to problem (3.94) with $(\vec{f},\vec{g}) = (\vec{0},\vec{0})$. Since $\lambda\notin\Sigma$, it must be $\vec{u} = \vec{0}$. If we show that $\|\vec{u}\|_V = 1$, we obtain our desired contradiction. However, as $\|\vec{u}_k\|_V = 1$ for any $k\in\mathbb{N}$, it is enough to show that $\vec{u}_k \to \vec{u}$ in V. From (3.99) we already know that $\vec{u}_k\to\vec{u}$ in $L^2(\Omega)$; hence to conclude the proof, it is enough to verify that $\|\nabla\vec{u}_k - \nabla\vec{u}\|_2 \to 0$. Using the ellipticity (i.e., (3.26)), definition of $B_{L,\sigma}$ and identities (3.97) and (3.105) we easily deduce the estimate

$$\begin{aligned} C_1\|\nabla\vec{u}_k - \nabla\vec{u}\|_2^2 &\le \int_\Omega \vec{\mathbb{A}}\nabla(\vec{u}_k-\vec{u})\cdot\nabla(\vec{u}_k-\vec{u})\,\mathrm{d}x\\ &= B_{L,\sigma}(\vec{u}_k-\vec{u},\vec{u}_k-\vec{u}) - \int_{\Gamma_3}\sigma(\vec{u}_k-\vec{u})\cdot(\vec{u}_k-\vec{u})\,\mathrm{d}S\\ &\quad - \int_\Omega\left(\vec{\mathfrak{c}}\cdot\nabla(\vec{u}_k-\vec{u})(\vec{u}_k-\vec{u}) - \vec{\mathfrak{d}}\cdot\nabla(\vec{u}_k-\vec{u})(\vec{u}_k-\vec{u})\right)\mathrm{d}x\\ &= \int_\Omega(\vec{f}_k + \lambda(\vec{u}_k-\vec{u}))\cdot(\vec{u}_k-\vec{u})\,\mathrm{d}x\\ &\quad + \int_{\Gamma_2\cup\Gamma_3}(\vec{g}_k + \lambda(\vec{u}_k-\vec{u}))\cdot(\vec{u}_k-\vec{u})\,\mathrm{d}S\\ &\quad - \int_{\Gamma_3}\sigma(\vec{u}_k-\vec{u})\cdot(\vec{u}_k-\vec{u})\,\mathrm{d}S\\ &\quad - \int_\Omega\left(\vec{\mathfrak{c}}\cdot\nabla(\vec{u}_k-\vec{u})(\vec{u}_k-\vec{u}) - \vec{\mathfrak{d}}\cdot\nabla(\vec{u}_k-\vec{u})(\vec{u}_k-\vec{u})\right)\mathrm{d}x. \end{aligned}$$

By virtue of Hölder's inequality and strong convergences (3.99) and (3.100) we may pass with $k\to\infty$ to show

$$\lim_{k\to\infty}\|\nabla\vec{u}_k - \nabla\vec{u}\|_2^2 = 0.$$

Thus, it holds due to (3.99) and assumptions on the sequence $\{\vec{u}_k\}$ that

$$1 = \lim_{k\to\infty}\|\vec{u}_k\|_V = \|\vec{u}\|_V$$

which leads to the contradiction. ■

3.6 Maximum principle for weak solutions

This section deals only with scalar equations, the situation for systems is quite different. If we know that solution to our problem is sufficiently regular and thus classical, we may use the maximum principle for classical solutions. We shall discuss this issue in the following sections and we shall see that this is not always true. The aim of this section is thus to show that the maximum principle holds also for weak solutions only.

We first recall Corollary 2.2.5 which says that if $u \in W^{1,p}(\Omega)$, then also $(u-M)^+ \in W^{1,p}(\Omega)$ for any $M \in \mathbb{R}$.

Let us consider the problem (in the weak setting)

$$\begin{aligned} -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) &= 0 \qquad \text{in } \Omega \\ u &= u_0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{3.106}$$

We know that under corresponding assumptions there exists unique weak solution $u \in W^{1,2}(\Omega)$ of this problem. More precisely,

$$\begin{aligned} &u - U_0 \in W_0^{1,2}(\Omega) \\ &\sum_{i,j=1}^{d} a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial \varphi}{\partial x_i}\,\mathrm{d}x = 0 \qquad \text{for any } \varphi \in W_0^{1,2}(\Omega). \end{aligned} \tag{3.107}$$

Let us now formulate the main result of this section.

Theorem 3.6.1 — Weak maximum principle for elliptic equations. Let u satisfy (3.107). Then

$$\|u\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\partial\Omega)}\,. \tag{3.108}$$

Proof. If $\|u_0\|_{L^\infty(\partial\Omega)} = \infty$, the claim of the theorem holds trivially true. Let us assume that the norm is finite and denote $M := \|u_0\|_{L^\infty(\partial\Omega)}$. We aim to show that

$$-M \le u(x) \le M \quad \text{almost everywhere in } \Omega. \tag{3.109}$$

Since $(u-M)^+$ is an element of $W^{1,2}(\Omega)$ and the trace of this function is zero, we see that $(u-M)^+ \in W_0^{1,2}(\Omega)$. This function is therefore an eligible test function in $(3.107)_2$. By virtue of $(3.107)_2$ and Corollary 2.2.5 we get

$$\begin{aligned} 0 = \int_\Omega a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial (u-M)^+}{\partial x_i}\,\mathrm{d}x &= \int_\Omega a_{ij}\frac{\partial (u-M)^+}{\partial x_j}\frac{\partial (u-M)^+}{\partial x_i}\,\mathrm{d}x \\ &\ge C_1 \left\|\nabla (u-M)^+\right\|_2^2 \ge \alpha \left\|(u-M)^+\right\|_{1,2}^2, \end{aligned}$$

where we used the ellipticity of the operator and the equivalence of norms on $W_0^{1,2}(\Omega)$. Therefore it holds

$$(u-M)^+ = 0 \quad \text{almost everywhere in } \Omega; \quad \text{or } u \le M \quad \text{almost everywhere in } \Omega.$$

Since $-u$ solves the same equation with the boundary condition $-u_0$, repeating the proof above implies

$$(-u-M)^+ = 0 \quad \text{almost everywhere in } \Omega; \quad \text{or } u \ge -M \quad \text{almost everywhere in } \Omega.$$

Inequalities in (3.109) are proved. ■

Remark 3.6.2. The same results also holds under suitable assumptions for more general elliptic problems. Assuming $b \in L^\infty(\Omega)$, $b \ge 0$, the weak maximum principle as in the theorem above also holds for the unique solution to problem

$$u - U_0 \in W_0^{1,2}(\Omega), \tag{3.110}$$

$$\int_\Omega \Big(\sum_{i,j=1}^{d} a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial \varphi}{\partial x_i} + bu\varphi\Big)\,\mathrm{d}x = 0 \quad \text{for any } \varphi \in W_0^{1,2}(\Omega). \tag{3.111}$$

3.7 Regularity of weak solutions

In what follows, we restrict ourselves on the scalar equation

$$\begin{aligned} Lu &= f \qquad \text{in } \Omega \\ u &= u_0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{3.112}$$

where

$$Lu = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big) + bu.$$

Assume that $u_0 \in W^{\frac{1}{2},2}(\partial\Omega)$ (the range of the trace operator in $W^{1,2}(\Omega)$; we rather work with $U_0 \in W^{1,2}(\Omega)$ such that $U_0 = u_0$ on $\partial\Omega$ in the sense of traces) and $f \in (W_0^{1,2}(\Omega))^*$, $\mathbb{A}$ satisfies the ellipticity condition

$$\begin{aligned} \sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq C_1|\boldsymbol{\xi}|^2 \qquad & \text{for all } \boldsymbol{\xi} \in \mathbb{R}^d \text{ almost everywhere in } \Omega \\ b(x) \geq 0 \qquad & \text{almost everywhere in } \Omega \end{aligned} \tag{3.113}$$

and for $i,j = 1,2,\ldots,d$ we have that $a_{ij} \in L^\infty(\Omega)$ and $b \in L^\infty(\Omega)$. Then the Lax–Milgram Theorem 3.4.1 yields existence of a unique weak solution to the following problem.

We look for $u \in W^{1,2}(\Omega)$ such that for a given $U_0 \in W^{1,2}$ we have $u - U_0 \in W_0^{1,2}(\Omega)$ and

$$\int_\Omega \Big(\sum_{i,j=1}^{d} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \varphi}{\partial x_i} + bu\varphi \Big) \,\mathrm{d}x = \langle f, \varphi \rangle_{W_0^{1,2}(\Omega)}$$

for all $\varphi \in W_0^{1,2}(\Omega)$.

The regularity theory studies the following problem: Let the data of the problem are smoother (better) than it is required by the existence theorem. In addition to (3.113) we assume

$$\forall i,j \in \{1,2,\ldots,d\}\colon\ a_{ij} \in W^{1,\infty}(\Omega),\ b \in L^\infty(\Omega),\ f \in L^2(\Omega),\ U_0 \in W^{2,2}(\Omega),\ \Omega \in \mathcal{C}^{1,1}. \tag{3.114}$$

Can we show that the unique weak solution is more regular, i.e., $u \in W^{2,2}(\Omega)$?

Or even more generally, for some $k \geq 2$, we assume

$$\forall i,j \in \{1,2,\ldots,d\}\colon\ a_{ij} \in W^{k-1,\infty}(\Omega),\ b \in W^{k-2,\infty}(\Omega), f \in W^{k-2,2}(\Omega),\ U_0 \in W^{k,2}(\Omega),\ \Omega \in \mathcal{C}^{k-1,1}. \tag{3.115}$$

Does it hold that $u \in W^{k,2}(\Omega)$?

In particular, assuming

$$\forall i,j \in \{1,2,\ldots,d\}\colon\ a_{ij} \in \mathcal{C}^\infty(\overline{\Omega}),\ b \in \mathcal{C}^\infty(\overline{\Omega}),\ f \in C^\infty(\overline{\Omega}),\ U_0 \in \mathcal{C}^\infty(\overline{\Omega}),\ \Omega \in \mathcal{C}^\infty. \tag{3.116}$$

Does it imply that $u \in \mathcal{C}^\infty(\overline{\Omega})$?

We first explain in detail the case $k = 2$ (i.e., extra conditions (3.114)), then in less detail the general case $k \geq 2$ (i.e., condition (3.115)). The case (3.116) is then a direct consequence of embedding of Sobolev spaces, provided we can show the previous result for arbitrary $k \in \mathbb{N}$, $k \geq 2$.

The proof is based on the following procedure. We first consider the problem in the full $\mathbb{R}^d$. This subsequently implies, using a suitable localization, the regularity of the solution in arbitrary $\Omega' \subset \overline{\Omega'} \subset \Omega$ ($\Omega' \subset\subset \Omega$). This result is independent of the regularity of U_0 as well as of Ω.

Next we deal with the question of regularity near the boundary. Here, the regularity of U_0 as well as of Ω come into play. The proof is based on the local description of the boundary together with the corresponding partition of unity which reduces the regularity question to a special situation with our solution having a very particular support located in $V_r^+ \cup \Lambda_r$. We flatten the boundary (here, the regularity of Ω comes into play) and show first the regularity in the tangential direction and then, using the weak formulation, also in the normal direction.

Finally, we collect all the results together and obtain the regularity in Ω, up to the boundary. Let us therefore consider problem (3.112) with (3.113) and (3.114) (we aim to show that $u \in W^{2,2}(\Omega)$).

However, before doing so, let us consider the problem in the full space. Here the situation is slightly more complex since the unbounded domain causes small troubles.

We consider

$$-\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial u}{\partial x_j} \right) + bu = f \quad \text{in } \mathbb{R}^d. \tag{3.117}$$

We have the following result.

Lemma 3.7.1 — Existence of weak solutions in the full space. Let $f \in L^2(\mathbb{R}^d)$, $a_{ij} \in L^\infty(\mathbb{R}^d)$ for any $i,j = 1,2,\ldots,d$, $b \in L^\infty(\mathbb{R}^d)$, $b \geq 0$ almost everywhere in $\mathbb{R}^d$, $\sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j \geq C_1|\boldsymbol{\xi}|^2$ for almost every $x \in \mathbb{R}^d$ and all $\boldsymbol{\xi} \in \mathbb{R}^d$, $C_1 > 0$. Let either $b(x) \geq b_0 > 0$ almost everywhere in $\mathbb{R}^d$ or let f and b have bounded support.

Then there exists a weak solution to the problem

$$\int_{\mathbb{R}^d} \sum_{i,j=1}^{d} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \varphi}{\partial x_i} \,\mathrm{d}x + \int_{\mathbb{R}^d} bu\varphi \,\mathrm{d}x = \int_{\mathbb{R}^d} f\varphi \,\mathrm{d}x \tag{3.118}$$

in the sense specified below. Moreover, the solution is unique provided $d \geq 3$ or if $b(x) \geq b_0 > 0$ almost everywhere in a nontrivial subset of $\mathbb{R}^d$.

Remark 3.7.2. In fact, if $b = 0$, we may always formally add a constant and get the same solution. However, we fix the constant by requiring that for $d \geq 3$ the solution is globally integrable with the power corresponding to the Gagliardo–Nirenberg inequality. For $d = 2$ the solution may logarithmically grow at infinity and we have no such power available.

Proof of Lemma 3.7.1. The main point is to choose correctly the weak formulation. If $b(x) \geq b_0 > 0$ almost everywhere in $\mathbb{R}^d$, we look for $u \in W^{1,2}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} \Big(\sum_{i,j=1}^d a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \varphi}{\partial x_i} + bu\varphi \Big) \,\mathrm{d}x = \int_{\mathbb{R}^d} f\varphi \,\mathrm{d}x \tag{3.119}$$

for all $\varphi \in W^{1,2}(\mathbb{R}^d)$. The existence and uniqueness of the weak solution is a direct consequence of the Lax–Milgram Theorem 3.4.1.

Assume now the other situation, i.e., $f \in L^2(\mathbb{R}^d)$ with a bounded support and $b \geq 0$ almost everywhere in $\mathbb{R}^d$, $b \in L^\infty(\Omega)$ with a bounded support. Let $d \geq 3$. Then we set

$$V = \big\{ u \in L^{\frac{2d}{d-2}}(\mathbb{R}^d) \,|\, \nabla u \in L^2(\mathbb{R}^d;\mathbb{R}^d) \big\}$$

with

$$\|u\|_V = \|u\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} + \|\nabla u\|_{L^2(\mathbb{R}^d;\mathbb{R}^d)}.$$

We look for $u \in V$ such that (3.118) holds for any $\varphi \in V$. As

$$\Big| \int_{\mathbb{R}^d} f\varphi \,\mathrm{d}x \Big| \leq \Big| \int_{B_R} f\varphi \,\mathrm{d}x \Big| \leq \|f\|_{L^2(B_R)} \|\varphi\|_{L^2(B_R)} \leq C(R) \|f\|_{L^2(\mathbb{R}^d)} \|\varphi\|_V,$$

similarly for the integral $\int_{\mathbb{R}^d} bu\varphi \,\mathrm{d}x$ and

$$\int_{\mathbb{R}^d} \Big(\sum_{i,j=1}^d a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_i} \,\mathrm{d}x + bu^2 \Big) \,\mathrm{d}x \geq C_1 \|\nabla u\|^2_{L^2(\mathbb{R}^d;\mathbb{R}^d)} \geq C \|u\|^2_V,$$

we can again use the Lax–Milgram Theorem 3.4.1 to prove the existence and uniqueness of a solution to our problem.

Assume now $d = 2$, $b(x) \geq b_0 > 0$ in Ω^*, a nontrivial subset of $\mathbb{R}^d$ (i.e., the two-dimensional measure of Ω^* is larger than zero). Then we define

$$V = \big\{ u \in L^2(\Omega^*) \,|\, \nabla u \in L^2(\mathbb{R}^d;\mathbb{R}^d) \big\}$$

with

$$\|u\|_V = \|u\|_{L^2(\Omega^*)} + \|\nabla u\|_{L^2(\mathbb{R}^d;\mathbb{R}^d)}.$$

Since the support of f is bounded, we get

$$\begin{aligned} \Big| \int_{\mathbb{R}^d} f\varphi \,\mathrm{d}x \Big| &\leq \|f\|_{L^2(B_R)} \|\varphi\|_{L^2(B_R)} \leq \|f\|_{L^2(B_R)} \|\varphi\|_{W^{1,2}(B_R)} \\ &\leq C \|f\|_{L^2(B_R)} (\|\varphi\|_{L^2(\Omega^*)} + \|\nabla\varphi\|_{L^2(B_R;\mathbb{R}^d)}) \leq C \|f\|_{L^2(B_R)} \|\varphi\|_V, \end{aligned}$$

similarly for the integral $\int_{\mathbb{R}^d} bu\varphi \,\mathrm{d}x$ and

$$\int_{\mathbb{R}^d} \Big(\sum_{i,j=1}^d a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_i} \,\mathrm{d}x + bu^2 \Big) \,\mathrm{d}x \geq C_1 \|\nabla u\|^2_{L^2(\mathbb{R}^d;\mathbb{R}^d)} + b_0 \|u\|^2_{L^2(\Omega^*)} \geq C \|u\|^2_V.$$

Finally, if $d = 2$ and $b = 0$ almost everywhere in $\mathbb{R}^d$, then the solution is clearly non-unique. However, we may fix the additive constant by choosing

$$V = \Big\{ u \in L^1_{\mathrm{loc}}(\mathbb{R}^d) \,|\, \nabla u \in L^2(\mathbb{R}^d;\mathbb{R}^d), \int_{B_R(0)} u \,\mathrm{d}x = 0 \Big\},$$

where $\operatorname{supp} f \subset \overline{B_R(0)}$. Then $\|u\|_V = \|\nabla u\|_{L^2(\mathbb{R}^d)}$ and we get

$$\Big| \int_{\mathbb{R}^d} f\varphi \,\mathrm{d}x \Big| \leq \|f\|_{L^2(B_R)} \|\varphi\|_{L^2(B_R)} \leq C \|f\|_{L^2(\mathbb{R}^d)} \|\nabla\varphi\|_{L^2(\mathbb{R}^d;\mathbb{R}^d)}$$

for any $\varphi \in V$. Then we can also use the Lax–Milgram Theorem 3.4.1 to complete the proof. Clearly, adding any constant to u and changing the space V in this direction we get that the solution is non-unique. ■

Let us now show that under the assumptions

$$\forall i,j = 1,2,\dots,d:\ a_{ij} \in W^{1,\infty}(\mathbb{R}^d),\ b \in L^\infty(\mathbb{R}^d) \quad \text{and } f \in L^2(\mathbb{R}^d) \tag{3.120}$$

together with the assumptions of the previous lemma we get that the unique weak solution to the corresponding weak formulation satisfies that the solution is more regular, i.e., $\nabla u \in W^{1,2}(\mathbb{R}^d;\mathbb{R}^d)$. Recall that the solution from Lemma 3.7.1 fulfils
(i) $b(x) \geq b_0 > 0$ almost everywhere in $\mathbb{R}^d$

$$\|u\|_{W^{1,2}(\mathbb{R}^d)} \leq C\|f\|_{L^2(\mathbb{R}^d)}$$

(ii) f and b have bounded support, $d \geq 3$

$$\|\nabla u\|_{L^2(\mathbb{R}^d;\mathbb{R}^d)} + \|u\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} \leq C\|f\|_{L^2(\mathbb{R}^d)}$$

(iii) f and b have bounded support, $d = 2$, b is nontrivial

$$\|\nabla u\|_{L^2(\mathbb{R}^d;\mathbb{R}^d)} + \|u\|_{L^2(B_R)} \leq C\|f\|_{L^2(\mathbb{R}^d)},$$

where we assume that $\operatorname{supp} f$ and $\operatorname{supp} b$ lie in $\overline{B_R(0)}$
. (iv) f has bounded support, $d = 2$, $b = 0$ in $\mathbb{R}^d$

$$\|\nabla u\|_{L^2(\mathbb{R}^d;\mathbb{R}^d)} \leq C\|f\|_{L^2(\mathbb{R}^d)}.$$

Theorem 3.7.3 — Regularity in the full space. Let assumptions of Lemma 3.7.1 be satisfied. Let moreover (3.120) hold for $\{a_{ij}\}_{i,j=1}^d$, b and f. Then the uniquely defined weak solution $u \in W^{1,2}_{\text{loc}}(\mathbb{R}^d)$ (or $W^{1,2}(\mathbb{R}^d)$, respectively) to problem (3.117) is such that $\nabla u \in W^{1,2}(\mathbb{R}^d;\mathbb{R}^d)$ and it holds

$$\|\nabla u\|_{W^{1,2}(\mathbb{R}^d;\mathbb{R}^d)} \leq C\,\|f\|_{L^2(\mathbb{R}^d)}\,. \tag{3.121}$$

Moreover, if $b(x) \geq b_0 > 0$ almost everywhere in $\mathbb{R}^d$, then also

$$\|u\|_{W^{2,2}(\mathbb{R}^d)} \leq C\,\|f\|_{L^2(\mathbb{R}^d)}\,, \tag{3.122}$$

otherwise only

$$\|u\|_{L^2(B_R(0))} \leq C\,\|f\|_2\,. \tag{3.123}$$

Proof: formal, incorrect, but fast and clear idea. We fix $l \in \{1,2,\dots,d\}$ and for a function Z we denote $Z' = \frac{\partial Z}{\partial x_l}$. We differentiate our equation

$$Lu = f \qquad \text{in } \mathbb{R}^d$$

with respect to x_l. It leads to equation

$$-\sum_{i,j=1}^d \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u'}{\partial x_j}\right) = \sum_{i,j=1}^d \frac{\partial}{\partial x_i}\left(a'_{ij}\frac{\partial u}{\partial x_j}\right) - (bu)' + f'.$$

We multiply this equation on u', integrate over $\mathbb{R}^d$ and perform integration by parts (apply the Green theorem). It yields

$$\int_{\mathbb{R}^d}\sum_{i,j=1}^d a_{ij}\frac{\partial u'}{\partial x_j}\frac{\partial u'}{\partial x_i}\,\mathrm{d}x = -\int_{\mathbb{R}^d}\sum_{i,j=1}^d a'_{ij}(x)\frac{\partial u}{\partial x_j}\frac{\partial u'}{\partial x_i} - \int_{\mathbb{R}^d}(bu)'u'\,\mathrm{d}x + \int_{\mathbb{R}^d} f'u'\,\mathrm{d}x.$$

We now integrate by part in the last two integrals on the right-hand side to get

$$\int_{\mathbb{R}^d}\sum_{i,j=1}^d a_{ij}\frac{\partial u'}{\partial x_j}\frac{\partial u'}{\partial x_i}\,\mathrm{d}x = -\int_{\mathbb{R}^d}\sum_{i,j=1}^d a'_{ij}(x)\frac{\partial u}{\partial x_j}\frac{\partial u'}{\partial x_i} + \int_{\mathbb{R}^d} buu''\,\mathrm{d}x - \int_{\mathbb{R}^d} fu''\,\mathrm{d}x.$$

The left-hand side is due to the ellipticity condition bounded from below by $C_1\,\|\nabla u'\|^2_{L^2(\mathbb{R}^d;\mathbb{R}^d)}$. We now estimate the right-hand side by means of the Hölder inequality

$$\begin{aligned}
C_1\|\nabla u'\|^2_{L^2(\mathbb{R}^d;\mathbb{R}^d)} &\leq \|f\|_{L^2(\mathbb{R}^d)}\,\|u''\|_{L^2(\mathbb{R}^d)} + \|b\|_{L^\infty(\mathbb{R}^d)}\|u\|_{L^2(\operatorname{supp} b)}\|u''\|_{L^2(\mathbb{R}^d)} \\
&\quad + \|\mathbb{A}'\|_{L^\infty(\mathbb{R}^d;\mathbb{R}^{d\times d})}\|\nabla u'\|_{L^2(\mathbb{R}^d;\mathbb{R}^d)}\|\nabla u\|_{L^2(\mathbb{R}^d;\mathbb{R}^d)} \\
&\leq \|f\|_{L^2(\mathbb{R}^d)}\|\nabla u'\|_{L^2(\mathbb{R}^d)} + C(\|b\|_{L^\infty(\mathbb{R}^d)}, \|\mathbb{A}'\|_{L^\infty(\mathbb{R}^d;\mathbb{R}^{d\times d})})\|f\|_{L^2(\mathbb{R}^d)}\|\nabla u'\|_{L^2(\mathbb{R}^d;\mathbb{R}^d)} \\
&\leq \frac{C_1}{2}\|\nabla u'\|^2_{L^2(\mathbb{R}^d;\mathbb{R}^d)} + C(\|b\|_{L^\infty(\mathbb{R}^d)}, \|\mathbb{A}'\|_{L^\infty(\mathbb{R}^d;\mathbb{R}^{d\times d})})\|f\|^2_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

Thus we get the desired bound

$$\|\nabla u'\|_{L^2(\mathbb{R}^d;\mathbb{R}^d)} \le C(C_1, \|b\|_{L^\infty(\mathbb{R}^d)}, \|\mathbb{A}'\|_{L^\infty(\mathbb{R}^d;\mathbb{R}^{d\times d})})\|f\|_{L^2(\mathbb{R}^d)}.$$

■

This proof is formal (incorrect), because it was based on the classical formulation of the problem. We, however, do not know whether the weak solution $u \in W^{1,2}(\mathbb{R}^d)$ satisfies the corresponding equation almost everywhere (or pointwise) in $\mathbb{R}^d$, the less we know that we can differentiate the equation. Moreover, as in intermediate step we also needed differentiability of b. The correct proof must be based on the weak formulation of our problem.

Proof: rigorous considerations. Theorem 2.3.1 claims that

$$u \in W^{1,2}(\mathbb{R}^d) \Longleftrightarrow u \in L^2(\mathbb{R}^d) \ \& \ \int_{\mathbb{R}^d} \left|\frac{u(x+h\mathbf{e}_k) - u(x)}{h}\right|^2 \mathrm{d}x \le C < \infty \quad \forall |h| \le h_0,\ k \in \{1,2,\dots,d\}.$$

We want to verify that $\nabla u \in W^{1,2}(\mathbb{R}^d;\mathbb{R}^d)$. It is enough to show (since we know that $\nabla u \in L^2(\mathbb{R}^d;\mathbb{R}^d)$)

$$\int_{\mathbb{R}^d} \frac{|\nabla u(x+h\mathbf{e}_k) - \nabla u(x)|^2}{|h|^2} \mathrm{d}x \le C\,\|f\|_2^2 \quad \forall |h| \le h_0,\ k = 1,2,\dots,d, \tag{3.124}$$

where the vector $\mathbf{e}_k = (0,\dots,1,0\dots,0)$ is the k-th vector of the canonical basis in $\mathbb{R}^d$.

Estimate (3.121) then follows from Theorem 2.3.1, Claim 2. Recall the notation $\Delta_k^h u(x) = \frac{u(x+h\mathbf{e}_k)-u(x)}{h}$, where k and h are fixed. Since $\nabla u \in L^2(\mathbb{R}^d;\mathbb{R}^d)$, then also $\Delta_k^h \nabla u \in L^2(\mathbb{R}^d;\mathbb{R}^d)$. Moreover, if $b(x) \ge b_0 > 0$ almost everywhere in $\mathbb{R}^d$, then even $\Delta_k^h u \in W^{1,2}(\mathbb{R}^d)$, if not, then at least $\Delta_k^h u \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ and we also have that $\operatorname{supp} f$ and $\operatorname{supp} b$ are bounded.

In order to obtain our estimate (3.124), we first use as a test function in the weak formulation $r_k^{-h}\varphi := \varphi(x - h\mathbf{e}_k)$. After a standard change of variable the weak formulation transforms into

$$\int_{\mathbb{R}^d} \sum_{i,j=1}^d a_{ij}(x+h\mathbf{e}_k)\frac{\partial u(x+h\mathbf{e}_k)}{\partial x_j}\frac{\partial \varphi(x)}{\partial x_i}\,\mathrm{d}x + \int_{\mathbb{R}^d} b(x+h\mathbf{e}_k)u(x+h\mathbf{e}_k)\varphi(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} f(x+h\mathbf{e}_k)\varphi(x)\,\mathrm{d}x. \tag{3.125}$$

Note that all integrals are finite. We now subtract the weak formulation with the function φ from (3.125) and we plug $\varphi := \Delta_k^h u$ into the resulted equality.

$$\begin{aligned}\int_{\mathbb{R}^d} \sum_{i,j=1}^d a_{ij}(x)\frac{\partial \Delta_k^h u(x)}{\partial x_j}\frac{\partial \Delta_k^h u(x)}{\partial x_i}\,\mathrm{d}x = &- \int_{\mathbb{R}^d} \frac{1}{h}\sum_{i,j=1}^d \big(a_{ij}(x+h\mathbf{e}_k) - a_{ij}(x)\big)\frac{\partial u(x+h\mathbf{e}_k)}{\partial x_j}\frac{\partial \Delta_k^h u(x)}{\partial x_i}\,\mathrm{d}x \\ &- \int_{\mathbb{R}^d} \frac{1}{h}\big(b(x+h\mathbf{e}_k)u(x+h\mathbf{e}_k) - b(x)u(x)\big)\Delta_k^h u(x)\,\mathrm{d}x \\ &+ \int_{\mathbb{R}^d} \frac{1}{h}\big(f(x+h\mathbf{e}_k) - f(x)\big)\Delta_k^h u(x)\,\mathrm{d}x.\end{aligned}$$

The last term is transformed by the change of variables to

$$\int_{\mathbb{R}^d} f(x)\Delta_k^{-h}(\Delta_k^h u(x))\,\mathrm{d}x = -\int_{\mathbb{R}^d} f(x)\frac{u(x+h\mathbf{e}_k) - 2u(x) + u(x-h\mathbf{e}_k)}{h^2}\,\mathrm{d}x.$$

Since $\Delta_k^h u$ is bounded in $L^2(B_R)$ independently of h (however, the bound may depend on R in case the function f does not have a bounded support) it follows from Theorem 2.3.1 that (we set $g = \Delta_k^h u$)

$$\left\|\Delta_k^{-h}(\Delta_k^h u)\right\|_2 = \left\|\Delta_k^{-h} g\right\|_2 \le c\,\|\nabla g\|_2 = c\left\|\nabla(\Delta_k^h u)\right\|_2.$$

The last term in (3.125) is therefore estimated by

$$C\,\|f\|_2\left\|\nabla(\Delta_k^h u)\right\|_2.$$

Similarly we get

$$\left|\int_{\mathbb{R}^d} \frac{1}{h}\big(b(x+h\mathbf{e}_k)u(x+h\mathbf{e}_k) - b(x)u(x)\big)\Delta_k^h u(x)\,\mathrm{d}x\right| \le C\|b\|_{L^\infty(\mathbb{R}^d)}\|u\|_{L^2(\operatorname{supp} b)}\|\nabla\Delta_k^h u\|_{L^2(\mathbb{R}^d)}.$$

Other terms in (3.125) are estimated similarly as in the formal proof. We therefore get

$$\begin{aligned}C_1\left\|\nabla(\Delta_k^h u)\right\|_2^2 &\le C\big(\|\Delta_k^h\mathbb{A}\|_\infty\|\nabla u\|_2\|\nabla(\Delta_k^h u)\|_2 + \|b\|_\infty\|u\|_2\|\nabla(\Delta_k^h u)\|_2 + \|f\|_2\|\nabla(\Delta_k^h u)\|_2\big) \\ &\le C(\|\mathbb{A}\|_{1,\infty},\|b\|_\infty)\big(\|f\|_2^2 + \|\nabla u\|_2^2 + \|u\|_{L^2(B_R(0))}\big) + \frac{C_1}{2}\|\nabla(\Delta_k^h u)\|_2^2.\end{aligned}$$

Above, $B_R(0)$ is chosen so that $\operatorname{supp} f$, $\operatorname{supp} g \subset B_R(0)$. Thus, inequality (3.124) is proved. ■

We now apply the result from the previous theorem to obtain the interior regularity result.

Theorem 3.7.4 — Interior regularity in bounded domains. Let Ω be a bounded domain (in particular, we have no regularity assumptions on Ω), let $\{a_{ij}\}_{i,j=1}^d$, b and f satisfy (3.113)–(3.114). Let $u \in W^{1,2}(\Omega)$, unique weak solution to Problem 3.112, i.e.,

$$\int_\Omega \sum_{i,j=1}^d a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \varphi}{\partial x_i} \, \mathrm{d}x + \int_\Omega b u \varphi \, \mathrm{d}x = \int_\Omega f \varphi \, \mathrm{d}x \quad \forall \varphi \in W_0^{1,2}(\Omega). \tag{3.126}$$

Then for any $\Omega' \subset \overline{\Omega'} \subset \Omega$ we have

$$\|u\|_{W^{2,2}(\Omega')} \leq C(\Omega') \, \|f\|_{L^2(\Omega)} \, .$$

Proof. Recall that we have $\|u\|_{W^{1,2}(\Omega)} \leq C\|f\|_{L^2(\Omega)}$ and it is enough to show the estimates of the second derivative. We transfer our problem to the situation of Theorem 3.7.3 as follows. We replace φ by $\varphi|\xi|^2$ in (3.126), where $\xi \in \mathcal{C}_0^\infty(\Omega)$, $\xi \equiv 1$ in Ω', $0 \leq \xi \leq 1$. We rewrite (3.126) for the quantity $w = u\xi$. Then we extend w by zero outside of Ω. Thus our equation can be understood as equation in $\mathbb{R}^d$. Even though b is not strictly positive in $\mathbb{R}^d$, we see that after extension by zero all functions have compact support in $\mathbb{R}^d$. We have

$$a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial}{\partial x_i} (\varphi|\xi|^2) = a_{ij} \frac{\partial (u\xi)}{\partial x_j} \frac{\partial (\varphi\xi)}{\partial x_i} + a_{ij} \frac{\partial u}{\partial x_j} \varphi\xi \frac{\partial \xi}{\partial x_i} - a_{ij} u \frac{\partial \xi}{\partial x_j} \frac{\partial (\varphi\xi)}{\partial x_i}.$$

Thus we get

$$\begin{aligned} \int_\Omega \Big(\sum_{i,j=1}^d a_{ij} \frac{\partial (u\xi)}{\partial x_j} \frac{\partial (\varphi\xi)}{\partial x_i} + b(u\xi)(\varphi\xi) \Big) \, \mathrm{d}x = &- \int_\Omega \Big(\sum_{i,j=1}^d a_{ij} \frac{\partial u}{\partial x_j} \varphi\xi \frac{\partial \xi}{\partial x_i} + \frac{\partial}{\partial x_i} \Big(a_{ij} u \frac{\partial \xi}{\partial x_j} \Big) \varphi\xi \Big) \, \mathrm{d}x \\ &+ \int_\Omega (f\xi)(\varphi\xi) \, \mathrm{d}x. \end{aligned}$$

Defining $w := u\xi$, $\psi := \varphi\xi$ and $g := f\xi - \sum_{i,j=1}^d \big(a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \xi}{\partial x_i} + \frac{\partial}{\partial x_i} \big(a_{ij} u \frac{\partial \xi}{\partial x_j} \big) \big)$ and extending all these functions by zero outside of Ω, we end up with

$$\int_{\mathbb{R}^d} \Big(\sum_{i,j=1}^d a_{ij} \frac{\partial w}{\partial x_j} \frac{\partial \psi}{\partial x_i} + b w \psi \Big) \, \mathrm{d}x = \int_{\mathbb{R}^d} g \psi \, \mathrm{d}x \qquad \text{for all } \psi \in W^{1,2}(\mathbb{R}^d), \operatorname{supp} \psi \subset \Omega.$$

Recall that both $\operatorname{supp} g$, $\operatorname{supp} b \subset \Omega$. Since we know that

$$\|u\|_{1,2} \leq C\|f\|_2,$$

we may now successfully apply Theorem 3.7.3 which yields

$$\|w\|_{W^{2,2}(\mathbb{R}^d)} \leq C(\|g\|_{L^2(\mathbb{R}^d)} + \|f\|_{L^2(\mathbb{R}^d)}) \leq C\|f\|_{L^2(\mathbb{R}^d)}.$$

Whence

$$\|u\|_{W^{2,2}(\Omega')} \leq C\|f\|_{L^2(\Omega)}$$

and the proof is complete. ∎

Remark 3.7.5. Note that in the previous theorem we do not need any information about any regularity of the boundary (we only need the basic estimate for the weak solution which may require the Lipschitz continuity) and we do "not see" the boundary conditions. The same procedure as above can justify the same result as above for only

$$f \in W^{-1,2}(\Omega) \cap L^2_{\mathrm{loc}}(\Omega), \quad a_{ij} \in L^\infty(\Omega) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega), \quad b \in L^p(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$$

for $p \geq d$ if $d > 2$ and $p > 2$ for $d = 2$.

We now consider the boundary regularity. Let us first look at a very particular situation, namely for the domain $C^+ = (-1,1)^{d-1} \times (0,1)$ with the main part of the boundary $\Lambda = (-1,1)^{d-1} \times \{0\}$ we study the following problem.
Let $u - U_0 \in W^{1,2}(\Omega)$, $\operatorname{supp} u \subset C^+ \cup \Lambda$ and

$$\int_{C^+} \Big(\sum_{i,j=1}^d a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \varphi}{\partial x_i} + b u \varphi \Big) \, \mathrm{d}x = \int_{C^+} f \varphi \, \mathrm{d}x \qquad \text{for all } \varphi \in W_0^{1,2}(C^+), \tag{3.127}$$

$\operatorname{supp} f \subset C^+ \cup \Lambda$ as well.

Lemma 3.7.6 — Regularity in a cube. Let $U_0 \in W^{2,2}(C^+)$, $a_{ij} \in W^{1,\infty}(C^+)$ for all $i,j \in \{1,2,\dots,d\}$, $b \in L^\infty(C^+)$, $f \in L^2(C^+)$ and let u be a weak solution to (3.127) and both u and f have support in $C^+ \cup \Lambda$. Then $u \in W^{2,2}(C^+)$ and it holds

$$\|u\|_{W^{2,2}(C^+)} \le C(\|f\|_{L^2(C^+)} + \|U_0\|_{W^{2,2}(C^+)}).$$

Proof. Since u is the unique solution to (3.127), we have

$$\|u\|_{W^{1,2}(C^+)} \le C(\|f\|_{L^2(C^+)} + \|U_0\|_{W^{1,2}(C^+)}).$$

We now repeat the technique from the full space estimates (local regularity results), but this is possible only for the tangential derivatives (the normal derivatives may require knowledge of $u(x + h\mathbf{e}_d)$ which is for $x \in \Lambda$, $h < 0$ outside of C^+).

We first use as test function

$$\varphi(x) := \Delta_k^{-h}(\Delta_k^h(u - U_0)),\ 1 \le k \le d-1.$$

This test function is clearly eligible, it belongs to $W_0^{1,2}(C^+)$. Moreover, it is well defined as the differences are computed only along the boundary. We can now repeat the computations from the full space situation, the only difference is that on the right-hand side some extra terms appear which are not controlled by $\|f\|_{L^2(C^+)}$, but by $\|U_0\|_{W^{2,2}(C^+)}$. Let us first look at the most restrictive term (but also the term giving the main estimate)

$$\begin{aligned}
&\int_{C^+} \sum_{i,j=1}^d a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial}{\partial x_i} \big(\Delta_k^{-h}(\Delta_k^h(u - U_0))\big)\,\mathrm{d}x \\
&\quad = \int_{C^+} \sum_{i,j=1}^d a_{ij} \frac{\partial \Delta_k^h u}{\partial x_j} \frac{\partial \Delta_k^h u}{\partial x_i}\,\mathrm{d}x + \int_{C^+} \sum_{i,j=1}^d \frac{1}{h}(a_{ij}(x + h\mathbf{e}_k) - a_{ij}(x)) \frac{\partial u(x + h\mathbf{e}_k)}{\partial x_j} \frac{\partial \Delta_k^h u}{\partial x_i}\,\mathrm{d}x \\
&\quad - \int_{C^+} \sum_{i,j=1}^d a_{ij} \frac{\partial \Delta_k^h u}{\partial x_j} \frac{\partial \Delta_k^h U_0}{\partial x_i}\,\mathrm{d}x - \int_{C^+} \sum_{i,j=1}^d \frac{1}{h}(a_{ij}(x + h\mathbf{e}_k) - a_{ij}(x)) \frac{\partial u(x + h\mathbf{e}_k)}{\partial x_j} \frac{\partial \Delta_k^h U_0}{\partial x_i}\,\mathrm{d}x.
\end{aligned}$$

Thus we get

$$\begin{aligned}
&\int_{C^+} \sum_{i,j=1}^d a_{ij} \frac{\partial \Delta_k^h u}{\partial x_j} \frac{\partial \Delta_k^h u}{\partial x_i}\,\mathrm{d}x = -\int_{C^+} \sum_{i,j=1}^d \frac{1}{h}(a_{ij}(x + h\mathbf{e}_k) - a_{ij}(x)) \frac{\partial u(x + h\mathbf{e}_k)}{\partial x_j} \frac{\partial \Delta_k^h u}{\partial x_i}\,\mathrm{d}x \\
&- \int_{C^+} b(x)u(x)\Delta_k^{-h}(\Delta_k^h(u(x)))\,\mathrm{d}x + \int_{C^+} f(x)\Delta_k^{-h}(\Delta_k^h(u(x)))\,\mathrm{d}x \\
&+ \int_{C^+} \sum_{i,j=1}^d \frac{1}{h}(a_{ij}(x + h\mathbf{e}_k) - a_{ij}(x)) \frac{\partial u(x + h\mathbf{e}_k)}{\partial x_j} \frac{\partial \Delta_k^h U_0}{\partial x_i}\,\mathrm{d}x + \int_{C^+} \sum_{i,j=1}^d a_{ij} \frac{\partial \Delta_k^h u}{\partial x_j} \frac{\partial \Delta_k^h U_0}{\partial x_i}\,\mathrm{d}x \\
&+ \int_{C^+} b(x)u(x)\Delta_k^{-h}(\Delta_k^h(U_0(x)))\,\mathrm{d}x - \int_{C^+} f(x)\Delta_k^{-h}(\Delta_k^h(U_0(x)))\,\mathrm{d}x.
\end{aligned}$$

Using the estimates

$$\begin{aligned}
\|\Delta_k^{-h}(\Delta_k^h(U_0))\|_{L^2(C^+)} + \Big\|\frac{\partial \Delta_k^h U_0}{\partial x_i}\Big\|_{L^2(C^+)} &\le C\|\nabla^2 U_0\|_{L^2(\mathbb{R}^d;\mathbb{R}^{d\times d})} \\
\|\Delta_k^{-h}(\Delta_k^h(u))\|_{L^2(C^+)} &\le C\|\nabla(\Delta_k^h(u))\|_{L^2(C^+;\mathbb{R}^d)}
\end{aligned}$$

together with the ellipticity of the differential operator and standard Young's inequality (similarly as in the case of the full space estimates) we end up with

$$\|\nabla(\Delta_k^h(u))\|_{L^2(C^+;\mathbb{R}^d)} \le C(\|\mathbb{A}\|_{W^{1,\infty}(C^+;\mathbb{R}^{d\times d})}, \|b\|_{L^\infty(C^+)}, C_1)(\|f\|_{L^2(C^+)} + \|U_0\|_{W^{2,2}(C^+)}), \quad k = 1,2,\dots,d-1.$$

The only second order derivative we do not consider on the left-hand side is $\frac{\partial^2 u}{\partial x_d^2}$. The technique of differences does not work in this case. However, we can write for any $\varphi \in \mathcal{C}_0^\infty(\Omega)$

$$\begin{aligned}
\int_{C^+} a_{dd} \frac{\partial u}{\partial x_d} \frac{\partial \varphi}{\partial x_d}\,\mathrm{d}x &= \int_{C^+} \Big(f\varphi - bu\varphi - \sum_{\substack{i,j=1\\ i+j<2d}}^d a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \varphi}{\partial x_i}\Big)\,\mathrm{d}x \\
&= \int_{C^+} \Big(f\varphi - bu\varphi + \sum_{\substack{i,j=1\\ i+j<2d}}^d \frac{\partial}{\partial x_i}\Big(a_{ij} \frac{\partial u}{\partial x_j}\Big)\varphi\Big)\,\mathrm{d}x.
\end{aligned}$$

The ellipticity condition implies $a_{dd} > 0$ (for $\boldsymbol{\xi} = (0,0,\dots,1)$ we have $a_{dd}\xi_d^2 \geq C_1|\boldsymbol{\xi}|^2 = C_1\xi_d^2$). The formula above by the definition of the weak solution implies $\frac{\partial^2 u}{\partial x_d^2} \in L^2(C^+)$, together with the estimate

$$\Big\|\frac{\partial^2 u}{\partial x_d^2}\Big\|_{L^2(C^+)} \leq C\Big(\|f\|_{L^2(C^+)} + \sum_{\substack{|\alpha|\leq 2\\ \alpha\neq(0,0,\dots,2)}} \|D^\alpha u\|_{L^2(C^+)}\Big) \leq C(\|f\|_{L^2(C^+)} + \|U_0\|_{W^{2,2}(C^+)}).$$

The proof is complete. ■

We need to deal with the situation when $\operatorname{supp} u \subset V^+ \cup \Lambda$. We shall transform this situation to the case in the previous theorem, i.e., we flatten the boundary. To this aim, consider the following change of variables

$$\begin{aligned} x &= \boldsymbol{\Psi}(y), \qquad \boldsymbol{\Psi}\colon C^+ \to V^+ \qquad \text{and } (-1,1)^{d-1}\times\{0\} \to \Lambda \\ y &= \boldsymbol{\Phi}(x), \qquad \boldsymbol{\Phi}\colon V^+ \to C^+ \qquad \text{and } \Lambda \to (-1,1)^{d-1}\times\{0\}, \end{aligned}$$

where

$$\begin{aligned} y_i &= \frac{x_i}{\alpha},\ i=1,2,\dots,d-1, \qquad & x_i &= \alpha y_i,\ i=1,2,\dots,d-1 \\ y_d &= \frac{1}{\beta}x_d - \frac{1}{\beta}a(x'), \qquad & x_d &= \beta y_d + a(\alpha y'). \end{aligned}$$

Above, $x' = (x_1, x_2, \dots, x_{d-1})$, similarly for y'. Moreover, we have for the jacobians

$$\begin{aligned} \mathcal{J}_{\boldsymbol{\Psi}}(y) &= \det \nabla_y \boldsymbol{\Psi}(y) = \alpha^{d-1}\beta \\ \mathcal{J}_{\boldsymbol{\Phi}}(x) &= \det \nabla_x \boldsymbol{\Phi}(x) = \frac{1}{\alpha^{d-1}\beta}. \end{aligned}$$

Clearly, these changes of variables are inverse to each other and assuming $\Omega \in \mathcal{C}^{1,1}$, they are of class $W^{2,\infty}$. Moreover, we have for $\widetilde{u}(y) = u(\boldsymbol{\Psi}(y))$, $\widetilde{b}(y) = b(\boldsymbol{\Psi}(y))$, $\widetilde{f}(y) = f(\boldsymbol{\Psi}(y))$ and $\widetilde{a_{kl}}(y) = \sum_{m,n=1}^d a_{mn}(\boldsymbol{\Psi}(y))\frac{\partial\Phi_k(\boldsymbol{\Psi}(y))}{\partial x_m}\frac{\partial\Phi_l(\boldsymbol{\Psi}(y))}{\partial x_n}$ that

$$\int_{V^+} \sum_{i,j=1}^d a_{ij}(x)\frac{\partial u(x)}{\partial x_j}\frac{\partial\varphi(x)}{\partial x_i}\,\mathrm{d}x = \int_{C^+}\sum_{k,l=1}^d \widetilde{a_{kl}}(y)\frac{\partial\widetilde{u}(y)}{\partial y_l}\frac{\partial\widetilde{\varphi}(y)}{\partial y_k}\alpha^{d-1}\beta\,\mathrm{d}y.$$

Therefore we end up with the weak formulation

$$\int_{C^+}\sum_{k,l=1}^d \widetilde{a_{kl}}(y)\frac{\partial\widetilde{u}(y)}{\partial y_l}\frac{\partial\widetilde{\varphi}(y)}{\partial y_k}\,\mathrm{d}y + \int_{C^+}\widetilde{b}(y)\widetilde{u}(y)\widetilde{\varphi}(y)\,\mathrm{d}y = \int_{C^+}\widetilde{f}(y)\widetilde{\varphi}(y)\,\mathrm{d}y. \tag{3.128}$$

We need to verify that the differential operator in the weak formulation (3.128) is elliptic. However, it is easy to show that

$$\begin{aligned} \sum_{k,l=1}^d \widetilde{a_{kl}}(y)\xi_k\xi_l &= \sum_{k,l=1}^d\Big(\sum_{m,n=1}^d a_{mn}(\boldsymbol{\Psi}(y))\frac{\partial\Phi_k(\boldsymbol{\Psi}(y))}{\partial x_m}\frac{\partial\Phi_l(\boldsymbol{\Psi}(y))}{\partial x_n}\Big)\xi_k\xi_l \\ &= \sum_{m,n=1}^d a_{mn}(\boldsymbol{\Psi}(y))\eta_m\eta_l \geq C_1|\boldsymbol{\eta}|^2, \end{aligned}$$

where $\boldsymbol{\eta} = (\nabla\boldsymbol{\Phi})^T\boldsymbol{\xi}$, i.e., $\boldsymbol{\xi} = (\nabla\boldsymbol{\Psi})^T\boldsymbol{\eta}$. Thus $|\boldsymbol{\xi}| \leq C|\boldsymbol{\eta}|$ which yields the desired ellipticity estimate

$$\sum_{k,l=1}^d \widetilde{a_{kl}}(y)\xi_k\xi_l \geq C_1|\boldsymbol{\eta}|^2 \geq \widetilde{C}_1|\boldsymbol{\xi}|^2.$$

Since the mappings $x = \boldsymbol{\Psi}(y)$ and $y = \boldsymbol{\Phi}(x)$ are of class $\mathcal{C}^{1,1}$, thus $W^{2,\infty}$, we finally see that $\widetilde{a_{kl}} \in W^{1,\infty}(C^+)$, $\widetilde{b} \in L^\infty(C^+)$ and $\widetilde{f} \in L^2(C^+)$. In the last claim we used the form of the jacobians.

We have therefore proved

Theorem 3.7.7 — Regularity near the boundary. Let $u \in W_0^{1,2}(V^+)$, $u - U_0 \in W^{1,2}(V^+)$ satisfy in V^+ (the set from the definition of a domain with smooth boundary and u is the unique solution here)

$$\int_{V^+}\Big(\sum_{i,j=1}^d a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial\varphi}{\partial x_i} + bu\varphi\Big)\,\mathrm{d}x = \int_{V^+} f\varphi\,\mathrm{d}x \qquad \text{for all } \varphi \in W^{1,2}(V^+),$$

where $U_0 \in W^{2,2}(V^+)$. Assume that the support of all functions belong to $V^+ \cup \Lambda$. Let $\{a_{ij}\}_{i,j=1}^d \in W^{1,\infty}(V^+)$,

$b \in L^\infty(V^+)$ and $f \in L^2(V^+)$. Then $u \in W^{2,2}(V^+)$ and we have

$$\|u\|_{W^{2,2}(V^+)} \le C(\|f\|_{L^2(V^+)} + \|U_0\|_{W^{2,2}(V^+)}).$$

Proof. By the change of the variables specified above we showed that $\widetilde{u} \in W^{2,2}(C^+)$ together with the estimate (see also Lemma 3.7.6)

$$\|\widetilde{u}\|_{W^{2,2}(C^+)} \le C(\|\widetilde{f}\|_{L^2(C^+)} + \|\widetilde{U}_0\|_{W^{2,2}(C^+)}).$$

It is now enough to realize that by the inverse change of variables $x = \mathbf{\Psi}(y)$ we may return back to V^+ and since the mapping is of class $W^{2,\infty}(\Omega)$, we get the desired estimate in V^+. ■

Combining Theorems 3.7.4 and 3.7.7 we get

Theorem 3.7.8 — Regularity in the whole bounded domain. Let u be the unique weak solution to Problem (3.112) with the data satisfying (3.113)–(3.114). Then $u \in W^{2,2}(\Omega)$ and it holds

$$\|u\|_{W^{2,2}(\Omega)} \le C(\|f\|_{L^2(\Omega)} + \|U_0\|_{W^{2,2}(\Omega)}).$$

Proof. We use the partition of unity applied on the covering of $\overline{\Omega}$ by $\{V_r\}_{r=1}^M$ and by V_{M+1} (which lies strictly inside of Ω). To simplify, we ignore the possible rotation and shift expressed by the mappings T_r. We have

$$u = u\varphi_{M+1} + \sum_{r=1}^{M} u\varphi_r =: u_{M+1} + \sum_{r=1}^{M} u_r.$$

We now apply Theorem 3.7.4 on u_{M+1} and Theorem 3.7.7 on u_r, $r = 1, 2, \dots, M$ and sum up the estimates. Recall that the functions have exactly the same type of support as required in the corresponding theorems and satisfy the corresponding problems with lower order terms due to the presence of φ_r on the right-hand side; more precisely, we have

$$\int_{V_r^+} \Big(\sum_{i,j=1}^{d} a_{ij} \frac{\partial u_r}{\partial x_j} \frac{\partial \varphi}{\partial x_j} + b u_r \varphi \Big) \, \mathrm{d}x = \int_{V_r^+} f \varphi_r \varphi \, \mathrm{d}x + \int_{V_r^+} \sum_{i,j=1}^{d} a_{ij} u \frac{\partial \varphi_r}{\partial x_j} \frac{\partial \varphi}{\partial x_j} \, \mathrm{d}x,$$

$r = 1, 2, \dots, M$; similarly for u_{M+1}. We obtain

$$\|u_{M+1}\|_{W^{2,2}(V_{M+1})} \le C\|f\|_{L^2(\Omega)}$$
$$\|u_r\|_{W^{2,2}(V_r)} \le C(\|f\|_{L^2(V_r)} + \|U_0\|_{W^{2,2}(V_r)}), \quad r = 1, 2, \dots, M.$$

Summing up these estimates we end up with

$$\|u\|_{W^{2,2}(\Omega)} \le \|u_{M+1}\|_{W^{2,2}(V_{M+1})} + \sum_{r=1}^{M} \|u_r\|_{W^{2,2}(V_r)} \le C(\|f\|_{L^2(\Omega)} + \|U_0\|_{W^{2,2}(\Omega)}).$$

■

Remark 3.7.9. The whole procedure can be applied also in the situation when we have general elliptic operator and u is a solution which is generally non-unique (obtained, e.g., in the part devoted to the application of Fredholm alternative). Then, under the same assumptions as above we conclude with the estimate

$$\|u\|_{W^{2,2}(\Omega)} \le C(\|f\|_{L^2(\Omega)} + \|U_0\|_{W^{2,2}(\Omega)} + \|u\|_{L^2(\Omega)}). \tag{3.129}$$

We shall skip details of this proof. If the solution is unique then we know that we can estimate the last term on the right-hand side by the other two ones.

By induction of the previous process we can show

Theorem 3.7.10 — Higher regularity. Let u be the unique weak solution to Problem (3.112) and let the data satisfy (3.113) and (3.115) for some $k \ge 2$. Then $u \in W^{k,2}(\Omega)$ and it holds

$$\|u\|_{W^{k,2}(\Omega)} \le C(\|f\|_{W^{k-2,2}(\Omega)} + \|U_0\|_{W^{k,2}(\Omega)}).$$

Since the previous theorem holds for any $k \in \mathbb{N}$, $k \ge 2$, we have

Corollary 3.7.11 (Full regularity). Let u be the unique weak solution to Problem (3.112) and let the data satisfy (3.113) and (3.116). Then $u \in \mathcal{C}^\infty(\overline{\Omega})$.

Remark 3.7.12. Remark 3.7.9 also applies to Theorem 3.7.10 for any fixed $k \in \mathbb{N}$, $k \ge 2$.

Proof of Theorem 3.7.10. **Step 1:** Induction in interior regularity
We proceed by induction. We first consider the interior regularity (corresponding to u_{M+1}), where the situation is easier. Assume that we have for all $V \subset \overline{V} \subset \Omega$ and some $k \geq 2$ (under corresponding assumptions on the data of our problem, $f \in W^{k-2,2}(\Omega)$, $a_{ij} \in W^{k-1,\infty}(\Omega)$, $i,j = 1,2,\dots,d$ and $b \in W^{k-2,\infty}(\Omega)$)

$$\|u\|_{W^{k,2}(V)} \leq C\|f\|_{W^{k-2,2}(\Omega)},$$

the constant C may depend on V. Indeed, Theorem 3.7.4 corresponds to $k=2$.

Assume now that $f \in W^{k-1,2}(\Omega)$, $a_{ij} \in W^{k,\infty}(\Omega)$, $i,j = 1,2,\dots,d$ and $b \in W^{k-1,\infty}(\Omega)$. Take any multiindex α such that $|\alpha| = k-1$ and choose $\widetilde{v} \in C_0^\infty(V)$ for some $V \subset \overline{V} \subset \Omega$. Use as a test function in the weak formulation the function $\varphi := (-1)^\alpha D^\alpha \widetilde{v}$. Using integration by parts we obtain

$$\int_\Omega \Big(\sum_{i,j=1}^d a_{ij} \frac{\partial \widetilde{u}}{\partial x_j} \frac{\partial \widetilde{v}}{\partial x_i} + b\widetilde{u}\widetilde{v} \Big) \,\mathrm{d}x = \int_\Omega \widetilde{f}\widetilde{v} \,\mathrm{d}x, \tag{3.130}$$

where $\widetilde{u} = D^\alpha u \in W^{1,2}(V)$ and

$$\widetilde{f} = D^\alpha f - \sum_{\substack{\beta\leq\alpha\\ \beta\neq\alpha}} \binom{\alpha}{\beta} \Big[-\frac{\partial}{\partial x_i} \Big(\sum_{i,j=1}^d D^{\alpha-\beta} a_{ij} \frac{\partial (D^\beta u)}{\partial x_j} \Big) \Big] - \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} D^{\alpha-\beta} b D^\beta u.$$

As (3.130) holds for any $\widetilde{v} \in C_0^\infty(V)$, we see that $\widetilde{u}$ is a weak solution to

$$\int_V \Big(\sum_{i,j=1}^d a_{ij} \frac{\partial \widetilde{u}}{\partial x_j} \frac{\partial \widetilde{v}}{\partial x_i} + b\widetilde{u}\widetilde{v} \Big) \,\mathrm{d}x = \int_V \widetilde{f}\widetilde{v} \,\mathrm{d}x,$$

where $\widetilde{f} \in L^2(V)$ with

$$\|\widetilde{f}\|_{L^2(V)} \leq C\|f\|_{W^{k-1,2}(V)} + C(\|\mathbb{A}\|_{W^{k,\infty}(\Omega;\mathbb{R}^{d\times d})}, \|b\|_{W^{k-1,\infty}(\Omega)})\|u\|_{W^{k,2}(\Omega)} \leq C\|f\|_{W^{k-1,2}(V)}.$$

We now apply Theorem 3.7.4 and get

$$\|\widetilde{u}\|_{W^{2,2}(W)} \leq C\|f\|_{W^{k-1,2}(V)},$$

where $C = C(\|\mathbb{A}\|_{W^{k,\infty}(\Omega)}, \|b\|_{W^{k-1,\infty}(\Omega)}, C_1)$ and $W \subset \overline{W} \subset V$. Thus $u \in W^{k+1,2}_{\mathrm{loc}}(\Omega)$.

Step 2: Induction for regularity near the boundary; flat part of the boundary, tangential derivatives
We now proceed with the regularity near the boundary. Let us first consider the case of the flat domain C^+. We proceed by induction. We aim to show that for $U_0 \in W^{k,2}(C^+)$, $u - U_0 \in W_0^{1,2}(C^+)$, $f \in W^{k-2,2}(C^+)$, $a_{ij} \in W^{k-1,\infty}(C^+)$, $i,j = 1,2,\dots,d$ and $b \in W^{k-2,\infty}(C^+)$ we get

$$\|u\|_{W^{k,2}(C^+)} \leq C(\|f\|_{W^{k-2,2}(C^+)} + \|U_0\|_{W^{k,2}(C^+)}),$$

where $C = C(\|\mathbb{A}\|_{W^{k-1,\infty}(C^+;\mathbb{R}^{d\times d})}, \|b\|_{W^{k-2,\infty}(C^+)}, C_1)$. We know that the inequality is true for $k=2$; this is exactly the claim of Lemma 3.7.6. Let us therefore assume that the claim holds for some $k \geq 2$ and let us show it for $k+1$.

Let α be a multiindex such that $|\alpha| = k-1$, but $\alpha_d = 0$. Then $\widetilde{u} := D^\alpha u$ belongs to $W^{1,2}(C^+)$. Similarly as in Step 1. it solves

$$\int_{C^+} \Big(\sum_{i,j=1}^d a_{ij} \frac{\partial \widetilde{u}}{\partial x_j} \frac{\partial \varphi}{\partial x_i} + b\widetilde{u}\varphi \Big) \,\mathrm{d}x = \int_{C^+} \widetilde{f}\varphi \,\mathrm{d}x,$$

where again

$$\widetilde{f} = D^\alpha f - \sum_{\substack{\beta\leq\alpha\\ \beta\neq\alpha}} \binom{\alpha}{\beta} \Big[-\frac{\partial}{\partial x_i} \Big(\sum_{i,j=1}^d D^{\alpha-\beta} a_{ij} \frac{\partial D^\beta u}{\partial x_j} \Big) \Big] - \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} D^{\alpha-\beta} b D^\beta u.$$

Then $\widetilde{f} \in L^2(C^+)$ and

$$\|f\|_{L^2(C^+)} \leq C(\|f\|_{W^{k-1,2}(C^+)} + \|U_0\|_{W^{k,2}(C^+)}).$$

Thus, exactly as in the proof of Lemma 3.7.6 we have

$$\|D^\beta u\|_{L^2(C^+)} \leq C(\|f\|_{W^{k-1,2}(C^+)} + \|U_0\|_{W^{k+1,2}(C^+)})$$

for any $|\beta| = k+1$, $\beta_d = 0, 1$ or 2 which we wanted to prove; however we miss the terms with $\beta_d > 2$.

Step 3: Induction for regularity near the boundary; flat part of the boundary, normal derivatives
We have to deal with $\beta_d \geq 3$. We proceed by induction. Assume that we have

$$\|D^\beta u\|_{L^2(C^+)} \leq C(\|f\|_{W^{k-1,2}(C^+)} + \|U_0\|_{W^{k+1,2}(C^+)})$$

for $\beta = k+1$ and $\beta_n = 1, 2, \dots, j$ for some $j \in \{2, 3, \dots, k\}$. Let now $|\beta| = k+1$ and $\beta_n = j+1$. We write $\beta = \gamma + \delta$, $\delta = (0, 0, \dots, 2)$ and $|\gamma| = k-1$. From Step 1. we know that $u \in W^{k+1,2}_{\mathrm{loc}}(\Omega)$ and thus

$$-\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big) + bu = f$$

holds almost everywhere in C^+. Then also

$$D^\gamma\Big(-\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big) + bu\Big) = D^\gamma f$$

almost everywhere in C^+. We write

$$D^\gamma\Big(-\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big) + bu\Big) = a_{dd} D^\beta u + g,$$

where g contains terms with at most j derivatives with respect to x_d and at most $k+1$ derivative overall. As $a_{dd} \geq C_1 > 0$, we get

$$\|D^\beta u\|_{L^2(C^+)} \leq C(\|f\|_{W^{k-1,2}(C^+)} + \|U_0\|_{W^{k+1,2}(C^+)}), \qquad |\beta| = k+1,\ \beta_d = j+1.$$

Thus by induction

$$\|u\|_{W^{k+1,2}(C^+)} \leq C(\|f\|_{W^{k-1,2}(C^+)} + \|U_0\|_{W^{k+1,2}(C^+)}).$$

Step 4: Regularity near the boundary; curved part of the boundary, special support
To obtain the same result in V^+, we proceed exactly as in the proof of Theorem 3.7.7, we flatten the boundary and apply Step 3. Here we use that $\Omega \in \mathcal{C}^{k-1,1}$ for estimates of the $W^{k,2}(V^+)$ norm.

Step 5: Regularity in the whole domain
To get the estimates in the whole domain Ω, we proceed exactly as in the proof of Theorem 3.7.8. We write $u = \sum_{r=1}^{M} u_r + u_{M+1}$ and apply Step 1. for u_{M+1} and Steps 2.–4. for u_r. ■

Remark 3.7.13. (i) Note that in the Step 1. we in fact show that $u \in W^{k,2}_{\mathrm{loc}}(\Omega)$ provided $f \in W^{k-2,2}_{\mathrm{loc}}(\Omega)$, $a_{ij} \in W^{k-1,\infty}_{\mathrm{loc}}(\Omega)$, $i, j = 1, 2, \dots, d$ and $b \in W^{k-2,2}_{\mathrm{loc}}(\Omega)$. Similarly, if f, a_{ij}, $i, j = 1, 2, \dots, d$ and $b \in C^\infty(\Omega)$, then also $u \in C^\infty(\Omega)$ (but generally u is not in $C^\infty(\overline{\Omega})$!).

(ii) If $d = 3$, then we get from Theorem 2.4.21 (continuous embedding) that $u \in C^{2,\frac{1}{2}}(\overline{\Omega})$ (for $k = 4$). Then u is a classical solution to our problem.

(iii) A similar result holds also for the Neumann problem. Under the same assumptions on $\{a_{ij}\}_{i,j=1}^{d}$, b, f and Ω as in Theorem 3.7.10 we have

$$g \in W^{k-\frac{3}{2},2}(\partial\Omega) \Rightarrow u \in W^{k,2}(\Omega), \quad k \geq 2.$$

(iv) We must be careful for mixed boundary conditions. A singularity may appear between Γ_1 and Γ_2.

3.8 Connection with calculus of variations — solvability for a symmetric coercive operator

We show in this part how for certain type of elliptic operators solvability of (3.31)–(3.34) is connected with minimization of quadratic functionals. In the whole section we thus do not consider a general elliptic operator L, but we restrict ourselves to symmetric (selfadjoint) operators L which are additionally coercive. We consider again the systems of elliptic equations.

Definition 3.8.1 — Symmmetric linear elliptic operator. Let L be an elliptic operator. We say that the operator L is symmetric, if it is given as

$$(L\vec{u})^\alpha := -\sum_{\beta=1}^{N}\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}^{\alpha\beta}\frac{\partial u^\beta}{\partial x_j}\right) + \sum_{\beta=1}^{N} b^{\alpha\beta}u^\beta \tag{3.131}$$

and for any α, $\beta \in \{1, \dots, N\}$ and $i, j \in \{1, \dots, d\}$ it holds

$$a_{ij}^{\alpha\beta} = a_{ji}^{\beta\alpha}, \quad b^{\alpha\beta} = b^{\beta\alpha} \quad \text{almost everywhere in } \Omega. \tag{3.132}$$

We can write such an operator in the shorten form

$$L\vec{u} := -\mathrm{div}\,(\vec{\mathbb{A}}\nabla\vec{u}) + \mathfrak{b}\vec{u}$$

and condition (3.131) as

$$\vec{\mathbb{A}} = \vec{\mathbb{A}}^T, \qquad \mathfrak{b} = \mathfrak{b}^T.$$

We consider instead of (3.27)–(3.30) a simplified problem

$$\begin{aligned} L\vec{u} &= \vec{f} && \text{in } \Omega \\ \vec{u} &= \vec{u}_0 && \text{on } \Gamma_1 \\ \vec{\mathbb{A}}\nabla\vec{u}\cdot\boldsymbol{\nu} &= \vec{g} && \text{on } \Gamma_2 \\ \vec{\mathbb{A}}\nabla\vec{u}\cdot\boldsymbol{\nu} + \mathbb{\sigma}\vec{u} &= \vec{g} && \text{on } \Gamma_3, \end{aligned} \tag{3.133}$$

where we additionally assume the symmetry of $\mathbb{\sigma}$, i.e.,

$$\sigma^{\alpha\beta} = \sigma^{\beta\alpha} \text{ for any } \alpha, \beta \in \{1, \dots, N\} \text{ almost everywhere on } \partial\Omega. \tag{3.134}$$

Let us further recall the notation from (3.56)

$$\begin{aligned} \widetilde{b}(x) &:= \inf_{\{\vec{z}\in\mathbb{R}^N;\,|\vec{z}|=1\}} \sum_{\alpha,\beta=1}^{N} b^{\alpha\beta}(x) z^\alpha z^\beta, && x \in \Omega \\ \tilde{\sigma}(x) &:= \inf_{\{\vec{z}\in\mathbb{R}^N;\,|\vec{z}|=1\}} \sum_{\alpha,\beta=1}^{N} \sigma^{\alpha\beta}(x) z^\alpha z^\beta, && x \in \Gamma_3 \end{aligned}$$

and in this section we always assume that

$$\widetilde{b}(x) \geq 0, \qquad \widetilde{\sigma}(x) \geq 0 \qquad \text{almost everywhere in } \Omega \text{ and on } \Gamma_3, \text{ respectively.} \tag{3.135}$$

The space of test functions we consider here is $V = \{\vec{v} \in W^{1,2}(\Omega;\mathbb{R}^N) \,|\, \vec{v} = \vec{0} \text{ in the sense of traces on } \Gamma_1\}$. Recall also that

$$W^{1,2}(\Omega;\mathbb{R}^N) := \{\vec{\varphi} = (\varphi_1, \dots, \varphi_N) \,|\, \varphi_\alpha \in W^{1,2}(\Omega) \text{ for any } \alpha \in \{1, \dots, N\}\}.$$

The corresponding bilinear form for problem (3.133) is

$$B_{L,\mathbb{\sigma}}(\vec{u}, \vec{\varphi}) := \int_\Omega \left(\vec{\mathbb{A}}\nabla\vec{u} : \nabla\vec{\varphi} + \mathfrak{b}\vec{u}\cdot\vec{\varphi}\right) \mathrm{d}x + \int_{\Gamma_3} \mathbb{\sigma}\vec{u}\cdot\vec{\varphi}\,\mathrm{d}S.$$

The weak solution to problem (3.133) (compare with Definition 3.3.4) is then a function $\vec{u}$ such that $\vec{u} - \vec{U}_0 \in V$, $\vec{U}_0 \in W^{1,2}(\Omega)$, the trace $\vec{U}_0 = \vec{u}_0$ on Γ_1, and for any $\vec{\varphi} \in V$ it holds

$$B_{L,\mathbb{\sigma}}(\vec{u}, \vec{\varphi}) = \langle \vec{f}, \vec{\varphi} \rangle_V + \langle \vec{g}, \vec{\varphi} \rangle_{W^{\frac{1}{2},2}(\Gamma_2\cup\Gamma_3;\mathbb{R}^N)}. \tag{3.136}$$

Note that due to our assumed symmetries it holds $B_{L,\mathbb{\sigma}}(\vec{u}, \vec{\varphi}) = B_{L,\mathbb{\sigma}}(\vec{\varphi}, \vec{u})$ and our problem is thus selfadjoint. This will allow us to formulate the problem of existence of a weak solution equivalently as looking for a minimum of a certain functional. Let us define the functional $\Phi\colon W^{1,2}(\Omega;\mathbb{R}^N) \to \mathbb{R}$ by the formula

$$\Phi_{L,\mathbb{\sigma}}(\vec{u}) := \frac{1}{2} B_{L,\mathbb{\sigma}}(\vec{u}, \vec{u}). \tag{3.137}$$

Then the following theorem holds true.

Theorem 3.8.2 — Connection between weak solution and minimum of a functional. Let $\Omega \in \mathcal{C}^{0,1}$ with the corresponding parts of the boundary $\{\Gamma_i\}_{i=1}^3$ and let L be a symmetric ellliptic operator in the sense of Definition 3.8.1. Let further for any $\alpha, \beta = 1, \dots, N$, $\sigma^{\alpha\beta} \in L^\infty(\Gamma_3)$ satisfy additionally (3.134) and together with $\mathfrak{b}$ also (3.135). Finally, let $\vec{u} \in W^{1,2}(\Omega;\mathbb{R}^N)$ be an arbitrary function satisfying $\vec{u} - \vec{U}_0 \in V$, $\vec{U}_0$, V as above. Then the following two claims are equivalent.

1. The function $\vec{u}$ is a weak solution, i.e., it satisfies (3.136).
2. For any $\vec{\varphi} \in V$ it holds that

$$\Phi_{L,\mathbb{\sigma}}(\vec{u} + \vec{\varphi}) - \Phi_{L,\mathbb{\sigma}}(\vec{u}) \geq \langle \vec{f}, \vec{\varphi} \rangle_V + \langle \vec{g}, \vec{\varphi} \rangle_{W^{\frac{1}{2},2}(\Gamma_2\cup\Gamma_3;\mathbb{R}^N)}. \tag{3.138}$$

Before we start with the proof of this theorem, let us present an equivalent form of 2. in the case when $\vec{f} \in (W^{1,2}(\Omega;\mathbb{R}^N))^*$. For this slightly more regular $\vec{f}$ we may reformulate 2. as minimization on a convex set and we get (the proof is left for a kind reader as a simple exercise) that to find $\vec{u}$ satisfying 2. is equivalent to finding $\vec{u}$ such that $\vec{u} - \vec{U}_0 \in V$ and it holds for any $\vec{v}$ satisfying $\vec{v} - \vec{U}_0 \in V$

$$\Phi_{L,\sigma}(\vec{u}) - \langle \vec{f}, \vec{u}\rangle_V - \langle \vec{g}, \vec{u}\rangle_{W^{\frac{1}{2},2}(\Gamma_2\cup\Gamma_3;\mathbb{R}^N)} \le \Phi_{L,\sigma}(\vec{v}) - \langle \vec{f}, \vec{v}\rangle_V + \langle \vec{g}, \vec{v}\rangle_{W^{\frac{1}{2},2}(\Gamma_2\cup\Gamma_3;\mathbb{R}^N)}.$$

Problem (3.136) is also called the *Euler–Lagrange equations* for the functional $\Phi_{L,\sigma}(\vec{u}) - \langle \vec{f}, \vec{u}\rangle_V - \langle \vec{g}, \vec{u}\rangle_{W^{\frac{1}{2},2}(\Gamma_2\cup\Gamma_3;\mathbb{R}^N)}$ and (3.136) can be obtained as

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi_{L,\sigma}(\vec{u}+t\vec{\varphi}) - \langle \vec{f}, \vec{u}+t\vec{\varphi}\rangle_V - \langle \vec{g}, \vec{u}+t\vec{\varphi}\rangle_{W^{\frac{1}{2},2}(\Gamma_2\cup\Gamma_3;\mathbb{R}^N)}\right)\right]_{t=0} = 0,$$

or $t=0$ is the critical point of the function $\psi(t) := \Phi_{L,\sigma}(\vec{u}+t\vec{\varphi}) - \langle \vec{f}, \vec{u}+t\vec{\varphi}\rangle_V - \langle \vec{g}, \vec{u}+t\vec{\varphi}\rangle_{W^{\frac{1}{2},2}(\Gamma_2\cup\Gamma_3;\mathbb{R}^N)}$ for any $\vec{\varphi} \in V$.

Proof of Theorem 3.8.2. We start with the implication 1. $\Longrightarrow$ 2. We get directly from the definition of $\Phi_{L,\sigma}$ (see (3.137)), from the fact that $\vec{u}$ solves (3.136) and from the fact that $B_{L,\sigma}$ is due to our assumption a symmetric bilinear form that

$$\begin{aligned}\Phi_{L,\sigma}(\vec{u}+\vec{\varphi}) - \Phi_{L,\sigma}(\vec{u}) &= \frac{1}{2}B_{L,\sigma}(\vec{u}+\vec{\varphi},\vec{u}+\vec{\varphi}) - \frac{1}{2}B_{L,\sigma}(\vec{u},\vec{u})\\ &= \frac{1}{2}B_{L,\sigma}(\vec{u},\vec{u}) + B_{L,\sigma}(\vec{u},\vec{\varphi}) + \frac{1}{2}B_{L,\sigma}(\vec{\varphi},\vec{\varphi}) - \frac{1}{2}B_{L,\sigma}(\vec{u},\vec{u})\\ &= \langle \vec{f}, \vec{\varphi}\rangle_V + \langle \vec{g}, \vec{\varphi}\rangle_{W^{\frac{1}{2},2}(\Gamma_2\cup\Gamma_3)} + \frac{1}{2}B_{L,\sigma}(\vec{\varphi},\vec{\varphi}).\end{aligned}$$

To finish the proof of the first implication it is enough to verify that $B_{L,\sigma}(\vec{\varphi},\vec{\varphi}) \ge 0$ for any $\vec{\varphi} \in V$. This is evident due to the assumption on the ellipticity of the matrix $\vec{\mathbb{A}}$ (see (3.14)) and non-negativity of $\widetilde{b}$ and $\widetilde{\sigma}$ (see (3.135)).

Let us now consider the other implication 2. $\Longrightarrow$ 1. Let $\vec{\psi} \in V$ be arbitrary. Using assumption 2. and the same computations as above we conclude that

$$\begin{aligned}0 &\le \Phi_{L,\sigma}(\vec{u}+\vec{\psi}) - \Phi_{L,\sigma}(\vec{u}) - \langle \vec{f}, \vec{\psi}\rangle_V - \langle \vec{g}, \vec{\psi}\rangle_{W^{\frac{1}{2},2}(\Gamma_2\cup\Gamma_3)}\\ &= B_{L,\sigma}(\vec{u},\vec{\psi}) + \frac{1}{2}B_{L,\sigma}(\vec{\psi},\vec{\psi}) - \langle \vec{f}, \vec{\psi}\rangle_V - \langle \vec{g}, \vec{\psi}\rangle_{W^{\frac{1}{2},2}(\Gamma_2\cup\Gamma_3)}.\end{aligned}$$

For arbitrary $t>0$ and $\vec{\varphi} \in V$ let us choose in the inequality above $\vec{\psi}: = t\vec{\varphi}$ which due to the bilinearity yields

$$0 \le tB_{L,\sigma}(\vec{u},\vec{\varphi}) + \frac{1}{2}t^2 B_{L,\sigma}(\vec{\varphi},\vec{\varphi}) - t\langle \vec{f}, \vec{\varphi}\rangle_V - t\langle \vec{g}, \vec{\varphi}\rangle_{W^{\frac{1}{2},2}(\Gamma_2\cup\Gamma_3)}.$$

Dividing by $t>0$ and passing with $t \to 0_+$ we thus get

$$\begin{aligned}0 &\le \lim_{t\to 0_+}\left(B_{L,\sigma}(\vec{u},\vec{\varphi}) + \frac{1}{2}tB_{L,\sigma}(\vec{\varphi},\vec{\varphi}) - \langle \vec{f}, \vec{\varphi}\rangle_V - \langle \vec{g}, \vec{\varphi}\rangle_{W^{\frac{1}{2},2}(\Gamma_2\cup\Gamma_3)}\right)\\ &= B_{L,\vec{\sigma}}(\vec{u},\vec{\varphi}) - \langle \vec{f}, \vec{\varphi}\rangle_V - \langle \vec{g}, \vec{\varphi}\rangle_{W^{\frac{1}{2},2}(\Gamma_2\cup\Gamma_3)}.\end{aligned}$$

This inequality holds for any $\vec{\varphi} \in V$ and thus also for $-\vec{\varphi}$ which due to the linearity of the inequality in $\vec{\varphi}$ yields that equality (3.136) holds and thus Claim 1. is proved. ■

We proved above that for certain types of problems (more precisely, for the symmetric ones) finding a weak solution is equivalent to finding a minimizer of a certain variational problem. We now show that if $\mathfrak{b} = \mathfrak{o}$ and $|\Gamma_3| = 0$, we may formulate the problem differently, but we anyway get to the same. This new formulation is called *dual formulation.* The word "dual" refers here that instead of variational formulation for $\vec{u}$ we consider variational formulation for $\vec{\mathbf{T}} = \vec{\mathbb{A}}\nabla\vec{u}$. To this aim we first define a suitable closed subset of $L^2(\Omega;\mathbb{R}^{d\times N})$

$$W := \left\{\vec{\mathbf{T}} \in L^2(\Omega;\mathbb{R}^{d\times N}) \,|\, \text{ for any } \vec{\varphi}\in V \text{ it holds } \int_\Omega \vec{\mathbf{T}} : \nabla\vec{\varphi}\,\mathrm{d}x = \langle \vec{f}, \vec{\varphi}\rangle_V + \langle \vec{g}, \vec{\varphi}\rangle_{W^{\frac{1}{2},2}(\Gamma_2\cup\Gamma_3)}\right\}. \tag{3.139}$$

Note that $\vec{\mathbf{T}} : \nabla\vec{\varphi} = \sum_{i=1}^d \sum_{\alpha=1}^N T_i^\alpha \frac{\partial \varphi^\alpha}{\partial x_i}$. This definition does not mean anything else that $\vec{\mathbf{T}} \in W$ solves (in the sense of distributions) the following problem

$$\begin{aligned}-\operatorname{div}\vec{\mathbf{T}} &= \vec{f} \quad \text{in } \Omega,\\ \vec{\mathbf{T}}\cdot\boldsymbol{\nu} &= \vec{g} \quad \text{on } \Gamma_2.\end{aligned} \tag{3.140}$$

Thus if $\vec{u}$ is a weak solution to the original problem, then automatically $\vec{\mathbf{T}} = \vec{\mathbb{A}}\nabla\vec{u} \in W$. Let us further introduce the dual functional

$$\Phi^*(\vec{\mathbf{T}}) = \frac{1}{2}\int_\Omega \vec{\mathbb{A}}^{-1}\vec{\mathbf{T}} : \vec{\mathbf{T}}\,\mathrm{d}x - \int_\Omega \nabla\vec{U}_0 : \vec{\mathbf{T}}\,\mathrm{d}x, \tag{3.141}$$

where $\vec{\mathbb{A}}^{-1}$ denotes the inverse matrix to $\vec{\mathbb{A}}$, i.e., it holds

$$\sum_{\gamma=1}^{N}\sum_{k=1}^{d}(\vec{\mathbb{A}})_{ik}^{\alpha\gamma}(\vec{\mathbb{A}}^{-1})_{kj}^{\gamma\beta}=\delta_{\alpha\beta}\delta_{ij}.$$

Note that due to the measurability, boundedness and ellipticity of the matrix $\vec{\mathbb{A}}$ (see (3.26)), the inverse matrix $\vec{\mathbb{A}}^{-1}$ exists and is again measurable, bounded and elliptic. We may finally formulate the theorem on the equivalence of the dual formulation and the existence of the weak solution.

Theorem 3.8.3 — Dual variational formulation. Let $\Omega \in \mathcal{C}^{0,1}$ and let $\{\Gamma_i\}_{i=1}^3$, be the corresponding parts of the boundary, $|\Gamma_3| = 0$ and let the operator L be symmetric elliptic in the sense of Definition 3.8.1. Let further $b^{\alpha\beta} = 0$ for any $\alpha, \beta \in \{1, \dots, N\}$. Then the following claims are equivalent

1. Function $\vec{\mathbf{T}} \in W$ is such that it holds for any $\vec{\mathbf{W}} \in W$
$$\Phi^*(\vec{\mathbf{T}}) \le \Phi^*(\vec{\mathbf{W}}).$$
2. $\vec{\mathbf{T}} = \vec{\mathbb{A}}\nabla\vec{u}$, where $\vec{u}$ is a weak solution to the corresponding elliptic problem, i.e., $\vec{u} - \vec{U}_0 \in V$ and it satisfies (3.136) with $\mathfrak{b} = \mathfrak{o}$.

Proof. **Step 1:** 2. $\Longrightarrow$ 1.
If $\vec{u}$ is a weak solution to our original problem, then it follows directly from the definition of V^* (see (3.139)) and the weak formulation of (3.136) that $\vec{\mathbf{T}} = \vec{\mathbb{A}}\nabla\vec{u} \in W$. Let now $\vec{\mathbf{W}} \in W$ be arbitrary. Due to the ellipticity of the matrix $\vec{\mathbb{A}}^{-1}$ and definition of Φ^* we get

$$\begin{aligned}
\Phi^*(\vec{\mathbf{W}}) - \Phi^*(\vec{\mathbf{T}}) &= \frac{1}{2}\int_\Omega \Big(\vec{\mathbb{A}}^{-1}\vec{\mathbf{W}} : \vec{\mathbf{W}} - \vec{\mathbb{A}}^{-1}\vec{\mathbf{T}} : \vec{\mathbf{T}}\Big)\,\mathrm{d}x - \int_\Omega \nabla\vec{U}_0 : (\vec{\mathbf{W}} - \vec{\mathbf{T}})\Big)\,\mathrm{d}x \\
&= \frac{1}{2}\int_\Omega \vec{\mathbb{A}}^{-1}(\vec{\mathbf{W}} - \vec{\mathbf{T}}) : (\vec{\mathbf{W}} - \vec{\mathbf{T}})\,\mathrm{d}x + \int_\Omega \vec{\mathbb{A}}^{-1}\vec{\mathbf{T}} : (\vec{\mathbf{W}} - \vec{\mathbf{T}})\,\mathrm{d}x \\
&\quad - \int_\Omega \nabla\vec{U}_0 : (\vec{\mathbf{W}} - \vec{\mathbf{T}})\,\mathrm{d}x \\
&\ge \int_\Omega \vec{\mathbb{A}}^{-1}\vec{\mathbf{T}} : (\vec{\mathbf{W}} - \vec{\mathbf{T}})\,\mathrm{d}x - \int_\Omega \nabla\vec{U}_0 : (\vec{\mathbf{W}} - \vec{\mathbf{T}})\,\mathrm{d}x \\
&= \int_\Omega \nabla\vec{u} : (\vec{\mathbf{W}} - \vec{\mathbf{T}})\,\mathrm{d}x - \int_\Omega \nabla\vec{U}_0 : (\vec{\mathbf{W}} - \vec{\mathbf{T}})\,\mathrm{d}x \\
&= \int_\Omega (\vec{\mathbf{W}} - \vec{\mathbf{T}}) : \nabla(\vec{u} - \vec{U}_0)\,\mathrm{d}x = 0,
\end{aligned}$$

where the last equality is based on the facts that $\vec{u} - \vec{U}_0 \in V$ and $\vec{\mathbf{T}}$, $\vec{\mathbf{W}} \in W$. We thus see that $\Phi^*(\vec{\mathbf{W}}) \ge \Phi^*(\vec{\mathbf{T}})$ for any $\vec{\mathbf{W}} \in W$.

Step 2: 1. $\Longrightarrow$ 2.
. We first deduce the *Euler–Lagrange equations* for problem formulated in claim 1. Therefore, let $\vec{\mathbf{T}} \in W$ be a minimizer of Φ^* over W. Let further $\vec{\mathbf{Z}} \in L^2(\Omega; \mathbb{R}^{d\times N})$ be an arbitrary function satisfying for every $\vec{\varphi} \in V$

$$\int_\Omega \vec{\mathbf{Z}} : \nabla\vec{\varphi}\,\mathrm{d}x = 0. \tag{3.142}$$

We set for arbitrary $t > 0$, $\vec{\mathbf{W}} := \vec{\mathbf{T}} + t\vec{\mathbf{Z}}$. Since $\vec{\mathbf{T}} \in W$ and $\vec{\mathbf{Z}}$ satisfies (3.142), it is then evident that $\vec{\mathbf{W}} \in W$ and it can be used in the variational formulation. From Claim 1. we thus have

$$\begin{aligned}
0 &\le \frac{1}{t}(\Phi^*(\vec{\mathbf{W}}) - \Phi^*(\vec{\mathbf{T}})) = \frac{1}{t}\big(\Phi^*(\vec{\mathbf{T}} + t\vec{\mathbf{Z}}) - \Phi^*(\vec{\mathbf{T}})\big) \\
&= \frac{1}{t}\int_\Omega \Big(\frac{1}{2}\vec{\mathbb{A}}^{-1}(\vec{\mathbf{T}} + t\vec{\mathbf{Z}}) : (\vec{\mathbf{T}} + t\vec{\mathbf{Z}}) - \frac{1}{2}\vec{\mathbb{A}}^{-1}\vec{\mathbf{T}} : \vec{\mathbf{T}} - t\nabla\vec{U}_0 : \vec{\mathbf{Z}}\Big)\,\mathrm{d}x \\
&= \int_\Omega \Big(\vec{\mathbb{A}}^{-1}\vec{\mathbf{T}} : \vec{\mathbf{Z}} + \frac{t}{2}\vec{\mathbb{A}}^{-1}\vec{\mathbf{Z}} : \vec{\mathbf{Z}} - \nabla\vec{U}_0 : \vec{\mathbf{Z}}\Big)\,\mathrm{d}x \\
&\overset{t\to 0_+}{\to} \int_\Omega (\vec{\mathbb{A}}^{-1}\vec{\mathbf{T}} - \nabla\vec{U}_0) : \vec{\mathbf{Z}}\,\mathrm{d}x.
\end{aligned}$$

Since $(-\vec{\mathbf{Z}})$ satisfies also (3.142), we get the opposite inequality, and thus for any $\vec{\mathbf{Z}} \in L^2(\Omega; \mathbb{R}^{d\times N})$ satisfying (3.142) it holds

$$\int_\Omega (\vec{\mathbb{A}}^{-1}\vec{\mathbf{T}} - \nabla\vec{U}_0) : \vec{\mathbf{Z}}\,\mathrm{d}x = 0. \tag{3.143}$$

Equations (3.143) are the Euler–Lagrange equations to the minimization problem for Φ^*.

We show in the second step of the proof that the validity of (3.143) requires further properties of $\vec{\mathbf{T}}$; in particular existence of $\vec{u}$ such that $\vec{u}-\vec{U}_0 \in V$ and $\vec{\mathbf{T}} = \vec{\mathbb{A}}\nabla\vec{u}$. If such $\vec{u}$ really exists, it must fulfil $\nabla\vec{u} = \vec{\mathbb{A}}^{-1}\vec{\mathbf{T}}$ almost everywhere in Ω. Instead of this, at the first glance strong assumption, we look for $\vec{u}$ such that $\vec{u} - \vec{U}_0 \in V$ and

$$\int_\Omega \nabla\vec{u} : \nabla\vec{\varphi}\,\mathrm{d}x = \int_\Omega \vec{\mathbb{A}}^{-1}\vec{\mathbf{T}} : \nabla\vec{\varphi}\,\mathrm{d}x \quad \text{for every } \vec{\varphi} \in V. \tag{3.144}$$

Since $\vec{\mathbb{A}}^{-1}$ is bounded, we may define $\langle \vec{f}, \vec{\varphi}\rangle_V := \int_\Omega \vec{\mathbb{A}}^{-1}\vec{\mathbf{T}} \cdot \nabla\vec{\varphi}\,\mathrm{d}x$ and see that we look for a weak solution of an elliptic problem. Due to Theorem 3.4.2 (verifying of the assumptions is left as an exercise for a kind reader) we know that there exists a unique solution to problem (3.144) (in the case of $|\Gamma_1| = 0$ we have the Neumann problem and the solution is unique up to an additive constant). We now show how from (3.143) and (3.144) Claim 2. follows.

It holds due to (3.144) that $\vec{\mathbf{Z}}$ defined by $\vec{\mathbf{Z}} := \vec{\mathbb{A}}^{-1}\vec{\mathbf{T}} - \nabla\vec{u}$ satisfies (3.142) and thus can be used in (3.143). We get the identity

$$\int_\Omega \vec{\mathbb{A}}^{-1}\vec{\mathbf{T}} : (\vec{\mathbb{A}}^{-1}\vec{\mathbf{T}} - \nabla\vec{u})\,\mathrm{d}x = \int_\Omega \nabla\vec{U}_0 : (\vec{\mathbb{A}}^{-1}\vec{\mathbf{T}} - \nabla\vec{u})\,\mathrm{d}x. \tag{3.145}$$

Further, the choice $\vec{\varphi} := \vec{u} - \vec{U}_0$ in (3.144) yields

$$\int_\Omega \nabla\vec{u} : (\nabla\vec{u} - \nabla\vec{U}_0)\,\mathrm{d}x = \int_\Omega \vec{\mathbb{A}}^{-1}\vec{\mathbf{T}} : (\nabla\vec{u} - \nabla\vec{U}_0)\,\mathrm{d}x. \tag{3.146}$$

From (3.145) and (3.146) we get (we also use the symmetry of $\vec{\mathbb{A}}$)

$$\begin{aligned} 0 &= \int_\Omega \Big(\vec{\mathbb{A}}^{-1}\vec{\mathbf{T}} : (\vec{\mathbb{A}}^{-1}\vec{\mathbf{T}} - \nabla\vec{u}) + \nabla\vec{u} : (\nabla\vec{u} - \nabla\vec{U}_0) + \nabla\vec{U}_0 : \nabla\vec{u} - \vec{\mathbb{A}}^{-1}\vec{\mathbf{T}} : \nabla\vec{u}\Big)\,\mathrm{d}x \\ &= \int_\Omega \Big(|\vec{\mathbb{A}}^{-1}\vec{\mathbf{T}}|^2 - 2\vec{\mathbb{A}}^{-1}\vec{\mathbf{T}} : \nabla\vec{u} + |\nabla\vec{u}|^2\Big)\,\mathrm{d}x = \int_\Omega |\vec{\mathbb{A}}^{-1}\vec{\mathbf{T}} - \nabla\vec{u}|^2\,\mathrm{d}x. \end{aligned}$$

Therefore it holds that $\vec{\mathbf{T}} = \vec{\mathbb{A}}\nabla\vec{u}$ almost everywhere in Ω and $\vec{u} - \vec{U}_0 \in V$. Furthermore, since $\vec{\mathbf{T}} \in W$, then $\vec{u}$ must fulfil (we use that fact that $\vec{\mathbb{A}}\nabla\vec{u} = \vec{\mathbf{T}} \in W$) the weak formulation (3.136) and thus it is also a weak solution. The proof is therefore finished. ■

Remark 3.8.4. We saw in the text above that the primer variational problem (minimizer for Φ) and the dual variational problem (minimizer for Φ^*) are in a certain sense equivalent for symmetric operator without absolute terms, i.e., in the cases when $\mathfrak{b} = \mathfrak{c} = \mathfrak{o}$, and lead to the same solution. The difference lies in the fact that by minimizing Φ we enforce the validity of the relation $L\vec{u} = \vec{f}$, while by minimizing Φ^* we enforce the validity of relation $\vec{\mathbf{T}} = \vec{\mathbb{A}}\nabla\vec{u}$. From the point of view of applications it is more useful to solve the problem for Φ, if we are interested in $\vec{u}$, and the problem for Φ^*, if we are interested in the form of $\vec{\mathbf{T}} = \vec{\mathbb{A}}\nabla\vec{u}$.

The first part of this chapter dealt with existence of weak solution to the elliptic problems based on the Lax–Milgram theorem. We now show that the proof of existence of weak solutions is much easier by looking for minima of a certain functionals, provided the operators are not only V–elliptic, but also symmetric.

Example 3.8.5 (Existence by means of minimizing certain functionals)**.** Show the existence of a solution to (3.133) by showing existence of $\vec{u}$ satisfying (3.138).

Solution. Let us now consider the elliptic problem with symmetric operator and assume that (3.135) holds and $B_{L,\mathfrak{c}}$ is V-elliptic. We show how it is possible to solve problem (3.133). We look the solution in the form $\vec{u} = \vec{U}_0 + \vec{w}$, where $\vec{w} \in V$. Theorem 3.8.2 claims that $\vec{u}$ is a weak solution, if and only if

$$\Phi_{L,\mathfrak{c}}(\vec{u} + \vec{\varphi}) - \Phi_{L,\mathfrak{c}}(\vec{u}) \geq \langle \vec{f}, \vec{\varphi}\rangle_V + \langle \vec{g}, \vec{\varphi}\rangle_{W^{\frac{1}{2},2}(\Gamma_2\cup\Gamma_3;\mathbb{R}^N)}.$$

Let $\vec{\psi} \in V$ be arbitrary and choose $\vec{\varphi} := \vec{\psi} - \vec{w} \in V$, where $\vec{w}$ is given by $\vec{u} = \vec{U}_0 + \vec{w}$. Then the above given inequality changes into the form

$$\Phi_{L,\mathfrak{c}}(\vec{U}_0 + \vec{w}) - \langle \vec{f}, \vec{w}\rangle_V - \langle \vec{g}, \vec{w}\rangle_{W^{\frac{1}{2},2}(\Gamma_2\cup\Gamma_3;\mathbb{R}^N)} \leq \Phi_{L,\mathfrak{c}}(\vec{U}_0 + \vec{\psi}) - \langle \vec{f}, \vec{\psi}\rangle_V - \langle \vec{g}, \vec{\psi}\rangle_{W^{\frac{1}{2},2}(\Gamma_2\cup\Gamma_3;\mathbb{R}^N)}.$$

Since $\vec{\psi} \in V$ is arbitrary, we in fact look for a minimizer of a certain functional. Let us consider

$$I := \inf_{\vec{w}\in V} \Big(\Phi_{L,\mathfrak{c}}(\vec{U}_0 + \vec{w}) - \langle \vec{f}, \vec{w}\rangle_V - \langle \vec{g}, \vec{w}\rangle_{W^{\frac{1}{2},2}(\Gamma_2\cup\Gamma_3;\mathbb{R}^N)}\Big).$$

It follows from the definition of the infimum that it holds for any $\vec{\psi} \in V$ that

$$I \leq \Phi_{L,\mathfrak{c}}(\vec{U}_0 + \vec{\psi}) - \langle \vec{f}, \vec{\psi}\rangle_V - \langle \vec{g}, \vec{\psi}\rangle_{W^{\frac{1}{2},2}(\Gamma_2\cup\Gamma_3;\mathbb{R}^N)}$$

and choosing $\vec{\psi} = \vec{0}$ we thus have

$$I \leq \Phi_{L,\sigma}(\vec{U}_0) < \infty.$$

The definition of the infimum implies that we can find a sequence $\{\vec{w}^n\}_{n\in\mathbb{N}} \subset V$ such that

$$I = \lim_{n\to\infty} \left(\Phi_{L,\sigma}(\vec{U}_0 + \vec{w}^n) - \langle \vec{f}, \vec{w}^n \rangle_V - \langle \vec{g}, \vec{w}^n \rangle_{W^{\frac{1}{2},2}(\Gamma_2\cup\Gamma_3;\mathbb{R}^N)} \right)$$

and furthermore that there exists n_0 such that for any $n \geq n_0$

$$\Phi_{L,\sigma}(\vec{U}_0 + \vec{w}^n) - \langle \vec{f}, \vec{w}^n \rangle_V - \langle \vec{g}, \vec{w}^n \rangle_{W^{\frac{1}{2},2}(\Gamma_2\cup\Gamma_3;\mathbb{R}^N)} \leq I + 1 \leq \Phi(\vec{U}_0) + 1. \tag{3.147}$$

Hence we see that the sequence $\vec{w}^n$ is bounded in V. The continuity of the trace operator also yields that the sequence is bounded in $W^{\frac{1}{2},2}(\partial\Omega;\mathbb{R}^N)$. Due to the reflexivity (we again relabel the subsequence)

$$\begin{aligned} \vec{w}^n &\rightharpoonup \vec{w} \quad \text{weakly in } V, \\ \vec{w}^n &\rightharpoonup \vec{w} \quad \text{weakly in } W^{\frac{1}{2},2}(\partial\Omega;\mathbb{R}^N). \end{aligned} \tag{3.148}$$

Using the definition of Φ and the V-ellipticity, symmetry and bilinearity of $B_{L,\sigma}$ we get the following estimate

$$\begin{aligned} \Phi_{L,\sigma}(\vec{U}_0 + \vec{w}^n) &= \frac{1}{2} B_{L,\sigma}(\vec{U}_0 + \vec{w}^n, \vec{U}_0 + \vec{w}^n) \\ &= \frac{1}{2} B_{L,\sigma}(\vec{U}_0 + \vec{w}, \vec{U}_0 + \vec{w}^n) + \frac{1}{2} B_{L,\sigma}(\vec{w}^n - \vec{w}, \vec{U}_0 + \vec{w}) \\ &\quad + \frac{1}{2} B_{L,\sigma}(\vec{w}^n - \vec{w}, \vec{w}^n - \vec{w}) \\ &= \frac{1}{2} B_{L,\sigma}(\vec{U}_0 + \vec{w}, \vec{U}_0 + \vec{w}) + B_{L,\sigma}(\vec{U}_0 + \vec{w}, \vec{w}^n - \vec{w}) \\ &\quad + \frac{1}{2} B_{L,\sigma}(\vec{w}^n - \vec{w}, \vec{w}^n - \vec{w}) \\ &\geq \Phi_{L,\sigma}(\vec{U}_0 + \vec{w}) + B_{L,\sigma}(\vec{w}^n - \vec{w}, \vec{U}_0 + \vec{w}). \end{aligned}$$

Next, due to the V-ellipticity we know that there exists $\vec{F} \in V^*$ such that $\vec{F} = B_{L,\sigma}(\cdot, \vec{U}_0 + \vec{w})$. We get, using the definition of the infimum I and the weak convergences (3.148) that

$$\begin{aligned} I &= \lim_{n\to\infty} \left(\Phi_{L,\sigma}(\vec{U}_0 + \vec{w}^n) - \langle \vec{f}, \vec{w}^n \rangle_V - \langle \vec{g}, \vec{w}^n \rangle_{W^{\frac{1}{2},2}(\Gamma_2\cup\Gamma_3;\mathbb{R}^N)} \right) \\ &\geq \Phi_{L,\sigma}(\vec{U}_0 + \vec{w}) + \lim_{n\to\infty} \left(\langle \vec{F}, \vec{w}^n - \vec{w} \rangle - \langle \vec{f}, \vec{w}^n \rangle_V - \langle \vec{g}, \vec{w}^n \rangle_{W^{\frac{1}{2},2}(\Gamma_2\cup\Gamma_3;\mathbb{R}^N)} \right) \\ &= \Phi_{L,\sigma}(\vec{U}_0 + \vec{w}) - \langle \vec{f}, \vec{w} \rangle_V - \langle \vec{g}, \vec{w} \rangle_{W^{\frac{1}{2},2}(\Gamma_2\cup\Gamma_3;\mathbb{R}^N)}. \end{aligned}$$

On the other hand, the definition of the infimum of I implies that

$$I \leq \Phi(\vec{U}_0 + \vec{w}) - \langle \vec{f}, \vec{w} \rangle_V - \langle \vec{g}, \vec{w} \rangle_{W^{\frac{1}{2},2}(\Gamma_2\cup\Gamma_3;\mathbb{R}^N)}$$

and thus necessarily

$$\begin{aligned} I &= \Phi(\vec{U}_0 + \vec{w}) - \langle \vec{f}, \vec{w} \rangle_V - \langle \vec{g}, \vec{w} \rangle_{W^{\frac{1}{2},2}(\Gamma_2\cup\Gamma_3;\mathbb{R}^N)} \\ &\leq \Phi(\vec{U}_0 + \vec{\varphi}) - \langle \vec{f}, \vec{\varphi} \rangle_V - \langle \vec{g}, \vec{\varphi} \rangle_{W^{\frac{1}{2},2}(\Gamma_2\cup\Gamma_3;\mathbb{R}^N)} \end{aligned}$$

for any $\vec{\varphi} \in V$. The function $\vec{w}$ is thus a minimizer and further, due to Theorem 3.8.2, $\vec{u} = \vec{U}_0 + \vec{w}$ is a weak solution to our problem. $\square$

Chapter 4

An easy guide to Lebesgue–Bochner and Sobolev–Bochner spaces

In this chapter, we briefly introduce the necessary function spaces and tools needed for the correct formulation of the evolutionary partial differential equations in the weak setting. As in the case of Sobolev spaces, we only introduce the basic concepts and present more details, including most of the proofs, in Chapter 8.

We aim at studying mappings

$$f\colon I \subset \mathbb{R} \to X,$$

where X is a Banach space. Most of the results in this section remain true even if we replace the one-dimensional interval by a measurable subset of $\mathbb{R}^d$, however, since we do not need this level of generality, we remain in $\mathbb{R}$. Moreover, instead of the Lebesgue measure on $\mathbb{R}$ we may as well consider a more general measure μ. The space X is always a Banach space; sometimes, however, we will require more properties (separability, reflexivity etc.). For simplicity, we always assume in this chapter that $I \subset \mathbb{R}$ is open and bounded, even though most of the results remain true for arbitrary interval in $\mathbb{R}$; however, the proofs must be slightly modified in this case. The presentation below follows in the first part nicely written thesis Kreuter (2015), the second part is then based on Boyer and Fabrie (2006) or more precisely, on Pokorný (2022).

4.1 Bochner integral

First, we construct a new type of integral, in order to be able to integrate functions with values in general Banach spaces.

Definition 4.1.1 — Simple function. A function $s\colon I \to X$ is called a simple function, if we can write

$$s(t) = \sum_{i=1}^{n} x_i \chi_{E_i}(t), \tag{4.1}$$

where $x_i \in X$, E_i are pairwise disjoint, measurable, and $\sum_{i=1}^{n} \lambda_1(E_i) < +\infty$.

Remark 4.1.2. Assuming $x_i \neq x_j$ for $i \neq j$ and $\cup_{i=1}^{n} E_i = I$ (it also means that we define s to be the zero element on the set of one dimensional Lebesgue measure zero, where the function is possibly not defined), then the representation of the function s in the form (4.1) is unique.

Definition 4.1.3 — Strong and weak measurability.

1. A function $f\colon I \to X$ is (strongly) measurable, if there exists a sequence of simple functions $\{s_n\}_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \|s_n(t) - f(t)\|_X = 0$$

for almost all $t \in I$.

2. A function $f\colon I \to X$ is weakly measurable, if

$$\langle x', f\rangle_X$$

is measurable for all $x' \in X^*$, where X^* denotes the dual space to X.

Definition 4.1.4 — Almost separably valued function. A function $f\colon I \to X$ is called almost separably valued, if there exists a set $N \subset I$, $\lambda_1(N) = 0$ such that $f(I \setminus N)$ is separable. If N is empty, then we say that the function f is separably valued.

We have the following important result

Theorem 4.1.5 — Pettis. A function $f\colon I \to X$ is measurable, if and only if f is weakly measurable and almost separably valued.

The theorem has several corollaries.

Corollary 4.1.6. Let $f\colon I \to X$ be continuous. Then f is measurable.

Corollary 4.1.7. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of measurable functions from I to X such that $f_n(t) \rightharpoonup f(t)$ in X for almost every $t \in I$. Then f is measurable.

We are now ready to define the Bochner integral. For a simple function s, we may evidently set

$$\int_I s \,\mathrm{d}\lambda_1 := \sum_{i=1}^{n} x_i \lambda_1(E_i),$$

where $s = \sum_{i=1}^{n} x_i \chi_{E_i}(t)$. It can be easily shown, similarly as for the Lebesgue integral, that this definition is independent of the representative of s and that

$$\Big\| \int_I s \,\mathrm{d}\lambda_1 \Big\|_X \le \int_I \|s\|_X \,\mathrm{d}t.$$

We now extend the definition to measurable functions.

Definition 4.1.8 — Bochner integral. A measurable function $f\colon I \to X$ is Bochner integrable, if there exists a sequence $\{s_n\}_{n\in\mathbb{N}}$ of simple functions converging to f almost everywhere such that

$$\lim_{n\to\infty} \int_I \|f - s_n\|_X \,\mathrm{d}t = 0.$$

Then for such a function f we define

$$\int_I f \,\mathrm{d}\lambda_1 := \lim_{n\to\infty} \int_I s_n \,\mathrm{d}\lambda_1.$$

Note that the definition is correct in the sense that

$$\Big\| \int_I s_n \,\mathrm{d}\lambda_1 - \int_I s_m \,\mathrm{d}\lambda_1 \Big\|_X \le \int_I \|s_n - s_m\|_X \,\mathrm{d}t \le \int_I \|s_n - f\|_X \,\mathrm{d}t + \int_I \|s_m - f\|_X \,\mathrm{d}t \to 0 \qquad \text{as } n, m \to \infty.$$

Hence $\int_I s_n \,\mathrm{d}\lambda_1$ is a Cauchy sequence and due to the completeness of X it is convergent. If $\{\widetilde{s}_n\}_{n\in\mathbb{N}}$ is another sequence of simple functions approximating f in X almost everywhere together with $\lim_{n\to\infty} \int_I \|f - \widetilde{s}_n\|_X \,\mathrm{d}t = 0$, it is easy to see that the limits of sequences $\left\{ \int_I s_n \,\mathrm{d}\lambda_1 \right\}_{n\in\mathbb{N}}$ and $\left\{ \int_I \widetilde{s}_n \,\mathrm{d}\lambda_1 \right\}_{n\in\mathbb{N}}$ are the same which shows that the definition of the integral is independent of the approximating sequence of the simple functions. The linearity of the Bochner integral is a consequence of the linearity of the integral for simple functions.

The following fundamental theorem yields the correspondence between the Bochner and Lebesgue integrals.

Theorem 4.1.9 — Bochner. Let $f\colon I \to X$ be a measurable function. Then f is Bochner integrable over I, if and only if $\|f\|_X$ is Lebesgue integrable over I. Moreover,

$$\Big\| \int_I f \,\mathrm{d}\lambda_1 \Big\|_X \le \int_I \|f\|_X \,\mathrm{d}t. \tag{4.2}$$

Corollary 4.1.10 (Dominated convergence Theorem). Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of Bochner integrable functions and let f be such that $f_n \to f$ almost everywhere in I. Let $g \in L^1(I;\mathbb{R})$ be such that $\|f_n\|_X \le g$ for all $n \in \mathbb{N}$ almost everywhere in I. Then f is Bochner integrable and

$$\int_I f \,\mathrm{d}\lambda_1 = \lim_{n\to\infty} \int_I f_n \,\mathrm{d}\lambda_1.$$

Corollary 4.1.11. Let f be Bochner integrable. Then the sequence s_n of simple functions converging to f can be chosen so that $\|s_n(t)\|_X \le 2\|f(t)\|_X$ holds for almost every $t \in I$.

Corollary 4.1.12. Let $x' \in X^*$ and let f be Bochner integrable over I. Then

$$\int_I \langle x', f\rangle_X \,\mathrm{d}t = \Big\langle x', \int_I f \,\mathrm{d}\lambda_1 \Big\rangle_X.$$

Corollary 4.1.13. Let f be Bochner integrable over I. Then

$$\lim_{\lambda_1(J)\to 0^+, J\subset I} \int_J f \,\mathrm{d}\lambda_1 = 0 \in X.$$

We finish this section by a generalization of the Fubini theorem for the Bochner integral.

Theorem 4.1.14 — Fubini. Let $J = I_1 \times I_2$ be a product measure space with respect to the measure $\lambda_1 \otimes \lambda_1 = \lambda_2$ and let $f\colon J \to X$ be measurable. Assume that the integral

$$\int_{I_1} \Big(\int_{I_2} \|f(t_1, t_2)\|_X \,\mathrm{d}t_2 \Big) \,\mathrm{d}t_1 \tag{4.3}$$

exists and is finite. Then f is Bochner integrable over $I_1 \times I_2$ and we have

$$\int_{I_1\times I_2} f \,\mathrm{d}\lambda_2 = \int_{I_1} \Big(\int_{I_2} f(t_1, t_2) \,\mathrm{d}t_2 \Big) \,\mathrm{d}t_1 = \int_{I_2} \Big(\int_{I_1} f(t_1, t_2) \,\mathrm{d}t_1 \Big) \,\mathrm{d}t_2. \tag{4.4}$$

Conversely, if f is Bochner integrable over $I_1 \times I_2$, then the integrals above exist and the equalities hold.

4.2 The spaces $L^p(I;X)$ (Lebesgue–Bochner spaces)

We now introduce an analogue of the Lebesgue spaces.

Definition 4.2.1 — Lebesgue–Bochner spaces. A measurable function $f \in L^p(I;X)$, $1 \le p \le \infty$, if for $1 \le p < \infty$

$$\int_I \|f\|_X^p \,\mathrm{d}t < \infty,$$

and for $p = \infty$

$$\operatorname*{ess\,sup}_I \|f\|_X < \infty.$$

Remark 4.2.2. Note that a function from $L^p(I;X)$ is (recall that I is bounded) Bochner integrable. In case we would accept the possibility that I is unbounded, then the function is at least locally Bochner integrable over I. We also introduce the notation

$$\|f\|_{L^p(I;X)} := \begin{cases} \Big(\int_I \|f\|_X^p \,\mathrm{d}t \Big)^{\frac{1}{p}} & \text{if } 1 \le p < \infty \\ \operatorname{ess\,sup}_I \|f\|_X & \text{if } p = \infty. \end{cases}$$

We now present several results which are more or less similar to the standard results for Lebesgue spaces and which will finally lead to the result that the spaces defined above (often called Lebesgue–Bochner spaces) are complete normed spaces.

Lemma 4.2.3 Let $1 \le p \le \infty$, $f \in L^p(I;X)$ and $g \in L^{p'}(I;\mathbb{R})$, $\frac{1}{p} + \frac{1}{p'} = 1$. Then $f \cdot g \in L^1(I;X)$ and

$$\Big\| \int_I fg \,\mathrm{d}\lambda_1 \Big\|_X \le \int_I \|f\|_X |g| \,\mathrm{d}t \le \|f\|_{L^p(I;X)} \|g\|_{L^{p'}(I)}.$$

Lemma 4.2.4 Let $1 \le p \le \infty$, $f \in L^p(I;X)$ and $g \in L^{p'}(I;X^*)$, $\frac{1}{p} + \frac{1}{p'} = 1$ and X^* is the dual space to X. Then $\langle g, f\rangle_X \in L^1(I;\mathbb{R})$ and

$$\Big| \int_I \langle g, f\rangle_X \,\mathrm{d}t \Big| \le \|f\|_{L^p(I;X)} \|g\|_{L^{p'}(I;X^*)}.$$

Corollary 4.2.5. We have for $1 \le p \le q \le \infty$ that $L^q(I;X) \hookrightarrow L^p(I;X)$ and

$$\|f\|_{L^p(I;X)} \le \lambda_1(I)^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^q(I;X)}.$$

In general, for I possibly unbounded, we have at least that any $f \in L^q(I;X)$ is locally integrable over I and belongs to $L^1_{\mathrm{loc}}(I;X)$.

Lemma 4.2.6 Let $1 \le p \le \infty$ and let $f_n \to f$ in $L^p(I;X)$. Then there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ which converges to f almost everywhere in I (in the norm of X).

As in the scalar case we therefore have

Theorem 4.2.7 — Properties of Lebesgue–Bochner spaces. The spaces $L^p(I;X)$ are Banach spaces with respect to the norms $\|f\|_{L^p(I;X)}$ introduced above, where we consider two functions identical, $f_1 = f_2$, provided $f_1(t) = f_2(t)$ for almost every $t \in I$ (in the sense of equality in X).

If $p = 2$ and X is a Hilbert space, then $L^2(I;X)$ is a Hilbert space with respect to the scalar product

$$(f,g)_{L^2(I;X)} := \int_I (f,g)_X \,\mathrm{d}t.$$

We now study the question of the density for different classes of functions in the spaces $L^p(I;X)$ and the related question of the separability of these function spaces. We denote (recall, I is a bounded, open interval in $\mathbb{R}$)

$$\begin{aligned}
\mathcal{C}(I;X) &= \{f\colon I \to X \mid \text{ continuous in } I \text{ with values in } X\}\\
\mathcal{C}^k(I;X) &= \{f\colon I \to X \mid f, f', \dots, f^{(k)} \in \mathcal{C}(I;X)\}\\
\mathcal{C}(\overline{I};X) &= \{f\colon I \to X \mid \text{ continuous in } I \text{ with values in } X \text{ up to the endpoints}\}\\
\mathcal{C}^k(\overline{I};X) &= \{f\colon I \to X \mid f, f', \dots, f^{(k)} \in \mathcal{C}(\overline{I};X)\}\\
\mathcal{C}_0(I;X) &= \{f\colon I \to X \mid f \in \mathcal{C}(I;X), f \text{ is compactly supported in } I\},\\
\mathcal{C}^\infty(\overline{I};X) &= \bigcap_{k\in\mathbb{N}} \mathcal{C}^k(\overline{I};X)\\
\mathcal{C}_0^\infty(I;X) &= \mathcal{C}^\infty(I;X) \bigcap \mathcal{C}_0(I;X).
\end{aligned}$$

Theorem 4.2.8 — Dense subsets of Lebesgue–Bochner spaces. Let $1 \le p < \infty$. Then

1. simple functions are dense in $L^p(I;X)$
2. functions of the form $s(t) = \sum_{j=1}^n \varphi_j(t)x_j$, $\varphi_j \in \mathcal{C}_0^\infty(I;\mathbb{R})$, $x_j \in X$, are dense in $L^p(I;X)$
3. if the set Y is dense in X, then $\mathcal{C}_0^\infty(I;Y)$ is dense in $L^p(I;X)$
4. let η_ε be the regularizing kernel in $\mathbb{R}$ and suppose f is extended by zero outside I; then $f \star \eta_\varepsilon \to f$ in $L^p(I;X)$ for $\varepsilon \to 0_+$.

Corollary 4.2.9. Let X be a separable Banach space, then $L^p(I;X)$ is separable for $1 \le p < \infty$.

Theorem 4.2.10 — Lebesgue points. Let $f \in L^1_{\mathrm{loc}}(I;X)$. Then

$$\frac{1}{2h}\int_{-h}^{h} \|f(t+s) - f(t)\|_X \,\mathrm{d}s \to 0$$

for $h \to 0_+$ for almost every $t \in I$. In particular,

$$f(t) = \lim_{h\to 0_+} \frac{1}{2h}\int_{-h}^{h} f(t+s)\,\mathrm{d}\lambda_1(s)$$

almost everywhere in I.

Finally, we have

Lemma 4.2.11 Let f_n be a sequence of functions from $L^p(I;X)$, $1 \le p \le \infty$ such that $\|f_n\|_{L^p(I;X)} \le C < +\infty$ for all $n \in \mathbb{N}$. Let there exist $f\colon I \to X$ such that for almost every $t \in I$

$$f_n(t) \rightharpoonup f(t) \quad \text{ in } X.$$

Then $f \in L^p(I;X)$ and $\|f\|_{L^p(I;X)} \le C$.

We might expect that the dual space to $L^p(I;X)$ for $p < \infty$ is $L^{p'}(I;X^*)$. However, this is in general not true for arbitrary Banach space and we have to add some properties of X. On the other hand, using Lemma 4.2.4, we at least know that for arbitrary Banach space X we have

$$L^{p'}(I;X^*) \hookrightarrow \big(L^p(I;X)\big)^*.$$

Moreover, for any $g \in L^{p'}(I;X^*)$, it also holds:

$$\|g\|_{(L^p(I;X))^*} \le \|g\|_{L^{p'}(I;X)}. \tag{4.5}$$

In fact, we also have:

Proposition 4.2.12. For $1 \le p < \infty$ the inclusion mapping $I\colon L^{p'}(I;X^*) \to \big(L^p(I;X)\big)^*$ is an isometry, i.e., equality in (4.5) holds.

We can show the following.

Theorem 4.2.13 — Duality for Lebesgue–Bochner spaces. Let either X be a reflexive Banach space or let X^* be separable. Then $\big(L^p(I;X)\big)^*$ is isometrically isomorphic to the space $L^{p'}(I;X^*)$ for $1 \le p < \infty$.

Corollary 4.2.14. If X is reflexive and $1 < p < \infty$, then $L^p(I;X)$ is reflexive.

4.3 Spaces $W^{1,p}(I;X)$ (Sobolev–Bochner spaces)

Definition 4.3.1 — Weak time derivative. Let $u \in L^1_{\text{loc}}(I;X)$ and $g \in L^1_{\text{loc}}(I;X)$. We say that g is weak derivative of u with respect to t, i.e., $g = u'$, if

$$\int_I u(t)\varphi'(t)\,\mathrm{d}\lambda_1(t) = -\int_I g(t)\varphi(t)\,\mathrm{d}\lambda_1(t)$$

holds for all $\varphi \in \mathcal{C}_0^\infty(I;\mathbb{R})$.

In what follows, in particular in Chapters 5 and 9, we will often use the notation $u' = \partial_t u$.
We have

Lemma 4.3.2 Let $g \in L^1_{\text{loc}}(I;X)$, $t_0 \in I$ and let I be an open interval. Let

$$f(t) := \int_{t_0}^t g(s)\,\mathrm{d}\lambda_1(s).$$

Then $f \in \mathcal{C}(I;X)$ and

$$f' = g$$

in the weak sense and almost everywhere in I.

Further

Corollary 4.3.3. Let $f \in L^1_{\text{loc}}(I;X)$ be such that $f' = 0$ almost everywhere in I. Then there exists $x_0 \in X$ such that $f = x_0$ almost everywhere in I.

Definition 4.3.4 — Sobolev–Bochner spaces. Let $1 \le p \le \infty$ and $u \in L^p(I;X)$ be such that $u' \in L^p(I;X)$. Then we say that u belongs to the Sobolev–Bochner space $W^{1,p}(I;X)$ and we define

$$\|u\|_{W^{1,p}(I;X)} := \|u\|_{L^p(I;X)} + \|u'\|_{L^p(I;X)}.$$

Proposition 4.3.5. If $1 \le p \le \infty$, then $W^{1,p}(I;X)$ is with respect to the norm defined above a Banach space. If X is a Hilbert space, then $W^{1,2}(I;X)$ is a Hilbert space with respect to the scalar product

$$(u,v)_{W^{1,2}(I;X)} := \int_I (u,v)_X\,\mathrm{d}\lambda_1 + \int_I (u',v')_X\,\mathrm{d}\lambda_1$$

for any u, $v \in W^{1,2}(I;X)$. The associated norm is an equivalent norm on $W^{1,2}(I;X)$.

Proposition 4.3.6. If X is reflexive, then $W^{1,p}(I;X)$ is for $1 < p < \infty$ reflexive. If $1 \le p < \infty$ and X is separable, then $W^{1,p}(I;X)$ is separable.

Proposition 4.3.7. Let $1 \le p \le \infty$ and $u \in W^{1,p}(I;X)$. Then there exists $t_0 \in I$ such that for almost all $t \in I$

$$u(t) = u(t_0) + \int_{t_0}^t u'(s)\,\mathrm{d}\lambda_1(s).$$

In fact, Proposition 4.3.7 yields existence of $N \subset I$, a null set, such that $u(t) = u(t_0) + \int_{t_0}^t u'(s)\,\mathrm{d}\lambda_1(s)$ for all t, $t_0 \in I \backslash N$. Let $I = (a,b)$. If $g \in L^1(a,b;X)$ and $f(t) = \int_a^t g(s)\,\mathrm{d}\lambda_1(s)$, then also $\|g\|_X \in L^1((a,b))$ and $f \in \mathcal{AC}([a,b];X)$. Recall that $f\colon [a,b] \to X$ is absolutely continuous on $[a,b]$, if $\forall \varepsilon > 0\, \exists \delta > 0$: $\sum_{i=1}^n \|f(b_i) - f(a_i)\|_X < \varepsilon$ for all

$\{[a_i, b_i]\}_{i=1}^n$, disjoint collection of subintervals of I with the total length less than δ. Therefore, if $u \in W^{1,p}(a, b; X)$, then there exists a representative of u, a function $\widetilde{u} \in \mathcal{AC}([a, b]; X)$ such that $\widetilde{u}$ is differentiable almost everywhere, $\widetilde{u}' = u'$ and $\widetilde{u} = u$ almost everywhere in $[a, b]$. Unless stated differently, we shall always assume that we work directly with this representative.

We further have

Proposition 4.3.8. Let $1 \le p \le \infty$. Then $W^{1,p}(I; X) \hookrightarrow \mathcal{C}(\overline{I}; X)$, i.e., working with the representative from above, we have

$$\|u\|_{\mathcal{C}(\overline{I};X)} \le C\|u\|_{W^{1,p}(I;X)}.$$

The previous results are summarized in the following theorem.

Theorem 4.3.9 — Properties of weak derivative. Let $u \in L^p(I; X)$, $1 \le p \le \infty$. Then the following assertions are equivalent:

1. $u \in W^{1,p}(I; X)$
2. u is absolutely continuous on $\overline{I}$ and $u' \in L^p(I; X)$
3. there exists a function $v \in L^p(I; X)$ such that for all $x' \in X^*$ the function $\psi(t) := \langle x', u(t)\rangle_X$ is absolutely continuous on I and $\psi'(t) = \langle x', v(t)\rangle_X$ almost everywhere in I.

Furthermore, we have

Theorem 4.3.10 — Density of smooth functions in time. Let $1 \le p < \infty$. Then $\mathcal{C}^\infty(\overline{I}; X)$ is dense in $W^{1,p}(I; X)$.

4.4 Gelfand triple and properties of corresponding Bochner spaces

We next consider a more general setting, where the time derivative in general belongs to a different space than the function itself.

Definition 4.4.1 — Gelfand triple. Let X be a separable reflexive Banach space such that there exists a Hilbert space H, where $X \hookrightarrow H$ densely. Then we call the triple X, $H \cong H^*$ and X^* the Gelfand triple.

In what follows we shall prove that

$$X \hookrightarrow H \cong H^* \hookrightarrow X^*,$$

where both embeddings are dense. The identification of H and H^* is through the Riesz representation Theorem. Let $x \in X$ and $Ix \in H$, where $I\colon X \to H$ represents the embedding. Let $\Phi\colon H^* \to H$ be the mapping which represents the Riesz representation Theorem. Then we may define $i\colon X \to X^*$ as follows

$$\langle ix_0, x\rangle_X := (Ix_0, Ix)_H = \langle \Phi^{-1} Ix_0, Ix\rangle_H,$$

where x, $x_0 \in X$. The mapping $i\colon X \to X^*$ is injective and $i(X)$ is dense in X^* (due to the fact that both embeddings are dense).

Example 4.4.2. Let $X = W_0^{1,2}(\Omega)$ and $H = L^2(\Omega)$. As $W_0^{1,2}(\Omega)$ is densely embedded into $L^2(\Omega)$, the spaces $W_0^{1,2}(\Omega)$, $L^2(\Omega)$ and $W^{-1,2}(\Omega) = \big(W_0^{1,2}(\Omega)\big)^*$ form the Gelfand triple. By the Riesz representation Theorem, for any $f \in W^{-1,2}(\Omega)$ there exists unique $u_f \in W_0^{1,2}(\Omega)$ such that

$$\langle f, v\rangle_{W_0^{1,2}(\Omega)} = ((u_f, v))_{W_0^{1,2}(\Omega)} := \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x,$$

where the scalar product is equivalent to the standard scalar product on $W^{1,2}(\Omega)$. Now, if $u \in C_0^\infty(\Omega)$ (which is dense in $W_0^{1,2}(\Omega)$), then

$$((u, v))_{W_0^{1,2}(\Omega)} = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x = -\int_\Omega \Delta u v \,\mathrm{d}x$$

for any $v \in W_0^{1,2}(\Omega)$. Hence for any $f \in W^{-1,2}(\Omega)$ there exists $\{u_n\}_{n\in\mathbb{N}}$ such that $u_n \to u_f$ in $W_0^{1,2}(\Omega)$. Then

$$\begin{aligned}\langle f, v\rangle_{W_0^{1,2}(\Omega)} &= \lim_{n\to\infty} \int_\Omega -\Delta u_n v \,\mathrm{d}x = \lim_{n\to\infty} (-\Delta u_n, v)_{L^2(\Omega)} \\ &= \lim_{n\to\infty} \langle i(-\Delta u_n), v\rangle_{W_0^{1,2}(\Omega)} =: \lim_{n\to\infty} \langle i(f_n), v\rangle_{W_0^{1,2}(\Omega)},\end{aligned}$$

where $f_n \in C_0^\infty(\Omega)$. Clearly, i is injective by definition and $i(W_0^{1,2}(\Omega))$ is dense in $L^2(\Omega)$. Moreover, if $f \in L^2(\Omega)$, then $-\Delta u_n \to f$ in $L^2(\Omega)$ and we have that

$$\langle f, v\rangle_{W_0^{1,2}(\Omega)} = \int_\Omega f v \, \mathrm{d}x$$

for any $v \in W_0^{1,2}(\Omega)$.

In this setting, we can generalize the definition of the time derivative.

Definition 4.4.3 — Generalized weak time derivative. Let $u \in L^p(I;X)$, where X, H and X^* form the Gelfand triple. Then we say that $v \in L^q(I;X^*)$ is the time derivative of u, if

$$\int_I \langle v, w\rangle_X \psi \, \mathrm{d}t = -\int_I (u, w)_H \psi' \, \mathrm{d}t$$

holds for all $w \in X$, $\psi \in \mathcal{C}_0^\infty(I;\mathbb{R})$. We denote $u' := v$.

We have the following important result.

Theorem 4.4.4 — Continuous representative for Gelfand triple. Let X, H, X^* form the Gelfand triple, and let $u \in L^p(I;X)$ with $u' \in L^{p'}(I;X^*)$, $1 < p < \infty$. Then $u = \widetilde{u}$ almost everywhere in I, where $\widetilde{u} \in \mathcal{C}(\overline{I};H)$. Moreover, $t \mapsto \|u(t)\|_H^2$ is weakly differentiable, and

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_H^2 = 2\langle u'(t), u(t)\rangle_X$$

almost everywhere in I. In particular,

$$\|\widetilde{u}(t_2)\|_H^2 = \|\widetilde{u}(t_1)\|_H^2 + \int_{t_1}^{t_2} 2\langle u'(s), u(s)\rangle_X \, \mathrm{d}s.$$

Remark 4.4.5. Similarly as in Theorem 4.4.4 we may also show that if u, $v \in L^p(I;X)$, u', $v' \in L^{p'}(I;X^*)$, then for $\psi \in \mathcal{C}_0^\infty(I;\mathbb{R})$

$$\int_I \Big(\langle u', v\rangle_X + \langle v', u\rangle_X\Big)\psi \, \mathrm{d}t = -\int_I (u, v)_H \psi' \, \mathrm{d}t$$

as well as

$$\int_{t_1}^{t_2} \Big(\langle u', v\rangle_X + \langle v', u\rangle_X\Big)\psi \, \mathrm{d}t = (u(t_2), v(t_2))_H \psi(t_2) - (u(t_1), v(t_1))_H \psi(t_1) - \int_{t_1}^{t_2} (u, v)_H \psi' \, \mathrm{d}t$$

for any $\psi \in C^\infty(\overline{I};\mathbb{R})$ and any $t_1, t_2 \in \overline{I}$.

The proof of the theorem requires several auxiliary results which are also important independently.

Lemma 4.4.6 Let $u \in L^p(I;X)$ and $u' \in L^q(I;Y)$, $1 \le p, q < \infty$, where X, Y are Banach spaces, $X \hookrightarrow Y$. Then there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{C}^\infty(\overline{I};X)$ such that $\{u_n'\}_{n\in\mathbb{N}} \subset \mathcal{C}^\infty(\overline{I};Y)$, and

$$u_n \to u \quad \text{in } L^p(I;X) \quad \text{and} \quad u_n' \to u' \quad \text{in } L^q(I;Y).$$

Lemma 4.4.7 Let X be a reflexive Banach space, Y a Banach space and $X \hookrightarrow Y$ densely. Then $Y^* \hookrightarrow X^*$ densely.

Definition 4.4.8 — Weak continuity. Let X be a Banach space. We say that $u\colon I \to X$ is continuous in the weak topology of X (weakly continuous in X), $u \in \mathcal{C}(I;X_w)$, if the mapping

$$t \mapsto \langle x', u(t)\rangle_X$$

is continuous in I for all $x' \in X^*$.

Clearly, if $u \in \mathcal{C}(I;X)$, then $u \in \mathcal{C}(I;X_w)$, the other implication is generally true only if X is finite dimensional.

Lemma 4.4.9 Let X, Y be two Banach spaces, X reflexive, $X \hookrightarrow Y$ densely, $I \subset \mathbb{R}$ be bounded, open. Let $\varphi \in L^\infty(I;X)$ and $\varphi \in \mathcal{C}(\overline{I};Y_w)$. Then $\varphi \in \mathcal{C}(\overline{I};X_w)$.

4.5 Compact embedding of spaces with time derivative

We close this chapter by presenting a result which will replace for the evolutionary equations the result on compact embedding of Sobolev spaces and thus will allow to solve certain classes of nonlinear problems. We consider for X_0 and X_1 Banach spaces

$$W = W^{\alpha_0,\alpha_1}_{X_0,X_1} = \{v \in L^{\alpha_0}(I;X_0) \,|\, v' \in L^{\alpha_1}(I;X_1)\},$$

where $I \subset \mathbb{R}$ is an open bounded interval. We define

$$\|u\|_W := \|u\|_{L^{\alpha_0}(I;X_0)} + \|u'\|_{L^{\alpha_1}(I;X_1)}.$$

The main result is

Theorem 4.5.1 — Aubin–Lions. Let X_0, X_1 and X be Banach spaces such that $X_0 \hookrightarrow\hookrightarrow X \hookrightarrow X_1$. Let X_0, X_1 be reflexive, $1 < \alpha_0, \alpha_1 < \infty$ and let I be open, bounded. Then $W \hookrightarrow\hookrightarrow L^{\alpha_0}(I;X)$.

Chapter 5

Linear evolutionary equations

We shall now study problems which describe evolution of certain quantities in time. In this chapter we restrict ourselves to linear problems which are easier to study and we present existence of weak solutions to linear parabolic equation (as, e.g., the heat equation) and linear hyperbolic problems (as, e.g., the wave equation). We also study regularity of weak solutions as well as some of their further properties. In the whole chapter, we restrict ourselves to scalar equations. Another method, based on the theory of semigroups, will be presented in Chapter 9.

5.1 Second order parabolic equations

Let $\Omega \subset \mathbb{R}^d$ be open, bounded and let $T \in (0, \infty)$. We set $Q_T := (0,T) \times \Omega$ and consider

$$\begin{aligned} \partial_t u + Lu &= f && \text{in } Q_T \\ u &= 0 && \text{on } (0,T) \times \partial\Omega \\ u(0,\cdot) &= g(\cdot) && \text{in } \Omega, \end{aligned} \tag{5.1}$$

where f: $Q_T \to \mathbb{R}$ and g: $\Omega \to \mathbb{R}$ are given, and u: $Q_T \to \mathbb{R}$ is unknown. Moreover, we assume that

$$Lu := -\sum_{i,j=1}^d \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big) + \sum_{i=1}^d c_i \frac{\partial u}{\partial x_i} + bu,$$

where $\{a_{ij}\}_{i,j=1}^d$, $\{c_i\}_{i=1}^d$ and b: $Q_T \to \mathbb{R}$ are given functions.

Definition 5.1.1 — Parabolic operator. We say that the operator $\partial_t + L$ is parabolic if there exists $C_1 > 0$ such that

$$\sum_{i,j=1}^d a_{ij}(t,x)\xi_i\xi_j \geq C_1|\boldsymbol{\xi}|^2$$

for all $\boldsymbol{\xi} \in \mathbb{R}^d$ and almost every $(t,x) \in Q_T$. Moreover, the functions $\{a_{ij}\}_{i,j=1}^d$, $\{c_i\}_{i=1}^d$ and b are from $L^\infty(Q_T)$.

Example 5.1.2. Taking $a_{ij} = \delta_{ij}$, $\mathbf{c} = \mathbf{0}$ and $b = 0$ we get that $Lu = -\Delta u$ and the corresponding equation $(5.1)_1$ reduces to the heat equation

$$\partial_t u - \Delta u = f.$$

Keeping $\mathbf{c} = \mathbf{0}$ and $b = 0$, but considering $Lu = -\sum_{i,j=1}^d \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big)$ we get a generalized heat equation (corresponding to an inhomogeneous material with respect to the heat conductivity)

$$\partial_t u - \sum_{i,j=1}^d \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big) = f.$$

5.1.1 Existence of weak solutions to linear parabolic problems

In order to obtain the weak formulation for our problem, we define

$$B[u,v](t) := \int_\Omega \Big[\sum_{i,j=1}^d a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i} + \sum_{i=1}^d c_i\frac{\partial u}{\partial x_i}v + buv\Big](t,\cdot)\,\mathrm{d}x$$

which is well defined for almost every $t \in (0,T)$ if u, $v \in W^{1,2}(\Omega)$ and $\{a_{ij}\}_{i,j=1}^d$, $\{c_i\}_{i=1}^d$, $b \in L^\infty(Q_T)$. We first reformulate our problem (5.1) into the setting we introduced in Chapter 4.

We consider the mapping
$$u\colon [0,T] \to W_0^{1,2}(\Omega)$$
defined by
$$[u(t)](x) := u(t,x), \qquad x \in \Omega,\, t \in [0,T],$$
and similarly
$$f\colon [0,T] \to L^2(\Omega) \ \text{ or } \ W^{-1,2}(\Omega).$$

Since we consider $u \in L^2(0,T;W^{1,2}(\Omega))$ and a_{ij}, c_i and $b \in L^\infty(Q_T)$, using
$$\partial_t u = \sum_{i,j=1}^d \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big) - \sum_{i=1}^d c_i \frac{\partial u}{\partial x_i} - bu + f,$$
we may expect that $\partial_t u \in L^2(0,T;W^{-1,2}(\Omega))$. For simplicity, we first consider only the homogeneous Dirichlet boundary conditions. A more complex choice of boundary conditions will be presented later. We have

Definition 5.1.3 — Weak solution to linear parabolic problem. Let $T \in (0,\infty)$, $f \in L^2(0,T;W^{-1,2}(\Omega))$, $g \in L^2(\Omega)$ and let the operator $\partial_t + L$ be parabolic in the sense of Definition 5.1.1. Then $u \in L^2(0,T;W_0^{1,2}(\Omega))$ with $\partial_t u \in L^2(0,T;W^{-1,2}(\Omega))$ is a weak solution to our problem (5.1), provided
$$\langle \partial_t u, v\rangle_{W_0^{1,2}(\Omega)} + B[u,v](t) = \langle f, v\rangle_{W_0^{1,2}(\Omega)}$$
holds for almost every $t \in (0,T)$ and all $v \in W_0^{1,2}(\Omega)$ and
$$u(0,\cdot) = g(\cdot).$$

Remark 5.1.4. If $u \in L^2(0,T;W_0^{1,2}(\Omega))$ and $\partial_t u \in L^2(0,T;W^{-1,2}(\Omega))$, then $u \in \mathcal{C}([0,T];L^2(\Omega))$, as $W_0^{1,2}(\Omega)$, $L^2(\Omega)$ and $W^{-1,2}(\Omega)$ form a Gelfand triple. Therefore the initial condition at $t=0$ is satisfied in the sense
$$\lim_{t\to 0_+} \|u(t,\cdot) - g\|_{L^2(\Omega)} = 0,$$
where we work with the continuous representative from Theorem 4.4.4.

Before proving existence of the unique weak solution, let us show several important auxiliary results.

Lemma 5.1.5 — Properties of projection to eigenfunctions of Laplace equation I. Let $\{w_k\}_{k=1}^\infty$ be the eigenfunctions to Laplace equation with Dirichlet boundary conditions normalized in $L^2(\Omega)$, i.e., for any $k \in \mathbb{N}$ the function $w_k \in W_0^{1,2}(\Omega)$ and
$$\int_\Omega \nabla w_k \cdot \nabla v \,\mathrm{d}x = \lambda_k \int_\Omega w_k v \,\mathrm{d}x$$
holds for any $v \in W_0^{1,2}(\Omega)$, where $\lambda_k > 0$ are the corresponding eigenvalues, $\|w_k\|_2 = 1$. We set for arbitrary $u \in L^2(\Omega)$ and $n \in \mathbb{N}$
$$u_n := \sum_{j=1}^n a_j w_j,$$
where $a_j := (u, w_k)_2 = \int_\Omega u w_j \,\mathrm{d}x$. Then it holds.

1. We have for any $n \in \mathbb{N}$
$$\|u_n\|_2 \le \|u\|_2. \tag{5.2}$$
2. If further $u \in W_0^{1,2}(\Omega)$, then also for any $n \in \mathbb{N}$
$$\|\nabla u_n\|_2 \le \|\nabla u\|_2. \tag{5.3}$$
3. Moreover, there exists a positive constant C such that if $\Omega \in C^{1,1}$ and $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$, for any $n \in \mathbb{N}$ it holds
$$\|u_n\|_{2,2} \le C\|u\|_{2,2}. \tag{5.4}$$

Proof. **Step 1:** Proof of Claim 1.
The proof follows from standard theory on abstract Fourier series, as $\{w_k\}_{k=1}^\infty$ forms an orthonormal complete system in $L^2(\Omega)$.

Step 2: Proof of Claim 2.
Recall that $\{\frac{\nabla w_k}{\sqrt{\lambda_k}}\}_{k=1}^\infty$ forms an orthonormal system in $W_0^{1,2}(\Omega)$ with respect to the equivalent norm in $W_0^{1,2}(\Omega)$ defined

as $\|u\|_{W_0^{1,2}(\Omega)} := \|\nabla u\|_{L^2(\Omega;\mathbb{R}^d)}$. We then compute (recall that $(\nabla w_k, \nabla w_l)_2 = 0$ if $k \neq l$)

$$\|\nabla u_n\|_2^2 = \Big\| \sum_{k=1}^n (u, w_k)_2 \nabla w_k \Big\|_2^2 = \sum_{k=1}^n (u, w_k)_2^2 \|\nabla w_k\|_2^2 = \sum_{k=1}^n \frac{1}{\lambda_k^2} \|\nabla w_k\|_2^2 (\nabla u, \nabla w_k)_2^2$$
$$= \sum_{k=1}^n \frac{1}{\lambda_k} (\nabla u, \nabla w_k)_2^2 = \sum_{k=1}^n \Big(\nabla u, \frac{\nabla w_k}{\sqrt{\lambda_k}}\Big)_2^2 \leq \|\nabla u\|_2^2,$$

where in the last step we used the Bessel inequality.

Step 3: Proof of Claim 3.
Note first that under the assumption on regularity of Ω there exist positive constants C_1 and C_2 such that for any $v \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ we have

$$C_1 \|v\|_{2,2} \leq \|\Delta v\|_2 \leq C_2 \|v\|_{2,2}. \tag{5.5}$$

This follows directly from Theorem 3.7.8. Indeed, let $f \in L^2(\Omega)$ be such that

$$\Delta v = f.$$

Since $v \in W_0^{1,2}(\Omega)$, we may view the problem above as a weak solution to the Laplace equation with the homogeneous Dirichlet boundary condition. Whence by Theorem 3.7.8 we know that

$$\|v\|_{2,2} \leq C \|f\|_2.$$

We thus compute

$$\|v\|_{2,2}^2 \leq \|f\|_2^2 = \|\Delta v\|_2^2.$$

The other inequality is trivial. Note further that again due to Theorem 3.7.8 and Remark 3.7.9 we know that for any $k \in \mathbb{N}$ $w_k \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. As

$$(\Delta w_k, \Delta w_l)_2 = \lambda_k \lambda_l (w_k, w_l)_2 = \lambda_k \lambda_l \delta_{kl},$$

we have

$$C_1 \|u_n\|_{2,2}^2 \leq \|\Delta u_n\|_2^2 = \Big\| \sum_{k=1}^n (u, w_k) \Delta w_k \Big\|_2^2 = \sum_{k=1}^n (u, w_k)_2^2 \|\Delta w_k\|_2^2$$
$$= \sum_{k=1}^n (u, w_k)_2^2 \lambda_k^2 = \sum_{k=1}^n (\nabla u, \nabla w_k)_2^2 = \sum_{k=1}^n (\Delta u, w_k)_2^2 \leq \|\Delta u\|_2^2 \leq C_2 \|u\|_{2,2}^2.$$

Above, we used twice inequality (5.5), properties of the eigenfunctions to the Laplace operator as well as Bessel inequality. ■

In case we study more general boundary conditions for the parabolic problem, the following generalization of Lemma 5.1.5 is useful. Note that we can even have $|\Gamma_1| = 0$ (i.e., $\lambda_1 = 0$ and $w_1 = const = \frac{1}{\sqrt{|\Omega|}}$).

Lemma 5.1.6 — Properties of projection to eigenfunctions of Laplace equation II. Let

$$V = \{u \in W^{1,2}(\Omega) \,|\, u = 0 \text{ on } \Gamma_1 \subset \partial\Omega\},$$

$\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup M$, $|M|_{d-1} = 0$ and consider the following problem in the weak setting,

$$\begin{aligned} -\Delta w_k &= \lambda_k w_k && \text{in } \Omega \\ w_k &= 0 && \text{on } \Gamma_1 \\ \frac{\partial w_k}{\partial \boldsymbol{\nu}} &= 0 && \text{on } \Gamma_2, \end{aligned}$$

i.e., we look for a nontrivial $w_k \in V$, $\|w_k\|_2 = 1$, such that

$$\int_\Omega \nabla w_k \nabla v \,\mathrm{d}x = \lambda_k \int_\Omega w_k v \,\mathrm{d}x$$

holds for all $v \in V$. We set as above $u_n = \sum_{k=1}^n (u, w_k) w_k$, where $u \in L^2(\Omega)$.

1. We have for any $n \in \mathbb{N}$
$$\|u_n\|_2 \leq \|u\|_2.$$
2. If additionally $u \in V$, then also
$$\|\nabla u_n\|_2 \leq \|\nabla u\|_2.$$

Proof. The proof is exactly the same as in the case of Lemma 5.1.5, Claims 1. and 2. ■

Next we recall

Lemma 5.1.7 — Gronwall lemma. Let $\eta \in \mathcal{C}([0,T))$, g, $h \in L^1(0,T)$, g nonnegative almost everywhere in $(0,T)$ be such that

$$\eta'(t) \leq \eta(t)g(t) + h(t) \tag{5.6}$$

holds almost everywhere in $(0,T)$. Then for any $t \in (0,T]$

$$\eta(t) \leq \mathrm{e}^{\int_0^t g(s)\,\mathrm{d}s}\Big(\eta(0) + \int_0^t h(s)\,\mathrm{d}s\Big). \tag{5.7}$$

Exercise 5.1.8. Prove the Gronwall inequality (5.7).

We now aim at showing the following fundamental result

Theorem 5.1.9 — Existence and uniqueness of weak solutions to linear parabolic problem. Under the assumptions stated in Definition 5.1.3, for any $T < \infty$, there exists a unique weak solution to Problem (5.1) in the sense of Definition 5.1.3.

Proof. **Step 1:** Galerkin approximation
We assume that $\{w_k\}_{k\in\mathbb{N}}$ is a the orthonormal basis in $L^2(\Omega)$ and orthogonal in $W^{1,2}(\Omega)$ formed by eigenfunctions of the Laplace equation with homogeneous Dirichlet boundary condition. We have that $w_k \in \mathcal{C}^\infty(\Omega) \cap W_0^{1,2}(\Omega)$ for any $k \in \mathbb{N}$. We now look for the approximate solution in the form

$$u_n(t,x) := \sum_{k=1}^n d_i^n(t) w_k(x) \tag{5.8}$$

and we have that

$$u_n(0,x) = \sum_{k=1}^n d_k^n(0) w_k(x), \tag{5.9}$$

where

$$d_k^n(0) := \int_\Omega g w_k \,\mathrm{d}x = (g, w_k)_{L^2(\Omega)}. \tag{5.10}$$

Further, we assume that u_n solves our problem for test functions from a finite dimensional subspace of $W_0^{1,2}(\Omega)$ only, i.e., from the linear hull of $\{w_k\}_{k=1}^n$. Then we can write our approximate problem as follows

$$(\partial_t u_n(t,\cdot), w_k)_{L^2(\Omega)} + B[u_n, w_k](t) = \langle f(t,\cdot), w_k\rangle_{W_0^{1,2}(\Omega)}, \tag{5.11}$$

$k = 1, 2, \ldots, n$, where $u_n(0)$ fulfils (5.9)–(5.10). The main point is that we may rewrite (5.11) as a system of ordinary differential equations for the unknown functions $\{d_k^n\}_{k=1}^n$ as follows

$$\begin{aligned} {d_k^n}'(t) + B\Big[\sum_{l=1}^n d_l^n(t) w_l, w_k\Big](t) = \langle f(t,\cdot), w_k\rangle_{W_0^{1,2}(\Omega)} \\ d_k^n(0) = (g, w_k)_{L^2(\Omega)}. \end{aligned} \tag{5.12}$$

Note that (5.12) is a system of linear ordinary differential equations of the first order, where the right-hand side belongs to $L^2(0,T)$. Therefore we may apply the Carathéodory theory to deduce existence of a unique vector-valued absolutely continuous function $\vec{d}$ solving (5.12) a.e. in $(0,T)$. (If the reader prefers to use the classical theory for ODEs, it is possible to mollify the function f in time which we, however, will not do in these Lecture Notes.)

Step 2: Energy estimates
We intend to pass with $n \to \infty$, so we need to control the sequence of approximate solutions independently of n, i.e., we need to prove the so called *a priori estimates.* To this aim, we multiply the Galerkin approximation tested by w_k by $d_k^n(t)$ and sum up with respect to k, from 1 to n. It is in fact the same as using as a test function the solution itself. We get

$$\int_\Omega \partial_t u_n u_n \,\mathrm{d}x + \int_\Omega \Big(\sum_{i,j=1}^d a_{ij} \frac{\partial u_n}{\partial x_i}\frac{\partial u_n}{\partial x_j} + \sum_{i=1}^d c_i \frac{\partial u_n}{\partial x_i} u_n + b|u_n|^2\Big)\,\mathrm{d}x = \langle f, u_n\rangle_{W_0^{1,2}(\Omega)}. \tag{5.13}$$

We have

$$\int_\Omega \partial_t u_n u_n \,\mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u_n|^2 \,\mathrm{d}x$$

and
$$B[u_n,u_n](t) \geq C_1\|\nabla u_n\|^2_{L^2(\Omega;\mathbb{R}^d)} - C\|\nabla u_n\|_{L^2(\Omega;\mathbb{R}^d)}\|u_n\|_{L^2(\Omega)} - C\|u_n\|^2_{L^2(\Omega)}.$$
Therefore
$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_n\|^2_{L^2(\Omega)} + \widetilde{C}_1\|u_n\|^2_{W^{1,2}(\Omega)} \leq C\big(\|u_n\|^2_{L^2(\Omega)} + \|f\|^2_{W^{-1,2}(\Omega)}\big). \tag{5.14}$$

We now employ the Gronwall Lemma 5.1.7. Inequality
$$\eta'(t) \leq C_1\eta(t) + C_2F(t)$$
in $(0,T)$ implies
$$\eta(t) \leq \mathrm{e}^{C_1t}\Big(\eta(0) + C_2\int_0^t F(s)\,\mathrm{d}s\Big)$$
in $(0,T)$. Denoting $\eta(t) := \|u_n\|^2_{L^2(\Omega)}$, $F(t) := \|f(t)\|^2_{W^{-1,2}(\Omega)}$, we end up with
$$\|u_n(t,\cdot)\|^2_{L^2(\Omega)} \leq C(T)\big(\|u_n(0,\cdot)\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(0,t;W^{-1,2}(\Omega))}\big)$$
for arbitrary $t \in (0,T)$. Due to the Bessel inequality (see also Lemma 5.1.5, Case 1.)
$$\|u_n(0,\cdot)\|_{L^2(\Omega)} \leq \|g\|^2_{L^2(\Omega)},$$
hence we get
$$\max_{t\in[0,T]}\|u_n(t,\cdot)\|_{L^2(\Omega)} \leq C\big(\|g\|_{L^2(\Omega)} + \|f\|_{L^2(0,T;W^{-1,2}(\Omega))}\big).$$
We now return back to (5.14) and read from it
$$\widetilde{C}_1\int_0^T \|u_n\|^2_{W^{1,2}(\Omega)}\,\mathrm{d}t \leq C\Big(\int_0^T \|u_n\|^2_{L^2(\Omega)}\,\mathrm{d}t + \int_0^t \|f\|^2_{W^{-1,2}(\Omega)}\,\mathrm{d}t\Big) + \|u_n(0,\cdot)\|^2_{L^2(\Omega)},$$
therefore, together with the estimate above
$$\|u_n\|_{L^\infty(0,T;L^2(\Omega))} + \|u_n\|_{L^2(0,T;W^{1,2}(\Omega))} \leq C\Big(\|f\|_{L^2(0,T;W^{-1,2}(\Omega))} + \|g\|_{L^2(\Omega)}\Big).$$
We, however, also need an estimate of the time derivative. For arbitrary $v \in W^{1,2}_0(\Omega)$ we can write
$$v = v_n^1 + v_n^2,$$
where v_n^1 belongs to the linear hull of $\{w_k\}_{k=1}^n$, and v_n^2 is perpendicular to this set in $L^2(\Omega)$. We easily get
$$\int_\Omega \partial_t u_n v\,\mathrm{d}x = \int_\Omega \partial_t u_n(v_n^1 + v_n^2)\,\mathrm{d}x = \int_\Omega \partial_t u_n v_n^1\,\mathrm{d}x + 0$$
and we may use the Galerkin approximation for u_n. It yields (recall that $W^{1,2}_0(\Omega)$, $L^2(\Omega)$ and $W^{-1,2}(\Omega)$ form the Gelfand triple)
$$\langle\partial_t u_n, v_n^1\rangle_{W^{1,2}_0(\Omega)} = \int_\Omega \partial_t u_n v_n^1\,\mathrm{d}x = \langle f, v_n^1\rangle_{W^{1,2}_0(\Omega)} - B[u_n, v_n^1](t),$$
and therefore
$$\begin{aligned}
\|\partial_t u_n\|_{W^{-1,2}(\Omega)} &= \sup_{\substack{v\in W^{1,2}_0(\Omega)\\ \|v\|_{W^{1,2}(\Omega)}\leq 1}} \langle\partial_t u_n, v\rangle_{W^{1,2}_0(\Omega)} = \sup_{\substack{v\in W^{1,2}_0(\Omega)\\ \|v\|_{W^{1,2}(\Omega)}\leq 1}} \int_\Omega \partial_t u_n v\,\mathrm{d}x \\
&= \sup_{\substack{v\in W^{1,2}_0(\Omega)\\ \|v\|_{W^{1,2}(\Omega)}\leq 1}} \int_\Omega \partial_t u_n v_n^1\,\mathrm{d}x = \sup_{\substack{v\in W^{1,2}_0(\Omega)\\ \|v\|_{W^{1,2}(\Omega)}\leq 1}} \Big(\langle f, v_n^1\rangle_{W^{1,2}_0(\Omega)} - B[u_n, v_n^1]\Big) \\
&\leq \sup_{\substack{v\in W^{1,2}_0(\Omega)\\ \|v\|_{W^{1,2}(\Omega)}\leq 1}} \big(\|f\|_{W^{-1,2}(\Omega)} + C\|u_n\|_{W^{1,2}(\Omega)}\big)\|v_n^1\|_{W^{1,2}(\Omega)} \\
&\leq C\big(\|f\|_{W^{-1,2}(\Omega)} + \|u_n\|_{W^{1,2}(\Omega)}\big),
\end{aligned}$$
where we used that $\|v_n^1\|_{W^{1,2}(\Omega)} \leq \|v\|_{W^{1,2}(\Omega)}$, see Lemma 5.1.5, Claim 2. Therefore we have
$$\begin{aligned}
\|\partial_t u_n\|_{L^2(0,T;W^{-1,2}(\Omega))} &\leq C\big(\|f\|_{L^2(0,T;W^{-1,2}(\Omega))} + \|u_n\|_{L^2(0,T;W^{1,2}(\Omega))}\big) \\
&\leq C\big(\|f\|_{L^2(0,T;W^{-1,2}(\Omega))} + \|g\|_{L^2(\Omega)}\big).
\end{aligned}$$

Step 3: Limit passage
Next we want to let $n \to \infty$. As the spaces $L^2(0,T;W_0^{1,2}(\Omega))$ and $L^2(0,T;W^{-1,2}(\Omega))$ are Hilbert spaces and the space $L^\infty(0,T;L^2(\Omega))$ has a separable predual space, there exists $u \in L^2(0,T;W_0^{1,2}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ with $\partial_t u \in L^2(0,T;W^{-1,2}(\Omega))$ such that for a chosen subsequence $\{n_k\}_{k\in\mathbb{N}} \subset \{n\}_{n\in\mathbb{N}}$ we have for $k \to \infty$

$$\begin{aligned} u_{n_k} &\rightharpoonup u && \text{in } L^2(0,T;W_0^{1,2}(\Omega)) \\ u_{n_k} &\rightharpoonup^* u && \text{in } L^\infty(0,T;L^2(\Omega)) \\ \partial_t u_{n_k} &\rightharpoonup \partial_t u && \text{in } L^2(0,T;W^{-1,2}(\Omega)). \end{aligned}$$

Passing to the limit in the modified Galerkin approximation

$$\int_0^T \langle \partial_t u_{n_k}, w_l\rangle_{W_0^{1,2}(\Omega)} \psi \,\mathrm{d}t + \int_0^T B[u_{n_k}, w_l]\psi \,\mathrm{d}t = \int_0^T \langle f, w_l\rangle_{W_0^{1,2}(\Omega)} \psi \,\mathrm{d}t,$$

where $\psi \in \mathcal{C}_0^\infty((0,T))$, we get

$$\int_0^T \langle \partial_t u, w_l\rangle_{W_0^{1,2}(\Omega)} \psi \,\mathrm{d}t + \int_0^T B[u, w_l]\psi \,\mathrm{d}t = \int_0^T \langle f, w_l\rangle_{W_0^{1,2}(\Omega)} \psi \,\mathrm{d}t$$

for all $l \in \mathbb{N}$ and all $\psi \in \mathcal{C}_0^\infty((0,T))$. As the linear hull of $\{w_l\}_{l\in\mathbb{N}}$ is dense in $W_0^{1,2}(\Omega)$, we easily show

$$\int_0^T \langle \partial_t u, v\rangle_{W_0^{1,2}(\Omega)} \psi \,\mathrm{d}t + \int_0^T B[u, v]\psi \,\mathrm{d}t = \int_0^T \langle f, v\rangle_{W_0^{1,2}(\Omega)} \psi \,\mathrm{d}t \tag{5.15}$$

for all $v \in W_0^{1,2}(\Omega)$ and all $\psi \in \mathcal{C}_0^\infty((0,T))$. Hence

$$\langle \partial_t u, v\rangle_{W_0^{1,2}(\Omega)} + B[u,v]\psi = \langle f, v\rangle_{W_0^{1,2}(\Omega)}$$

for all $v \in W^{1,2}(\Omega)$ almost everywhere in $(0,T)$. Note that v does not depend on time and the set of times, where the equality holds, is connected to the set of Lebesgue points of the three terms above. Thus the equality holds on the same set for all functions $v \in W_0^{1,2}(\Omega)$.

Let us now verify that $\lim_{t\to 0_+} \|u(t) - g\|_{L^2(\Omega)} = 0$. First, we have from (5.15) due to the definition of the time derivative

$$-\int_0^T (u,v)_{L^2(\Omega)} \psi' \,\mathrm{d}t + \int_0^T B[u,v](t)\psi \,\mathrm{d}t = \int_0^T \langle f, v\rangle_{W_0^{1,2}(\Omega)} \psi \,\mathrm{d}t$$

for all $v \in W_0^{1,2}(\Omega)$ and all $\psi \in \mathcal{C}_0^\infty((0,T))$. As $(u,v)_{L^2(\Omega)} \in \mathcal{C}([0,T])$ for any $v \in W_0^{1,2}(\Omega)$, we may take a sequence of $\psi_n \in \mathcal{C}_0^\infty((0,T))$ such that it converges to a function $\psi \in \mathcal{C}_0^\infty([0,T))$ locally uniformly in $(0,T)$ (and ψ' converges in distributions to $\psi' + \psi(0)\delta$). Then we get

$$-\int_0^T (u,v)_{L^2(\Omega)} \psi' \,\mathrm{d}t - (u(0,\cdot), v(\cdot))_{L^2(\Omega)}\psi(0) + \int_0^T B[u,v](t)\psi \,\mathrm{d}t = \int_0^T \langle f, v\rangle_{W_0^{1,2}(\Omega)} \psi \,\mathrm{d}t \tag{5.16}$$

for all $v \in W_0^{1,2}(\Omega)$ and $\psi \in \mathcal{C}_0^\infty([0,T))$. Similarly, starting now from the Galerkin approximation, we have (we multiply the k-th equation by $\psi \in \mathcal{C}_0^\infty([0,T))$, integrate over the time interval and use the integration by parts with respect to time)

$$-\int_0^T (u_{n_k}, w_l)_{L^2(\Omega)} \psi' \,\mathrm{d}t - (u_{n_k}(0,\cdot), w_l(\cdot))_{L^2(\Omega)}\psi(0) + \int_0^T B[u_{n_k}, w_l](t)\psi \,\mathrm{d}t = \int_0^T \langle f, w_l\rangle_{W_0^{1,2}(\Omega)} \psi \,\mathrm{d}t$$

for any $l \in \{1,2,\dots,n_k\}$. Using the fact that $(u_{n_k}(0), w_l)_{L^2(\Omega)} = (g, w_l)_{L^2(\Omega)}$ for any $l \le n_k$, we may pass to the limit $k \to \infty$ to get

$$-\int_0^T (u, w_l)_{L^2(\Omega)} \psi' \,\mathrm{d}t - (g, w_l)_{L^2(\Omega)}\psi(0) + \int_0^T B[u, w_l](t)\psi \,\mathrm{d}t = \int_0^T \langle f, w_l\rangle_{W_0^{1,2}(\Omega)} \psi \,\mathrm{d}t$$

for any $l \in \mathbb{N}$ and any $\psi \in \mathcal{C}_0^\infty([0,T))$. Next, using the density of the linear hull of $\{w_l\}_{l\in\mathbb{N}}$ in $W_0^{1,2}(\Omega)$ we easily get

$$-\int_0^T (u, v)_{L^2(\Omega)} \psi' \,\mathrm{d}t - (g, v)_{L^2(\Omega)}\psi(0) + \int_0^T B[u, v](t)\psi \,\mathrm{d}t = \int_0^T \langle f, v\rangle_{W_0^{1,2}(\Omega)} \psi \,\mathrm{d}t$$

for any $v \in W_0^{1,2}(\Omega)$ and any $\psi \in \mathcal{C}_0^\infty([0,T))$. Hence

$$(u(0,\cdot), v(\cdot))_{L^2(\Omega)} = (g, v)_{L^2(\Omega)} \qquad \forall v \in W_0^{1,2}(\Omega) \supset C_0^\infty(\Omega)$$

which yields

$$u(0,\cdot) = g(\cdot).$$

Since $u \in \mathcal{C}([0,T];L^2(\Omega))$, we also have

$$\lim_{t\to 0_+} \|u(t)-g\|_{L^2(\Omega)} = 0.$$

Step 4: Uniqueness
Since the problem is linear, it is enough to verify that unique solution to our problem with $f \equiv 0$ and $g \equiv 0$ is $u \equiv 0$. As the solution u is an appropriate test function, we have for almost every $t \in (0,T)$

$$\langle u', u\rangle_{W_0^{1,2}(\Omega)} + B[u,u] = 0.$$

Using now Theorem 4.4.4 and the properties of the bilinear form, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{L^2(\Omega)} + \frac{\widetilde{C}_1}{2}\|u\|^2_{W^{1,2}(\Omega)} \le C\|u\|^2_{L^2(\Omega)}.$$

The Gronwall Lemma 5.1.7 together with the fact that the initial condition is zero yields

$$\|u(t,\cdot)\|^2_{L^2(\Omega)} = 0 \qquad \text{for any } t \in [0,T],$$

whence $u = 0$ almost everywhere in Q_T. ■

We now consider a more general initial-boundary value problem (with the same parabolic operator as above)

$$\begin{aligned} \partial_t u + Lu &= f && \text{in } Q_T \\ u &= u_0 && \text{on } (0,T)\times\Gamma_1 \\ (\mathbb{A}\nabla u)\cdot \boldsymbol{\nu} &= h && \text{on } (0,T)\times\Gamma_2 \\ (\mathbb{A}\nabla u)\cdot \boldsymbol{\nu} + \sigma u &= h && \text{on } (0,T)\times\Gamma_3 \\ u(\cdot,0) &= g(\cdot) && \text{in } \Omega. \end{aligned} \tag{5.17}$$

We set

$$V = \{v \in W^{1,2} \,|\, v = 0 \text{ on } \Gamma_1 \text{ in the sense of traces}\}.$$

We assume that there exists

$$U_0 \in L^2(0,T;W^{1,2}(\Omega)) \quad \text{with } \partial_t U_0 \in L^2(0,T;(W^{1,2}(\Omega))^*), \qquad U_0|_{\Gamma_1} = u_0 \tag{5.18}$$

and further, we take

$$f \in L^2(0,T;V^*), \qquad h \in L^2(0,T;(W^{\frac{1}{2},2}(\Gamma_2\cup\Gamma_3))^*), \qquad \sigma \in L^\infty((0,T)\times\Gamma_3) \qquad g \in L^2(\Omega). \tag{5.19}$$

Definition 5.1.10 We say that $u \in L^2(0,T;W^{1,2}(\Omega))$ such that $u - U_0 \in L^2(0,T;V)$, $\partial_t(u-U_0) \in L^2(0,T;V^*)$ is a weak solution to problem (5.17), if

$$\langle \partial_t u, v\rangle_V + B[u,v](t) + \int_{\Gamma_3} \sigma u v \,\mathrm{d}S = \langle f,v\rangle_V + \langle h, v\rangle_{W^{\frac{1}{2},2}(\Gamma_2\cup\Gamma_3)}$$

for all $v \in V$ and almost every $t \in (0,T)$, together with $u(0,\cdot) = g(\cdot)$.

Remark 5.1.11. Note that $V \hookrightarrow H = L^2(\Omega) = (L^2(\Omega))^* \hookrightarrow V^*$, where the embeddings are dense, form the Gelfand triple. Furthermore $u - U_0 \in \mathcal{C}([0,T];L^2(\Omega))$. Since also $U_0 \in \mathcal{C}([0,T];L^2(\Omega))$, we have $u \in \mathcal{C}([0,T];L^2(\Omega))$. Therefore the initial condition is well defined.

Theorem 5.1.12 — Existence and uniqueness of weak solutions to linear parabolic problem with general boundary conditions. Let $\Omega \in \mathcal{C}^{0,1}$ and $\{\Gamma_i\}_{i=1}^3$ be as in the elliptic problem. Let the operator $\partial_t + L$ be parabolic. Then under assumptions (5.18)–(5.19) there exists unique weak solution to (5.17) in the sense of Definition 5.1.10.

Proof. We take $\{w_k\}_{k\in\mathbb{N}}$, formed by the eigenfunction to the following problem

$$\begin{aligned} -\Delta w_k &= \lambda_k w_k && \text{in } \Omega \\ w_k &= 0 && \text{on } \Gamma_1 \\ \frac{\partial w_k}{\partial \boldsymbol{\nu}} &= 0 && \text{on } \Gamma_2\cup\Gamma_3. \end{aligned}$$

We now look for u_n in the form

$$u_n(t,x) = U_0(t,x) + \sum_{k=1}^{n} d_k^n(t) w_k(x)$$

with initial condition

$$(u_n - U_0)(0,x) = \sum_{k=1}^{n} (u_n - U_0, w_k)_{L^2(\Omega)}(t) w_k(x)$$

such that

$$(\partial_t u_n, w_k)_{L^2(\Omega)} + B[u_n, w_k] + \int_{\Gamma_3} \sigma u_n w_k \,\mathrm{d}S = \langle f, w_k \rangle_V + \langle h, w_k \rangle_{W^{\frac{1}{2},2}(\Gamma_2 \cup \Gamma_3)}$$

for all $k = 1, 2, \dots, n$. The existence of a unique solution to this problem can be shown exactly as in Theorem 5.1.9. For the uniform estimates, we use as a test function $u_n - U_0$ (which belongs to the linear hull of $\{w_k\}_{k=1}^n$) and get

$$(\partial_t u_n, u_n - U_0)_{L^2(\Omega)} + B[u_n, u_n - U_0] + \int_{\Gamma_3} \sigma u_n (u_n - U_0) \,\mathrm{d}S = \langle f, u_n - U_0 \rangle_V + \langle h, u_n - U_0 \rangle_{W^{\frac{1}{2},2}(\Gamma_2 \cup \Gamma_3)}.$$

This can be rewritten as

$$\begin{aligned}(\partial_t (u_n - U_0), u_n - U_0)_{L^2(\Omega)} + B[u_n - U_0, u_n - U_0] + \int_{\Gamma_3} \sigma (u_n - U_0)^2 \,\mathrm{d}S = -\langle \partial_t U_0, u_n - U_0 \rangle_{W_0^{1,2}(\Omega)} \\ - B[U_0, u_n - U_0] - \int_{\Gamma_3} \sigma U_0 (u_n - U_0) \,\mathrm{d}S + \langle f, u_n - U_0 \rangle_V + \langle h, u_n - U_0 \rangle_{W^{\frac{1}{2},2}(\Gamma_2 \cup \Gamma_3)}.\end{aligned}$$

As in Theorem 5.1.9, by means of the Gronwall argument, we get

$$\begin{aligned}&\max_{t\in[0,T]} \|(u_n - U_0)(t,\cdot)\|_{L^2(\Omega)} + \frac{\widetilde{C}_1}{2} \|u_n - U_0\|_{L^2(0,T;W^{1,2}(\Omega))} \\ &\le C\big(\|f\|_{L^2(0,T;V^*)} + \|U_0\|_{L^2(0,T;W^{1,2}(\Omega))} + \|\partial_t U_0\|_{L^2(0,T;(W^{1,2}(\Omega)))^*} + \|h\|_{(W^{\frac{1}{2},2}(\Gamma_2\cup\Gamma_3))^*} + \|(u_n - U_0)(0,\cdot)\|_{L^2(\Omega)}\big) \\ &\le C\big(\|f\|_{L^2(0,T;V^*)} + \|U_0\|_{L^2(0,T;W^{1,2}(\Omega))} + \|\partial_t U_0\|_{L^2(0,T;(W^{1,2}(\Omega)))^*} + \|h\|_{(W^{\frac{1}{2},2}(\Gamma_2\cup\Gamma_3))^*} + \|g\|_{L^2(\Omega)}\big).\end{aligned}$$

The difficulty is connected with the surface integral over Γ_3, where we need to estimate

$$\|u_n - U_0\|_{L^2(\Gamma_3)} \le C \|u_n - U_0\|_{L^2(\Omega)}^{\frac{1}{2}} \|u_n - U_0\|_{W^{1,2}(\Omega)}^{\frac{1}{2}}$$

which follows from the properties of the trace operator.

Similarly as in Theorem 5.1.9 we also get

$$\begin{aligned}&\|\partial_t (u_n - U_0)\|_{L^2(0,T;V^*)} \\ &\quad \le C\big(\|f\|_{L^2(0,T;V^*)} + \|U_0\|_{L^2(0,T;W^{1,2}(\Omega))} + \|\partial_t U_0\|_{L^2(0,T;(W^{1,2}(\Omega)))^*} + \|h\|_{(W^{\frac{1}{2},2}(\Gamma_2\cup\Gamma_3))^*} + \|g\|_{L^2(\Omega)}\big).\end{aligned}$$

The limit passage $n \to \infty$ is the same as above in Theorem 5.1.9, similarly also the satisfaction of the initial condition. Finally, for uniqueness, we again need to verify that for zero data, the only solution is zero. We have

$$\langle \partial_t u, v \rangle_V + B[u,v](t) + \int_{\Gamma_3} \sigma u v \,\mathrm{d}S = 0$$

with $u(0) = 0$. We plug in $v := u$ and get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^2(\Omega)}^2 + \frac{\widetilde{C}_1}{2}\|u\|_{W^{1,2}(\Omega)} \le C(\|u\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Gamma_3)}^2).$$

We again employ the interpolation inequality

$$\|u\|_{L^2(\Gamma_3)} \le C \|u\|_{L^2(\Omega)}^{\frac{1}{2}} \|u\|_{W^{1,2}(\Omega)}^{\frac{1}{2}}$$

together with the Young inequality and the Gronwall inequality yields due to the zero initial condition that $u = 0$ almost everywhere in Q_T. The proof is complete. ■

5.1.2 Regularity of weak solutions to linear parabolic problems

Let us now look at the problem of the regularity of the solution. We first present formal argument why we may expect a better regularity than we get from the definition of the weak solution. For the sake of simplicity, we consider only the heat equation in $\mathbb{R}^d$, i.e.,

$$\begin{aligned} \partial_t u - \Delta u = f \qquad & \text{in } (0,T)\times\mathbb{R}^d, \\ u(0,\cdot) = g(\cdot) \qquad & \text{in } \mathbb{R}^d, \end{aligned}$$

and let $f \in L^2((0,T)\times\mathbb{R}^d) = L^2(0,T;L^2(\mathbb{R}^d))$, $g \in W^{1,2}(\mathbb{R}^d) = W^{1,2}_0(\mathbb{R}^d)$. Furthermore, we assume that $u \to 0$ when $|x| \to \infty$ so that all integrations by parts we use below are allowed. Then we have

$$\begin{aligned} \int_{\mathbb{R}^d} f^2 \,\mathrm{d}x &= \int_{\mathbb{R}^d} (\partial_t u - \Delta u)^2 \,\mathrm{d}x = \int_{\mathbb{R}^d} \Big((\partial_t u)^2 + (\Delta u)^2 - 2\Delta u \partial_t u \Big) \,\mathrm{d}x \\ &= \int_{\mathbb{R}^d} \Big((\partial_t u)^2 \,\mathrm{d}x + \int_{\mathbb{R}^d} (\Delta u)^2 \,\mathrm{d}x + 2\int_{\mathbb{R}^d} \partial_t \nabla u \cdot \nabla u \,\mathrm{d}x. \end{aligned}$$

The second integral can be rewritten as

$$\begin{aligned} \int_{\mathbb{R}^d} (\Delta u)^2 \,\mathrm{d}x &= \int_{\mathbb{R}^d} \Big(\sum_{k=1}^d \frac{\partial^2 u}{\partial u_k^2} \Big) \Big(\sum_{l=1}^d \frac{\partial^2 u}{\partial x_l^2} \Big) \,\mathrm{d}x \\ &= \sum_{k,l=1}^d \int_{\mathbb{R}^d} \frac{\partial^2 u}{\partial x_k \partial x_l} \frac{\partial^2 u}{\partial x_k \partial x_l} \,\mathrm{d}x = \int_{\mathbb{R}^d} |\nabla^2 u|^2 \,\mathrm{d}x \end{aligned}$$

and the third integral

$$2\int_{\mathbb{R}^d} \partial_t \nabla u \cdot \nabla u \,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} |\nabla u|^2 \,\mathrm{d}x.$$

Therefore

$$\begin{aligned} \max_{t\in[0,T]} \int_{\mathbb{R}^d} |\nabla u|^2 \,\mathrm{d}x &+ \int_0^T \int_{\mathbb{R}^d} \Big(|\nabla^2 u|^2 + (\partial_t u)^2 \Big) \,\mathrm{d}x \,\mathrm{d}t \\ &\le \int_{\mathbb{R}^d} |\nabla u(0)|^2 \,\mathrm{d}x + \int_0^T \int_{\mathbb{R}^d} f^2 \,\mathrm{d}x \,\mathrm{d}t \\ &= \int_{\mathbb{R}^d} |\nabla g|^2 \,\mathrm{d}x + \int_0^T \int_{\mathbb{R}^d} f^2 \,\mathrm{d}x \,\mathrm{d}t. \end{aligned}$$

Next we differentiate the equation with respect to time. We get

$$\partial^2_{tt} u - \Delta \partial_t u = \partial_t f,$$

hence testing the equation by $\partial_t u$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|\partial_t u\|^2_{L^2(\mathbb{R}^d)} + \int_{\mathbb{R}^d} |\nabla \partial_t u|^2 \,\mathrm{d}x = \int_{\mathbb{R}^d} \partial_t f \partial_t u \,\mathrm{d}x.$$

Thus, using also the Gronwall lemma 5.1.7 and the fact that $\partial_t u(0,\cdot) = \Delta u(0,\cdot) + f(0,\cdot)$,

$$\max_{t\in[0,T]} \|\partial_t u\|^2_{L^2(\mathbb{R}^d)} + \|\nabla \partial_t u\|^2_{L^2((0,T)\times\mathbb{R}^d;\mathbb{R}^d)} \le \|\partial_t f\|^2_{L^2((0,T)\times\mathbb{R}^d)} + \|\nabla^2 g\|^2_{L^2(\mathbb{R}^d;\mathbb{R}^{d\times d})} + \|f(0,\cdot)\|^2_{L^2(\mathbb{R}^d)}.$$

Since

$$\|f(0,\cdot)\|_{L^2(\mathbb{R}^d)} \le C\big(\|\partial_t f\|_{L^2((0,T)\times\mathbb{R}^d)} + \|f\|_{L^2((0,T)\times\mathbb{R}^d)} \big),$$

we get

$$\max_{t\in[0,T]} \|\partial_t u\|_{L^2(\mathbb{R}^d)} + \|\nabla \partial_t u\|_{L^2((0,T)\times\mathbb{R}^d;\mathbb{R}^d)} \le C\big(\|f\|_{W^{1,2}(0,T;L^2(\mathbb{R}^d))} + \|\nabla^2 g\|_{L^2(\mathbb{R}^d;\mathbb{R}^{d\times d})} \big).$$

Combining all the estimates together, we finish with

$$\begin{aligned} \max_{t\in[0,T]} (\|\partial_t u\|_{L^2(\mathbb{R}^d)} + \|\nabla u\|_{L^2(\mathbb{R}^d;\mathbb{R}^d)}) + \|\nabla \partial_t u\|_{L^2((0,T)\times\mathbb{R}^d;\mathbb{R}^d)} + \|\nabla^2 u\|_{L^2((0,T)\times\mathbb{R}^d;\mathbb{R}^{d\times d})} \\ \le C\big(\|f\|_{W^{1,2}(0,T;L^2(\mathbb{R}^d))} + \|\nabla g\|_{W^{1,2}(\mathbb{R}^d;\mathbb{R}^d)} \big). \end{aligned}$$

We shall now show the estimates rigorously, for our general parabolic operator. As in the elliptic part, we consider only homogeneous Dirichlet boundary conditions. We shall assume the following:

$$\begin{aligned} \{a_{ij}\}_{i,j=1}^d \in W^{1,\infty}(Q_T), \qquad & \{c_i\}_{i=1}^d, b \in L^\infty(Q_T), \\ \Omega \in \mathcal{C}^{1,1}, \qquad & f \in L^2(Q_T),\ g \in W^{1,2}_0(\Omega) \end{aligned} \tag{5.20}$$

and, for higher regularity,

$$\begin{aligned} \{a_{ij}\}_{i,j=1}^d \in W^{1,\infty}(Q_T), \qquad & \{c_i\}_{i=1}^d, b \in W^{1,\infty}(0,T;L^\infty(\Omega)), \\ \Omega \in \mathcal{C}^{1,1}, \qquad & f \in W^{1,2}(0,T;L^2(\Omega)),\ g \in W^{2,2}(\Omega)\cap W^{1,2}_0(\Omega). \end{aligned} \tag{5.21}$$

We have

Theorem 5.1.13 — Regularity of weak solutions to parabolic equations I. Let $u \in L^2(0,T;W_0^{1,2}(\Omega))$ with $\partial_t u \in L^2(0,T;W^{-1,2}(\Omega))$ be a weak solution to our parabolic problem in the sense of Definition 5.1.3. Let assumptions (5.20) be fulfilled. Then $u \in L^2(0,T;W^{2,2}(\Omega)) \cap L^\infty(0,T;W_0^{1,2}(\Omega))$ with $\partial_t u \in L^2((0,T)\times\Omega)$ and

$$\operatorname*{ess\,sup}_{t\in[0,T]} \|\nabla u\|_{L^2(\Omega;\mathbb{R}^d)} + \|\partial_t u\|_{L^2(Q_T)} + \|\nabla^2 u\|_{L^2(Q_T;\mathbb{R}^{d\times d})} \leq C\big(\|f\|_{L^2(Q_T)} + \|\nabla g\|_{L^2(\Omega;\mathbb{R}^d)}\big),$$

where C depends on T, Ω and coefficients of L. If (5.21) is fulfilled, then additionally $u \in L^\infty(0,T;W^{2,2}(\Omega))$ with $\partial_t u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W_0^{1,2}(\Omega))$, $\partial^2_{tt} u \in L^2(0,T;W^{-1,2}(\Omega))$, and

$$\operatorname*{ess\,sup}_{t\in[0,T]} \big(\|\nabla^2 u\|_{L^2(\Omega;\mathbb{R}^{d\times d})} + \|\partial_t u\|_{L^2(\Omega)}\big) + \|\partial_t u\|_{L^2(0,T;W^{1,2}(\Omega))} + \|\partial^2_{tt} u\|_{L^2(0,T;W^{-1,2}(\Omega))}$$
$$\leq C\big(\|f\|_{W^{1,2}(0,T;L^2(\Omega))} + \|g\|_{W^{2,2}(\Omega)}\big).$$

Proof. Recall that due to the assumptions of Theorem 5.1.13, there exists unique weak solution to the parabolic problem in the sense of Definition 5.1.3. Therefore we may work either with the Galerkin approximation or with the weak formulation.

Step 1: First time derivative estimate

We start with the Galerkin approximation

$$\int_\Omega \partial_t u_n w_k \,\mathrm{d}x + \int_\Omega \Big(\sum_{k=1}^d a_{ij}\frac{\partial u_n}{\partial x_j}\frac{\partial w_k}{\partial x_i} + \sum_{i=1}^d c_i \frac{\partial u_n}{\partial x_i} w_k + b u_n w_k\Big)\,\mathrm{d}x = \int_\Omega f w_k \,\mathrm{d}x,$$

$k = 1,2,\dots,n$. We multiply the k-th equation by $\partial_t d_k^n(t)$ and sum up for k from 1 to n (i.e., we test by the time derivative of the approximate solution). We get

$$\int_\Omega (\partial_t u_n)^2 \,\mathrm{d}x + \int_\Omega \Big(\sum_{i,j=1}^d a_{ij}\frac{\partial u_n}{\partial x_j}\frac{\partial(\partial_t u_n)}{\partial x_i}\,\mathrm{d}x + \sum_{i=1}^d c_i \frac{\partial u_n}{\partial x_i}\partial_t u_n + b u_n \partial_t u_n\Big)\,\mathrm{d}x = \int_\Omega f \partial_t u_n \,\mathrm{d}x.$$

If $\mathbb{A} = \mathbb{A}^T$, we could proceed directly, since

$$\int_\Omega \Big(\sum_{i,j=1}^d a_{ij}\frac{\partial u_n}{\partial x_j}\frac{\partial(\partial_t u_n)}{\partial x_i}\,\mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \sum_{i,j=1}^d a_{ij}\frac{\partial u_n}{\partial x_j}\frac{\partial u_n}{\partial x_i}\,\mathrm{d}x - \frac{1}{2}\sum_{i,j=1}^d \partial_t a_{ij}\frac{\partial u_n}{\partial x_j}\frac{\partial u_n}{\partial x_i}\,\mathrm{d}x.$$

In the general case we write

$$\begin{aligned}\int_\Omega \sum_{i,j=1}^d a_{ij}\frac{\partial u_n}{\partial x_j}\frac{\partial(\partial_t u_n)}{\partial x_i}\,\mathrm{d}x &= \int_\Omega \Big(\sum_{i,j=1}^d \frac{1}{2}(a_{ij}+a_{ji})\frac{\partial u_n}{\partial x_j}\frac{\partial(\partial_t u_n)}{\partial x_i} + \sum_{i,j=1}^d \frac{1}{2}(a_{ij}-a_{ji})\frac{\partial u_n}{\partial x_j}\frac{\partial(\partial_t u_n)}{\partial x_i}\Big)\,\mathrm{d}x \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \sum_{i,j=1}^d \frac{1}{2}(a_{ij}+a_{ji})\frac{\partial u_n}{\partial x_j}\frac{\partial u_n}{\partial x_i}\,\mathrm{d}x - \int_\Omega \sum_{i,j=1}^d \frac{1}{2}\partial_t(a_{ij}+a_{ji})\frac{\partial u_n}{\partial x_j}\frac{\partial u_n}{\partial x_i}\,\mathrm{d}x \\ &\quad - \int_\Omega \sum_{i,j=1}^d \frac{1}{2}\frac{\partial}{\partial x_i}(a_{ij}-a_{ji})\frac{\partial u_n}{\partial x_j}\partial_t u_n\,\mathrm{d}x.\end{aligned}$$

Above, we used in the last term integration by parts together with the fact that $\Omega \in \mathcal{C}^{1,1}$, the eigenfunctions of the Laplace equation with homogeneous Dirichlet boundary condition are of the class $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ and the extra term

$$\int_\Omega \sum_{i,j=1}^d \frac{1}{2}(a_{ij}-a_{ji})\frac{\partial^2 u_n}{\partial x_i \partial x_j}\partial_t u_n\,\mathrm{d}x = 0,$$

since the second derivative is symmetric and the term $a_{ij} - a_{ji}$ is skew symmetric. We end up with (note that $\sum_{i,j=1}^d \frac{1}{2}(a_{ij}+a_{ji})\frac{\partial u_n}{\partial x_j}\frac{\partial u_n}{\partial x_i} = \sum_{i,j=1}^d a_{ij}\frac{\partial u_n}{\partial x_j}\frac{\partial u_n}{\partial x_i}$ due to the symmetry)

$$\begin{aligned}\|\partial_t u_n\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \sum_{i,j=1}^d a_{ij}\frac{\partial u_n}{\partial x_j}\frac{\partial u_n}{\partial x_i}\,\mathrm{d}x &= \int_\Omega f\partial_t u_n\,\mathrm{d}x - \int_\Omega \sum_{i=1}^d c_i\frac{\partial u_n}{\partial x_i}\partial_t u_n\,\mathrm{d}x - \int_\Omega b u_n \partial_t u_n\,\mathrm{d}x \\ &+ \frac{1}{2}\int_\Omega \sum_{i,j=1}^d \partial_t a_{ij}\frac{\partial u_n}{\partial x_j}\frac{\partial u_n}{\partial x_i}\,\mathrm{d}x + \int_\Omega \sum_{i,j=1}^d \frac{1}{2}\frac{\partial}{\partial x_i}(a_{ij}-a_{ji})\frac{\partial u_n}{\partial x_j}\partial_t u_n\,\mathrm{d}x.\end{aligned}$$

Due to our assumptions we can estimate the right-hand side as

$$|RHS| \leq \|f\|_{L^2(\Omega)}\|\partial_t u_n\|_{L^2(\Omega)} + C(L)\big(\|\nabla u_n\|_{L^2(\Omega;\mathbb{R}^d)}\|\partial_t u_n\|_{L^2(\Omega)} + \|u_n\|_{L^2(\Omega)}\|\partial_t u_n\|_{L^2(\Omega)} + \|\nabla u_n\|^2_{L^2(\Omega;\mathbb{R}^d)}\big)$$

and due to the bounds from Theorem 5.1.9 and parabolicity of the operator $\partial_t + L$; i.e.,

$$C_1 \|\nabla u\|^2_{L^2(\Omega;\mathbb{R}^d)} \le \int_\Omega \sum_{i,j=1}^d a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_i} \,\mathrm{d}x$$

we get

$$\operatorname*{ess\,sup}_{t\in[0,T]} \int_\Omega \sum_{i,j=1}^d a_{ij} \frac{\partial u_n}{\partial x_j} \frac{\partial u_n}{\partial x_i} \,\mathrm{d}x + \|\partial_t u_n\|^2_{L^2(Q_T)} \le C\big(\|f\|^2_{L^2(Q_T)} + \|g\|^2_{L^2(\Omega)} + \|\nabla u_n(0,\cdot)\|^2_{L^2(\Omega)}\big).$$

Recall that due to Claim 2. from Lemma 5.1.5 $\|\nabla u_n(0,\cdot)\|_{L^2(\Omega;\mathbb{R}^d)} \le \|\nabla g\|_{L^2(\Omega;\mathbb{R}^d)}$. Then

$$\|\nabla u_n\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^d))} + \|\partial_t u_n\|_{L^2(Q_T)} \le C\big(\|f\|_{L^2(Q_T)} + \|g\|_{W^{1,2}(\Omega)}\big),$$

where the constant C depends on the coefficients of L, on T and Ω. We may therefore let $n \to \infty$ to get

$$\|\nabla u\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^d))} + \|\partial_t u\|_{L^2(Q_T)} \le C\big(\|f\|_{L^2(Q_T)} + \|g\|_{W^{1,2}(\Omega)}\big).$$

Step 2: Regularity in spatial variables
We may write now

$$\int_\Omega \sum_{i,j=1}^d \Big(a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i}\Big)(t,\cdot)\,\mathrm{d}x = \int_\Omega f(t,\cdot)v(\cdot)\,\mathrm{d}x - \int_\Omega \partial_t u(t,\cdot)v(\cdot)\,\mathrm{d}x - \int_\Omega \Big(\sum_{i=1}^d c_i \frac{\partial u}{\partial x_i} v - buv\Big)(t,\cdot)\,\mathrm{d}x$$

for all $v \in W^{1,2}(\Omega)$ and almost every $t \in (0,T)$. Therefore, using the elliptic regularity, we get

$$\|u(t,\cdot)\|_{W^{2,2}(\Omega)} \le C\big(\|f(t,\cdot)\|_{L^2(\Omega)} + \|\partial_t u(t,\cdot)\|_{L^2(\Omega)} + \|u(t,\cdot)\|_{W^{1,2}(\Omega)}\big),$$

hence, integrating over $(0,T)$,

$$\|u\|_{L^2(0,T;W^{2,2}(\Omega))} \le C\big(\|f\|_{L^2(Q_T)} + \|g\|_{W^{1,2}(\Omega)}\big).$$

Above, the constant C depends again on T, on the smoothness of Ω and on the coefficients of L.
Step 3: Higher order time derivative estimate
We now continue with the second part of the theorem. We return back to the Galerkin approximation and differentiate the corresponding system of ordinary differential equations with respect to time. This is, due to the regularity theory for ODEs, possible, and we get

$$\begin{aligned}\int_\Omega \partial^2_{tt} u_n w_k \,\mathrm{d}x + B[\partial_t u_n, w_k](t) &= \int_\Omega \partial_t f w_k \,\mathrm{d}x - \int_\Omega \sum_{i,j=1}^d \partial_t a_{ij} \frac{\partial u_n}{\partial x_j} \frac{\partial w_k}{\partial x_i} \,\mathrm{d}x \\ &\quad - \int_\Omega \Big(\sum_{i=1}^d \partial_t c_i \frac{\partial u_n}{\partial x_i} w_k + \partial_t b u_n w_k\Big)\,\mathrm{d}x.\end{aligned} \tag{5.22}$$

We multiply the k-th equation by $\partial_t d_k^n(t)$ and sum for k from 1 to n (i.e., we test by $\partial_t u_n$). We get

$$\begin{aligned}\int_\Omega \partial^2_{tt} u_n \partial_t u_n \,\mathrm{d}x + B[\partial_t u_n, \partial_t u_n](t) &= \int_\Omega \partial_t f \partial_t u_n \,\mathrm{d}x \\ - \int_\Omega \sum_{i,j=1}^d \partial_t a_{ij} \frac{\partial u_n}{\partial x_j} \frac{\partial(\partial_t u_n)}{\partial x_i} \,\mathrm{d}x &- \int_\Omega \Big(\sum_{i=1}^d \partial_t c_i \frac{\partial u_n}{\partial x_i} \partial_t u_n + \partial_t b u_n \partial_t u_n\Big)\,\mathrm{d}x.\end{aligned}$$

Therefore we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t u_n\|^2_{L^2(\Omega)} + \|\nabla \partial_t u_n\|^2_{L^2(\Omega;\mathbb{R}^d)} &\le \|\partial_t f\|_{L^2(\Omega)} \|\partial_t u_n\|_{L^2(\Omega)} \\ + C\|\nabla u_n\|_{L^2(\Omega;\mathbb{R}^d)} \|\nabla \partial_t u_n\|_{L^2(\Omega;\mathbb{R}^d)} &+ C\|\nabla u_n\|_{L^2(\Omega;\mathbb{R}^d)} \|\partial_t u_n\|_{L^2(\Omega)} + C\|u_n\|_{L^2(\Omega)} \|\partial_t u_n\|_{L^2(\Omega)}\end{aligned}$$

which after employing the Gronwall and Young inequalities provides the estimate

$$\sup_{t\in[0,T]} \|\partial_t u_n\|^2_{L^2(\Omega)} + \int_0^T \|\nabla \partial_t u_n\|^2_{L^2(\Omega;\mathbb{R}^d)} \,\mathrm{d}t \le C\big(\|\partial_t f\|^2_{L^2(0,T;L^2(\Omega))} + \|\partial_t u_n(0,\cdot)\|^2_{L^2(\Omega)}\big).$$

Looking once more at the Galerkin approximation we see that

$$\|\partial_t u_n(0,\cdot)\|^2_{L^2(\Omega)} \le \|f(0,\cdot)\|^2_{L^2(\Omega)} + C\|u_n(0,\cdot)\|^2_{W^{2,2}(\Omega)} \le C\big(\|f\|^2_{W^{1,2}(0,T;L^2(\Omega))} + \|g\|_{W^{2,2}(\Omega)}\big),$$

where we used that $\|u_n(0,\cdot)\|_{W^{2,2}(\Omega)} \leq C\|g\|_{W^{2,2}(\Omega)}$, see Lemma 5.1.5, Claim 3. We can therefore let $n \to \infty$ and get

$$\|\partial_t u\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla \partial_t u\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^d))} \leq C\big(\|\partial_t f\|_{W^{1,2}(0,T;L^2(\Omega))} + \|g\|_{W^{2,2}(\Omega)}\big).$$

Step 4: Another time regularity
We look again at the problem

$$B[u,v] = \int_\Omega f v \,\mathrm{d}x - \int_\Omega \partial_t u v \,\mathrm{d}x$$

for any $v \in W_0^{1,2}(\Omega)$ and a.e. $t \in (0,T)$, and get, similarly as above

$$\|u\|_{W^{2,2}(\Omega)} \leq C\big(\|f\|_{L^2(\Omega)} + \|\partial_t u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\big)$$

due to the elliptic regularity. Therefore

$$\begin{aligned}\|u\|_{L^\infty(0,T;W^{2,2}(\Omega))} &\leq C\big(\|f\|_{L^\infty(0,T;L^2(\Omega))} + \|\partial_t u\|_{L^\infty(0,T;L^2(\Omega))} + \|u\|_{L^\infty(0,T;L^2(\Omega))}\big)\\ &\leq C\big(\|f\|_{W^{1,2}(0,T;L^2(\Omega))} + \|g\|_{W^{2,2}(\Omega)} + \|u\|_{L^\infty(0,T;L^2(\Omega))}\big).\end{aligned}$$

Step 5: Second order time derivative estimate
The last step is to show that $\partial^2_{tt} u \in L^2(0,T;W^{-1,2}(\Omega))$. Since (5.22) holds, we may proceed exactly as in the proof of the existence of a solution in Theorem 5.1.9 that $\partial_t u_n$ is bounded in $L^2(0,T;W^{-1,2}(\Omega))$. ■

Remark 5.1.14. Note that if $\mathbb{A}$ is symmetric, it is enough to assume, both in (5.20) and (5.21), that $\{a_{ij}\}_{i,j=1}^d \in W^{1,\infty}(0,T;L^\infty(\Omega))$.

Similarly as in the case of the elliptic regularity, we may improve the smoothness provided all the data are smooth enough. In order to simplify the presentation, we assume here that the coefficients of the operator L are independent of time and sufficiently smooth in the spatial variable.

Theorem 5.1.15 — Regularity of weak solutions to parabolic equations II. Let Ω be of class $C^{2m+1,1}$. Let $g \in W^{2m+1,2}(\Omega)$ and $\frac{\partial^k f}{\partial t^k} \in L^2(0,T;W^{2m-2k,2}(\Omega))$, $k = 0,1,2,\dots,m$. Let the coefficients of L be sufficiently smooth in the spatial variables and independent of time, and let $g_0 := g \in W_0^{1,2}(\Omega)$, $g_1 = f(0,\cdot) - Lg_0 \in W_0^{1,2}(\Omega)$, $\dots$, $g_m := \frac{\partial^{m-1} f(0,\cdot)}{\partial t^{m-1}} - Lg_{m-1} \in W_0^{1,2}(\Omega)$. Then $\frac{\partial^k u}{\partial t^k} \in L^2(0,T;W^{2m+2-2k}(\Omega))$, $k = 0,1,\dots,m+1$ and

$$\sum_{k=0}^{m+1} \Big\|\frac{\partial^k u}{\partial t^k}\Big\|_{L^2(0,T;W^{2m+2-2k,2}(\Omega))} \leq C\Big(\sum_{k=0}^{m} \Big\|\frac{\partial^k f}{\partial t^k}\Big\|_{L^2(0,T;W^{2m-2k,2}(\Omega))} + \|g\|_{W^{2m+1,2}(\Omega)}\Big),$$

where the constant C depends on T, Ω, m and on the coefficients of L.

Remark 5.1.16. Let us indicate, where do the compatibility conditions come from. Formally, $\partial_t u =: w$ solves

$$\begin{aligned}\partial_t w + Lw &= \partial_t f && \text{in } Q_T\\ w &= 0 && \text{on } (0,T)\times\partial\Omega\\ w(0) &= f(0) - Lg && \text{in } \Omega\end{aligned}$$

and recall that we needed above that the initial condition belongs to $W_0^{1,2}(\Omega)$. Hence we must require that $f(0,\cdot) - Lg \in W_0^{1,2}(\Omega)$ which is the second compatibility condition. Similarly for higher order estimates.

Consequently, we have

Theorem 5.1.17 — Regularity of weak solutions to parabolic equations III. Let $g \in \mathcal{C}^\infty(\overline{\Omega})$, $f \in \mathcal{C}^\infty(\overline{Q_T})$, $\Omega \in \mathcal{C}^\infty$, the coefficients of L are time independent and of class $\mathcal{C}^\infty(\overline{\Omega})$ and let infinitely many compatibility conditions from Theorem 5.1.15 above hold. Then the solution to our problem is smooth, i.e.,

$$u \in \mathcal{C}^\infty(\overline{Q_T}).$$

Proof of Theorem 5.1.15. The proof will be based on induction. The case $m = 0$ corresponds to Theorem 5.1.13, more precisely to its first part.

Step 1: Estimates for highest time regularity
Assume that Theorem 5.1.15 is valid for some nonnegative integer m and suppose that

$$g \in W^{2m+3,2}(\Omega), \qquad \frac{\partial^k f}{\partial t^k} \in L^2(0,T;W^{2m+2-2k,2}(\Omega)), \quad k = 0,1,\dots,m+1.$$

Assume that the $(m+1)$th compatibility condition holds. We differentiate our equation with respect to t and consider ($\widetilde{u} = \partial_t u$)

$$\begin{aligned}\partial_t \widetilde{u} + L\widetilde{u} &= \widetilde{f} && \text{in } Q_T\\ \widetilde{u} &= 0 && \text{on } (0,T)\times\partial\Omega\\ \widetilde{u}(0,\cdot) &= \widetilde{g}(\cdot) && \text{in } \Omega,\end{aligned} \tag{5.23}$$

where $\widetilde{f} = \partial_t f$ and $\widetilde{g}(\cdot) = f(\cdot,0) - (Lg)(\cdot)$. By Theorem 5.1.13 we have that $\widetilde{u} \in L^2(0,T;W_0^{1,2}(\Omega))$, $\partial_t\widetilde{u} \in L^2(0,T;W^{-1,2}(\Omega))$, where $\widetilde{u}$ is the unique solution to (5.23). Since f and g satisfy the $(m+1)$th compatibility condition, functions $\widetilde{f}$ and $\widetilde{g}$ satisfy the mth compatibility condition. Thus by induction

$$\frac{\partial^k \widetilde{u}}{\partial t^k} \in L^2(0,T;W^{2m+2-2k,2}(\Omega)), \quad k = 0,1,\dots,m+1$$

and

$$\sum_{k=0}^{m+1} \Big\| \frac{\partial^k \widetilde{u}}{\partial t^k} \Big\|_{L^2(0,T;W^{2m+2-2k,2}(\Omega))} \le C\Big(\sum_{k=0}^{m} \Big\| \frac{\partial^k \widetilde{f}}{\partial t^k} \Big\|_{L^2(0,T;W^{2m-2k,2}(\Omega))} + \|\widetilde{g}\|_{W^{2m+1,2}(\Omega)} \Big).$$

Since $\widetilde{u} = \partial_t u$, we can write

$$\begin{aligned}\sum_{k=1}^{m+2} \Big\| \frac{\partial^k u}{\partial t^k} \Big\|_{L^2(0,T;W^{2m+4-2k,2}(\Omega))} &\le C\Big(\sum_{k=1}^{m+1} \Big\| \frac{\partial^k f}{\partial t^k} \Big\|_{L^2(0,T;W^{2m+2-2k,2}(\Omega))} + \|f(0,\cdot)\|_{W^{2m+1,2}(\Omega)} + \|Lg\|_{W^{2m+1,2}(\Omega)} \Big) \\ &\le C\Big(\sum_{k=0}^{m+1} \Big\| \frac{\partial^k f}{\partial t^k} \Big\|_{L^2(0,T;W^{2m+2-2k,2}(\Omega))} + \|g\|_{W^{2m+3,2}(\Omega)} \Big);\end{aligned} \tag{5.24}$$

above we used that

$$\|f(0,\cdot)\|_{W^{2m+1,2}(\Omega)} \le C\Big(\|f\|_{L^2(0,T;W^{2m+2,2}(\Omega))} + \Big\| \frac{\partial f}{\partial t} \Big\|_{L^2(0,T;W^{2m,2}(\Omega))} \Big)$$

which is a consequence of (Evans, 1998, Theorem 4 Section 5.9.2).

Step 2: Estimates for highest spatial regularity

For $0 \le t \le T$ fixed we write $Lu = f - \partial_t u = h$. By the corresponding theorem on elliptic regularity (Theorem 3.7.10) we have that

$$\begin{aligned}\|u\|_{W^{2m+4,2}(\Omega)} &\le C(\|h\|_{W^{2m+2,2}(\Omega)} + \|u\|_{L^2(\Omega)}) \\ &\le C(\|f\|_{W^{2m+2,2}(\Omega)} + \|\partial_t u\|_{W^{2m+2,2}(\Omega)} + \|u\|_{L^2(\Omega)}).\end{aligned}$$

Integrating this inequality over $t \in (0,T)$ and adding it to estimate (5.24) we end up with

$$\sum_{k=0}^{m+2} \Big\| \frac{\partial^k u}{\partial t^k} \Big\|_{L^2(0,T;W^{2m+4-2k}(\Omega))} \le C\Big(\sum_{k=0}^{m+1} \Big\| \frac{\partial^k f}{\partial t^k} \Big\|_{L^2(0,T;W^{2m+2-2k,2}(\Omega))} + \|g\|_{W^{2m+3,2}(\Omega)} + \|u\|_{L^2(Q_T)} \Big).$$

But we already know that $\|u\|_{L^2(Q_T)} \le C(\|f\|_{L^2(Q_T)} + \|g\|_{L^2(\Omega)})$ which finishes the proof of the theorem for $m+1$. ■

5.1.3 Maximum principles for parabolic problems

We would like to prove the maximum principle for our parabolic problem and for weak solutions only. Recall that for the classical solution, this issue is discussed in Pokorný (2025) or in Evans (1998). We restrict ourselves to the case when $b \ge 0$ a.e. in Q_T, $-\operatorname{div} \mathbf{c} \ge 0$ in the weak sense and $f \le 0$ in the following sense

Definition 5.1.18 A distribution $f \in L^2(0,T;W^{-1,2}(\Omega)) \ge 0$ (or ≤ 0), provided

$$\int_0^T \langle f, \varphi \rangle_{W_0^{1,2}(\Omega)} \ge 0 \qquad (\text{or } \le 0)$$

for all $\varphi \in L^2(0,T;W_0^{1,2}(\Omega))$ which are almost everywhere in Q_T nonnegative.

We first present the heuristic idea of the proof. Assume that the function $\chi_+ := \chi_{\{(t,x)\in(0,T)\times\Omega \,|\, u(t,x)>0\}}$ is an appropriate test function. Then (recall that $u^+ := \max\{0,u\}$)

$$\langle \partial_t u, \chi_+ \rangle_{W_0^{1,2}(\Omega)} = \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u^+ \,\mathrm{d}x,$$

$$\int_\Omega \sum_{i,j=1}^d a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \chi_+}{\partial x_i} \,\mathrm{d}x = 0,$$

$$\int_\Omega \sum_{i=1}^d c_i \frac{\partial u}{\partial x_i} \chi_+ \,\mathrm{d}x = -\int_\Omega \operatorname{div} \mathbf{c} u^+ \,\mathrm{d}x \ge 0,$$

$$\int_\Omega b u \chi_+ \,\mathrm{d}x = \int_\Omega b u^+ \,\mathrm{d}x \ge 0$$

$$\langle f, \chi_+ \rangle \le 0,$$

therefore we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^+\,\mathrm{d}x \le 0,$$

hence $u^+(t,\cdot) \le 0$ provided $u^+(0,\cdot) \le 0$. However, we need several technical tools to justify the computations above.

Note that for the function $z \mapsto z^+$ there exists a sequence of functions $z \mapsto \psi_n(z)$ such that $0 \le \psi_n \le z^+$ in $\mathbb{R}$, $\psi_n \nearrow z^+$ in $\mathbb{R}$ and ψ_n', $\psi_n'' \ge 0$ and they are uniformly bounded on $\mathbb{R}$ (not necessarily with respect to n).

Exercise 5.1.19. Show that the function $\psi_n(z) := z^+\mathrm{arctg}^2(nz)\left(\frac{2}{\pi}\right)^2$ satisfies all conditions above.

We may therefore use the following lemma

Lemma 5.1.20 Let $\psi\colon \mathbb{R} \to \mathbb{R}$ be such that $\psi \in \mathcal{C}^1(\mathbb{R})$, $\psi(z) = 0$ for $z \le 0$ and $0 \le \psi(z) \le z$ for $z \ge 0$.

1. If $u \in W_0^{1,2}(\Omega)$, then $\psi(u) \in W_0^{1,2}(\Omega)$ and

$$\nabla\psi(u) = \psi'(u)\nabla u$$

in the weak sense and $u \mapsto \psi(u)$ is continuous from $W_0^{1,2}(\Omega) \to W_0^{1,2}(\Omega)$.

2. If $u \in L^2(0,T;W_0^{1,2}(\Omega))$, $\partial_t u \in L^2(0,T;W^{-1,2}(\Omega))$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \psi(u(t))\,\mathrm{d}x = \langle\partial_t u, \psi'(u(t))\rangle_{W_0^{1,2}(\Omega)}$$

almost everywhere in $(0,T)$.

Proof. Claim 1. follows from Theorem 2.2.4. We therefore prove Claim 2. Let first $u \in \mathcal{C}^1([0,T];W_0^{1,2}(\Omega))$. Then for t_1, $t_2 \in [0,T]$

$$u(t_2) - u(t_1) = \int_{t_1}^{t_2} \partial_t u(s)\,\mathrm{d}s \qquad \text{in } W_0^{1,2}(\Omega),$$

i.e., $u(t_2,x) - u(t_1,x) = \int_{t_1}^{t_2} \partial_t u(s,x)\,\mathrm{d}s$ for almost every $x \in \Omega$. Therefore, for almost every $x \in \Omega$

$$\partial_t\psi(u(t,x)) = \psi'(u(t,x))\partial_t u(t,x),$$

and

$$\psi(u(t_2,x)) - \psi(u(t_1,x)) = \int_{t_1}^{t_2} \psi'(u(s,x))\partial_t u(s,x)\,\mathrm{d}s \qquad \text{for almost every } x \in \Omega.$$

Then, integrating the identity above over Ω and using the Fubini Theorem and properties of the Gelfand triple

$$\begin{aligned}\int_\Omega \psi(u(t_2,x))\,\mathrm{d}x - \int_\Omega \psi(u(t_1,x))\,\mathrm{d}x &= \int_{t_1}^{t_2}\int_\Omega \psi'(u(s,x))\partial_t u(s,x)\,\mathrm{d}x\,\mathrm{d}s \\ &= \int_{t_1}^{t_2} \langle\partial_t u(s,x), \psi'(u(s,x))\rangle_{W_0^{1,2}(\Omega)}\,\mathrm{d}s.\end{aligned}$$

Now, if $u \in L^2(0,T;W_0^{1,2}(\Omega))$ and $\partial_t u \in L^2(0,T;W^{-1,2}(\Omega))$, there exists a sequence $u_n \in \mathcal{C}^1([0,T];W_0^{1,2}(\Omega))$ such that $u_n \to u$ in $L^2(0,T;W_0^{1,2}(\Omega))$ and $\partial_t u_n \to \partial_t u$ in $L^2(0,T;W^{-1,2}(\Omega))$, see Lemma 4.4.6. Moreover, due to Theorem 4.4.4, we also have $u_n(t) \to u(t)$ in $L^2(\Omega)$ for all $t \in [0,T]$ (where we take u as the continuous representative). Then

$$\int_\Omega \psi(u_n(t_2,x))\,\mathrm{d}x - \int_\Omega \psi(u_n(t_1,x))\,\mathrm{d}x = \int_{t_1}^{t_2} \langle\partial_t u_n(s), \psi'(u_n(s))\rangle_{W_0^{1,2}(\Omega)}\,\mathrm{d}s.$$

Letting $n \to \infty$ and recalling that ψ grows at most linearly we end up with the desired formula

$$\int_\Omega \psi(u(t_2,x))\,\mathrm{d}x - \int_\Omega \psi(u(t_1,x))\,\mathrm{d}x = \int_{t_1}^{t_2} \langle\partial_t u(s), \psi'(u(s))\rangle_{W_0^{1,2}(\Omega)}\,\mathrm{d}s.$$

Hence we can differentiate, e.g., with respect to t_2 in the sense of distributions. ∎

Theorem 5.1.21 — Weak maximum principle for parabolic problem. Let $u \in L^2(0,T;W_0^{1,2}(\Omega))$ with $\partial_t u \in L^2(0,T;W^{-1,2}(\Omega))$ be a weak solution to our parabolic problem, where $\partial_t + L$ is a parabolic operator, $\operatorname{div}\mathbf{c} \le 0$ (in the weak sense on Q_T), $b \ge 0$ a.e. in Q_T, $f \le 0$ in the sense of Definition 5.1.18 and $g \le 0$ a.e. in Ω. Then

$$u(t) \le 0 \qquad \text{almost everywhere in } (0,T)\times\Omega.$$

Proof. We consider the sequence ψ_n approximating the function $z \mapsto z^+$ as presented above. We use $\psi_n'(u)$ as a test function in the weak formulation of our parabolic problem. Whence

$$\langle \partial_t u, \psi_n'(u)\rangle_{W_0^{1,2}(\Omega)} + \int_\Omega \sum_{i,j=1}^d a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial \psi_n'(u)}{\partial x_i}\,\mathrm{d}x + \int_\Omega \sum_{i=1}^d c_i \frac{\partial u}{\partial x_i}\psi_n'(u)\,\mathrm{d}x + \int_\Omega bu\psi_n'(u)\,\mathrm{d}x = \langle f, \psi_n'(u)\rangle_{W_0^{1,2}(\Omega)}.$$

Now, due to Lemma 5.1.20, our assumptions and properties of the approximation ψ_n we have

$$\begin{aligned}
\langle \partial_t u, \psi_n'(u)\rangle_{W_0^{1,2}(\Omega)} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \psi_n(u(t))\,\mathrm{d}x \\
\int_\Omega \sum_{i,j=1}^d a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial \psi_n'(u)}{\partial x_i}\,\mathrm{d}x &= \int_\Omega \sum_{i,j=1}^d a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_i}\psi_n''(u)\,\mathrm{d}x \geq 0 \\
\int_\Omega \sum_1^d c_i \frac{\partial u}{\partial x_i}\psi_n'(u)\,\mathrm{d}x &= \int_\Omega \sum_1^d c_i \frac{\partial \psi_n(u)}{\partial x_i}\,\mathrm{d}x \geq 0 \\
\int_\Omega bu\psi_n'(u)\,\mathrm{d}x &\geq 0 \\
\langle f, \psi_n'(u)\rangle_{W_0^{1,2}(\Omega)} &\leq 0.
\end{aligned}$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \psi_n(u(t,\cdot))\,\mathrm{d}x \leq 0,$$

hence

$$\int_\Omega \psi_n(u(t,\cdot))\,\mathrm{d}x \leq \int_\Omega \psi_n(g)\,\mathrm{d}x \leq 0$$

for all $t \in (0,T]$ (provided we consider the continuous representative of u). Letting $n \to \infty$ and using the Lebesgue Monotone convergence Theorem we end up with

$$\int_\Omega u^+(t,\cdot)\,\mathrm{d}x \leq 0$$

for all $t \in [0,T]$, hence $u(t,x) \leq 0$ almost everywhere in $[0,T] \times \Omega$. ■

5.2 Second order hyperbolic equation

We will consider the following problem of the hyperbolic type

$$\begin{aligned}
\partial_{tt}u + Lu &= f && \text{in } Q_T \\
u(0,\cdot) &= g(\cdot) && \text{in } \Omega \\
\partial_t u(0,\cdot) &= h(\cdot) && \text{in } \Omega \\
u &= 0 && \text{on } (0,T) \times \partial\Omega,
\end{aligned} \tag{5.25}$$

where

$$Lu = -\sum_{i,j=1}^d \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big) + \sum_{i=1}^d c_i \frac{\partial u}{\partial x_i} + bu.$$

Definition 5.2.1 We say that the operator $\partial_{tt} + L$ is (uniformly) hyperbolic, if there exists a constant $C_1 > 0$ such that

$$\sum_{i,j=1}^d a_{ij}\xi_i\xi_j \geq C_1|\boldsymbol{\xi}|^2 \qquad \text{for all } \boldsymbol{\xi} \in \mathbb{R}^d \qquad \text{a.e. in } Q_T.$$

Moreover, we further assume that $\{a_{ij}\}_{i,j=1}^d$, $\{c_i\}_{i=1}^d$ and $b \in L^\infty(Q_T)$.

Example 5.2.2. Assuming $a_{ij} = \delta_{ij}$, $c_i = 0$, $i = 1, 2, \ldots, d$ and $b = 0$, we get the classical wave equation

$$\partial_{tt}u - \Delta u = f.$$

As in the parabolic case, we introduce

$$B[u,v](t) = \int_\Omega \Big(\sum_{i,j=1}^d a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + \sum_{i=1}^d c_i \frac{\partial u}{\partial x_i} v + buv \Big)(t,\cdot)\,\mathrm{d}x$$

for all u, $v \in W_0^{1,2}(\Omega)$.

We expect the weak formulation in the form

$$\langle \partial_{tt} u, v\rangle_{W_0^{1,2}(\Omega)} + B[u,v](t) = (f,v)_{L^2(\Omega)} \qquad \text{for all } v \in W_0^{1,2}(\Omega),$$

where the reason why we prefer to have more regular right-hand side than in the parabolic setting will be explained later. Therefore we expect, by analogy with the parabolic case, that u should belong to $L^2(0,T;W_0^{1,2}(\Omega))$, while $\partial_{tt} u$ should belong to $L^2(0,T;W^{-1,2}(\Omega))$.

5.2.1 Existence of a weak solution to linear hyperbolic problem

We assume

$$\begin{aligned} &a_{ij} \in W^{1,\infty}(Q_T), \quad c_i, b \in L^\infty(Q_T) \quad \text{for all } i,j = 1,2,\dots,d \\ &h \in L^2(\Omega), \quad g \in W_0^{1,2}(\Omega), \quad f \in L^2(Q_T). \end{aligned} \tag{5.26}$$

Definition 5.2.3 — Weak solution for hyperbolic equation. We say that $u \in L^2(0,T;W_0^{1,2}(\Omega))$ with $\partial_t u \in L^2(Q_T)$ and $\partial_{tt} u \in L^2(0,T;W^{-1,2}(\Omega))$ is a weak solution to the hyperbolic initial-boundary value problem problem provided

$$\langle \partial_{tt} u, v\rangle_{W_0^{1,2}(\Omega)} + B[u,v](t) = (f,v)_{L^2(\Omega)}$$

for all $v \in W_0^{1,2}(\Omega)$ and a.e. $t \in (0,T)$, and

$$u(0,\cdot) = g(\cdot), \qquad \partial_t u(0,\cdot) = h(\cdot).$$

Remark 5.2.4. Recall that the assumptions above imply that (after a possible change of the function on the time levels of zero one dimensional Lebesgue measure) $u \in \mathcal{C}([0,T];L^2(\Omega))$ and $\partial_t u \in \mathcal{C}([0,T];W^{-1,2}(\Omega))$. The fact why we require higher integrability of g and h will be explained later.

Theorem 5.2.5 — Existence and uniqueness of weak solution to hyperbolic equation. Under assumptions (5.26) there exists a weak solution to our hyperbolic problem in the sense of Definition 5.2.3. If furthermore $\nabla \mathbf{c} \in L^2(Q_T;\mathbb{R}^{d\times d})$, the solution is unique.

Remark 5.2.6. If the matrix $\mathbb{A}$ is symmetric almost everywhere in Q_T, it is enough to assume that $\{a_{ij}\}_{i,j=1}^d \in W^{1,\infty}(0,T;L^\infty(\Omega))$. The reason is similar to the proof of regularity for parabolic problems.

Proof of Theorem 5.2.5. **Step 1:** Galerkin approximation
We proceed similarly as in the case of the parabolic problem. We take $\{w_k\}_{k=1}^\infty$ the orthonormal basis in $L^2(\Omega)$ and orthogonal basis in $W_0^{1,2}(\Omega)$ formed by eigenfunctions of the Laplace operator with the homogeneous Dirichlet condition. We write

$$u_n(t,x) = \sum_{k=1}^n d_i^n(t) w_k(x),$$

where

$$(\partial_{tt} u_n, w_l)_{L^2(\Omega)} + B[u_n, w_l](t) = (f, w_l)_{L^2(\Omega)}, \qquad l = 1,2,\dots,n,$$

i.e.,

$$\begin{aligned} d_l^{n\prime\prime}(t) + B\Big[\sum_{k=1}^n d_k^n(t) w_k, w_l\Big](t) &= (f,w_l)_{L^2(\Omega)} \\ d_l^n(0) &= (g,w_l)_{L^2(\Omega)} \\ d_l^{n\prime}(0) &= (h,w_l)_{L^2(\Omega)}, \end{aligned} \tag{5.27}$$

$l = 1,2,\dots,n$. We obtained a system of the second order linear ODEs (n equations) which can be easily transformed into a system of $2n$ ODEs of the first order. We may then apply the Carathéodory theory which yields the existence of a unique generalized solution on the time interval $[0,T]$ such that $d_l^{n\prime} \in \mathcal{AC}([0,T])$, $l = 1,2,\dots,n$. The fact that the solution is global follows from the fact that the system is linear.

Step 2: Energy estimate
We now multiply (5.27) by $d_l^{n\prime}$, sum for $l = 1,2,\dots,n$ and get

$$(\partial_{tt} u_n, \partial_t u_n)_{L^2(\Omega)} + B[u_n, \partial_t u_n] = (f, \partial_t u_n)_{L^2(\Omega)}.$$

First, we have

$$(\partial_{tt} u_n, \partial_t u_n)_{L^2(\Omega)} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t u_n\|^2_{L^2(\Omega)}.$$

Next

$$\begin{aligned}
B[u_n, \partial_t u_n] &= \int_\Omega \Big(\sum_{i,j=1}^d a_{ij} \frac{\partial u_n}{\partial x_j}\frac{\partial \partial_t u_n}{\partial x_i} + \sum_{i=1}^d c_i \frac{\partial u_n}{\partial x_i}\partial_t u_n + b u_n \partial_t u_n\Big)\,\mathrm{d}x \\
&= \int_\Omega \Big(\sum_{i,j=1}^d \frac{1}{2}(a_{ij}+a_{ji})\frac{\partial u_n}{\partial x_j}\frac{\partial(\partial_t u_n)}{\partial x_i} + \sum_{i,j=1}^d \frac{1}{2}(a_{ij}-a_{ji})\frac{\partial u_n}{\partial x_j}\frac{\partial(\partial_t u_n)}{\partial x_i} + \sum_{i=1}^d c_i \frac{\partial u_n}{\partial x_i}\partial_t u_n + b u_n \partial_t u_n\Big)\,\mathrm{d}x \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \sum_{i,j=1}^d \frac{1}{2}(a_{ij}+a_{ji})\frac{\partial u_n}{\partial x_j}\frac{\partial u_n}{\partial x_i}\,\mathrm{d}x - \frac{1}{2}\int_\Omega \sum_{i,j=1}^d \partial_t(a_{ij}+a_{ji})\frac{\partial u_n}{\partial x_j}\frac{\partial u_n}{\partial x_i}\,\mathrm{d}x \\
&\quad - \int_\Omega \sum_{i,j=1}^d \frac{1}{2}\frac{\partial}{\partial x_i}(a_{ij}-a_{ji})\frac{\partial u_n}{\partial x_j}\partial_t u_n\,\mathrm{d}x + \int_\Omega \Big(\sum_{i=1}^d c_i\frac{\partial u_n}{\partial x_i}\partial_t u_n + b u_n \partial_t u_n\Big)\,\mathrm{d}x.
\end{aligned}$$

We used above that the term

$$\int_\Omega \sum_{i,j=1}^d \frac{1}{2}(a_{ij}-a_{ji})\frac{\partial^2 u_n}{\partial x_i \partial x_j}\partial_t u_n\,\mathrm{d}x = 0,$$

as $(a_{ij}-a_{ji})$ is skew symmetric and $\frac{\partial^2 u_n}{\partial x_i \partial x_j}$ is symmetric. We use now the fact that u_n can be approximation by smooth compactly supported functions. We continue

$$\begin{aligned}
B[u_n, \partial_t u_n] &\geq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \sum_{i,j=1}^d a_{ij}\frac{\partial u_n}{\partial x_j}\frac{\partial u_n}{\partial x_i}\,\mathrm{d}x \\
&\quad - C(\|\mathbb{A}\|_{W^{1,\infty}(Q_T)}, \|b\|_{L^\infty(Q_T)}, \|\mathbf{c}\|_{L^\infty(Q_T)})\Big(\int_\Omega \sum_{i,j=1}^d a_{ij}\frac{\partial u_n}{\partial x_j}\frac{\partial u_n}{\partial x_i}\,\mathrm{d}x + \|\partial_t u_n\|^2_{L^2(\Omega)}\Big),
\end{aligned}$$

where we used the symmetry in the term with highest spatial derivatives. Employing the Gronwall Lemma 5.1.7 and the hyperbolicity of the operator, we get (this is precisely the reason why we need higher regularity of g and h than expected by the continuity of the solution)

$$\|\partial_t u_n(t,\cdot)\|^2_{L^2(\Omega)} + \|u_n(t,\cdot)\|^2_{W^{1,2}_0(\Omega)} \leq C\big(\|g\|^2_{W^{1,2}_0(\Omega)} + \|h\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(0,T;L^2(\Omega))}\big),$$

hence

$$\max_{[0,T]}\big(\|\partial_t u_n(t,\cdot)\|_{L^2(\Omega)} + \|u_n(t,\cdot)\|_{W^{1,2}_0(\Omega)}\big) \leq C\big(\|g\|_{W^{1,2}_0(\Omega)} + \|h\|_{L^2(\Omega)} + \|f\|_{L^2(0,T;L^2(\Omega))}\big).$$

In order to estimate the second order time derivative we proceed as in the parabolic problem. We write for $v \in W^{1,2}_0(\Omega)$ that $v = v^1_n + v^2_n$, where v^1_n belongs to the linear hull of $\{w_l\}_{l=1}^n$ and v^2_n is perpendicular to it (both in $L^2(\Omega)$ and $W^{1,2}_0(\Omega)$). Recall that (see Lemma 5.1.5) $\|v^1_n\|_{L^2(\Omega)} \leq \|v\|_{L^2(\Omega)}$ and $\|v^1_n\|_{W^{1,2}_0(\Omega)} \leq \|v\|_{W^{1,2}_0(\Omega)}$. Then

$$\langle \partial_{tt} u_n, v\rangle_{W^{1,2}_0(\Omega)} = (\partial_{tt} u_n, v)_{L^2(\Omega)} = (\partial_{tt} u_n, v^1_n)_{L^2(\Omega)} = -B[u_n, v^1_n](t) + (f, v^1_n)_{L^2(\Omega)}.$$

Thus

$$\|\partial_{tt} u_n\|_{W^{-1,2}(\Omega)} = \sup_{\|v\|_{W^{1,2}_0(\Omega)}\leq 1} \langle \partial_{tt} u_n, v\rangle_{W^{1,2}_0(\Omega)} \leq C(\|u_n\|_{W^{1,2}_0(\Omega)} + \|f\|_{L^2(\Omega)}),$$

hence

$$\|\partial_{tt} u_n\|_{L^2(0,T;W^{-1,2}(\Omega))} \leq C\big(\|f\|_{L^2(0,T;L^2(\Omega))} + \|g\|_{W^{1,2}_0(\Omega)} + \|h\|_{L^2(\Omega)}\big).$$

Step 3: Limit passage

We now let $n \to \infty$. As in the parabolic problem, for suitably chosen subsequence

$$\begin{aligned}
u_{n_k} &\rightharpoonup^* u && \text{in } L^\infty(0,T;W^{1,2}_0(\Omega)) \\
\partial_t u_{n_k} &\rightharpoonup^* \partial_t u && \text{in } L^\infty(0,T;L^2(\Omega)) \\
\partial_{tt} u_{n_k} &\rightharpoonup \partial_{tt} u && \text{in } L^2(0,T;W^{-1,2}(\Omega))
\end{aligned}$$

and we can pass to the limit in the modified weak formulation to get

$$\int_0^T \langle \partial_{tt} u, w_l\rangle_{W^{1,2}_0(\Omega)}\psi\,\mathrm{d}t + \int_0^T B[u, w_l]\psi\,\mathrm{d}t = \int_0^T (f, w_l)_{L^2(\Omega)}\psi\,\mathrm{d}t$$

for all $l \in \mathbb{N}$ and all $\psi \in \mathcal{C}_0^\infty((0,T))$. Hence, exactly as for the parabolic problem

$$\int_0^T \langle \partial_{tt} u, v\rangle_{W_0^{1,2}(\Omega)} \psi \,\mathrm{d}t + \int_0^T B[u,v]\psi \,\mathrm{d}t = \int_0^T (f,v)_{L^2(\Omega)}\psi \,\mathrm{d}t$$

for all $v \in W_0^{1,2}(\Omega)$ and all $\psi \in \mathcal{C}_0^\infty((0,T))$, i.e.,

$$\langle \partial_{tt} u, v\rangle_{W_0^{1,2}(\Omega)} + B[u,v] = (f,v)_{L^2(\Omega)}$$

for all $v \in W_0^{1,2}(\Omega)$ at almost every $t \in (0,T)$.

Next we look at the initial condition. Here, the problem is slightly more complex than for the parabolic problem. First, using the definition of the first order time derivative of u we have

$$\int_0^T \langle \partial_{tt} u, v\rangle_{W_0^{1,2}(\Omega)} \psi \,\mathrm{d}t = -\int_0^T (\partial_t u, v)_{L^2(\Omega)}\psi' \,\mathrm{d}t$$

for all $v \in W_0^{1,2}(\Omega)$ and $\psi \in \mathcal{C}_0^\infty((0,T))$. Therefore, taking a sequence $\psi_n \in \mathcal{C}_0^\infty((0,T))$ such that ψ_n converges uniformly to $\psi \in \mathcal{C}_0^\infty([0,T))$ we end up with ($\psi_n' \to \psi' + \psi(0)\delta$ in the sense of distributions)

$$-\int_0^T (\partial_t u, v)_{L^2(\Omega)}\psi' \,\mathrm{d}t + \int_0^T B[u,v]\psi \,\mathrm{d}t = \int_0^T (f,v)_{L^2(\Omega)}\psi \,\mathrm{d}t + \langle \partial_t u(0,\cdot), v(\cdot)\rangle_{W_0^{1,2}(\Omega)}\psi(0)$$

for all $\psi \in \mathcal{C}_c^\infty([0,T))$. Now, as $u \in \mathcal{AC}([0,T]; L^2(\Omega))$, we finally have

$$\int_0^T (u,v)\psi'' \,\mathrm{d}t + \int_0^T B[u,v]\psi \,\mathrm{d}t = \int_0^T (f,v)_{L^2(\Omega)}\psi \,\mathrm{d}t + \langle \partial_t u(0,\cdot), v(\cdot)\rangle_{W_0^{1,2}(\Omega)}\psi(0) - (u(0,\cdot), v(\cdot))_{L^2(\Omega)}\psi'(0).$$

We now return to the Galerkin approximation and multiply it by $\psi \in \mathcal{C}_0^\infty([0,T))$ and integrate over the time interval. We get

$$\int_0^T (\partial_{tt} u_n, w_l)_{L^2(\Omega)}\psi \,\mathrm{d}t + \int_0^T B[u_n, w_l]\psi \,\mathrm{d}t = \int_0^T (f, w_l)_{L^2(\Omega)}\psi \,\mathrm{d}t.$$

We integrate twice by parts in the first term and get (recall that $\partial_t u_n \in \mathcal{AC}\left([0,T]; W_0^{1,2}(\Omega)\right)$)

$$\int_0^T (u_n, w_l)\psi'' \,\mathrm{d}t + \int_0^T B[u_n, w_l]\psi \,\mathrm{d}t = \int_0^T (f, w_l)_{L^2(\Omega)}\psi \,\mathrm{d}t + (h, w_l)_{L^2(\Omega)}\psi(0) - (g, w_l)_{L^2(\Omega)}\psi'(0).$$

We let $n \to \infty$ and then use the density of finite linear combination of $\{w_l\}_{l\in\mathbb{N}}$ in $W_0^{1,2}(\Omega)$ to get

$$\int_0^T (u,v)\psi'' \,\mathrm{d}t + \int_0^T B[u,v]\psi \,\mathrm{d}t = \int_0^T (f,v)_{L^2(\Omega)}\psi \,\mathrm{d}t + (h,v)_{L^2(\Omega)}\psi(0) - (g,v)_{L^2(\Omega)}\psi'(0)$$

for all $v \in W_0^{1,2}(\Omega)$ and $\psi \in \mathcal{C}_0^\infty([0,T))$. Therefore

$$\begin{aligned} \langle \partial_t u(0,\cdot), v(\cdot)\rangle_{W_0^{1,2}(\Omega)} &= (h,v)_{L^2(\Omega)} \\ (u(0,\cdot), v(\cdot))_{L^2(\Omega)} &= (g,v)_{L^2(\Omega)} \end{aligned}$$

for all $v \in W_0^{1,2}(\Omega)$. Due to the density of $W_0^{1,2}(\Omega)$ in $L^2(\Omega)$ we get the desired equality. Note that in fact, we also have $(\partial_t u(0,\cdot), v(\cdot))_{L^2(\Omega)} = (h,v)_{L^2(\Omega)}$ as $\partial_t \in \mathcal{C}([0,T]; L^2_w(\Omega))$, see Lemma 4.4.9.

Step 4: Uniqueness

We now show the uniqueness of weak solutions to the hyperbolic equation. As for the parabolic case, it is enough to show that if $f = g = h = 0$, then the only solution is $u = 0$. However, we cannot proceed as in the part devoted to the a priori estimates, as the function $\partial_t u$ cannot be, due to the lack of regularity, used as test function. Therefore we use the "minus first" derivative. To this aim, let us define

$$v(t) := \begin{cases} \displaystyle\int_t^s u(\tau)\,\mathrm{d}\tau, & 0 \le t \le s \\ 0, & s \le t \le T, \end{cases}$$

where u solves our hyperbolic problem with $f = g = h = 0$. Then $v(t,\cdot) \in W_0^{1,2}(\Omega)$ for almost every $t \in (0,T)$ (we even have $\partial_t v \in L^2(0,T; W^{1,2}(\Omega))$) and

$$\langle \partial_{tt} u, v\rangle_{W_0^{1,2}(\Omega)} + B[u,v] = 0$$

almost everywhere in $(0,T)$, thus

$$\int_0^s \left[\langle \partial_{tt} u, v\rangle_{W_0^{1,2}(\Omega)} + B[u,v]\right] \mathrm{d}t = 0.$$

As $\partial_t v = -u$, $\partial_t u(0) = v(s) = 0$, we get using the definition of the time derivative (and consequently, by approximation, using as test function in the time variable a constant function $\psi \equiv 1$)

$$\int_0^s \Big[-(\partial_t u, \partial_t v)_{L^2(\Omega)} + B[u,v]\Big](t)\,\mathrm{d}t = 0,$$

hence

$$\int_0^s \Big[(\partial_t u, u)_{L^2(\Omega)} - B[\partial_t v, v]\Big](t)\,\mathrm{d}t = 0.$$

Therefore we have (we use again the fact that the product of symmetric and skew-symmetric term cancel out as well as that the symmetric part of the matrix can be replaced by the matrix itself if the other term is symmetric)

$$\int_0^s \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|u\|^2_{L^2(\Omega)} - \int_\Omega \sum_{i,j=1}^d a_{ij}\frac{\partial v}{\partial x_j}\frac{\partial v}{\partial x_i}\Big)\,\mathrm{d}t = -\int_0^s\int_\Omega \sum_{i,j=1}^d \partial_t a_{ij}\frac{\partial v}{\partial x_j}\frac{\partial v}{\partial x_i}\,\mathrm{d}x\,\mathrm{d}t$$
$$-\int_0^s\int_\Omega \sum_{i,j=1}^d \frac{\partial}{\partial x_i}(a_{ij}-a_{ji})\frac{\partial v}{\partial x_j}\partial_t v\,\mathrm{d}x\,\mathrm{d}t + \int_0^s\int_\Omega \Big(bv\partial_t v + \sum_{i=1}^d c_i\frac{\partial^2 v}{\partial t\partial x_i}v\Big)\,\mathrm{d}x\,\mathrm{d}t.$$

We rewrite the last term as

$$\int_\Omega \sum_{i=1}^d c_i \frac{\partial^2 v}{\partial t\partial x_i}v\,\mathrm{d}x = -\int_\Omega \sum_{i=1}^d \frac{\partial c_i}{\partial x_i}uv\,\mathrm{d}x - \int_\Omega\sum_{i=1}^d c_i u\frac{\partial v}{\partial x_i}\,\mathrm{d}x.$$

Then

$$\|u(s,\cdot)\|^2_{L^2(\Omega)} + \|v(0,\cdot)\|^2_{W_0^{1,2}(\Omega)} \le C\int_0^s \big(\|v\|^2_{W_0^{1,2}(\Omega)} + \|u\|^2_{L^2(\Omega)}\big)\,\mathrm{d}t.$$

We denote

$$w(t) := \int_0^t u(\tau)\,\mathrm{d}\tau, \qquad 0\le t\le T;$$

then

$$\|u(s,\cdot)\|^2_{L^2(\Omega)} + \|w(s,\cdot)\|^2_{W_0^{1,2}(\Omega)} \le C\int_0^s \Big(\|w(s,\cdot)-w(t,\cdot)\|^2_{W^{1,2}(\Omega)} + \|u(t,\cdot)\|^2_{L^2(\Omega)}\Big)\,\mathrm{d}t$$
$$\le C\int_0^s \Big(2(\|w(s,\cdot)\|^2_{W^{1,2}(\Omega)} + \|w(t,\cdot)\|^2_{W^{1,2}(\Omega)}) + \|u(t,\cdot)\|^2_{L^2(\Omega)}\Big)\,\mathrm{d}t.$$

Hence

$$\|u(s.\cdot)\|^2_{L^2(\Omega)} + (1-2Cs)\|w(s,\cdot)\|^2_{W_0^{1,2}(\Omega)} \le C\int_0^s \Big(\|w(t,\cdot)\|^2_{W_0^{1,2}(\Omega)} + \|u(t,\cdot)\|^2_{L^2(\Omega)}\Big)\,\mathrm{d}t.$$

We choose T_1 so small that

$$2CT_1 = \frac{1}{2}.$$

Then on $[0,T_1]$ we have

$$\|u(s,\cdot)\|^2_{L^2(\Omega)} + \|w(s,\cdot)\|^2_{W_0^{1,2}(\Omega)} \le C\int_0^s \Big(\|w(t,\cdot)\|^2_{W_0^{1,2}(\Omega)} + \|u(t,\cdot)\|^2_{L^2(\Omega)}\Big)\,\mathrm{d}t$$

which yields that $u=0$ on the possibly short time interval $[0,T_1]$. We may now use $u(T_1)$ as a new initial value. We proceed as above and prove that $u=0$ on the time interval $[T_1,2T_1]$. After finite number of steps we reach the time instant T. The proof is complete. ■

Remark 5.2.7. More complex boundary conditions can be treated similarly as in the case of the parabolic problems. Some technicalities are different, but the main points are similar modulo changes in the proof above for homogeneous Dirichlet boundary conditions with respect to the parabolic problem.

5.2.2 Regularity of weak solutions to hyperbolic equations

Next we study the regularity of the solution. As for the parabolic problem, we start with a simple model case: here it is the wave equation in $\mathbb{R}^d$. We consider

$$\begin{aligned}
\partial_{tt}u - \Delta u &= f && \text{in } (0,T)\times\mathbb{R}^d\\
u(0,\cdot) &= g(\cdot) && \text{in } \mathbb{R}^d\\
\partial_t u(0,\cdot) &= h(\cdot) && \text{in } \mathbb{R}^d.
\end{aligned}$$

We compute

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\big(|\partial_t u|^2+|\nabla u|^2\big)\,\mathrm{d}x=\int_{\mathbb{R}^d}\big(\partial_t u\partial_{tt}u+\nabla u\cdot\partial_t\nabla u\big)\,\mathrm{d}x=\int_{\mathbb{R}^d}\partial_t u\big(\partial_{tt}u-\Delta u\big)\,\mathrm{d}x=\int_{\mathbb{R}^d}f\partial_t u\,\mathrm{d}x.$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\big(|\partial_t u|^2+|\nabla u|^2\big)\,\mathrm{d}x\le\int_{\mathbb{R}^d}f^2\,\mathrm{d}x+\int_{\mathbb{R}^d}|\partial_t u|^2\,\mathrm{d}x$$

which after employing the Gronwall inequality yields

$$\max_{t\in[0,T]}\big(\|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R}^d)}+\|\nabla u(t,\cdot)\|^2_{L^2(\mathbb{R}^d)}\big)\le C\big(\|f\|^2_{L^2((0,T)\times\mathbb{R}^d)}+\|h\|^2_{L^2(\mathbb{R}^d)}+\|g\|^2_{W^{1,2}_0(\mathbb{R}^d)}\big).$$

Next, denote $\widetilde{u}:=\partial_t u$. Then

$$\begin{aligned}\partial_{tt}\widetilde{u}-\Delta\widetilde{u}&=\widetilde{f} && \text{in } (0,T)\times\mathbb{R}^d\\ \widetilde{u}(0,\cdot)=\partial_t u(0,\cdot)&=h(\cdot) && \text{in } \mathbb{R}^d\\ \partial_t\widetilde{u}(0,\cdot)=\partial_{tt}u(0,\cdot)&=\Delta u(0,\cdot)+f(0,\cdot)=\Delta g+f(0,\cdot) && \text{in } \mathbb{R}^d.\end{aligned}$$

As

$$\max_{[0,T]}\|f(t,\cdot)\|^2_{L^2(\mathbb{R}^d)}\le C\big(\|f\|_{L^2((0,T)\times\mathbb{R}^d)}+\|\partial_t f\|_{L^2((0,T)\times\mathbb{R}^d)}\big),$$

we get

$$\max_{t\in[0,T]}\big(\|\partial_{tt}u(t,\cdot)\|^2_{L^2(\mathbb{R}^d)}+\|\partial_t\nabla u(t,\cdot)\|^2_{L^2(\mathbb{R}^d)}\big)\le C\big(\|f\|^2_{L^2((0,T)\times\mathbb{R}^d)}+\|\partial_t f\|^2_{L^2((0,T)\times\mathbb{R}^d)}+\|\nabla h\|^2_{L^2(\mathbb{R}^d)}+\|\Delta g\|^2_{L^2(\mathbb{R}^d)}\big).$$

Finally, we write

$$\Delta u=\partial_{tt}u-f$$

and we have due to the elliptic regularity

$$\|\nabla^2 u\|_{L^\infty(0,T;L^2(\mathbb{R}^d))}\le C\big(\|f\|_{L^2(0,T;L^2(\mathbb{R}^d))}+\|\partial_t f\|_{L^2(0,T;L^2(\mathbb{R}^d))}+\|h\|_{W^{1,2}_0(\mathbb{R}^d)}+\|g\|_{W^{2,2}(\mathbb{R}^d)}\big).$$

We now show these bounds rigorously. We assume

$$\begin{aligned}&\{a_{ij}\}_{i,j=1}^d\in W^{1,\infty}(Q_T),\quad b\in L^\infty(Q_T),\{c_i\}_{i=1}^d\in L^\infty(0,T;W^{1,\infty}(\Omega))\\ &h\in L^2(\Omega),\quad g\in W^{1,2}_0(\Omega),\quad f\in L^2(Q_T)\end{aligned}\tag{5.28}$$

and for the higher regularity

$$\begin{aligned}&\{a_{ij}\}_{i,j=1}^d\in W^{1,\infty}(\Omega),\ \text{independent of time},\quad b\in W^{1,\infty}(0,T;L^\infty(\Omega)),\{c_i\}_{i=1}^d\in W^{1,\infty}(Q_T)\\ &f\in W^{1,2}(0,T;L^2(\Omega)),\qquad g\in W^{2,2}(\Omega),\qquad h\in W^{1,2}_0(\Omega),\quad \Omega\in\mathcal{C}^{1,1}.\end{aligned}\tag{5.29}$$

We have

Theorem 5.2.8 — Regularity of weak solutions for hyperbolic problem I. Let conditions (5.28) be fulfilled. Let $u\in L^2(0,T;W^{1,2}_0(\Omega))$ with $\partial_t u\in L^2(0,T;L^2(\Omega))$ and $\partial_{tt}u\in L^2(0,T;W^{-1,2}(\Omega))$ be the unique weak solution to

$$\begin{aligned}\partial_{tt}u+Lu&=f && \text{in } (0,T)\times\Omega,\\ u(0,\cdot)&=g(\cdot) && \text{in } \Omega,\\ \partial_t u(0,\cdot)&=h(\cdot) && \text{in } \Omega,\\ u&=0 && \text{on } (0,T)\times\partial\Omega.\end{aligned}$$

Then $u\in L^\infty(0,T;W^{1,2}_0(\Omega))$ with $\partial_t u\in L^\infty(0,T;L^2(\Omega))$ and

$$\|u\|_{L^\infty(0,T;W^{1,2}_0(\Omega))}+\|\partial_t u\|_{L^\infty(Q_T)}\le C\big(\|f\|_{L^2(Q_T)}+\|g\|_{W^{1,2}_0(\Omega)}+\|h\|_{L^2(\Omega)}\big).$$

Moreover, if (5.29) holds, then $u\in L^\infty(0,T;W^{2,2}(\Omega))$, $\partial_t u\in L^\infty(0,T;W^{1,2}_0(\Omega))$, $\partial_{tt}u\in L^\infty(Q_T)$, $\partial_{ttt}u\in L^2(0,T;W^{-1,2}(\Omega))$ and

$$\begin{aligned}&\|u\|_{L^\infty(0,T;W^{2,2}(\Omega))}+\|\partial_t u\|_{L^\infty(0,T;W^{1,2}_0(\Omega))}+\|\partial_{tt}u\|_{L^\infty(0,T;L^2(\Omega))}+\|\partial_{ttt}u\|_{L^2(0,T;W^{-1,2}(\Omega))}\\ &\qquad\le C\big(\|f\|_{W^{1,2}(0,T;L^2(\Omega))}+\|g\|_{W^{2,2}(\Omega)}+\|h\|_{W^{1,2}_0(\Omega)}\big).\end{aligned}$$

Proof. The first estimate was shown in the existence proof, since our assumptions ensure uniqueness of weak solutions. We therefore concentrate only on the second one. Due to the uniqueness of the weak solution under our assumptions, we use both the Galerkin approximation and the weak formulation.

Step 1: Second order time derivative estimate
We take the Galerkin approximation $u_n(t,x) = \sum_{i=1}^n d_i^n(t) w_i(x)$ which satisfies

$$(\partial_{tt} u_n, w_l)_{L^2(\Omega)} + B[u_n, w_l] = (f, w_l)_{L^2(\Omega)}, \qquad l = 1, 2, \dots, n,$$

where $\{w_l\}_{l=1}^\infty$ is the complete orthonormal system in $L^2(\Omega)$ and complete orthogonal system in $W_0^{1,2}(\Omega)$ formed by eigenfunctions of the Laplace operator with homogeneous Dirichlet conditions. We differentiate the equations with respect to time and get (we use the regularity properties of ODEs)

$$(\partial_{ttt} u_n, w_l)_{L^2(\Omega)} + B[\partial_t u_n, w_l] = (\partial_t f, w_l)_{L^2(\Omega)} - \int_\Omega \sum_{i=1}^d \partial_t c_i \frac{\partial u_n}{\partial x_j} w_l \, \mathrm{d}x - \int_\Omega \partial_t b u_n w_l \, \mathrm{d}x.$$

We may multiply the l-th equation by $d_l^{n\prime\prime}(t)$, sum up for $l = 1$ to n and integrate over $(0, T)$ (i.e., we test the equation by $\partial_{tt} u_n$). We get

$$(\partial_{ttt} u_n, \partial_{tt} u_n)_{L^2(\Omega)} + B[\partial_t u_n, \partial_{tt} u_n] = (\partial_t f, \partial_{tt} u_n)_{L^2(\Omega)} - \int_\Omega \sum_{i=1}^d \partial_t c_i \frac{\partial u_n}{\partial x_j} \partial_{tt} u_n \, \mathrm{d}x - \int_\Omega \partial_t b u_n \partial_{tt} u_n \, \mathrm{d}x.$$

We need to work slightly more with the highest order spatial derivative term. We have (recall that $\{a_{ij}\}_{i,j=1}^d$ are independent of time)

$$\begin{aligned}\int_\Omega \sum_{i,j=1}^d a_{ij} \frac{\partial(\partial_t u_n)}{\partial x_j} \frac{\partial(\partial_{tt} u_n)}{\partial x_i} \, \mathrm{d}x &= \int_\Omega \sum_{i,j=1}^d \frac{1}{2}(a_{ij} + a_{ji}) \frac{\partial(\partial_t u_n)}{\partial x_j} \frac{\partial(\partial_{tt} u_n)}{\partial x_i} \, \mathrm{d}x + \int_\Omega \sum_{i,j=1}^d \frac{1}{2}(a_{ij} - a_{ji}) \frac{\partial(\partial_t u_n)}{\partial x_j} \frac{\partial(\partial_{tt} u_n)}{\partial x_i} \, \mathrm{d}x \\ &= \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \sum_{i,j=1}^d \frac{1}{2}(a_{ij} + a_{ji}) \frac{\partial(\partial_t u_n)}{\partial x_j} \frac{\partial(\partial_t u_n)}{\partial x_i} \, \mathrm{d}x - \int_\Omega \sum_{i,j=1}^d \frac{1}{2} \frac{\partial}{\partial x_i}(a_{ij} - a_{ji}) \frac{\partial(\partial_t u_n)}{\partial x_j} \partial_{tt} u_n \, \mathrm{d}x,\end{aligned}$$

where we used that the other term in the integration by parts disappears due to smoothness of the orthogonal system and the symmetry/skew symmetry of the corresponding terms. We end up (due to the symmetry, we do not have to write the symmetric part of $\mathbb{A}$ any more)

$$\begin{aligned}&\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \Big(\|\partial_{tt} u_n\|_{L^2(\Omega)}^2 + \int_\Omega a_{ij} \frac{\partial(\partial_t u_n)}{\partial x_j} \frac{\partial(\partial_t u_n)}{\partial x_i} \, \mathrm{d}x \Big) \le C\Big(\|\partial_t \nabla u_n\|_{L^2(\Omega;\mathbb{R}^d)} \|\partial_{tt} u_n\|_{L^2(\Omega)} + \|\partial_{tt} u_n\|_{L^2(\Omega)} \|\partial_t u_n\|_{L^2(\Omega)} \\ &+ \|\nabla u_n\|_{L^2(\Omega;\mathbb{R}^d)} \|\partial_{tt} u_n\|_{L^2(\Omega)} + \|u_n\|_{L^2(\Omega)} \|\partial_{tt} u_n\|_{L^2(\Omega)} + \|\partial_t f\|_{L^2(\Omega)} \|\partial_{tt} u_n\|_{L^2(\Omega)} + \|\nabla \partial_t u_n\|_{L^2(\Omega;\mathbb{R}^d)} \|\partial_{tt} u_n\|_{L^2(\Omega)} \Big) \\ &\le C\Big(\|\partial_t f\|_{L^2(\Omega)}^2 + \|f\|_{L^2(\Omega)}^2 + \|\nabla \partial_t u_n\|_{L^2(\Omega;\mathbb{R}^d)}^2 + \|\partial_{tt} u_n\|_{L^2(\Omega)}^2 \Big).\end{aligned}$$

The Gronwall inequality yields

$$\begin{aligned}&\|\partial_{tt} u_n\|_{L^\infty(0,T;L^2(\Omega))} + \|\partial_t \nabla u_n\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^d))} \\ &\le C\Big(\|\partial_{tt} u_n(0,\cdot)\|_{L^2(\Omega)} + \|\partial_t \nabla u_n(0,\cdot)\|_{L^2(\Omega;\mathbb{R}^d)} + \|f\|_{W^{1,2}(0,T;L^2(\Omega))} + \|g\|_{W_0^{1,2}(\Omega)} + \|h\|_{L^2(\Omega)} \Big).\end{aligned}$$

As

$$\|\partial_t \nabla u_n(0,\cdot)\|_{L^2(\Omega;\mathbb{R}^d)} = \|\nabla \partial_t u_n(0,\cdot)\|_{L^2(\Omega;\mathbb{R}^d)} \le \|\nabla h\|_{L^2(\Omega;\mathbb{R}^d)}$$

and ($v_n^1 = P_n v$)

$$\begin{aligned}\|\partial_{tt} u_n(0,\cdot)\|_{L^2(\Omega)} &\le \sup_{\|v\|_{L^2(\Omega)} \le 1} \Big((f(0,\cdot), v_n^1)_{L^2(\Omega)} - B[u_n, v_1^n](0) \Big) \\ &\le C(\|u_n(0,\cdot)\|_{W^{2,2}(\Omega)} + \|f(0,\cdot)\|_{L^2(\Omega)} \le C\big(\|g\|_{W^{2,2}(\Omega)} + \|f\|_{W^{1,2}(0,T;L^2(\Omega))} \big),\end{aligned}$$

we get

$$\|\partial_{tt} u_n\|_{L^\infty(0,T;L^2(\Omega))} + \|\partial_t \nabla u_n\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^d))} \le C\big(\|g\|_{W^{2,2}(\Omega)} + \|f\|_{W^{1,2}(0,T;L^2(\Omega))} + \|h\|_{W_0^{1,2}(\Omega)} \big)$$

and passing to the limit we obtain the desired estimate.

Step 2: Elliptic regularity
Next, we write

$$\int_\Omega \sum_{i,j=1}^d a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} \, \mathrm{d}x = \int_\Omega \Big(-\partial_{tt} u + f - \sum_{i=1}^d c_i \frac{\partial u}{\partial x_i} - bu \Big) v \, \mathrm{d}x$$

and using the elliptic regularity we have

$$\|u(t,\cdot)\|_{W^{2,2}(\Omega)} \le C\big(\|\partial_{tt}u(t,\cdot)\|_{L^2(\Omega)} + \|f(t,\cdot)\|_{L^2(\Omega)} + \|\nabla u(t,\cdot)\|_{L^2(\Omega;\mathbb{R}^d)} + \|u(t,\cdot)\|_{L^2(\Omega)}\big).$$

Hence

$$\begin{aligned}\|u\|_{L^\infty(0,T;W^{2,2}(\Omega))} &\le C\big(\|\partial_{tt}u\|_{L^\infty(0,T;L^2(\Omega))} + \|f\|_{L^\infty(0,T;L^2(\Omega))} + \|u\|_{L^\infty(0,T;W^{1,2}(\Omega))}\big)\\ &\le C\big(\|f\|_{W^{1,2}(0,T;L^2(\Omega))} + \|g\|_{W^{2,2}(\Omega)} + \|h\|_{W^{1,2}(\Omega)}\big).\end{aligned}$$

Step 3: Third order time derivative estimate
We now return back to the Galerkin approximation and differentiate it with respect to time. We get

$$(\partial_{ttt}u_n, w_l)_{L^2(\Omega)} + B[\partial_t u_n, w_l] = (\partial_t f, w_l)_{L^2(\Omega)} - \int_\Omega \sum_{i=1}^d \partial_t c_i \frac{\partial u_n}{\partial x_j} w_l \,\mathrm{d}x - \int_\Omega \partial_t b u_n w_l.$$

Similarly as in the existence part we may now compute that

$$\begin{aligned}\|\partial_{ttt}u_n\|_{L^2(0,T;W^{-1,2}(\Omega))} &\le C\big(\|\partial_t f\|_{L^2(0,T;L^2(\Omega))} + \|\partial_t u_n\|_{L^2(0,T;W^{1,2}(\Omega))} + \|u_n\|_{L^2(0,T;W^{1,2}(\Omega))}\big)\\ &\le C\big(\|f\|_{W^{1,2}(0,T;L^2(\Omega))} + \|g\|_{W^{2,2}(\Omega)} + \|h\|_{W^{1,2}(\Omega)}\big).\end{aligned}$$

■

Exactly as in the parabolic problem we may proceed further. We will again not specify the precise assumptions on the regularity of the coefficients of L.

Theorem 5.2.9 — Regularity of weak solutions for hyperbolic problem II. Let $\{a_{ij}\}_{i,j=1}^d$, $\{c_i\}_{i=1}^d$, b be independent of time and sufficiently smooth in Ω. Let $g \in W^{m+1,2}(\Omega)$, $h \in W^{m,2}(\Omega)$ and $\frac{\partial^k f}{\partial t^k} \in L^2(0,T;W^{m-k,2}(\Omega))$, $k = 0,1,\dots,m$. Let $\Omega \in \mathcal{C}^{m,1}$. Let the following compatibility conditions hold

$$\begin{aligned} &g_0 := g \in W_0^{1,2}(\Omega), h_1 := h \in W_0^{1,2}(\Omega), \dots, \\ &g_{2l} := \frac{\partial^{2l-2} f}{\partial t^{2l-2}}(0,\cdot) - Lg_{2l-2} \in W_0^{1,2}(\Omega) \qquad \text{if } m = 2l, \\ &h_{2l+1} := \frac{\partial^{2l-1} f}{\partial t^{2l-1}}(0,\cdot) - Lh_{2l-1} \in W_0^{1,2}(\Omega) \qquad \text{if } m = 2l+1. \end{aligned}$$

Then

$$\frac{\partial^k u}{\partial t^k} \in L^\infty(0,T;W^{m+1-k,2}(\Omega)), \qquad k = 0,1,\dots,m+1,$$

and

$$\operatorname*{ess\,sup}_{t\in(0,T)} \sum_{k=0}^{m+1} \Big\|\frac{\partial^k u}{\partial t^k}\Big\|_{W^{m+1-k,2}(\Omega)} \le C\Big(\sum_{k=0}^{m} \Big\|\frac{\partial^k f}{\partial t^k}\Big\|_{L^2(0,T;W^{m-k,2}(\Omega))} + \|g\|_{W^{m+1,2}(\Omega)} + \|h\|_{W^{m,2}(\Omega)}\Big).$$

Remark 5.2.10. The role of the compatibility conditions is similar as in the case of parabolic problems and we saw its use above.

Since Theorem 5.2.9 hold for any $n \in \mathbb{N}$, we also have

Theorem 5.2.11 — Regularity of weak solutions for hyperbolic problem III. Let $\{a_{ij}\}_{i,j=1}^d$, $\{c_i\}_{i=1}^d$, b be $C^\infty(\overline{\Omega})$ (and independent of time), $f \in \mathcal{C}^\infty(\overline{Q_T})$, g and $h \in \mathcal{C}^\infty(\overline{\Omega})$, $\Omega \in \mathcal{C}^\infty$ and let the compatibility conditions from Theorem 5.2.9 hold for any $m \in \mathbb{N}$. Then the unique solution to the linear hyperbolic problem is such that $u \in \mathcal{C}^\infty(\overline{Q_T})$.

Proof of Theorem 5.2.9. The proof can be performed similarly as for the parabolic problem, by induction in m. The case $m = 0$ follows directly from Theorem 5.2.9.

Step 1: Highest time derivative estimate
Assume the theorem to be satisfied for some $m \in \mathbb{N}_0$ and suppose $g \in W^{2m+2}(\Omega)$, $h \in W^{2m+1}(\Omega)$ and the right-hand side $\frac{\partial^k f}{\partial t^k} \in L^2(0,T;W^{m+1-k,2}(\Omega))$, $k = 0,1,\dots,m+1$. Suppose the $(m+1)$th compatibility condition holds true. We set $\widetilde{u} = \partial_t u$ and differentiate the equation with respect to time. Then we see that $\widetilde{u}$ is the unique weak solution to the following problem

$$\begin{aligned} \partial_{tt}\widetilde{u} + L\widetilde{u} &= \widetilde{f} && \text{in } Q_T\\ \widetilde{u} &= 0 && \text{on } (0,T)\times\partial\Omega\\ \widetilde{u}(0,\cdot) &= \widetilde{g}(\cdot) && \text{in } \Omega\\ \partial_t\widetilde{u}(0,\cdot) &= \widetilde{h}(\cdot) && \text{in } \Omega,\end{aligned}$$

where $\widetilde{f} = \partial_t f$, $\widetilde{g} = h$, $\widetilde{h}(\cdot) = f(0,\cdot) - Lg(\cdot)$. For $m = 0$ we know that $\widetilde{u} \in L^2(0,T;W_0^{1,2}(\Omega))$, $\partial_t\widetilde{u} \in L^2(Q_T)$ and $\partial_{tt}\widetilde{u} \in L^2(0,T;W^{-1,2}(\Omega))$.

Since f, g and h satisfy the $(m+1)$th compatibility condition, $\widetilde{f}$, $\widetilde{g}$ and $\widetilde{h}$ satisfy the mth compatibility condition. Thus applying the induction assumption we get

$$\frac{\partial^k \widetilde{u}}{\partial t^k} \in L^\infty(0,T;W^{m+1-k,2}(\Omega)), \quad k=0,1,\dots,m+1$$

together with the estimate

$$\operatorname*{ess\,sup}_{t\in(0,T)} \sum_{k=0}^{m+1} \Big\|\frac{\partial^k \widetilde{u}}{\partial t^k}\Big\|_{L^2(0,T;W^{m+1-k,2}(\Omega))} \le C\Big(\sum_{k=0}^{m} \Big\|\frac{\partial^k \widetilde{f}}{\partial t^k}\Big\|_{W^{m+1-k,2}(\Omega)} + \|\widetilde{g}\|_{W^{m+1,2}(\Omega)} + \|\widetilde{h}\|_{W^{m,2}(\Omega)}\Big).$$

As $\widetilde{u} = \partial_t u$, we can rewrite the estimate above as

$$\begin{aligned}
&\operatorname*{ess\,sup}_{t\in(0,T)} \sum_{k=1}^{m+2} \Big\|\frac{\partial^k u}{\partial t^k}\Big\|_{W^{m+2-k,2}(\Omega)} \\
&\le C\Big(\sum_{k=1}^{m+1} \Big\|\frac{\partial^k f}{\partial t^k}\Big\|_{L^2(0,T;W^{m+1-k,2}(\Omega))} + \|g\|_{W^{m+2,2}(\Omega)} + \|h\|_{W^{m+1,2}(\Omega)} + \|f(0,\cdot)\|_{W^{m,2}(\Omega)}\Big) \qquad (5.30)\\
&\le C\Big(\sum_{k=0}^{m+1} \Big\|\frac{\partial^k f}{\partial t^k}\Big\|_{L^2(0,T;W^{m+1-k,2)}(\Omega)} + \|g\|_{W^{m+2,2}(\Omega)} + \|h\|_{W^{m+1,2}(\Omega)}\Big),
\end{aligned}$$

where we used the estimate $\|f(0,\cdot)\|_{W^{m,2}(\Omega)} \le C(\|f\|_{L^2(0,T;W^{m,2}(\Omega))} + \|\partial_t f\|_{L^2(0,T;W^{m,2}(\Omega))}$.

Step 2: Highest spatial derivative estimate

We now write for almost all $t\in(0,T)$

$$Lu = f - \partial_{tt}u =: z.$$

We have by elliptic regularity

$$\|u\|_{W^{m+2,2}(\Omega)} \le C(\|z\|_{W^{m,2}(\Omega)} + \|u\|_{L^2(\Omega)}) \le C(\|f\|_{W^{m,2}(\Omega)} + \|\partial_{tt}u\|_{W^{m,2}(\Omega)} + \|u\|_{L^2(\Omega)}).$$

Taking the essential supremum over $(0,T)$ and combining the resulted inequality with (5.30) we end up with

$$\operatorname*{ess\,sup}_{t\in(0,T)} \sum_{k=0}^{m+2} \Big\|\frac{\partial^k u}{\partial t^k}\Big\|_{W^{m+2-k,2}(\Omega)} \le C\Big(\sum_{k=0}^{m+1} \Big\|\frac{\partial^k f}{\partial t^k}\Big\|_{L^2(0,T;W^{m+1-k,2}(\Omega))} + \|g\|_{W^{m+2,2}(\Omega)} + \|h\|_{W^{m+1,2}(\Omega)}\Big).$$

This is the claim of the theorem for $m+1$. ■

5.2.3 Finite speed of propagation for weak solutions to hyperbolic problems

We proved this result for classical solutions in Pokorný (2025), see also Evans (1998). We now try to extend this result for weak solutions only.

Theorem 5.2.12 — Finite speed of propagation for wave equation. Let the function $u \in L^2(0,T;W^{1,2}_{\mathrm{loc}}(\Omega)) \cap W^{1,2}(0,T;L^2_{\mathrm{loc}}(\Omega)) \cap W^{2,2}(0,T;W^{-1,2}_{\mathrm{loc}}(\Omega))$ be a weak solution to

$$\langle \partial_{tt}u, v\rangle_{W_0^{1,2}(\Omega)} + c^2 \int_\Omega \nabla u\cdot\nabla v\,\mathrm{d}x = 0 \qquad (5.31)$$

for all $v\in W^{1,2}(\Omega)$, $v=0$ in the neighbourhood of the $\partial\Omega$, for almost every $t\in(0,T)$. Let $\overline{B_1(0)}\subset\Omega$ and $u(0,\cdot) = \partial_t u(0,\cdot) = 0$ almost everywhere in $B_1(0)$. Then $u(t,x)=0$ almost everywhere in the set $\{(t,x)\,|\,t>0,\ |x|+\frac{t}{c}<1\}$.

Remark 5.2.13. Note that c can be interpreted as the speed of propagation of the signal.

Proof of Theorem 5.2.12. We first assume that $c=1$, then we transform the general situation to this special case. We integrate (5.31) with $c=1$ over time, from 0 to t. We get, using also the Fubini Theorem

$$\langle \partial_t u, v\rangle_{W_0^{1,2}(\Omega)} + \int_\Omega \Big(\int_0^t \nabla u(\tau,\cdot)\mathrm{d}\tau\Big)\cdot\nabla v\,\mathrm{d}x = \langle \partial_t u(0,\cdot), v\rangle_{W_0^{1,2}(\Omega)}. \qquad (5.32)$$

Assume that g is a Lipschitz function, zero around $\partial\Omega$ such that moreover
(i) $g(t,x)=0$ for all x such that $|x|\ge 1$
(ii) $g(1,x)=0$ for all $x\in\mathbb{R}^d$.

We set $v(x) := (ug)(t,x)$ in (5.32) and integrate the resulted identity over time, from 0 to 1. We end up with

$$\int_0^1 \langle \partial_t u, ug\rangle_{W_0^{1,2}(\Omega)}\,\mathrm{d}t + \int_0^1\int_\Omega \Big(\int_0^t \nabla u(\tau,\cdot)\mathrm{d}\tau\Big)\cdot\nabla(ug)\,\mathrm{d}x\,\mathrm{d}t = \int_0^1 \langle \partial_t u(0,\cdot), ug\rangle_{W_0^{1,2}(\Omega)}\,\mathrm{d}t. \qquad (5.33)$$

First, note that the term on the right-hand side is zero. This follows from the fact that $\partial_t u(0,x) = 0$ for $|x| \leq 1$, $g(t,x) = 0$ for $|x| > 1$, hence also $\langle \partial_t u(0,\cdot), ug\rangle_{W_0^{1,2}(\Omega)} = 0$ for almost every $t \in (0,T)$.

Next we consider the first term on the left-hand side. Since $\partial_t u \in L^2(0,T; L^2_{\mathrm{loc}}(\Omega))$, we may write

$$\begin{aligned}\int_0^1 \langle \partial_t u, ug\rangle_{W_0^{1,2}(\Omega)} \,\mathrm{d}t &= \int_0^1 \int_\Omega \partial_t u u g \,\mathrm{d}x \,\mathrm{d}t = \frac{1}{2}\int_0^1 \int_\Omega \partial_t (u^2) g \,\mathrm{d}x\,\mathrm{d}t \\ &= -\frac{1}{2}\int_0^1 \int_\Omega u^2 \partial_t g \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_\Omega \big(u^2(1,\cdot)g(1,\cdot) - u^2(0,\cdot)g(0,\cdot)\big)\,\mathrm{d}x.\end{aligned}$$

Since $u^2(1,x)g(1,x) = 0$ by (ii) and $u^2(0,x)g(0,x) = 0$ by assumptions of the theorem and by (i), we end up with

$$\int_0^1 \langle \partial_t u, ug\rangle_{W_0^{1,2}(\Omega)} \,\mathrm{d}t = -\frac{1}{2}\int_0^1 \int_\Omega u^2 \partial_t g \,\mathrm{d}x\,\mathrm{d}t.$$

The second term can be computed as

$$\begin{aligned}\int_0^1 \int_\Omega \Big(\int_0^t \nabla u(\tau,\cdot)\mathrm{d}\tau\Big)\cdot \nabla(ug)\,\mathrm{d}x\,\mathrm{d}t &= \int_0^1 \int_\Omega \Big(\int_0^t \nabla u(\tau,\cdot)\mathrm{d}\tau\Big)\cdot (\nabla u g + \nabla g u)\,\mathrm{d}x\,\mathrm{d}t \\ &= \frac{1}{2}\int_0^1\int_\Omega \partial_t\Big(\int_0^t \nabla u(\tau,\cdot)\mathrm{d}\tau\Big)^2 g\,\mathrm{d}x\,\mathrm{d}t + \int_0^1\int_\Omega \Big(\int_0^t \nabla u(\tau,\cdot)\mathrm{d}\tau\Big)\cdot \nabla g u\,\mathrm{d}x\,\mathrm{d}t \\ &= -\frac{1}{2}\int_0^1\int_\Omega \Big(\int_0^t \nabla u(\tau,\cdot)\mathrm{d}\tau\Big)^2 \partial_t g\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \frac{1}{2}\int_\Omega \Big(\int_0^1 \nabla u(\tau,\cdot)\mathrm{d}\tau\Big)^2 g(1,\cdot)\,\mathrm{d}x - \frac{1}{2}\int_\Omega \Big(\int_0^0 \nabla u(\tau,\cdot)\mathrm{d}\tau\Big)^2 g(0,\cdot)\,\mathrm{d}x \\ &\quad + \int_0^1\int_\Omega \Big(\int_0^t \nabla u(\tau,\cdot)\mathrm{d}\tau\Big)\cdot \nabla g u\,\mathrm{d}x\,\mathrm{d}t \\ &= -\frac{1}{2}\int_0^1\int_\Omega \Big(\int_0^t \nabla u(\tau,\cdot)\mathrm{d}\tau\Big)^2 \partial_t g\,\mathrm{d}x\,\mathrm{d}t + \int_0^1\int_\Omega \Big(\int_0^t \nabla u(\tau,\cdot)\mathrm{d}\tau\Big)\cdot \nabla g u\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Plugging these computations into (5.33) we end up with

$$\begin{aligned}-\frac{1}{2}\int_0^1\int_\Omega \Big(u^2 + \Big(\int_0^t \nabla u(\tau,\cdot)\mathrm{d}\tau\Big)^2\Big)\partial_t g\,\mathrm{d}x\,\mathrm{d}t &= -\int_0^1\int_\Omega \Big(\int_0^t \nabla u(\tau,\cdot)\mathrm{d}\tau\Big)\cdot \nabla g u\,\mathrm{d}x\,\mathrm{d}t \\ &\leq \frac{1}{2}\int_0^1\int_\Omega |\nabla g|\Big(u^2 + \Big(\int_0^t \nabla u(\tau,\cdot)\mathrm{d}\tau\Big)^2\Big)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

We now suitably choose the function g. We take for $\varepsilon \in (0,1)$

$$g(t,x) = \max\{0, 1-\varepsilon - (1-\varepsilon)|x| - t\}.$$

This function is clearly nonnegative, Lipschitz, for ε sufficiently close to zero the function g is zero near the boundary of Ω. We further have $g(1,x) = 0$ for all $x \in \mathbb{R}^d$ and $g(0,x) = \max\{0, (1-\varepsilon)(1-|x|)\} = 0$ for $|x| \geq 1$. Thus our assumptions are satisfied.

We further have

$$\begin{aligned}\partial_t g &= -\chi_{\{(t,x)\in\mathbb{R}^+\times\Omega \,|\, t<(1-\varepsilon)(1-|x|)\}} =: -\chi_\varepsilon \\ \nabla g &= (1-\varepsilon)\chi_{\{(t,x)\in\mathbb{R}^+\times\Omega \,|\, t<(1-\varepsilon)(1-|x|)\}} = (1-\varepsilon)\chi_\varepsilon.\end{aligned}$$

Altogether, we have

$$\frac{1}{2}\int_0^1\int_\Omega \Big(u^2 + \Big(\int_0^t \nabla u(\tau,\cdot)\mathrm{d}\tau\Big)^2\Big)\chi_\varepsilon\,\mathrm{d}x\,\mathrm{d}t \leq \frac{1-\varepsilon}{2}\frac{1}{2}\int_0^1\int_\Omega \Big(u^2 + \Big(\int_0^t \nabla u(\tau,\cdot)\mathrm{d}\tau\Big)^2\Big)\chi_\varepsilon\,\mathrm{d}x\,\mathrm{d}t$$

which implies

$$u\chi_\varepsilon = 0 \qquad \text{almost everywhere in } (0,T)\times\Omega,$$

whence $u(t,x) = 0$ almost everywhere in the set $\{(t,x) \in \mathbb{R}^+\times\Omega \,|\, t < (1-\varepsilon)(1-|x|)\}$. Passing with $\varepsilon \to 0_+$ we get $u(t,x) = 0$ almost everywhere in the set $\{(t,x) \in \mathbb{R}^+\times\Omega \,|\, t + |x| < 1\}$. The case $c = 1$ is finished.

Let us assume that $c \neq 0$ (but positive). We have

$$\langle \partial_{tt} u, v\rangle_{W_0^{1,2}(\Omega)} + c^2 \int_\Omega \nabla u\cdot\nabla v\,\mathrm{d}x = 0$$

for all $v \in W_0^{1,2}(\Omega)$ which is zero near the boundary and almost every $t \in (0,T)$. Further, we assume $u(0,x) = \partial_t u(0,x) = 0$ in $B_1(0)$. We define $\widetilde{u}(t,x) := u(ct,x)$. Then we have

$$\langle \partial_{tt}\widetilde{u}, v\rangle_{W_0^{1,2}(\Omega)} + \int_\Omega \nabla\widetilde{u}\cdot\nabla v\,\mathrm{d}x = 0$$

for all $v \in W_0^{1,2}(\Omega)$ which is zero near the boundary and almost every $t \in (0,T)$. Since we did not change the condition at $t = 0$, using the procedure above we conclude that $\widetilde{u}(t,x) = 0$ almost everywhere in the set $\{(t,x) \in \mathbb{R}^+ \times \Omega \,|\, t + |x| < 1\}$. Thus also $u(ct,x) = 0$ almost everywhere in the set $\{(t,x) \in \mathbb{R}^+ \times \Omega \,|\, t + |x| < 1\}$ which implies $u(t,x) = 0$ almost everywhere in the set $\{(t,x) \in \mathbb{R}^+ \times \Omega \,|\, \frac{t}{c} + |x| < 1\}$. The proof is complete. ■

Chapter 6

A more detailed guide to Sobolev spaces

We know from the introductory chapter that the classical solution for different kinds of partial differential equations may not exist or the classical formulation may not be the appropriate type of solution. On the other hand, we also know that in many situations it is possible to speak about a generalized solution which we called *weak solutions*. For the steady problems, they are based on the Sobolev spaces $W^{k,p}(\Omega)$ which play the main role in the modern theory of partial differential equations. In Chapter 2 we only introduced them and explained their most important properties. However, deeper understanding of these spaces in unavoidable as soon as we touch nonlinear problems. This is precisely the aim of this chapter. For the sake of completeness, we repeat in this chapter also those parts which were already presented relatively carefully in Chapter 2.

As before, we assume in this chapter that the reader knows the elements of the theory of Lebesgue integral and Lebesgue spaces and knows the main properties of spaces of continuous, Hölder continuous and continuously differentiable functions. For completeness and for the reader's convenience we present a short overview of these results in Appendix A. We also assume that the reader has sufficient knowledge of functional analysis; a short overview of the main important results is presented in Appendix B

6.1 Definitions, basic properties

Let us first recall the definition of the multiindex.

Notation 6.1.1 (Multiindex)**.** The ordered d–tuple $\alpha = (\alpha_1, \ldots, \alpha_d)$, $\alpha_i \in \mathbb{N}_0$, is called a multiindex. The length of the multiindex is denoted $|\alpha|$ and is defined as $|\alpha| = \alpha_1 + \cdots + \alpha_d$.

In what follows we also use a shorten notation for partial derivatives.

Notation 6.1.2 (Partial derivative written by a multiindex)**.** The symbol $D^\alpha \phi$ denotes the partial derivative of a function ϕ

$$D^\alpha \phi(x) := \frac{\partial^{|\alpha|}\phi(x)}{\partial x_1^{\alpha_1} \ \ldots \ \partial x_d^{\alpha_d}}.$$

We can now introduce the basic notion in the modern theory of partial differential equations[1].

Definition 6.1.3 — Weak derivative. Let $\Omega \subset \mathbb{R}^d$ be an open (possibly also unbounded) set and $\alpha = (\alpha_1, \ldots, \alpha_d)$ be a multiindex. Let $u, v_\alpha \in L^1_{\mathrm{loc}}(\Omega)$. We say that v_α is a weak derivative of u with respect to

[1] The weak derivative is a special case of a more general notion — the distributional derivative. If $u \in L^1_{\mathrm{loc}}(\Omega)$, we may assign to the function u the regular distribution T_u defined as

$$\forall \varphi \in \mathcal{D}(\Omega): \ \ \langle T_u, \varphi \rangle := \int_\Omega u\varphi \, \mathrm{d}x,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $(\mathcal{D}(\Omega))^*$ and $\mathcal{D}(\Omega)$. Every distribution can be differentiated infinitely many times. A distribution G is the derivative of a distribution T with respect to x^α, if

$$\forall \varphi \in \mathcal{D}(\Omega): \ \ \langle T, D^\alpha \varphi \rangle = (-1)^{|\alpha|} \langle G, \varphi \rangle .$$

In particular, if $G = G_v$ and $T = T_u$ are regular distributions, it holds

$$\forall \varphi \in \mathcal{D}(\Omega): \ \ \int_\Omega u(x) D^\alpha \varphi(x) \, \mathrm{d}x = \langle T_u, D^\alpha \varphi \rangle = (-1)^{|\alpha|} \langle G_v, \varphi \rangle = (-1)^{|\alpha|} \int_\Omega v(x)\varphi(x) \, \mathrm{d}x;$$

it means that $v = D^\alpha u$ in the weak sense. While a derivative of the distribution exists for any order, it may not be the case for weak derivatives. On the other hand, if the weak derivative of the second (or higher) order exists and $\Omega \subset \mathbb{R}^d$ with $d \geq 2$, then the derivative is independent of the order of differentiation, see also Exercise 2.1.10 below.

x^α, if it holds for any $\phi \in \mathcal{C}_0^\infty(\Omega)$

$$\int_\Omega u(x) D^\alpha \phi(x)\,\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega v_\alpha(x)\phi(x)\,\mathrm{d}x.$$

The following properties of the weak derivative are more or less evident.

Lemma 6.1.4 — Connection between weak and classical derivative I. The following claims hold.

1. Let $u \in \mathcal{C}^k(\Omega)$. Then for any $|\alpha| \le k$ the classical and weak derivatives coincide.
2. The weak derivative is (in the sense of equality in $L^1_{\mathrm{loc}}(\Omega)$, thus almost everywhere) given uniquely.

Proof. We leave the proof of these claims to a kind reader as a useful exercise. ■

Remark 6.1.5. If the classical derivative is continuous, it is necessarily equal to the weak derivative. Therefore, we shall use the same notation for both; if v_α is a weak derivative of u with respect to x^α, we shall write $D^\alpha u = v_\alpha$.

We are ready to present the most important definition of this chapter.

Definition 6.1.6 — Sobolev spaces. Let $\Omega \subset \mathbb{R}^d$ be an open set, $k \in \mathbb{N}$ and $p \in [1,\infty]$. The Sobolev space $W^{k,p}(\Omega)$ is defined as

$$W^{k,p}(\Omega) := \{u \in L^p(\Omega) \,|\, \forall |\alpha| \le k \,|\, D^\alpha u \in L^p(\Omega)\}.$$

We endow this space with the norm

$$\|u\|_{k,p} = \|u\|_{W^{k,p}(\Omega)} := \begin{cases} \left(\sum_{|\alpha|\le k} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}} & \text{for } p \in [1,\infty), \\ \max_{|\alpha|\le k} \|D^\alpha u\|_{L^\infty(\Omega)} & \text{for } p = \infty. \end{cases}$$

If it is clear from the context on which set we work, we shall use the shorten notation $\|\cdot\|_{k,p}$. If there is a danger of ambiguity, we shall use the full notation $\|\cdot\|_{W^{k,p}(\Omega)}$. Similarly as in the case of the $L^p(\Omega)$ spaces, the elements of $W^{k,p}(\Omega)$ are in fact classes of functions which differ on a set of measure zero.

Remark 6.1.7. It is possible to define the Sobolev spaces for $k \in \mathbb{N}_0$ which means that the case $k = 0$ is included. For $k = 0$ we identify the Sobolev space with the Lebesgue space, i.e.,

$$W^{0,p}(\Omega) := L^p(\Omega).$$

Remark 6.1.8. We also often shorten the notation for partial derivatives of u. For $u \in W^{k,p}(\Omega)$, we define for $m = 1,\dots,k$ the vector (the tensor of the m-th order) $\nabla^m u : \Omega \to \mathbb{R}^{d^m}$ as follows

$$[\nabla^m u]_{i_1\cdots i_m} := \frac{\partial^m u}{\partial x_{i_1}\cdots \partial x_{i_m}}, \qquad \text{where } i_l = 1,\dots,d.$$

If $m = 1$, we shorten $\nabla u := \nabla^1 u$.

The correctness of Definition 6.1.6 is summarized in the following theorem.

Theorem 6.1.9 — Sobolev norm. The space $W^{k,p}(\Omega)$ is a normed linear space.

Proof. The space $W^{k,p}(\Omega)$ is clearly a linear space (cf. Exercise 6.1.10). It is therefore enough to verify that $\|\cdot\|_{k,p}$ is a norm. We consider only the case $p \in [1,\infty)$, the proof for $p = \infty$ is left for a reader as a useful exercise. We check step by step that $\|\cdot\|_{k,p}$ satisfies all the axioms of a norm.

Step 1: Property 1. of the norm

It evidently holds for any $u \in W^{k,p}(\Omega)$ that

$$0 \le \|u\|_{k,p} < \infty.$$

Moreover, if $\|u\|_{k,p} = 0$, then also $\|u\|_p = 0$ and therefore (property of the $\|\cdot\|_{L^p(\Omega)}$-norm) we also have $u = 0$ almost everywhere in Ω, i.e., u is equivalent to a zero function. The opposite implication is straightforward. Whence it holds

$$u = 0 \quad \Longleftrightarrow \quad \|u\|_{k,p} = 0.$$

Step 2: Property 2. of the norm

The weak derivative satisfies $D^\alpha(\lambda u) = \lambda D^\alpha u$ (cf. Exercise 6.1.10), further also $\|\lambda D^\alpha u\|_p = |\lambda| \|D^\alpha u\|_p$. Altogether, we have

$$\|\lambda u\|_{k,p} = \left(\sum_{|\alpha|\le k} \|D^\alpha(\lambda u)\|_p^p \right)^{\frac{1}{p}} = \left(|\lambda|^p \sum_{|\alpha|\le k} \|D^\alpha u\|_p^p \right)^{\frac{1}{p}}$$
$$= |\lambda| \|u\|_{k,p},$$

and we verified that the proposed norm is positively 1-homogeneous.

Step 3: Property 3. of the norm
Weak derivative is clearly linear $D^\alpha(u+v) = D^\alpha u + D^\alpha v$ (cf. Exercise 6.1.10) and for the L^p-norm, the Minkowski (triangle) inequality holds (cf. Theorem A.3.10)

$$\|u+v\|_p \le \|u\|_p + \|v\|_p.$$

Moreover, also the "discrete" Minkowski inequality holds, i.e., we have for any non-negative $\{a_n, b_n\}_{n=0}^m$

$$\left(\sum_{n=0}^m (a_n+b_n)^p\right)^{\frac{1}{p}} \le \left(\sum_{n=0}^m a_n^p\right)^{\frac{1}{p}} + \left(\sum_{n=0}^m b_n^p\right)^{\frac{1}{p}}.$$

This inequality yields

$$\begin{aligned}\|u+v\|_{k,p} &= \left(\sum_{|\alpha|\le k} \|D^\alpha u + D^\alpha v\|_p^p\right)^{\frac{1}{p}} \le \left(\sum_{|\alpha|\le k} \left(\|D^\alpha u\|_p + \|D^\alpha v\|_p\right)^p\right)^{\frac{1}{p}} \\ &\le \left(\sum_{|\alpha|\le k} \|D^\alpha u\|_p^p\right)^{\frac{1}{p}} + \left(\sum_{|\alpha|\le k} \|D^\alpha u\|_p^p\right)^{\frac{1}{p}} = \|u\|_{k,p} + \|v\|_{k,p};\end{aligned}$$

we thus verified the triangle inequality. It follows from Steps 1–3 that $\|\cdot\|_{k,p}$ is a norm. ■

The following exercise contains elementary properties of the weak derivative. Their proofs are easy, however, we recommend the reader to perform them in detail.

Exercise 6.1.10 (Properties of weak derivative)**.** Show that it holds for arbitrary two functions $u, v \in W^{k,p}(\Omega)$, where $k \in \mathbb{N}$, and an arbitrary multiindex α satisfying $|\alpha| \le k$:

1. $D^\alpha u \in W^{k-|\alpha|,p}(\Omega)$ and $D^\alpha(D^\beta u) = D^\beta(D^\alpha u) = D^{\alpha+\beta}u$, whenever $|\alpha| + |\beta| \le k$
2. $\lambda u + \mu v \in W^{k,p}(\Omega)$ and $D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v$ whenever $\lambda, \mu \in \mathbb{R}$
3. if $\widetilde{\Omega} \subset \Omega$ is open, then $u \in W^{k,p}(\widetilde{\Omega})$
4. if $\eta \in \mathcal{C}^\infty(\overline{\Omega})$, then $\eta u \in W^{k,p}(\Omega)$ and it holds that

$$D^\alpha(\eta u) = \sum_{\{\beta \,|\, \forall i=1,\dots,d\ \beta_i \le \alpha_i\}} \binom{\alpha}{\beta} D^\beta \eta D^{\alpha-\beta} u,$$

where $\binom{\alpha}{\beta} := \prod_{i=1}^d \binom{\alpha_i}{\beta_i}$.

Let us now present several typical examples showing which functions belong or do not belong to the spaces $W^{k,p}(\Omega)$. The first example illustrates that Sobolev functions cannot have a jump across a $(d-1)$–dimensional manifold.

Example 6.1.11. The function

$$u(x) := \begin{cases} x & \text{in } (0,1) \\ 2 & \text{in } [1,2) \end{cases}$$

is not an element of $W^{1,p}((0,2))$, because the weak derivative, if it had existed, would have been equal to the classical one in the intervals $(0,1)$ and $(1,2)$; the classical derivative is the function

$$v(x) := \begin{cases} 1 & \text{in } (0,1) \\ 0 & \text{in } (1,2). \end{cases}$$

This function, however, is not a weak derivative of[2] u, but it is a weak derivative of a function

$$\tilde{u}(x) = \begin{cases} x & \text{in } (0,1) \\ 1 & \text{in } [1,2). \end{cases}$$

Generally, a function which has a jump discontinuity across a $(d-1)$–dimensional manifold in $\Omega \subset \mathbb{R}^d$, does not have a weak derivative in Ω.

[2]The function u indeed possesses a distributional derivative. It is equal to the distribution $T_{\chi_{(0,1)}} + \delta_1$, where χ_I is the characteristic function of the interval I and δ_s is the Dirac distribution with the support at the point s. But this distribution is not regular and the function u does not possess a weak derivative. The function u, however, belongs to the space $BV((0,2))$, i.e., to the space of functions with bounded variation. The space $BV(\Omega)$ is defined as a subspace of the function space $L^1(\Omega)$ for which all distributional partial derivatives of the first order are Radon measures, cf. (Lukeš and Malý, 1995, Section 21).

The second example shows a typical behaviour near a singularity.

Example 6.1.12. Let $\Omega = B_1(0) \subset \mathbb{R}^d$. Then $u(x) := \frac{1}{|x|^\alpha} \in W^{1,p}(\Omega) \Leftrightarrow \alpha < \frac{d-p}{p}$. We see that also unbounded functions belong to some $W^{1,p}(\Omega)$. Note that for $\alpha < \frac{d-p}{p}$ the function $u \in L^q(\Omega)$ for every $q \in [1, p^*)$, where $p^* := \frac{dp}{d-p}$ (compare with the Embedding Theorem 6.5.1).

<u>*Solution*</u>. Consider the function

$$u_i(x) := -\alpha \frac{x_i}{|x|^{\alpha+2}}$$

and show that $u_i(x) = \frac{\partial}{\partial x_i} \frac{1}{|x|^\alpha}$ (in the weak sense). To this aim, apply the definition of the weak derivative and consider any $\varphi \in \mathcal{C}_0^\infty(\Omega)$ (it is in fact enough to take $\varphi \in \mathcal{C}_0^1(\Omega)$)

$$-\int_\Omega u(x) \frac{\partial \varphi(x)}{\partial x_i} \,\mathrm{d}x = -\lim_{\varepsilon \to 0_+} \int_{\Omega \setminus B_\varepsilon(0)} u(x) \frac{\partial \varphi(x)}{\partial x_i} \,\mathrm{d}x$$

and apply the Green formula (integration by parts in higher dimensions) on the second integral; (it can now be applied as both functions are sufficiently smooth on $\Omega \setminus B_\varepsilon(0)$). Then compute the limit $\varepsilon \to 0_+$. □

The last example illustrates the fact that the set of points, where the Sobolev function is discontinuous or unbounded, can be even dense in Ω.

Example 6.1.13. Let $\{r_k\}_{i=1}^\infty$ be a dense countable subset in $B_1(0)$. We define for $x \in B_1(0)$

$$u(x) = \sum_{i=1}^\infty \frac{1}{2^i} |x - r_i|^{-\alpha}.$$

If $p < d$ and $\alpha \in (0, \frac{d-p}{p})$, then $u \in W^{1,p}(B_1(0))$, but the function is not bounded on any open subset of $B_1(0)$.

The basic important properties of Sobolev spaces as completeness, separability and reflexivity are summarized in the following theorem.

Theorem 6.1.14 — On properties of Sobolev spaces. For every $k \in \mathbb{N}_0$ and $p \in [1, \infty]$ the space $W^{k,p}(\Omega)$ is a Banach space. For $p \in [1, \infty)$ the space $W^{k,p}(\Omega)$ is separable and for $p \in (1, \infty)$ the space is reflexive. For $p = 2$ the space $W^{k,2}(\Omega)$ is a Hilbert space with the scalar product

$$(u, v)_{W^{k,2}(\Omega)} = (u, v)_{k,2} := \sum_{|\alpha| \le k} \int_\Omega D^\alpha u D^\alpha v \,\mathrm{d}x. \tag{6.1}$$

<u>*Proof*</u>. **Step 1:** Completeness
The aim is to show that every Cauchy sequence in $W^{k,p}(\Omega)$ has a limit in $W^{k,p}(\Omega)$. Let $\{u_n\}_{n=1}^\infty \subset W^{k,p}(\Omega)$ be a Cauchy sequence, i.e.,

$$\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n, m \ge n_0 : \ \|u_n - u_m\|_{k,p} < \varepsilon.$$

The definition of $\|\cdot\|_{k,p}$ implies that for any multiindex α such that $|\alpha| \le k$ it holds $\|D^\alpha u_n - D^\alpha u_m\|_p < \varepsilon$. Therefore all sequences $\{D^\alpha u_n\}_{n=1}^\infty \subset L^p(\Omega)$ are Cauchy sequences. The spaces $L^p(\Omega)$ are complete (cf. Theorem A.3.11), and therefore there exist limits

$$\begin{aligned} u_n &\to u && \text{in } L^p(\Omega) \\ D^\alpha u_n &\to u_\alpha && \text{in } L^p(\Omega), \quad |\alpha| \le k. \end{aligned} \tag{6.2}$$

Since the limits of sequences $D^\alpha u_n$ were constructed separately, it is not clear whether we have $D^\alpha u = u_\alpha$. It remains to verify this claim. First, it holds $u_\alpha \in L^1_{\mathrm{loc}}(\Omega)$ (as $u_\alpha \in L^p(\Omega)$); we have verified the first property of the weak derivative. We take arbitrary α such that $|\alpha| \le k$. By virtue of the definition of the weak derivative it holds for every $\phi \in \mathcal{C}_0^\infty(\Omega)$

$$\int_\Omega u_n D^\alpha \phi \,\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega D^\alpha u_n \phi \,\mathrm{d}x.$$

We pass to the limit $n \to \infty$ on both sides of the equality. For the left-hand side we have due to $(2.2)_1$

$$\lim_{n \to \infty} \int_\Omega u_n D^\alpha \phi \,\mathrm{d}x = \int_\Omega u D^\alpha \phi \,\mathrm{d}x,$$

and for the right-hand side we obtain due to $(2.2)_2$

$$\lim_{n \to \infty} (-1)^{|\alpha|} \int_\Omega D^\alpha u_n \phi \,\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega u_\alpha \phi \,\mathrm{d}x.$$

Whence it must hold for any $\phi \in \mathcal{C}_0^\infty(\Omega)$ that $\int_\Omega u D^\alpha \phi \,\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega u_\alpha \phi \,\mathrm{d}x$. This implies $D^\alpha u = u_\alpha$.

Step 2: Reflexivity and separability
To show the reflexivity and separability we use the properties of the $L^p(\Omega)$ spaces, cf. Theorems A.3.34 and A.3.37. Denote $X = (L^p(\Omega))^\kappa$, where κ is the number of all multiindeces with the length equal or less than k. The space X is clearly reflexive (for $p \in (1, \infty)$) and separable (for $p \in [1, \infty)$).

We further define the mapping $I\colon W^{k,p}(\Omega) \to X$ as[3]

$$I(u) = [D^\alpha u]_{|\alpha| \le k} = \left[u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_d}, \dots, \frac{\partial^k u}{\partial x_d^k}\right].$$

Then I is an isomorphism between $W^{k,p}(\Omega)$ and $I(W^{k,p}(\Omega)) \subset X$. Due to the completeness of the space $W^{k,p}(\Omega)$, cf. Theorem 2.1.14, the set $I(W^{k,p}(\Omega))$ is a closed subset of X. Thus due to Theorem B.2.4 the space $W^{k,p}(\Omega)$ is separable, if $p \in [1, \infty)$, and reflexive, if $p \in (1, \infty)$.

Step 3: Case $p = 2$
We leave for a kind reader the verification that (2.1) is a scalar product. Since the associated norm is the standard norm in $W^{k,2}(\Omega)$, the space $W^{k,2}(\Omega)$ is a Hilbert space. ■

On the other hand, for the value $p = 1$ the Sobolev spaces (similarly as the Lebesgue ones) are not reflexive and for $p = \infty$ neither reflexive, nor separable.

Theorem 6.1.15 — On non-reflexivity and non-separability. The Sobolev space $W^{k,\infty}(\Omega)$ is not separable and the Sobolev spaces $W^{k,1}(\Omega)$ and $W^{k,\infty}(\Omega)$ are not reflexive.

Proof. The proof of the first claim is left for the reader, cf. the following Exercise 2.1.16. The proof of the second claim can be found in (Kufner et al., 1977, Theorems 5.2.4 and 5.2.6). ■

Exercise 6.1.16 ($W^{k,\infty}(\Omega)$ is not separable)**.** Let $\Omega \subset \mathbb{R}^d$ and let $\delta > 0$ be such that $B_\delta(x_0) \subset \Omega$ for a certain x_0. Consider for $\xi = (\xi_1, \dots, \xi_d) \in B_\delta(x_0)$ functions $\varphi_\xi = \min(1, |x_1 - \xi_1|)$. Show that φ_ξ is an uncountable system of functions from $W^{1,\infty}(\Omega)$ such that $\|\varphi_\xi - \varphi_{\widetilde{\xi}}\|_{W^{1,\infty}(\Omega)} \ge 1$ for $\xi_1 \ne \widetilde{\xi}_1$.

In what follows we introduce certain subspaces of Sobolev spaces whose elements "are zero" on the boundary Ω. These subspaces play an important role when we introduce the solution to certain boundary value problems in the theory of PDEs as well as at the rigorous justification of integration by parts for Sobolev functions.

Definition 6.1.17 — The space $W_0^{k,p}(\Omega)$. Let $\Omega \subset \mathbb{R}^d$ be an open set, $p \in [1, \infty)$ and $k \in \mathbb{N}$. Denote

$$W_0^{k,p}(\Omega) := \overline{\mathcal{C}_0^\infty(\Omega)}^{\|\cdot\|_{k,p}}.$$

Remark 6.1.18. If we allow $p = \infty$ in the definition above, we would get

$$\overline{\mathcal{C}_0^\infty(\Omega)}^{\|\cdot\|_{k,\infty}} \subseteq \left\{u \in \mathcal{C}^k(\overline{\Omega}) \,|\, \forall |\alpha| \le k, \forall x \in \partial\Omega\colon D^\alpha u(x) = 0\right\}$$

which follows directly from the definition of the convergence in the norm $\|\cdot\|_{k,\infty}$.

The following relation between $W^{k,p}(\Omega)$ and $W_0^{k,p}(\Omega)$ is left as an exercise for a kind reader.

Exercise 6.1.19. Show that $W_0^{k,p}(\Omega)$ is a subspace of $W^{k,p}(\Omega)$. Show further that $W_0^{k,p}(\Omega) \subsetneq W^{k,p}(\Omega)$ for an arbitrary open $\Omega \subsetneq \mathbb{R}^d$.

The spaces $W_0^{k,p}(\Omega)$ share almost all properties with $W^{k,p}(\Omega)$ which is formulated in the next theorem.

Theorem 6.1.20 — On properties of spaces $W_0^{k,p}(\Omega)$. For any $k \in \mathbb{N}$ and $p \in [1, \infty)$ the space $W_0^{k,p}(\Omega)$ is a Banach space. For $p \in [1, \infty)$ the space $W_0^{k,p}(\Omega)$ is separable and for $p \in (1, \infty)$ the space is reflexive. The space $W_0^{k,1}(\Omega)$ is not reflexive.

Proof. The proof is, similarly as the proof of Theorem 6.1.14, based on known properties of the Lebesgue spaces $L^p(\Omega)$. A more detailed proof can be found in (Kufner et al., 1977, Theorems 5.2.2, 5.2.4 and 5.2.6). ■

Remark 6.1.21. If we allow for $k = 0$ in the definition of the space $W_0^{k,p}(\Omega)$, then $W_0^{0,p}(\Omega) = L^p(\Omega)$ for $1 \le p < \infty$ and $\Omega \subset \mathbb{R}^d$ open, since the smooth compactly supported functions are dense in $L^p(\Omega)$ in this situation.

Finally, as a direct consequence, we deduce the formula for integration by parts for elements of Sobolev spaces $W_0^{1,p}(\Omega)$ and $W^{1,p'}(\Omega)$.

[3] The mapping I forms the vector of all possible (weak) partial derivatives of the order at most k.

Theorem 6.1.22 — On integration by parts I. Let $\Omega \subset \mathbb{R}^d$ be an open set, $k \in \mathbb{N}$ and $p \in [1, \infty)$. Then for any multiindex α such that $|\alpha| \leq k$, every $u \in W_0^{k,p}(\Omega)$ and every[a] $v \in W^{k,p'}(\Omega)$ it holds

$$\int_\Omega D^\alpha u \, v \, \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega u \, D^\alpha v \, \mathrm{d}x. \tag{6.3}$$

[a] Recall that $p' := \frac{p}{p-1}$ with the convention that for $p = 1$ we have $p' = \infty$.

Proof. By virtue of Hölder's inequality A.3.12 it is not difficult to verify that both integrals in (6.3) are finite. Furthermore, from the definition of the space $W_0^{k,p}(\Omega)$ we know that there exists a sequence $\{u_n\}_{n=1}^\infty \subset \mathcal{C}_0^\infty(\Omega)$ such that for any multiindex α, $|\alpha| \leq k$ we have

$$D^\alpha u_n \to D^\alpha u \qquad \text{in } L^p(\Omega).$$

As also $D^\alpha v \in L^{p'}(\Omega)$, we immediately obtain

$$\begin{aligned} \int_\Omega D^\alpha u \, v \, \mathrm{d}x &= \lim_{n\to\infty} \int_\Omega D^\alpha u_n \, v \, \mathrm{d}x \\ \int_\Omega u \, D^\alpha v \, \mathrm{d}x &= \lim_{n\to\infty} \int_\Omega u_n \, D^\alpha v \, \mathrm{d}x. \end{aligned} \tag{6.4}$$

Finally, directly from the definition of the weak derivative (recall that $u_n \in \mathcal{C}_0^\infty(\Omega)$) we deduce

$$\int_\Omega D^\alpha u_n \, v \, \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega u_n \, D^\alpha v \, \mathrm{d}x$$

and plugging this identity into (6.4) we get (6.3). ■

Inspired by Definition 6.1.17, we introduce at the end of this section yet other function spaces which we obtain as closure of smooth functions up to the boundary in the corresponding Sobolev norm.

Definition 6.1.23 — Sobolev spaces as closure. Let $\Omega \subset \mathbb{R}^d$ be an open set, $p \in [1, \infty)$ and $k \in \mathbb{N}$. The space $\widetilde{W}^{k,p}(\Omega)$ is defined as

$$\widetilde{W}^{k,p}(\Omega) := \overline{\mathcal{C}^\infty(\overline{\Omega})}^{\|\cdot\|_{k,p}}.$$

Remark 6.1.24. Analogously as in the case $W_0^{k,p}(\Omega)$ (cf. Remark 2.1.18) it does not make sense to define $\widetilde{W}^{k,\infty}(\Omega)$, because we would get due to the properties of $\|\cdot\|_{k,\infty}$ that $\widetilde{W}^{k,\infty}(\Omega) \subset \mathcal{C}^k(\overline{\Omega})$.

The following lemma summarizes the properties of the space defined as the closure of smooth functions up to the boundary in the Sobolev norm.

Lemma 6.1.25 — On properties of spaces $\widetilde{W}^{k,p}(\Omega)$. Let $k \in \mathbb{N}$ and $p \in [1, \infty)$. Then $\widetilde{W}^{k,p}(\Omega)$ is a closed subspace of $W^{k,p}(\Omega)$ (and thus a Banach space) which is separable and furthermore for $p \in (1, \infty)$ also reflexive. In particular, $\widetilde{W}^{k,p}(\Omega) \subset W^{k,p}(\Omega)$.

Proof. The fact that the space is closed follows directly from the definition. Other properties can by shown by virtue of Theorem 6.1.14. Their proof is left as a useful exercise for a kind reader. ■

The answer on the question when it holds $\widetilde{W}^{k,p}(\Omega) = W^{k,p}(\Omega)$ will be given in the next section. The validity of such claim will require certain assumptions on the qualitative properties of the set Ω. We now only present a counterexample of this claim for a sufficiently "ugly" open set Ω.

Exercise 6.1.26 ($\widetilde{W}^{k,p}(\Omega) \neq W^{k,p}(\Omega)$). We define the set $\Omega \subset \mathbb{R}^2$ as

$$\Omega := B_1(0) \setminus \{(x, 0) : x \in [0, 1)\}, \qquad \text{see also Figure 6.2 from Example 6.2.12.}$$

Consider a function u defined as

$$u(x, y) := \begin{cases} 0 & \text{if } x \leq 0, \\ 0 & \text{if } x > 0 \text{ a } y \geq 0, \\ x & \text{if } x > 0 \text{ a } y < 0. \end{cases}$$

Show that for every $p \in [1, \infty]$ it holds $u \in W^{1,p}(\Omega)$, but for $p < \infty$ we have $u \notin \widetilde{W}^{1,p}(\Omega)$.

Let us note that introducing the Sobolev spaces by Definitions 6.1.6 or 6.1.23 is not the only possibility. In the last subsection of this chapter we present an alternative, but fully equivalent definition based on the so-called Beppo Levi spaces.

6.2 Density of smooth functions

Next, we look at results concerning the density of different types of smooth functions. They are important since they allow the following proof strategy: we first show validity of certain integral identity for smooth functions and in the next step, based on the density argument, we show by suitable limit passage that the identity also holds for Sobolev functions. These results will be mostly based on mollification of Sobolev functions.

In what follows the notation η will be used exclusively for the mollification kernel, i.e., for smooth radially symmetric function supported in the unit ball with integral mean value equal to one, see also Definition A.3.28. The function $\eta_\varepsilon(x) := \varepsilon^{-d}\eta(\frac{x}{\varepsilon})$ stands then for its rescaling and finally for $u \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ we introduce the mollification of the function u, denoted u_ε, as (see also Definition A.3.30)

$$u_\varepsilon(x) := \eta_\varepsilon \star u(x) = \int_{\mathbb{R}^d} \eta_\varepsilon(x-y)u(y)\,\mathrm{d}y.$$

6.2.1 Local approximation of Sobolev functions

The first result concerns a direct application of the mollification of a Sobolev function.

Theorem 6.2.1 — On local approximation by smooth functions. Let $\Omega \subset \mathbb{R}^d$ be open, $p \in [1,\infty)$ and $u \in W^{k,p}(\Omega)$ be arbitrary. We define u as zero outside of Ω and let $u_\varepsilon := \eta_\varepsilon \star u$ denote the mollification of u. Then it holds:

1. $D^\alpha u_\varepsilon = (D^\alpha u)_\varepsilon$ almost everywhere in $\Omega_\varepsilon := \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$
2. $u_\varepsilon \to u$ in $W^{k,p}(\Omega')$ for every open $\Omega' \subset \overline{\Omega'} \subset \Omega$.

Proof. Based on the definition, we compute

$$D^\alpha u_\epsilon(x) = D^\alpha\left(\int_{\mathbb{R}^d} \eta_\epsilon(x-y)u(y)\,\mathrm{d}y\right) = \int_{\mathbb{R}^d} \left(D^\alpha_x \eta_\epsilon(x-y)\right)u(y)\,\mathrm{d}y.$$

The verification of assumptions of Theorem on the derivative of integral with respect to a parameter is here simple, since η_ε is a smooth function with compact support. We also used the symbol D^α_x to underline that the derivative is taken with respect to the variable x. We next use the fact that for any $x \in \Omega_\varepsilon$ and each $y \in \mathbb{R}^d \setminus \Omega$ it holds $\eta_\varepsilon(x-y) = 0$ and thus for $x \in \Omega_\varepsilon$ we have

$$\begin{aligned} D^\alpha u_\epsilon(x) &= \int_\Omega \left(D^\alpha_x \eta_\epsilon(x-y)\right)u(y)\,\mathrm{d}y = \int_\Omega \left((-1)^{|\alpha|}\, D^\alpha_y \eta_\epsilon(x-y)\right)u(y)\,\mathrm{d}y \\ &= \int_\Omega \eta_\epsilon(x-y)D^\alpha_y u(y)\,\mathrm{d}y = \int_{\mathbb{R}^d} \eta_\epsilon(x-y)D^\alpha_y u(y)\,\mathrm{d}y = \left(D^\alpha u\right)_\varepsilon(x); \end{aligned}$$

we used above the definition of the weak derivative as well as the fact that $\operatorname{supp}\eta_\varepsilon(x-y) \subset \Omega$. Claim 1. is proved. Claim 2. is then a simple consequence of Claim 1. and properties of mollification; we apply Claim 4. from Theorem A.3.33 subsequently to every $D^\alpha u$ with $|\alpha| \le k$. ■

This theorem, even though it concerns only local approximation, has several important corollaries. The first one is a precise characterization of $W^{k,p}(\mathbb{R}^d)$.

Lemma 6.2.2 — Connection of $W^{k,p}_0(\mathbb{R}^d)$ and $W^{k,p}(\mathbb{R}^d)$. Let $k \in \mathbb{N}$ a $p \in [1,\infty)$. Then $W^{k,p}(\mathbb{R}^d) = W^{k,p}_0(\mathbb{R}^d)$.

Proof. The inclusion $W^{k,p}_0(\mathbb{R}^d) \subset W^{k,p}(\mathbb{R}^d)$ is evident and follows directly from the definition of the space $W^{k,p}_0(\mathbb{R}^d)$.

Let us now deal with the opposite inclusion $W^{k,p}(\mathbb{R}^d) \subset W^{k,p}_0(\mathbb{R}^d)$. We have to show that for every $u \in W^{k,p}(\mathbb{R}^d)$ there exists a sequence $\{u_n\}_{n=1}^\infty \subset \mathcal{C}^\infty_0(\mathbb{R}^d)$ such that $\|u-u_n\|_{W^{k,p}(\mathbb{R}^d)} \overset{n\to\infty}{\to} 0$.

It is not difficult to see that for every $n \in \mathbb{N}$ there exists a non-negative $\xi_n \in \mathcal{C}^\infty_0(\mathbb{R}^d)$ such that $\xi_n = 1$ in $B_n(0)$, $\xi_n = 0$ in $\mathbb{R}^d \setminus B_{2n}(0)$ and which fulfils $\|\xi_n\|_{\mathcal{C}^k(\mathbb{R}^d)} \le C(k,d)$, where the constant $C(k,d)$ is independent of n. Let us define the function $u_n := u\xi_n$. Due to results of Exercise 6.1.10 we now that $u_n \in W^{k,p}(\mathbb{R}^d)$. Furthermore, using Claim 4. from the same exercise we immediately get the estimate

$$\begin{aligned} \|D^\alpha u - D^\alpha u_n\|^p_{L^p(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} |D^\alpha u - D^\alpha(u\xi_n)|^p\,\mathrm{d}x \\ &\le (C(k,d))^p \int_{\mathbb{R}^d\setminus B_n(0)} \sum_{|\beta|\le|\alpha|} |D^\beta u|^p\,\mathrm{d}x. \end{aligned}$$

As $u \in W^{k,p}(\mathbb{R}^d)$, it follows from the above stated inequality that for arbitrary $\rho > 0$ we may find $n \in \mathbb{N}$ such that

$$\|u-u_n\|_{W^{k,p}(\mathbb{R}^d)} < \frac{\rho}{2}.$$

Now, it is enough to mollify the function u_n which unlike the function u already has compact support $\operatorname{supp} u_n \subset B_{2n}(0)$; we apply Theorem 6.2.1. In other words, for any $\rho > 0$, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$ it holds

$$\|u_n - (u_n)_\varepsilon\|_{W^{k,p}(\mathbb{R}^d)} < \frac{\rho}{2}$$

and $(u_n)_\varepsilon \in \mathcal{C}_0^\infty(B_{2n+\varepsilon_0}(0)) \subset \mathcal{C}_0^\infty(\mathbb{R}^d)$. The triangle inequality yields

$$\|u - (u_n)_\varepsilon\|_{W^{k,p}(\mathbb{R}^d)} \leq \|u_n - (u_n)_\varepsilon\|_{W^{k,p}(\mathbb{R}^d)} + \|u - u_n\|_{W^{k,p}(\mathbb{R}^d)} < \rho$$

which is precisely what we wanted to prove. We construct the required sequence by taking $\rho_k := \frac{1}{k}$ for any $k \in \mathbb{N}$. ■

Another, "intuitively" straightforward corollary of the Theorem on local approximation by smooth functions 6.2.1 is the following claim dealing with Sobolev functions which have zero first order derivative almost everywhere (and, as follows from the theorem, the functions are constant).

Lemma 6.2.3 — On constant functions. Let $\Omega \subset \mathbb{R}^d$ be an open connected and $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$. Then the following two assertions are equivalent.

1. The function u is constant almost everywhere in Ω.
2. For any multiindex α of the length one it holds $D^\alpha u = 0$ almost everywhere in Ω.

Proof. Implication 1. $\Rightarrow$ 2. follows directly from the definition of the weak derivative. Hence, we concentrate on the proof 2. $\Rightarrow$ 1. Let $x_0 \in \Omega$ be arbitrary. As Ω is open, there exists ε_0 such that $B_{2\varepsilon_0}(x_0) \subset \Omega$. Let us define u by zero outside of Ω and applying the mollifier we obtain the function $u_\varepsilon \in \mathcal{C}_0^\infty(\mathbb{R}^d)$. Using Theorem 6.2.1 we get that for any $\varepsilon \in (0, \varepsilon_0)$ and any $x \in \overline{B_\varepsilon(x_0)}$ it holds

$$D^\alpha u_\varepsilon(x) = (D^\alpha u)_\varepsilon(x) \qquad \forall |\alpha| = 1.$$

Since $D^\alpha u \equiv 0$ almost everywhere, we see from the definition of $(D^\alpha u)_\varepsilon$ that for a suitable ε it holds for every $x \in \overline{B_{\varepsilon_0}(x_0)}$

$$D^\alpha u_\varepsilon(x) = (D^\alpha u)_\varepsilon(x) = 0.$$

Since u_ε is a smooth function, whose all first (classical) partial derivatives are zero, it must be constant in $\overline{B_{\varepsilon_0}(x_0)}$. Since $u \in W^{1,p}(\Omega)$, using Property 2. from the Theorem on properties of the mollifier A.3.33 we get that $const = u_\varepsilon \to u$ almost everywhere in $B_{\varepsilon_0}(x_0)$, hence $u = const$ in $B_{\varepsilon_0}(x_0)$. Since x_0 was arbitrary and Ω is connected and open, Claim 1. from our theorem holds true. ■

We showed in the previous lemma that a Sobolev function has zero gradient in an open connected set, if and only if the function is constant there. We now show that if a function is constant on a *measurable* set, then all derivatives of the first order are there zero almost everywhere. This will allow us to present a claim about composition of Lipschitz and Sobolev functions[4] Recall that χ_B denotes a characteristic function of a set B.

Theorem 6.2.4 — On the derivative of a composite function. Let Ω be open and $u \in W^{1,p}(\Omega)$ for some $p \in [1, \infty]$. Denote for arbitrary $a \in \mathbb{R}$

$$\Omega_a := \{x \in \Omega \,|\, u(x) = a\}.$$

Then for any $i \in \{1, \dots, d\}$ it holds that $\frac{\partial u}{\partial x_i} = 0$ almost everywhere in Ω_a.

Further, let $f \in \mathcal{C}^{0,1}(\mathbb{R})$ (note that $f' \in L^\infty(\mathbb{R})$). Then $f \circ u - f(0) \in W^{1,p}(\Omega)$ and it holds

$$\frac{\partial f(u(x))}{\partial x_i} = f'(u(x)) \frac{\partial u(x)}{\partial x_i} \chi_{\{x \in \Omega \,|\, u(x) \notin S_f\}} \quad \text{almost everywhere in } \Omega, \tag{6.5}$$

where $S_f := \{s \in \mathbb{R} \,|\,$ the classical derivative $f'(s)$ does not exist$\}$.

Note that the Rademacher Theorem A.2.16 ensures that the derivative f' exists almost everywhere in $\mathbb{R}$ and thus the set S_f is of zero measure. If moreover Ω has finite measure, then $f(0) \in W^{1,p}(\Omega)$ and thus $f(u) \in W^{1,p}(\Omega)$. Finally, as an easy corollary of Theorem 6.2.4 which is in fact a part of the proof, we get a claim which is often used in the theory of partial differential equations.

[4] It is easier to deduce these formulas, if we work with the equivalent definition of the Sobolev spaces based on so called Beppo Levi spaces.

Corollary 6.2.5. Let $u \in W^{1,1}(\Omega)$ be arbitrary and denote $u^+ := \max\{0, u\}$ and $u^- := -\min\{0, u\}$. Then we have for all $i \in \{1, \dots, d\}$ and almost everywhere in Ω that

$$\begin{aligned} \frac{\partial u^+}{\partial x_i} &= \frac{\partial u}{\partial x_i} \chi_{\{x \in \Omega \,|\, u(x) > 0\}} \\ \frac{\partial u^-}{\partial x_i} &= -\frac{\partial u}{\partial x_i} \chi_{\{x \in \Omega \,|\, u(x) < 0\}} \\ \frac{\partial |u|}{\partial x_i} &= \operatorname{sign} u \frac{\partial u}{\partial x_i} \chi_{\{x \in \Omega \,|\, u(x) \neq 0\}}. \end{aligned}$$

In particular, u^+, u^- and $|u| \in W^{1,1}(\Omega)$.

Proof of Theorem 6.2.4. Let us first denote $f_{Lip} := \|f'\|_{L^\infty(\mathbb{R})}$. Since f is continuous, $f(u)$ is measurable. Moreover, as f is globally Lipschitz, we have

$$|f(u(x)) - f(0)| \le f_{Lip}|u(x) - 0| \le f_{Lip}|u(x)|.$$

As $u \in L^p(\Omega)$, then also $f(u) - f(0) \in L^p(\Omega)$. Assume now that $f(u)$ is a Sobolev function and equality (6.5) holds. Then we also have $\|\frac{\partial f(u)}{\partial x_i}\|_p \le f_{Lip}\|\frac{\partial u}{\partial x_i}\|_p$ and thus $(f(u) - f(0)) \in W^{1,p}(\Omega)$. It remains to verify validity of (6.5).

We start with the case when additionally $f \in \mathcal{C}^1(\mathbb{R})$. We want to show that $f(u)$ has a weak derivative and the derivative is given by formula (6.5). Let $\varphi \in \mathcal{C}_0^\infty(\Omega)$ and u_ε be the mollification of u by Theorem 6.2.1 such that $\operatorname{dist}(\operatorname{supp} \varphi, \partial\Omega) > 2\varepsilon$. We may then apply classical theorem on the derivative of a composed function; Property 1. from Theorem 6.2.1 implies after integration by parts

$$-\int_\Omega f(u_\varepsilon) \frac{\partial \varphi}{\partial x_i} \,\mathrm{d}x = \int_\Omega f'(u_\varepsilon) \frac{\partial u_\varepsilon}{\partial x_i} \varphi \,\mathrm{d}x. \tag{6.6}$$

We now let $\varepsilon \to 0_+$. Let us first look at the left-hand side. As $u_\varepsilon \to u$ in $L^1(\Omega \cap \operatorname{supp} \varphi)$ and f is globally Lipschitz,

$$\lim_{\varepsilon \to 0_+} \left| \int_\Omega (f(u_\varepsilon) - f(u)) \frac{\partial \varphi}{\partial x_i} \,\mathrm{d}x \right| \le f_{Lip}\|\varphi\|_{1,\infty} \lim_{\varepsilon \to 0_+} \int_{\Omega \cap \operatorname{supp} \varphi} |u_\varepsilon - u| \,\mathrm{d}x = 0.$$

We estimate the right-hand side of (6.6) as follows.

$$\begin{aligned} &\left| \int_\Omega f'(u_\varepsilon) \frac{\partial u_\varepsilon}{\partial x_i} \varphi \,\mathrm{d}x - \int_\Omega f'(u) \frac{\partial u}{\partial x_i} \varphi \,\mathrm{d}x \right| \\ &\quad \le \int_\Omega |f'(u_\varepsilon)| \left| \frac{\partial u_\varepsilon}{\partial x_i} - \frac{\partial u}{\partial x_i} \right| |\varphi| \,\mathrm{d}x \\ &\qquad + \int_\Omega |f'(u_\varepsilon) - f'(u)| \left| \frac{\partial u}{\partial x_i} \right| |\varphi| \,\mathrm{d}x \\ &\quad \le f_{Lip}\|\varphi\|_\infty \int_{\Omega \cap \operatorname{supp} \varphi} \left| \frac{\partial u_\varepsilon}{\partial x_i} - \frac{\partial u}{\partial x_i} \right| \,\mathrm{d}x \\ &\qquad + \|\varphi\|_\infty \int_{\Omega \cap \operatorname{supp} \varphi} |f'(u_\varepsilon) - f'(u)| \left| \frac{\partial u}{\partial x_i} \right| \,\mathrm{d}x. \end{aligned}$$

The first term goes to zero due to the convergence $\nabla u_\varepsilon \to \nabla u$ in $L^1(\Omega \cap \operatorname{supp} \varphi)$. To show the convergence to zero of the second term, it is enough to realize that due to the continuity of f' and almost everywhere convergence of u_ε the whole integrand converges almost everywhere. To pass to the limit it is possible to apply the Lebesgue dominated convergence Theorem A.3.4, as

$$|f'(u_\varepsilon(x)) - f'(u(x))| \left| \frac{\partial u(x)}{\partial x_i} \right| \le 2f_{Lip}|\nabla u(x)| \in L^1(\Omega \cap \operatorname{supp} \varphi);$$

this gives us the integrable majorant. We thus have for every $\varphi \in \mathcal{C}_0^\infty(\Omega)$

$$-\int_\Omega f(u) \frac{\partial \varphi}{\partial x_i} \,\mathrm{d}x = \int_\Omega f'(u) \frac{\partial u}{\partial x_i} \varphi \,\mathrm{d}x \tag{6.7}$$

which we wanted to show.

Let turn our attention on the proof that $\nabla u = \mathbf{0}$ almost everywhere in Ω_0. Let us consider a special function

$$f(x) := \begin{cases} x & x > 0 \\ 0 & x \le 0, \end{cases}$$

i.e., $f(u) := u^+$. We approximate the function f by the sequence of functions

$$f_\varepsilon(x) := \begin{cases} \left(\varepsilon^2 + x^2\right)^{\frac{1}{2}} - \varepsilon & x > 0 \\ 0 & x \leq 0. \end{cases}$$

Clearly $f_\varepsilon(u) \to u^+$ almost everywhere in Ω and

$$\lim_{\varepsilon \to 0_+} f'_\varepsilon(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x \leq 0. \end{cases}$$

This immediately implies that

$$f'_\varepsilon(u) \to \chi_{\{x\in\Omega \,|\, u(x)>0\}} \text{ almost everywhere.}$$

However, the previous step implies

$$-\int_\Omega f_\varepsilon(u) \frac{\partial \varphi}{\partial x_i} \,\mathrm{d}x = \int_\Omega f'_\varepsilon(u) \frac{\partial u}{\partial x_i} \varphi \,\mathrm{d}x.$$

Due to the almost everywhere convergence of $f'_\varepsilon(u)$ and of $f_\varepsilon(u)$ and applying the Lebesgue dominated convergence Theorem A.3.4 we get

$$-\int_\Omega u^+ \frac{\partial \varphi}{\partial x_i} \,\mathrm{d}x = \int_\Omega \frac{\partial u}{\partial x_i} \varphi \chi_{\{x\in\Omega \,|\, u(x)>0\}} \,\mathrm{d}x.$$

We proved not only formula (6.5) for a particular function f, but also the first from the formulas in the second part of Corollary 6.2.5. Furthermore, as $u^- = (-u)^+$ and $|u| = u^+ + u^-$, the validity of other formulas from this corollary immediately follows.

Finally, since $u = u^+ - u^-$, then $\nabla u = \nabla u^+ - \nabla u^-$ almost everywhere. However, on Ω_0, the functions ∇u^+ and ∇u^- are equal to zero almost everywhere. Thus also $\nabla u = 0$ almost everywhere on Ω_0. Moreover, after adding an arbitrary constant, i.e., for $\tilde{u} := u + c$ we have $\nabla \tilde{u} = \nabla u$ and thus $\nabla u = 0$ almost everywhere on Ω_c for any $c \in \mathbb{R}$ and the first part of the theorem is shown.

Let us now show (6.5) for a general $f \in \mathcal{C}^{0,1}(\mathbb{R})$. Due to the properties of the mollifier (recall Theorem A.3.33) we know that there exist a sequence $f_\varepsilon \in \mathcal{C}^1(\mathbb{R})$ such that for $\varepsilon \to 0_+$[5]

$$\begin{aligned} f_\varepsilon &\rightrightarrows f && \text{in } \mathbb{R}, \\ f'_\varepsilon &\to f' && \text{almost everywhere in } \mathbb{R}. \end{aligned}$$

Furthermore, we have

$$|f_\varepsilon(x)| \leq |f(x)| + C\varepsilon \qquad \text{and} \qquad \|f'_\varepsilon\|_{L^\infty(\mathbb{R})} \leq f_{Lip}.$$

Since f_ε is a smooth function, we can use (6.7) and get

$$-\int_\Omega f_\varepsilon(u) \frac{\partial \varphi}{\partial x_i} \,\mathrm{d}x = \int_\Omega f'_\varepsilon(u) \frac{\partial u}{\partial x_i} \varphi \,\mathrm{d}x. \tag{6.8}$$

We have to perform the limit passage $\varepsilon \to 0_+$. The term on the left-hand side is the same as above. To pass to the limit on the right-hand side we would like to use again the Lebesgue dominated convergence Theorem A.3.4. Due to the estimate $\|f'_\varepsilon\|_\infty \leq f_{Lip}$ it is not difficult to find the majorant and it therefore remains to show

$$f'_\varepsilon(u(x))\nabla u(x) \to f'(u(x))\nabla u(x)\chi_{\{x\in\Omega \,|\, u(x)\notin S_f\}}$$

for almost every $x \in \Omega$. This is evident, whenever $u(x) \notin S_f$. Let now $a := u(x) \in S_f$. Then clearly the set $\{x \in \Omega \,|\, u(x) \in S_f\} \cap \{x \in \Omega \,|\, \nabla u(x) \neq \mathbf{0}\}$ is a null set, if S_f is at most countable. Since in general this might not be the case, we apply the argument from Giusti (2003). We use the Theorem on the coarea formula in the form

$$\int_L |\nabla u(x)| \,\mathrm{d}x = \int_{\mathbb{R}} \mathcal{H}_{d-1}(\partial U_t \cap L) \,\mathrm{d}t,$$

where $L = \{x \in \Omega \,|\, x \in S_f\}$ and $U_t = \{x \in \Omega \,|\, u(x) > t\}$, see, e.g., Federer (1969) for the proof. Then since $\partial U_t = \{x \in \Omega \,|\, u(x) = t\}$, we see that $\nabla u = \mathbf{0}$ a.e. in L. Thus $\nabla u(x)\chi_{\{x\in\Omega \,|\, u(x)\in S_f\}} = \mathbf{0}$ almost everywhere, which finishes the proof. ■

In general, a composition of two Sobolev functions may give a function which is not Sobolev any more.

Example 6.2.6. Consider the function $u(x) = x^3 \sin^3\left(\frac{1}{x}\right) \in W^{1,\infty}((-1,1))$ and further the function $f(z) = \sqrt[3]{z} \in W^{1,q}((-1,1))$ for $q < \frac{3}{2}$. Then the function $(f \circ u)(x) = x \sin\left(\frac{1}{x}\right)$ does not belong even to $W^{1,1}((-1,1))$.

[5] To prove the second convergence, it is necessary to combine the Rademacher Theorem A.2.16 and the Theorem on local approximation by smooth functions 6.2.1.

6.2.2 Global approximation of Sobolev functions by functions from $\mathcal{C}^\infty(\Omega)$

We now show that Theorem 6.2.1 can be significantly strengthen in the sense that the approximate sequence belongs to $\mathcal{C}^\infty(\Omega) \cap W^{k,p}(\Omega)$. Note that this stronger form does not require any extra assumptions on the set Ω, on the other hand, it does not say anything about the approximation by functions from $\mathcal{C}^\infty(\overline{\Omega})$. Such type of approximation will be studied in the next subsection.

To be able to show the desired approximation result, we need the following general version of the Lemma on partition of unity.

Lemma 6.2.7 — On partition of unity I. Let $\Omega \subset \mathbb{R}^d$ be open and $\{V_i\}_{i\in\mathcal{I}}$ its (generally uncountable) covering. Then there exists a countable system of functions $\{\varphi_j\}_{j\in\mathcal{J}}$ such that it holds:

1. $\varphi_j \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ for all $j \in \mathcal{J}$
2. for each $j \in \mathcal{J}$ there exists $i \in \mathcal{I}$ such that $\operatorname{supp} \varphi_j \subset V_i$
3. $0 \le \varphi_j \le 1$ for all $j \in \mathcal{J}$
4. for all $x \in \Omega$ the sum $\sum_{j\in\mathcal{J}} \varphi_j(x) = 1$ and moreover, for any compact $K \subset \Omega$ we have $\varphi_j \neq 0$ only for finite number of j.

Proof. For the proof see, e.g., Yosida (1980). ■

We can now come to the main result of this part.

Theorem 6.2.8 — On approximation by smooth functions in Ω. Let $\Omega \subset \mathbb{R}^d$ be a bounded open set, $p \in [1,\infty)$ and $u \in W^{k,p}(\Omega)$. Then there exists a sequence $\{u_n\}_{n=1}^\infty \subset \mathcal{C}^\infty(\Omega) \cap W^{k,p}(\Omega)$ such that $u_n \to u$ in $W^{k,p}(\Omega)$.

Proof. **Step 1:** Definition of the covering
Let us define the sets

$$\Omega_i := \left\{ x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \frac{1}{i} \right\}, \; i \in \mathbb{N}.$$

These sets are open, from a certain index i non-empty, satisfying $\Omega_k \subset \Omega_j \subset \overline{\Omega}_j \subset \Omega$ for $j > k$ and $\Omega = \bigcup_{i=1}^\infty \Omega_i$. We further define for each $i \in \mathbb{N}$ open sets

$$V_i := \Omega_{i+3} \setminus \overline{\Omega}_{i+1} = \left\{ x \in \Omega : \frac{1}{i+3} < \operatorname{dist}(x, \partial\Omega) < \frac{1}{i+1} \right\}.$$

Finally, we define V_0 suitably so that this set is open, $\overline{V}_0 \subset \Omega$ and so that it holds $\Omega = \bigcup_{i=0}^\infty V_i$. We evidently also have $\overline{V}_i \subset \Omega$. Due to Lemma 6.2.7, we construct to the covering $\{V_i\}_{i=0}^\infty$ the partition of unity $\{\varphi_i\}_{i=0}^\infty$.

Step 2: Interior approximation in V_i
For a given $u \in W^{k,p}(\Omega)$ and $i \in \mathbb{N}$ we define functions $u_i := u\varphi_i$. Evidently, $u_i \in W^{k,p}(\Omega)$ and $\operatorname{supp} u_i \subset V_j \subset \Omega$ for some $j \in \mathbb{N}$ (in what follows, for each i we consider such j). Let us now choose arbitrary $\rho > 0$. We find to this number sufficiently small ε_i such that it holds for the mollified function $(u_i)_{\varepsilon_i} = \eta_{\varepsilon_i} \star (\varphi_i u)$

$$\|u_i - (u_i)_{\varepsilon_i}\|_{W^{k,p}(V_j)} < \frac{\rho}{2^{i+1}};$$

this is surely possible, it is enough to apply the Theorem on local approximation by smooth functions 6.2.1. If necessary, we make ε_i smaller, in order to have $\operatorname{supp}(u_i)_{\varepsilon_i} \subset \Omega_{j+4} \setminus \overline{\Omega}_j$. It clearly holds that $(u_i)_{\varepsilon_i} \in \mathcal{C}^\infty(\Omega)$.

Step 3: Definition of the approximate function
We set

$$v := \sum_{i=0}^\infty (u_i)_{\varepsilon_i}.$$

This definition makes a good sense, since the sum is always (for fixed $x \in \Omega$) finite; the reason is that the covering $\{V_i\}$ we constructed is locally finite (Property 4. from Lemma 6.2.7, i.e., $\forall K \subset \Omega$, K compact, there is only a finite number of indeces j for which it holds $\varphi_j \neq 0$). Indeed,[6] $v \in \mathcal{C}^\infty(\Omega)$.

Step 4: Function v is a good approximation
It clearly holds $u = u\sum_{i=0}^\infty \varphi_i$. Let us now consider an arbitrary open set $\Omega' \subset \overline{\Omega'} \subset \Omega$. Then, due to the compactness of $\overline{\Omega'}$,

$$\begin{aligned}
\|v - u\|_{W^{k,p}(\Omega')} &= \left\| \sum_{i=0}^\infty (u_i)_{\varepsilon_i} - u \sum_{i=0}^\infty \varphi_i \right\|_{W^{k,p}(\Omega')} \\
&\le \sum_{i\in\mathbb{N}_0;\, V_{j(i)}\cap\Omega' \neq \emptyset} \|(u_i)_{\varepsilon_i} - u_i\|_{W^{k,p}(V_j)} < \rho \sum_{i=0}^\infty \frac{1}{2^{i+1}} = \rho.
\end{aligned}$$

[6] On the other hand, it is not difficult to check that in general it is not true that $v \in \mathcal{C}^\infty(\overline{\Omega})$.

Due to the continuous dependence of the integral on the domain it is now enough to come to the supremum over all sets Ω', i.e.„ $\Omega' \nearrow \Omega$, and we get

$$\|v - u\|_{W^{k,p}(\Omega)} \leq \rho.$$

This inequality implies that $v \in W^{k,p}(\Omega)$ which was not yet clear. Furthermore, it is now evident that we may approximate u arbitrarily closely (in dependence on ρ) by a function $v \in W^{k,p}(\Omega) \cap \mathcal{C}^\infty(\Omega)$ and the proof is finished. ■

6.2.3 Global approximation of Sobolev functions by functions from $\mathcal{C}^\infty(\overline{\Omega})$

We aim at obtaining global approximation of Sobolev functions by smooth functions up to the boundary. To this aim, we need to apply Theorem 6.2.1. Since the approximation must be done up to the boundary, it will already depend on the properties of the boundary. For general domains, this given result is not true (cf. Exercise 6.1.26), thus we must exclude such situations as in the above mentioned exercise. We will present two results of this type. The first one is connected with a special type of domains, so-called star-shaped ones for which the proof is rather straightforward. The other case covers more general class of domains, but the definition of the domain is much more involved. Indeed, also the proof is more complex.

Definition 6.2.9 — Star-shaped domain. We say that an open set $\Omega \subset \mathbb{R}^d$ is star-shaped (with respect to a point x_0), if there exists a point $x_0 \in \Omega$ such that for any $x \in \Omega$, $x \neq x_0$ the half line starting at x_0 and going through x has exactly one common point with the boundary of Ω. It means that

$$\left\{y \in \mathbb{R}^d \,|\, \exists \tau \in \mathbb{R}_+, y = \tau(x - x_0) + x_0\right\} \cap \partial\Omega \text{ contains exactly one point.}$$

An example of a star-shaped domain is a ball or cube in $\mathbb{R}^d$ or, as indicated by the name of the domain — a symmetric star in the plane. Recall that a star-shaped set is necessarily connected.

Theorem 6.2.10 — On approximation up to the boundary for star-shaped domains. Let Ω be a star-shaped domain and $u \in W^{k,p}(\Omega)$ for $p \in [1, \infty)$. Then there exists a sequence $\{u_n\}_{n=1}^\infty \subset \mathcal{C}^\infty(\overline{\Omega})$ such that $u_n \to u$ in $W^{k,p}(\Omega)$.

Proof. The main idea of the proof consists in the idea that for star-shaped domains we may relatively easily the function u "slide out" of Ω and then mollify this "slid" function. We may assume without loss of generality that Ω is star-shaped with respect to the origin, i.e., $x_0 = 0$ (in the general case we may use the change of variables $y = x - x_0$). Our goal is to show that for any $\rho > 0$ there exists $u_\rho \in \mathcal{C}^\infty\left(\overline{\Omega}\right)$ such that

$$\|u - u_\rho\|_{k,p} < \rho. \tag{6.9}$$

Step 1: Slid out

We define for $\tau \in (0, 1)$ the slid of the function $u_\tau(x) := u(\tau x)$ and denote the open set

$$\Omega_\tau := \left\{x \in \mathbb{R}^d \,|\, \tau x \in \Omega\right\}.$$

Since the set is star-shaped with respect to the origin, we have that for all $\tau \in (0, 1)$ we have $\overline{\Omega} \subset \Omega_\tau$. Evidently also $u_\tau \in W^{k,p}(\Omega_\tau)$ (verify in details!) and it holds for any $x \in \Omega_\tau$

$$D^\alpha(u_\tau)(x) = \tau^{|\alpha|} \left(D^\alpha u\right)_\tau(x).$$

Thanks to this it is rather easy to show (see Exercise A.3.27) that

$$u_\tau \to u \quad \text{in } L^p(\Omega) \quad \text{for } \tau \to 1_-. \tag{6.10}$$

Using the triangle inequality we then get

$$\begin{aligned}
\|D^\alpha (u - u_\tau)\|_{L^p(\Omega)} &= \left\|D^\alpha u - \tau^{|\alpha|}(D^\alpha u)_\tau\right\|_{L^p(\Omega)} \\
&= \left\|(D^\alpha u - (D^\alpha u)_\tau) + (1 - \tau^{|\alpha|})\,(D^\alpha u)_\tau\right\|_{L^p(\Omega)} \\
&\leq \left(1 - \tau^{|\alpha|}\right) \|(D^\alpha u)_\tau\|_{L^p(\Omega)} + \|D^\alpha u - (D^\alpha u)_\tau\|_{L^p(\Omega)}.
\end{aligned}$$

Due to (6.10) we see that for $\tau \to 1_-$ the right-hand side converges to zero and thus, for any $\rho > 0$ we can find $\tau \in (0, 1)$ (which will be from now on fixed) such that it holds

$$\|u - u_\tau\|_{W^{k,p}(\Omega)} < \frac{\rho}{2}.$$

Step 2: Mollification

The function u_τ belongs to $W^{k,p}(\Omega_\tau)$ and $\overline{\Omega} \subset \Omega_\tau$. We may use, for our fixed τ, the Theorem on local approximation by smooth functions 6.2.1 and we find for $\rho > 0$ a number $\varepsilon > 0$ such that it holds

$$\|u_\tau - (u_\tau)_\varepsilon\|_{W^{k,p}(\Omega)} < \frac{\rho}{2},$$

where $(u_\tau)_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^d)$ and thus also $(u_\tau)_\varepsilon \in \mathcal{C}^\infty(\overline{\Omega})$.

Step 3: Approximation

We finally define $u_\rho := (u_\tau)_\varepsilon$ and verify (6.9). Applying the triangle inequality we get

$$\|u - u_\rho\|_{W^{k,p}(\Omega)} \leq \|u - u_\tau\|_{W^{k,p}(\Omega)} + \|u_\tau - (u_\tau)_\varepsilon\|_{W^{k,p}(\Omega)} < \rho$$

which we wanted to show. ■

The star-shaped domains are too specific and do not include a big class of otherwise "nice" domains. Another class for which the approximation of Sobolev function by smooth functions holds true are so-called domains with continuous boundary. Since it is not more difficult to define domains with Hölder continuous (or even more general) boundary, we present the definition in full generality.

Definition 6.2.11 — Domain with $\mathcal{C}^{k,\mu}$-boundary. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, $k \in \mathbb{N}_0$ and $\mu \in [0,1]$[a]. We say that Ω is a domain with the $\mathcal{C}^{k,\mu}$-boundary (shortly domain of the type $\mathcal{C}^{k,\mu}$ and denote $\Omega \in \mathcal{C}^{k,\mu}$), if there exist positive numbers α, β and M cartesian coordinate systems, i.e., the coordinates of an arbitrary point $x \in \mathbb{R}^d$ in the r-th coordinate system are denoted as $x = (x_{r_1}, \dots, x_{r_d}) := (x'_r, x_{r_d})$, and M continuous functions $a_r\colon \Delta_r \to \mathbb{R}$ of class $\mathcal{C}^{k,\mu}$, where we define for any $r \in \{1, \dots, M\}$

$$\Delta_r := \left\{x'_r \in \mathbb{R}^{d-1} \,|\, i = 1, \dots, d-1 : \ |x_{r_i}| < \alpha\right\}$$

such that the following holds.

1. If we denote by T_r the mapping (rotation and shift) which describes the change of coordinates from the r-th cartesian coordinate system (x'_r, x_{r_d}) to the global coordinate system (x', x_d), then for any $x \in \partial\Omega$ there exists a coordinate system (i.e., there exists $r \in \{1, \dots, M\}$), such that $x = T_r(x'_r, a_r(x'_r))$ for some $x'_r \in \Delta_r$.

2. If we define

$$\begin{aligned} V_r^+ &:= \left\{(x'_r, x_{r_d}) \in \mathbb{R}^d \,|\, x'_r \in \Delta_r,\ a_r(x'_r) < x_{r_d} < a_r(x'_r) + \beta\right\} \\ V_r^- &:= \left\{(x'_r, x_{r_d}) \in \mathbb{R}^d \,|\, x'_r \in \Delta_r,\ a_r(x'_r) - \beta < x_{r_d} < a_r(x'_r)\right\} \\ \Lambda_r &:= \left\{(x'_r, x_{r_d}) \in \mathbb{R}^d \,|\, x'_r \in \Delta_r,\ a_r(x'_r) = x_{r_d}\right\}, \end{aligned}$$

then $T_r(V_r^+) \subset \Omega$, $T_r(V_r^-) \subset \mathbb{R}^d \setminus \overline{\Omega}$ and $T_r(\Lambda_r) \subset \partial\Omega$.

Property 1. implies furthermore that $\partial\Omega = \bigcup_{r=1}^M T_r(\Lambda_r)$. If we denote the open set $V_r := V_r^+ \cup V_r^- \cup \Lambda_r$, then $\partial\Omega \subset \bigcup_{r=1}^M T_r(V_r)$ and $\{T_r(V_r)\}_{r=1}^M$ is a finite covering of a certain neighbourhood of $\partial\Omega$.

[a] For $\mu = 0$ we speak about domains with $\mathcal{C}^k$-boundary; if further $k = 0$, then $\mathcal{C}^0$-domains are often called domains with continuous boundary and are also denoted as domains with $\mathcal{C}$-boundary.

Figure 6.1 displays the typical situation how the boundary should look like.

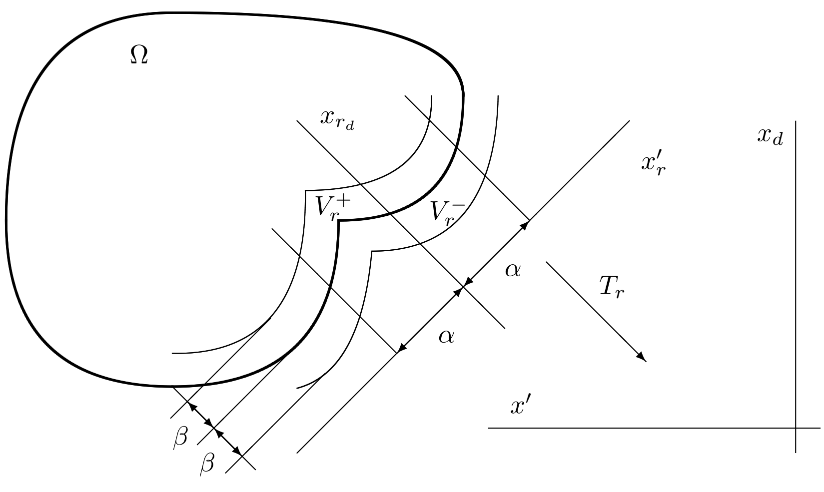

Figure 6.1: Domain Ω with the $\mathcal{C}^{0,1}$-boundary.

Example 6.2.12. Typical examples of different types of domains are as follows.

1. Domain defined as $(0,a)^d$, $a>0$ (i.e., the d-dimensional cube) is a domain with the $\mathcal{C}^{0,1}$-boundary (Lipschitz boundary).
2. The ball in $\mathbb{R}^d$ is a domain with the $\mathcal{C}^\infty$-boundary.
3. The ball in $\mathbb{R}^d$ without one line (see Figure 6.2) is even not a domain with the $\mathcal{C}$-boundary.

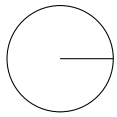

Figure 6.2: Domain which is not of the type $\mathcal{C}$.

Note that the third example is in fact a typical example of the domain assumed in Exercise 6.1.26 for which we do not have $\widetilde{W}^{k,p}(\Omega)=W^{k,p}(\Omega)$. On the other hand, as we shall see below, the equality holds for domains $\Omega\in\mathcal{C}^0$ and together with the continuity of the boundary, the key property is that we may speak about a well defined *interior* of the domain (represented by $T_r(V_r^+)$) and *exterior* of the domain (represented by $T_r(V_r^-)$) which allows us to modify the proof from the case of star-shaped domains.

Remark 6.2.13. Some of the domains have special names; the domain $\Omega\in\mathcal{C}$ is usually called the domain with continuous boundary, and $\Omega\in\mathcal{C}^{0,1}$ is called the domain with Lipschitz boundary.

Important tool in the proof of the theorem below which will also be very useful at several other situations is the following lemma.

Lemma 6.2.14 — On partition of unity II. Let $\{G_i\}_{i=1}^k$ be a finite number of open sets in $\mathbb{R}^d$ such that $\overline{\Omega}\subset\cup_{i=1}^k G_i$. Then there exist non-negative functions $\phi_i\in\mathcal{C}_0^\infty(G_i)$, $i=1,\dots,k$ such that for any i we have $\|\phi_i\|_{\mathcal{C}(\overline{\Omega})}\le 1$ and it holds for any $x\in\overline{\Omega}$

$$\sum_{i=1}^k \phi_i(x)=1.$$

Proof. We take open sets $G_i'\subset\subset G_i$, $i=1,2,\dots,k$ (i.e., $G_i'\subset\overline{G_i'}\subset G_i$) such that $\{G_i'\}_{i=1}^k$ still cover $\overline{\Omega}$. Then there exists an open set G_{k+1} such that $\cup_{i=1}^{k+1}G_i=\mathbb{R}^d$ and $\operatorname{dist}(G_{k+1},\Omega)>0$. We take $h>0$ sufficiently small so that $\operatorname{dist}(G_i',G_i)>h$ for all $i=1,2,\dots,k$ and $\operatorname{dist}(G_{k+1},\Omega)>h$. We set

$$\omega(x;h):=\begin{cases}\mathrm{e}^{\frac{|x|^2}{|x|^2-h^2}} & \text{pro } |x|<h\\ 0 & \text{pro } |x|\ge h.\end{cases}$$

We cover each $\overline{G_i'}$ and $\overline{G_{k+1}}$ by balls centred at y_{ij} lying in the given sets (this covering can be chosen finite for each $i=1,2,\dots,k$ and locally finite for G_{k+1}) and set

$$\psi_i(x)=\sum_j \omega(x-y_{ij};h)\qquad \text{for } i=1,2,\dots,k+1.$$

Clearly $\psi_i\in\mathcal{D}(G_i)$ for $i=1,2,\dots,k+1$ and for each $x\in\overline{\Omega}$ it is $\sum_{i=1}^k\psi_i(x)\neq 0$ and $\psi_{k+1}(x)=0$. We finally set

$$\phi_i(x):=\frac{\psi_i(x)}{\sum_{j=1}^{k+1}\psi_j(x)}.$$

Then the system of functions $\{\phi_i(x)\}_{i=1}^k$ forms the required partition of unity in $\overline{\Omega}$. ■

The main result of this subsection reads as follows.

Theorem 6.2.15 — On the approximation up to the boundary for $\Omega\in\mathcal{C}^0$. Let $\Omega\in\mathcal{C}^0$, $p\in[1,\infty)$, $k\in\mathbb{N}$ and $u\in W^{k,p}(\Omega)$. Then there exists a sequence $\{u_n\}_{n=1}^\infty$ of functions from $\mathcal{C}^\infty(\overline{\Omega})$ such that $u_n\to u$ in $W^{k,p}(\Omega)$. In other words, for $p\in[1,\infty)$, $k\in\mathbb{N}$ and $\Omega\in\mathcal{C}^0$ we have $\widetilde{W}^{k,p}(\Omega)=W^{k,p}(\Omega)$.

Proof. Our goal is to show that for any $u\in W^{k,p}(\Omega)$ and for arbitrary chosen $\rho>0$ we find $u^\rho\in\mathcal{C}^\infty(\overline{\Omega})$ such that

$$\|u-u^\rho\|_{k,p}<\rho.$$

In what follows, let u and ρ be fixed.

Step 1: Partition of unity
We denote for $r = 1, \dots, M$ the open sets from the definition of the set with continuous boundary $O_r := T_r(V_r)$. Then we can surely find an open set O_{M+1} such that $O_{M+1} \subset \overline{O_{M+1}} \subset \Omega$ and, moreover, $\overline{\Omega} \subset \cup_{r=1}^{M+1} O_r$. We apply the above shown Lemma on partition of unity II 6.2.14 on the system of sets $\{O_r\}_{r=1}^{M+1}$ and denote $u_r := u\phi_r \in W^{k,p}(\Omega)$. For $x \in \mathbb{R}^d \setminus \overline{\Omega}$ we define $u_r(x) := 0$.

Step 2: Approximation inside
The function $u_{M+1} \in W^{k,p}(\Omega)$ has compact support lying inside of Ω and since u_{M+1} is defined to be zero outside of Ω, we have $u_{M+1} \in W^{k,p}(\mathbb{R}^d)$. We can directly apply the Lemma on connection of $W_0^{k,p}(\mathbb{R}^d)$ and $W^{k,p}(\mathbb{R}^d)$ 6.2.2 and find for a given ρ the function $u^\rho_{M+1} \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ such that

$$\left\|u_{M+1} - u^\rho_{M+1}\right\|_{W^{k,p}(\Omega)} < \frac{\rho}{M+1}.$$

Step 3: "Slid out" of the function
For arbitrary $\delta \in (0, \frac{\beta}{2})$ we define $u_{r,\delta}(x) := u_r(T_r(x'_r, x_{r_d} + \delta))$; the function u_r is thus defined also in some neighbourhood of $\partial\Omega$, see Figure 6.3. Then main reason for this shift is the fact that the function u_r is defined in some neighbourhood of $\partial\Omega$ (here we also use the continuity of the boundary). Thus we shall be able to apply the Theorem on local approximation 6.2.1.

More precisely, as ϕ_r has compact support in $T_r(V_r \cap \Lambda_r)$, we see that by suitable choice of β, since u_r is defined to be zero outside of Ω, we have $u_r \in W^{k,p}(\Omega_r)$, where $\Omega_r := \mathbb{R}^d \setminus T_r(\overline{V^-_{\frac{r}{2}}})$ is open and

$$V^-_{\frac{r}{2}} := \left\{(x'_r, x_{r_d}) \in \mathbb{R}^d : \ x'_r \in \Delta_r, \ a_r(x'_r) - \frac{\beta}{2} < x_{r_d} < a_r(x'_r)\right\} \subset V^-_r.$$

If we additionally define $\Omega_{r,\delta} := \{y \in \mathbb{R}^d : \exists x \in \Omega_r, \ y = T_r(x'_r, x_{r_d} - \delta)\}$, then $u_{r,\delta} \in W^{k,p}(\Omega_{r,\delta})$. Evidently, for any multiindex α such that $|\alpha| \le k$ and each $x \in \Omega_{r,\delta}$ we have $D^\alpha u_{r,\delta}(x) = D^\alpha u_r(T_r(x'_r, x_{r_d} + \delta))$, whenever the derivative exists. Due to the Theorem on the p-mean continuity A.3.26 we can choose $\delta > 0$ ($\delta < \frac{\beta}{2}$) such that it holds

$$\|u_{r,\delta} - u_r\|_{W^{k,p}(\Omega_r \cap \Omega_{r,\delta})} < \frac{\rho}{2(M+1)}.$$

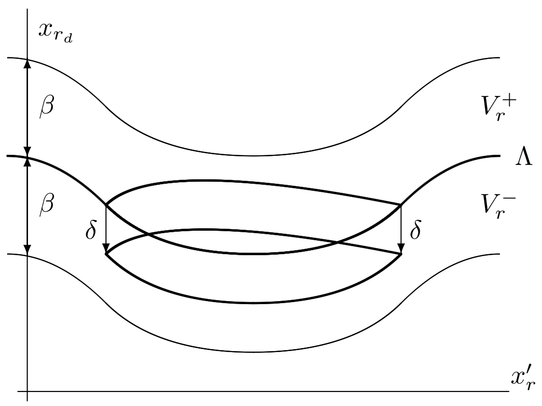

Figure 6.3: Shift of a function

Step 4: Mollification
To be able to apply the Theorem on local approximation 6.2.1, we first show that $\overline{\Omega} \subset \Omega_{r,\delta}$. To the contrary, assume that there exists $y \in \overline{\Omega}$ such that $y \notin \Omega_{r,\delta}$. In other words, for each $x = T_r(x'_r, x_{r_d}) \in \Omega_r$ it holds $y \ne T_r(x'_r, x_{r_d} - \delta)$. Describing y in the r-th local coordinate system, i.e., $y = T_r(y'_r, y_{r_d})$, then $y \ne T_r(x'_r, x_{r_d} - \delta)$ is equivalent with $(y'_r, y_{r_d} + \delta) \ne (x'_r, x_{r_d})$ which does not mean anything else than $(y'_r, y_{r_d} + \delta) \in \overline{V^-_{\frac{r}{2}}}$. This relation can be, due to the continuity of a and definition of $V^-_{\frac{r}{2}}$, written as

$$a_r(y'_r) - \frac{\beta}{2} \le y_{r_d} + \delta \le a_r(y'_r).$$

Since $\delta < \frac{\beta}{2}$, we see that

$$a_r(y'_r) - \beta < y_{r_d} < a_r(y'_r);$$

then $y \in T_r(V^-_r)$ and thus $y \notin \overline{\Omega}$ which leads to the contradiction.

We can now apply the Theorem on local approximation 6.2.1 and for $u_{r,\delta} \in W^{k,p}(\Omega_{r,\delta})$ we find $u^\rho_r \in \mathcal{C}^\infty(\overline{\Omega})$ (recall that $\overline{\Omega} \subset \Omega_{r,\delta}$) such that

$$\|u_{r,\delta} - u^\rho_r\|_{W^{k,p}(\Omega)} < \frac{\rho}{2(M+1)}.$$

Step 5: Construction of the approximate sequence
We set

$$u^\rho := \sum_{r=1}^{M+1} u_r^\rho.$$

Evidently $u^\rho \in \mathcal{C}^\infty(\overline{\Omega})$ and using Steps 1,2,3 (note that $\Omega \subset \Omega_{r,\delta} \cap \Omega_r$) and 4 as well as the triangle inequality we get

$$\begin{aligned}
\|u^\rho - u\|_{W^{k,p}(\Omega)} &\leq \left\| \sum_{r=1}^{M+1} (u_r^\rho - u_r) \right\|_{W^{k,p}(\Omega)} \leq \sum_{r=1}^{M+1} \|u_r^\rho - u_r\|_{W^{k,p}(\Omega)} \\
&< \frac{\rho}{M+1} + \sum_{r=1}^{M} \|u_r^\rho - u_r\|_{W^{k,p}(\Omega)} \\
&< \frac{\rho}{M+1} + \sum_{r=1}^{M} \|u_r^\rho - u_{r,\delta}\|_{W^{k,p}(\Omega)} + \sum_{r=1}^{M} \|u_{r,\delta} - u_r\|_{W^{k,p}(\Omega)} \\
&< \frac{\rho}{M+1} + 2\sum_{r=1}^{M} \frac{\rho}{2(M+1)} = \rho.
\end{aligned}$$

The proof is finished. ∎

Remark 6.2.16. Both fundamental properties of the set with continuous boundary appeared in Steps 3 and 4. On one hand we needed uniquely determined direction inside (outside) of Ω (Step 3), continuity of a_r was important in the description of $\overline{V_{\frac{r}{2}}^-}$.

6.3 Connection between weak derivative and differences

We saw in the previous subsection that the Sobolev functions can approximated arbitrarily close in the Sobolev norm by smooth functions. We also know that if a function has continuous classical derivatives, then they coincide with the weak derivatives. We shall now consider the opposite relation, namely if the weak derivative can be considered as a certain approximation of the classical derivative.

We denote for arbitrary $i = 1, \dots, d$ and $h \in \mathbb{R}$ the difference quotient

$$\Delta_i^h u(x) := \frac{u(x + h\mathbf{e}_i) - u(x)}{h}, \tag{6.11}$$

where $\mathbf{e}_i$ is the unit vector in the direction of the x_i axis. It is evident that if u has at the point x (classical) partial derivative with respect of the x_i variable, then

$$\lim_{h\to 0} \Delta_i^h u(x) = \frac{\partial u}{\partial x_i}.$$

We aim to clarify that something similar "in the mean" holds also for the weak derivative almost everwhere in Ω. The first result in this direction is the following.

Theorem 6.3.1 — On the connection between difference quotient and weak derivative I. Let Ω be open, $p \in [1, \infty]$ and $u \in L^p(\Omega)$. We denote for arbitrary $\delta > 0$

$$\Omega_\delta := \{x \in \Omega \,|\, \operatorname{dist}(x, \partial\Omega) > \delta\}.$$

Then we have the following.

1. If $u \in W^{1,p}(\Omega)$, then for any $i \in \{1, \dots, d\}$, $\delta \in (0, 1)$ and $|h| \in (0, \frac{\delta}{2})$ it holds

$$\left\|\Delta_i^h u\right\|_{L^p(\Omega_\delta)} \leq \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega)}.$$

2. If $p \in (1, \infty]$ and there exist constants $\{C_i\}_{i=1}^d$ such that for any $i \in \{1, \dots, d\}$, $\delta \in (0, 1)$ and $|h| \in (0, \frac{\delta}{2})$ it holds

$$\left\|\Delta_i^h u\right\|_{L^p(\Omega_\delta)} \leq C_i,$$

then $u \in W^{1,p}(\Omega)$. Moreover, we have for any $i \in \{1, \ldots, d\}$ the estimate

$$\left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega)} \leq C_i.$$

Proof. **Step 1:** Proof of Claim 1.
Let first $p \in [1, \infty)$ and $u \in W^{1,p}(\Omega)$ which we extend by zero[7] outside of Ω. From the Theorem on local approximation 6.2.1 we know that

$$u \star \eta_\varepsilon =: u_\varepsilon \overset{\varepsilon \to 0_+}{\to} u \text{ in } W^{1,p}(\Omega_{\frac{\delta}{2}}). \tag{6.12}$$

Since u_ε are smooth functions, we have for each $i \in \{1, \ldots, d\}$ and each $x \in \Omega_\delta$ ($|h| \leq \frac{\delta}{2}$)

$$\Delta_i^h u_\varepsilon(x) = \frac{1}{h} \int_0^h \frac{\partial u_\varepsilon}{\partial x_i}(x + t\mathbf{e}_i) \, \mathrm{d}t.$$

Using Hölder's inequality we thus get

$$\left|\Delta_i^h u_\varepsilon(x)\right|^p \leq \frac{1}{h^p} \left| \int_0^h \frac{\partial u_\varepsilon}{\partial x_i}(x + t\mathbf{e}_i) \, \mathrm{d}t \right|^p \leq \frac{1}{h} \int_0^h \left| \frac{\partial u_\varepsilon}{\partial x_i}(x + t\mathbf{e}_i) \right|^p \, \mathrm{d}t.$$

We integrate this inequality over Ω_δ and apply Fubini's theorem. As a result we get

$$\begin{aligned} \left\|\Delta_i^h u_\varepsilon(x)\right\|_{L^p(\Omega_\delta)}^p &\leq \frac{1}{h} \int_{\Omega_\delta} \left(\int_0^h \left| \frac{\partial u_\varepsilon}{\partial x_i}(x + t\mathbf{e}_i) \right|^p \, \mathrm{d}t \right) \mathrm{d}x \\ &= \frac{1}{h} \int_0^h \int_{\Omega_\delta} \left| \frac{\partial u_\varepsilon}{\partial x_i}(x + t\mathbf{e}_i) \right|^p \, \mathrm{d}x \, \mathrm{d}t. \end{aligned}$$

Using a suitable change of variables we may estimate the last integral (we increase the domain over which we integrate)

$$\frac{1}{h} \int_0^h \int_{\Omega_\delta} \left| \frac{\partial u_\varepsilon}{\partial x_i}(x + t\mathbf{e}_i) \right|^p \, \mathrm{d}x \, \mathrm{d}t \leq \frac{1}{h} \int_0^h \int_{\Omega_{\frac{\delta}{2}}} \left| \frac{\partial u_\varepsilon}{\partial x_i}(x) \right|^p \, \mathrm{d}x \, \mathrm{d}t = \left\| \frac{\partial u_\varepsilon}{\partial x_i} \right\|_{L^p(\Omega_{\frac{\delta}{2}})}^p.$$

We may now use the (strong) convergence (6.12) and combining the last two inequalities we obtain

$$\left\|\Delta_i^h u\right\|_{L^p(\Omega_\delta)}^p \leq \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega_{\frac{\delta}{2}})}^p \leq \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega)}^p.$$

The last inequality is just a simple enlarging of the domain over which we integrate. The proof of the first part for $p \in [1, \infty)$ is finished.

Let now $p = \infty$. Let us denote $\Omega^R := \Omega \cap B_R(0)$ which is again open and additionally bounded. If $u \in W^{1,\infty}(\Omega)$, then it must hold $u \in W^{1,p}(\Omega^R)$ for any $R > 0$ and $p \in [1, \infty)$. Denoting in a similar manner as above Ω_δ^R, we can use the previous step and obtain

$$\left\|\Delta_i^h u\right\|_{L^p(\Omega_\delta^R)} \leq \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega^R)}.$$

Now, we can let $p \to \infty$ (see Theorem A.3.17) and get

$$\left\|\Delta_i^h u\right\|_{L^\infty(\Omega_\delta^R)} \leq \left\| \frac{\partial u}{\partial x_i} \right\|_{L^\infty(\Omega^R)}.$$

Finally, the limit passage $R \to \infty$ finishes the proof of the first part also for $p = \infty$.

Step 2: Proof of Claim 2.
We first start with $p \in (1, \infty)$. We extend as above u by zero outside of Ω and take a sequence $\{h_n\}_{n=1}^\infty$ having zero limit. We then define functions

$$v_i^n := \Delta_i^{h_n} u \chi_{\Omega_{2|h_n|}}.$$

We see from the assumptions that it holds for any i that $\|v_i^n\|_{L^p(\Omega)} \leq C_i$. Since for our p's the spaces L^p are reflexive, we may choose a subsequence (which we relabel)

$$v_i^n \rightharpoonup v_i \text{ weakly in } L^p(\Omega).$$

[7] Note that after this extension the function u may not be a Sobolev function in $\mathbb{R}^d$.

Due to the weak lower semicontinuity of the norm (or also weak lower semicontinuity of convex functions, see Theorem A.3.43) it immediately follows

$$\|v_i\|_{L^p(\Omega)} \le C_i.$$

It remains to show that v_i are weak derivatives of u, i.e.,

$$v_i(x) = \frac{\partial u}{\partial x_i}(x) \text{ in the weak sense and almost everywhere in } \Omega,\ i = 1, 2, \ldots, d$$

which according to the definition means that it holds for any $\varphi \in \mathcal{C}_0^\infty(\Omega)$

$$\int_\Omega v_i \varphi \, \mathrm{d}x = -\int_\Omega u \frac{\partial \varphi}{\partial x_i} \, \mathrm{d}x. \tag{6.13}$$

By the uniqueness of the weak derivative we get that for any subsequence $v_i^{n_k} \rightharpoonup \frac{\partial u}{\partial x_i}$ and thus also for any sequence $h_n \to 0$. As φ has compact support, there exists n_0 such that for any $n \ge n_0$, $\operatorname{supp} \varphi \subset \Omega_{2|h_n|}$. The properties of φ yield

$$\begin{aligned}
\int_\Omega v_i \varphi \, \mathrm{d}x &= \lim_{n\to\infty} \int_\Omega v_i^n \varphi \, \mathrm{d}x = \lim_{n\to\infty} \int_{\Omega_{2|h_n|}} \Delta_i^{h_n} u \varphi \, \mathrm{d}x \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^d} \Delta_i^{h_n} u \varphi \, \mathrm{d}x = \lim_{n\to\infty} \int_{\mathbb{R}^d} u \Delta_i^{-h_n} \varphi \, \mathrm{d}x \\
&= -\lim_{n\to\infty} \int_{\mathbb{R}^d} u \frac{\varphi(x - h_n \mathbf{e}_i) - \varphi}{-h_n} \, \mathrm{d}x = -\int_\Omega u \frac{\partial \varphi}{\partial x_i} \, \mathrm{d}x;
\end{aligned}$$

we therefore verified (6.13) and the proof for $p \in (1, \infty)$ is finished. We used in the above weak convergence the Lebesgue dominated convergence Theorem A.3.4 for the last equality and also the identity

$$\int_{\mathbb{R}^d} \Delta_i^h u \varphi \, \mathrm{d}x = \int_{\mathbb{R}^d} u \Delta_i^{-h} \varphi \, \mathrm{d}x.$$

We leave the proof of this identity for a kind reader as an easy exercise.

The case $p = \infty$ can be shown similarly as in the previous step and is also left as an exercise. ■

Corollary 6.3.2. Note that (verify that the assumptions of the second part of Theorem 6.3.1 are satisfied) Lipschitz functions belong to $W^{1,\infty}(\Omega)$. We even have the embedding $\mathcal{C}^{0,1}(\overline{\Omega}) \hookrightarrow W^{1,\infty}(\Omega)$. The opposite inclusion requires certain regularity of the domain and will be discussed later.

Remark 6.3.3. The second part of Theorem 6.3.1 does not hold for $p = 1$, as a kind reader may verify for the function $u(x) := \operatorname{sign} x$ in the interval $(-1, 1)$. Nonetheless, the function fulfilling the assumptions of the second part of the theorem for $p = 1$ belongs to the space $BV(\Omega)$ — functions with bounded variation.

We showed in the previous theorem that the difference quotients converge to the weak derivative provided $p \in (1, \infty)$. We now strengthen the previous theorem in the sense that we will replace weak convergence by the strong one.

Theorem 6.3.4 — On the connection between difference quotient and weak derivative II. Let Ω be bounded, $p \in [1, \infty)$ and $u \in W^{1,p}(\Omega)$. We denote for $\delta > 0$

$$\Omega_\delta := \{x \in \Omega \,|\, \operatorname{dist}(x, \partial\Omega) > \delta\}.$$

Then it holds for any $\delta > 0$

$$\lim_{h\to 0} \left\| \Delta_i^h u - \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega_\delta)} = 0. \tag{6.14}$$

Proof. We shall use the same notation as in Theorem 6.3.1. Let $u \in W^{1,p}(\Omega)$. Similarly as above, we introduce the mollification u_ε for which we obtain

$$\Delta_i^h u_\varepsilon(x) = \frac{1}{h} \int_0^h \frac{\partial u_\varepsilon}{\partial x_i}(x + t\mathbf{e}_i) \, \mathrm{d}t.$$

Let now $\delta > 0$ be fixed; we also fix $h_1, h_2 \in (-\frac{\delta}{2}, \frac{\delta}{2})$. We get for the difference of the corresponding difference quotients, using its definition and the standard Theorem on the change of variables

$$\begin{aligned}
\Delta_i^{h_1} u_\varepsilon(x) - \Delta_i^{h_2} u_\varepsilon(x) &= \frac{1}{h_1} \int_0^{h_1} \frac{\partial u_\varepsilon}{\partial x_i}(x + t\mathbf{e}_i) \, \mathrm{d}t - \frac{1}{h_2} \int_0^{h_2} \frac{\partial u_\varepsilon}{\partial x_i}(x + t\mathbf{e}_i) \, \mathrm{d}t \\
&= \int_0^1 \left(\frac{\partial u_\varepsilon}{\partial x_i}(x + th_1\mathbf{e}_i) - \frac{\partial u_\varepsilon}{\partial x_i}(x + th_2\mathbf{e}_i) \right) \mathrm{d}t.
\end{aligned}$$

We can now estimate the L^p-norm of the term on the left-hand side by means of Fubini's theorem, Hölder's and triangle inequalities to get

$$\begin{aligned}
&\|\Delta_i^{h_1} u_\varepsilon - \Delta_i^{h_2} u_\varepsilon\|^p_{L^p(\Omega_\delta)} \\
&\leq \int_{\Omega_\delta}\int_0^1 \left|\frac{\partial u_\varepsilon}{\partial x_i}(x+th_1\mathbf{e}_i) - \frac{\partial u_\varepsilon}{\partial x_i}(x+th_2\mathbf{e}_i)\right|^p \,\mathrm{d}t\,\mathrm{d}x \\
&\leq \int_0^1 \left(\left\|\frac{\partial u_\varepsilon}{\partial x_i}(\cdot+th_1\mathbf{e}_i) - \frac{\partial u_\varepsilon}{\partial x_i}\right\|_{L^p(\Omega_\delta)} + \left\|\frac{\partial u_\varepsilon}{\partial x_i}(\cdot+th_2\mathbf{e}_i) - \frac{\partial u_\varepsilon}{\partial x_i}\right\|_{L^p(\Omega_\delta)}\right)^p \,\mathrm{d}t.
\end{aligned}$$

Due to the properties of the mollifications we may pass with $\varepsilon \to 0_+$ (recall that $|h_1|, |h_2| < \frac{\delta}{2}$) and obtain the estimate

$$\begin{aligned}
&\|\Delta_i^{h_1} u - \Delta_i^{h_2} u\|^p_{L^p(\Omega_\delta)} \\
&\leq \int_0^1 \left(\left\|\frac{\partial u}{\partial x_i}(\cdot+th_1\mathbf{e}_i) - \frac{\partial u}{\partial x_i}\right\|_{L^p(\Omega_\delta)} + \left\|\frac{\partial u}{\partial x_i}(\cdot+th_2\mathbf{e}_i) - \frac{\partial u}{\partial x_i}\right\|_{L^p(\Omega_\delta)}\right)^p \,\mathrm{d}t \\
&\leq 2^p \sup_{0<t<\max(|h_1|,|h_2|)} \int_{\Omega_\delta} \left|\frac{\partial u}{\partial x_i}(x+t\mathbf{e}_i) - \frac{\partial u}{\partial x_i}(x)\right|^p \,\mathrm{d}x.
\end{aligned}$$

We now verify, based on the inequality stated above, that the sequence of the differences is a Cauchy sequence. Let $\varepsilon > 0$ be arbitrary. Since $\frac{\partial u}{\partial x_i} \in L^p(\Omega)$, we may apply the Theorem on the p-mean continuity A.3.26 and find $h_\varepsilon \leq \frac{\delta}{2}$ such that for it holds for any $h \in (0, h_\varepsilon)$

$$\int_{\Omega_\delta} \left|\frac{\partial u}{\partial x_i}(x+h\mathbf{e}_i) - \frac{\partial u}{\partial x_i}(x)\right|^p \,\mathrm{d}x < \frac{\varepsilon^p}{2^p}.$$

Combining the last two inequalities we get that it holds for any $h_1, h_2 \in (0, h_\varepsilon)$

$$\|\Delta_i^{h_1} u - \Delta_i^{h_2} u\|_{L^p(\Omega_\delta)} < \varepsilon$$

and the sequence is a Cauchy one. Since the spaces L^p are complete, the sequence of the differences $\Delta_i^h u$ converges strongly in $L^p(\Omega_\delta)$ to some $v_i \in L^p(\Omega_\delta)$. Similarly as in the proof of the previous theorem it is possible to verify that

$$v_i = \frac{\partial u}{\partial x_i} \text{ almost everywhere in } \Omega_\delta$$

and the proof is complete. ■

Remark 6.3.5. Since the strong convergence implies almost everywhere convergence, at least for a chosen subsequence, Theorem 6.3.4 guarantees that for any subsequence $h_n \to 0$ there exists a set of zero Lebesgue measure $N \subset \Omega$ such that (for a chosen subsequence)

$$\lim_{h_n \to 0} \left|\Delta_i^{h_n} u(x) - \frac{\partial u}{\partial x_i}(x)\right| = 0 \qquad \text{for all } x \in \Omega \setminus N.$$

This claim, however, does not imply existence of the classical derivative, since the function u can be changed on a set of measure zero which can be dense in Ω and thus after this change the function u is not continuous at any point and cannot have the classical derivative anywhere. A different situation is for $p > d$, where already a continuous representative of the function exists. We shall discuss this case later.

On the other hand, we may strengthen Lemma 6.1.4. We relax the assumption on the continuity of the derivatives as follows.

Lemma 6.3.6 — Connection of weak and classical derivative II. Let Ω be an open set and $u \in W^{1,1}(\Omega)$. Let $D \subset \Omega$ be defined as

$$D := \{x \in \Omega \,|\, \text{ there exists the classical partial derivative of } u \text{ with respect to } x_i\}.$$

Then the weak and classical derivatives of u coincide almost everywhere in D.

Proof. The proof of this claim is based on the previous theorem and is left as an exercise for a kind reader. ■

Finally we present a small generalization of Theorem 6.3.1.

Lemma 6.3.7 — On the connection between difference quotient and weak derivative III. Let Ω be open, $p \in [1, \infty]$ and $u \in W^{1,p}(\Omega)$. We denote for $\delta > 0$, $h \in (-\frac{\delta}{2}, \frac{\delta}{2})$ and arbitrary unit vector $\mathbf{e} \in \mathbb{R}^d$

$$\Omega_\delta := \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \delta\}$$
$$\Delta_{\mathbf{e}}^h u(x) := \frac{u(x + h\mathbf{e}) - u(x)}{h}.$$

Then it holds

$$\left\|\Delta_{\mathbf{e}}^h u\right\|_{L^p(\Omega_{2|h|})} \le \|\nabla u \cdot \mathbf{e}\|_{L^p(\Omega)}.$$

Proof. The proof is also left as an exercise for a kind reader. ■

6.4 Extension from $W^{1,p}(\Omega)$ to $W^{1,p}(\mathbb{R}^d)$

In Theorem 6.2.15 we had to solve the mollification of a function from $W^{k,p}(\Omega)$ by "sliding out" the function, since unlike the Lebesgue spaces there is no simple extension which keeps the regularity of the function. We shall show in this section how to construct the extension; the prize is a certain smoothness of the boundary and the minimal requirement is Lipschitz continuity. The main result of this section is the following theorem.

Theorem 6.4.1 — Extension operator. Let $\Omega \in \mathcal{C}^{0,1}$ and $p \in [1, \infty]$. Then there exists a continuous linear operator

$$E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^d)$$

such that:

1. $Eu = u$ in Ω
2. Eu has compact support in $\mathbb{R}^d$
3. there exists $C = C(d, \Omega) > 0$ such that

$$\|Eu\|_{W^{1,p}(\mathbb{R}^d)} \le C \|u\|_{W^{1,p}(\Omega)}.$$

Remark 6.4.2. In what follows we shall call Eu extension of $u \in W^{1,p}(\Omega)$ to $u \in W^{1,p}(\mathbb{R}^d)$ and the operator E will be called the extension operator. The extension can be defined so that it preserves the $\mathcal{C}^1$ regularity. It is possible to preserve also higher regularity, but more regularity of the boundary of Ω must be required. This is due to the fact that the construction can be easily done for flat boundaries, while for more general non-flat boundaries we first need to flatten the boundary.

Let us assume that the boundary is described (for simplicity, we do not consider the change of variables caused by different coordinate systems)

$$x_d = a(x'),\ x' \in \Delta = \left\{x' \in \mathbb{R}^{d-1} \mid \forall i \in \{1, \dots, d-1\}\ |x_i| < \alpha\right\}.$$

Recall the notation from Definition 6.2.11

$$\begin{aligned} V^+ &= \left\{(x', x_d) \in \mathbb{R}^d \mid x' \in \Delta,\ a(x') < x_d < a(x') + \beta\right\} \\ V^- &= \left\{(x', x_d) \in \mathbb{R}^d \mid x' \in \Delta,\ a(x') - \beta < x_d < a(x')\right\} \\ \Lambda &= \left\{(x', x_d) \in \mathbb{R}^d \mid x' \in \Delta,\ a(x') = x_d\right\} \\ V &= V^+ \cup V^- \cup \Lambda. \end{aligned}$$

We now look at the change of variables which "flattens" the boundary. We define new variables (y', y_d) and the mapping $\mathbf{F}$: $(-1,1)^d \to V$ by means of

$$\begin{aligned} x' &= \alpha y' \\ x_d &= a(\alpha y') + \beta y_d, \quad \text{(i.e., } x = \mathbf{F}(y)) \end{aligned} \tag{6.15}$$

and

$$\begin{aligned} y' &= \frac{1}{\alpha} x' \\ y_d &= \frac{1}{\beta} x_d - \frac{1}{\beta} a(x'), \quad \text{(i.e., } y = \mathbf{F}^{-1}(x)), \end{aligned} \tag{6.16}$$

respectively. Denote further

$$\begin{aligned} C^+ &:= (-1,1)^{d-1} \times (0,1) \\ C^- &:= (-1,1)^{d-1} \times (-1,0). \end{aligned}$$

Then $\mathbf{F}$ maps C^+ onto V^+, C^- onto V^- and $(-1,1)^{d-1} \times \{0\}$ onto Λ. Furthermore, if a is at least Lipschitz, the following key lemma holds true.

Lemma 6.4.3 — Flattening of the boundary. Let $\mathbf{F}$ and $\mathbf{F}^{-1}$ be defined above by (6.15) and (6.16). Let a be a Lipschitz continuous function in $\overline{\Delta}$. Then the change of variables $\mathbf{F}$ and its inverse $\mathbf{F}^{-1}$ are Lipschitz continuous.

Moreover, there exist constants $C_1 = C_1(a,\alpha,\beta,d)$ and $C_2 = C_2(a,\alpha,\beta,d)$ such that for any $u \in W^{1,p}(V^+)$ with $p \in [1,\infty)$, the function $U := u \circ \mathbf{F} \in W^{1,p}(C^+)$ and it holds

$$C_1 \|u\|_{W^{1,p}(V^+)} \le \|U\|_{W^{1,p}(C^+)} \le C_2 \|u\|_{W^{1,p}(V^+)}. \tag{6.17}$$

Proof. The proof of the Lipschitz continuity of $\mathbf{F}(y) := (F_1(y),\dots,F_d(y))$ and $\mathbf{F}^{-1}(x) = (F_1^{-1}(x),\dots,F_d^{-1}(x))$ is left for a reader as an easy exercise.

Let us prove the remaining claims. First, note that due to the Lipschitz continuity of a we know (see Rademacher Theorem A.2.16) that a is differentiable almost everywhere in Δ and there exists a constant a_{Lip} such that we have for any $i = 1,\dots,d-1$

$$|a_i(x')| := \left|\frac{\partial a(x')}{\partial x_i}\right| \le \sup_{\{x',y'\in\Delta:\, x'\neq y'\}} \frac{|a(x')-a(y')|}{|x'-y'|} =: a_{\mathrm{Lip}}. \tag{6.18}$$

We may now compute the derivatives (almost everywhere in C^+ and V^+, respectively) of the mappings $\mathbf{F}$ and $\mathbf{F}^{-1}$ as follows

$$G_{ij}(y) := \frac{\partial F_i(y)}{\partial y_j} = \begin{cases} \alpha\delta_{ij} & \text{for } i = 1,\dots,d-1 \\ \alpha a_j(\alpha y') & \text{for } i = d \text{ and } j = 1,\dots,d-1 \\ \beta & \text{for } i = j = d \end{cases}$$

$$G^{-1}_{ij}(x) := \frac{\partial F_i^{-1}(x)}{\partial x_j} = \begin{cases} \alpha^{-1}\delta_{ij} & \text{for } i = 1,\dots,d-1 \\ -\beta^{-1}a_j(x') & \text{for } i = d \text{ and } j = 1,\dots,d-1 \\ \beta^{-1} & \text{for } i = j = d. \end{cases}$$

From these formulas, it is not difficult to show that it holds for the Jacobians of the mappings $\mathbf{F}$ and $\mathbf{F}^{-1}$

$$\begin{aligned} J_{\mathbf{F}}(y) &:= \det \mathbb{G}(y) = \alpha^{d-1}\beta, \\ J_{\mathbf{F}^{-1}}(x) &:= \det \mathbb{G}^{-1}(x) = \frac{1}{\alpha^{d-1}\beta}. \end{aligned}$$

Finally, since $\mathbf{F}$ is a bi-Lipschitz mapping and u is measurable, then also U is measurable and using the Theorem on change of variables (see (Lukeš and Malý, 1995, Theorem 34.18)) we get

$$\begin{aligned} \|U\|^p_{L^p(C^+)} &:= \int_{C^+} |U(y)|^p \,\mathrm{d}y = \int_{C^+} |u(\mathbf{F}(y))|^p \,\mathrm{d}y \\ &= \int_{V^+} |u(x)|^p J_{\mathbf{F}^{-1}}(x)\,\mathrm{d}x = \frac{\|u\|^p_{L^p(V^+)}}{\alpha^{d-1}\beta}. \end{aligned} \tag{6.19}$$

Assume now that the weak derivative of U exists and it holds

$$\frac{\partial U(y)}{\partial y_i} = \sum_{j=1}^d \frac{\partial u}{\partial x_j}(\mathbf{F}(y))G_{ji}(y) \qquad \text{almost everywhere in } C^+. \tag{6.20}$$

Note that due to measurability $D^\alpha u$ and properties of $\mathbf{F}$ the expression on the right-hand side is a measurable function. Furthermore, if u and a had continuous derivatives of the first order, then the identity (6.20) would be fulfilled everywhere in C^+ by the chain rule. Repeated application of the Theorem on the change of variables yields

$$\begin{aligned} \left\|\frac{\partial U}{\partial y_i}\right\|^p_{L^p(C^+)} &= \int_{C^+} \left|\frac{\partial U(y)}{\partial y_i}\right|^p \mathrm{d}y = \int_{C^+} \left|\sum_{j=1}^d \frac{\partial u}{\partial x_j}(\mathbf{F}(y))G_{ji}(y)\right|^p \mathrm{d}y \\ &\le d^{p-1}\sum_{j=1}^d \int_{C^+} \left|\frac{\partial u}{\partial x_j}(\mathbf{F}(y))\right|^p |G_{ji}(y)|^p \,\mathrm{d}y \\ &\le d^{p-1}\sum_{j=1}^d \|G_{ij}\|^p_\infty \int_{C^+} \left|\frac{\partial u}{\partial x_j}(\mathbf{F}(y))\right|^p \mathrm{d}y \\ &= d^{p-1}\alpha^{d-1}\beta\sum_{j=1}^d \|G_{ij}\|^p_\infty \int_{V^+} \left|\frac{\partial u(x)}{\partial x_j}\right|^p \mathrm{d}x \\ &\le C^p(\alpha,\beta,a,d)\|u\|^p_{W^{1,p}(V^+)}. \end{aligned}$$

Combining this estimate and (6.19) we get the second inequality from (6.17). Inverting the relation (6.20) (the matrix $\mathbb{G}$ is regular) and repeating the same procedure we also get the first inequality in (6.17).

It remains to verify that (6.20) holds in the weak sense. The idea of the proof is to approximate u by a smooth function for which it holds (6.20) and then pass to the limit. Since V^+ is a domain with Lipschitz boundary (verify carefully!), we may approximate u by means of $\{u^n\}_{n=1}^{\infty} \subset \mathcal{C}^\infty(\overline{V^+})$ so that $u^n \to u$ in $W^{1,p}(V^+)$. We may now introduce the approximation of U as

$$U^n(y) := u^n(\mathbf{F}(y)).$$

Since u^n is a smooth function and $\mathbf{F}$ is Lipschitz continuous (as a is Lipschitz continuous), then also U^n is Lipschitz continuous and by virtue of the Rademacher Theorem A.2.16 has almost everywhere *classical* derivative, i.e.,

$$\frac{\partial U^n(y)}{\partial y_i} = \sum_{j=1}^{d} \frac{\partial u^n}{\partial x_k}(\mathbf{F}(y))G_{ji}(y) \quad \text{almost everywhere in } C^+. \tag{6.21}$$

Furthermore, due to Corollary 6.3.2 we know that $U^n \in W^{1,\infty}(V^+)$ and therefore also $U^n \in W^{1,p}(V^+)$ for any $p \in [1,\infty)$. Finally, as the classical derivative exists almost everywhere and U^n is a Sobolev function, we may apply Lemma 6.3.6 to see that the classical derivative in (6.21) is equal to the weak derivative almost everywhere.

Repeating the procedure above we may deduce

$$\|U^n - U^m\|_{W^{1,p}(C^+)} \le C(a,\alpha,\beta,d)\|u^n - u^m\|_{W^{1,p}(V^+)};$$

as u^n is a Cauchy sequence, then also U^n is a Cauchy sequence. Due to the completeness of the Sobolev spaces (Theorem 6.1.14) there exists $U \in W^{1,p}(C^+)$ such that $U^n \to U$ in $W^{1,p}(C^+)$ and thus also

$$\frac{\partial U^n}{\partial y_i} \to \frac{\partial U}{\partial y_i} \quad \text{almost everywhere in } C^+.$$

To conclude, we pass to the limit in (6.21) (on the right-hand side, we use the strong convergence which implies also the almost everywhere convergence, at least for a subsequence, for ∇u^n) and get (6.20). ■

The following Lemma follows directly from the proof of Lemma 6.4.3 and is therefore left as a useful exercise for the kind reader.

Lemma 6.4.4 Let $\mathbf{F}$ and $\mathbf{F}^{-1}$ be defined by (6.15) and (6.16), a be a Lipschitz continuous function in $\overline{\Delta}$, $p \in [1,\infty)$ and functions $U\colon (-1,1)^d \to \mathbb{R}$ and $u\colon V \to \mathbb{R}$ satisfy $U := u \circ \mathbf{F}$. Then $U \in W^{1,p}(C)$, if and only if $u \in W^{1,p}(V)$. Moreover, there exist positive numbers $C_1 = C_1(a,\alpha,\beta,d)$ and $C_2 = C_2(a,\alpha,\beta,d)$ such that

$$C_1 \|u\|_{W^{1,p}(V)} \le \|U\|_{W^{1,p}(C)} \le C_2 \|u\|_{W^{1,p}(V)}. \tag{6.22}$$

We can now get to the proof of the main theorem from this section.

Proof of Theorem 6.4.1. Let us first assume $p \in [1,\infty)$ only.

Step 1: Smooth function on a cube:
Let $U \in \mathcal{C}^1(\overline{C^+})$ be such that $\operatorname{supp} U \cap \partial C^+ \subset (-1,1)^{d-1} \times \{0\}$, see Figure 6.4.

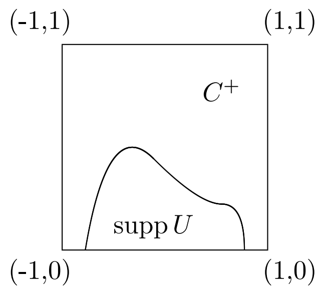

Figure 6.4: Support of the function U

For such a function U we define the extension as

$$EU(x) := \begin{cases} U(x) & x \in \overline{C^+}, \\ -3U(x_1,\dots,x_{d-1},-x_d) + 4U(x_1,\dots,x_{d-1},-\frac{x_d}{2}) & x \in \overline{C^-} \setminus \overline{C^+}, \\ 0 & x \in \mathbb{R}^d \setminus \overline{C}. \end{cases}$$

Clearly $\operatorname{supp} EU \subset C = (-1,1)^d$. We show that $EU \in \mathcal{C}^1(\mathbb{R}^d)$. It is enough to study the behaviour of the function EU at the common boundary of C^+ and C^-. It means to verify that for arbitrary $(x_1,\dots,x_{d-1})$ the limits $EU(x_1,\dots,x_d)$

and of all its partial derivatives are the same if $x_d \to 0_\pm$. We immediately get from the definition that (as U is from $\mathcal{C}^1(\overline{C^+})$)

$$\lim_{x_d\to 0_-} EU(x_1,\dots,x_{d-1},x_d) = (-3+4)U(x_1,\dots,x_{d-1},0) = U(x_1,\dots,x_{d-1},0)$$

and thus also $EU \in \mathcal{C}(\overline{C})$. Similarly we get for all $i = 1,\dots d-1$

$$\begin{aligned}
&\lim_{x_d\to 0_-} \frac{\partial(EU)}{\partial x_i}(x_1,\dots,x_{d-1},x_d)\\
&= -3\lim_{x_d\to 0_-}\frac{\partial U}{\partial x_i}(x_1,\dots,x_{d-1},-x_d) + 4\lim_{x_d\to 0_-}\frac{\partial U}{\partial x_i}\left(x_1,\dots,x_{d-1},-\frac{x_d}{2}\right)\\
&= \frac{\partial U}{\partial x_i}(x_1,\dots,x_{d-1},0).
\end{aligned}$$

Finally, for the derivative with respect to x_d we get directly from the definition of EU that (z_d denotes the last variable of U)

$$\begin{aligned}
&\lim_{x_d\to 0_-} \frac{\partial(EU)}{\partial x_d}(x_1,\dots,x_{d-1},x_d)\\
&= 3\lim_{x_d\to 0_-}\frac{\partial U}{\partial z_d}(x_1,\dots,x_{d-1},-x_d) - 2\lim_{x_d\to 0_-}\frac{\partial U}{\partial z_d}\left(x_1,\dots,x_{d-1},-\frac{x_d}{2}\right)\\
&= \frac{\partial U}{\partial z_d}(x_1,\dots,x_{d-1},0).
\end{aligned}$$

Moreover, it evidently holds

$$\|EU\|_{W^{1,p}(\mathbb{R}^d)} = \|EU\|_{W^{1,p}(C)} \le K\,\|U\|_{W^{1,p}(C^+)}\,, \tag{6.23}$$

where K is a fixed constant, independent of d and p.

Step 2: Functions from a Sobolev space on a cube
Let $U \in W^{1,p}(C^+)$ be such that $\operatorname{supp} U \cap \partial C^+ \subset (-1,1)^{d-1}\times\{0\}$, see Figure 6.4. Then by virtue of the Theorem on global approximation 6.2.15 there exists a sequence of functions $\{U_n\}_{n=1}^\infty \subset \mathcal{C}^\infty(\overline{C^+})$ such that $U_n \to U$ in $W^{1,p}(C^+)$. Moreover, it follows from the proof of Theorem 6.2.15 that $\operatorname{dist}(\operatorname{supp} U_n, \operatorname{supp} U) \to 0$. We then know for sufficiently large n that $\operatorname{supp} U_n \cap \partial C^+ \subset (-1,1)^{d-1}\times\{0\}$ and for functions U_n the previous step can be used. Hence there exist $EU_n \in \mathcal{C}^1(\mathbb{R}^d)$ such that $\operatorname{supp} EU_n \in C$. Due to the linearity of E, we then get from (6.23) that

$$\|EU_n - EU_m\|_{W^{1,p}(\mathbb{R}^d)} = \|EU_n - EU_m\|_{W^{1,p}(C)} \le K\,\|U_n - U_m\|_{W^{1,p}(C^+)}\,.$$

Therefore, if U_n is a Cauchy sequence, then also EU_n is a Cauchy sequence. We thus define as the constructed extension the limit of the sequence EU_n (recall that $W^{1,p}(\mathbb{R}^d)$ is complete, see Theorem 6.1.14). The extension has all required properties and the constant C from Claim 3. of the theorem is independent of d and p.

Step 3: Function from a Sobolev space on a part of the boundary
In what follows we use the notation from Definition 6.2.11. Let $u \in W^{1,p}(V^+)$ be such that $\operatorname{supp} u \cap \partial V^+ \subset \Lambda$. The situation is similar to the situation on the cube, only the boundary is not flat. We use the scheme of the flattening of the boundary and Lemma 6.4.3. We denote $U := u \circ \mathbf{F}$ and we define for the function U its extension EU as in the previous step. The extension of the function u can then be defined as $Eu := EU \circ \mathbf{F}^{-1}$. The extension has all required properties which follows from Lemma 6.4.4, and the constant c form the third claim of the theorem will now depend on a, α, β, d, or in other words, on d a V^+.

Step 4: Functions from a Sobolev space on Ω
Similarly as in the proof of Theorem 6.2.15 we denote for $r = 1,\dots,M$ the sets $O_r := T_r(V_r)$ a $O_r^+ := T_r(V_r^+)$, where T_r are mappings from the local coordinate system (x_r', x_{r_d}) to the fixed coordinate system (x', x_d). Then $\bigcup_{r=1}^M O_r$ is an open covering of a certain neighbourhood of the boundary nad we apply here the Lemma on partition of unity II (Lemma 6.2.14).

Let $\phi_r \in \mathcal{C}_0^\infty(O_r)$ are the functions forming the partition of unity on a certain neighbourhood of $\partial\Omega$. Denote $u_r = (u\phi_r)\circ T_r$. The function $u_r \in W^{1,p}(V^+)$ satisfies the assumptions of the previous step and thus there exists the corresponding extension Eu_r. We define the final extension operator E as

$$Eu := \sum_{r=1}^M (Eu_r)\circ T_r^{-1}.$$

It follows from Steps 1–3 that this extension has all required properties from Theorem 6.4.1, where the constant c from Claim 3. depends only on d and on Ω. The theorem is proved for $p \in [1,\infty)$.

Let now $p = \infty$. Since Ω is bounded, we have $W^{1,\infty}(\Omega) \subset W^{1,p}(\Omega)$ and due to the previous construction, there exists extension Eu of the function u for any $p < \infty$. We have

$$\|Eu\|_{W^{1,p}(\mathbb{R}^d)} \le c\,\|u\|_{W^{1,p}(\Omega)} \le c_1\,\|u\|_{W^{1,\infty}(\Omega)}\,,$$

where the constant c_1 can be chosen independently of p. It follows from Theorem A.3.17 that $Eu \in W^{1,\infty}(\mathbb{R}^d)$ and Eu satisfy the assumptions for o $p = \infty$. The proof is finished. ■

Exercise 6.4.5 (Extension operator $W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^d)$)**.** Show that for $\Omega \in \mathcal{C}^{k-1,1}$ there exists an extension operator E: $W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^d)$. Modify Step 1 of the previous proof so that $EU \in \mathcal{C}^k(\overline{C})$ if $U \in \mathcal{C}^k(\overline{C^+})$. Verify then that the whole proof remains almost unchanged, only in Step 3 we shall need smoother boundary.

Another important observation, based on the previous construction, is presented in the following exercise.

Exercise 6.4.6. Let $\Omega \in \mathcal{C}^1$. Show that the operator E defined in Theorem 6.4.1 is also a continuous linear operator from $\mathcal{C}^1(\overline{\Omega})$ to $\mathcal{C}^1_0(\mathbb{R}^d)$.

The extension presented in the first step of the proof of Theorem 6.4.1 (extension on a cube) was constructed as a $\mathcal{C}^1$ function. Nevertheless, if we are interested only in Sobolev spaces, we may proceed more directly as given in the following exercise.

Exercise 6.4.7. Modify the first step of the proof of Theorem 6.4.1 in the following way.

$$EU(x) := \begin{cases} U(x) & x \in \overline{C^+}, \\ U(x_1, \dots, x_{d-1}, -x_d) & x \in C^-, \\ 0 & x \in \mathbb{R}^d \setminus C. \end{cases}$$

Show that $EU \in W^{1,\infty}(C) \cap W^{1,1}_0(C)$. Check that the rest of the proof of Theorem 6.4.1 remains unchanged.

Remark 6.4.8. There is also another method of extension of Sobolev functions, so called Calderón method (the method from Theorem 6.4.1 is called Nikolskii method) and to construct the extension for any k and $p \in (1, \infty)$ it is enough to have $\Omega \in \mathcal{C}^{0,1}$, while the Nikolskii method requires, due to the flattening of the boundary, at least $\mathcal{C}^{k-1,1}$ boundary. Disadvantage of the Calderón method is that it does not work for $p = 1$ and $p = \infty$.[8]

6.5 Theorems on continuous and compact embeddings

We turn now our attention to different embeddings (continuous or compact) of Sobolev spaces. From the definition of Sobolev spaces it directly follows that $u \in W^{k,p}(\Omega)$ belongs also to $L^p(\Omega)$. The main goal is to show that the function u belongs to a "much better" space, provided Ω has a Lipschitz boundary.

Recall Example 6.1.12. We studied for which $\alpha \in \mathbb{R}$ the function

$$f(x) := |x|^{-\alpha}$$

belongs to $W^{1,p}(B_1(0))$ and to $L^q(B_1(0))$, respectively. We showed that if $\alpha < \frac{d-p}{p}$, then $f \in W^{1,p}(B_1(0))$, and, simultaneously, $f \in L^q(B_1(0))$ pro any $q \in [1, \frac{dp}{d-p}]$. This on one hand indicates that if $u \in W^{1,p}(\Omega)$, for $p < d$, then $u \in L^q(\Omega)$ for $q \in [1, \frac{dp}{d-p}]$, too. On the other hand, this examples shows that the best we may expect for $p < d$ is that $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, where $q \le \frac{dp}{d-p}$, higher exponent would contradict the above mentioned example.

Finally, for $p > d$, the assumption $u \in W^{1,p}(\Omega)$ implies that $-\alpha > 1 - \frac{d}{p} > 0$. Note that for such α we already have $f \in \mathcal{C}^{0,|\alpha|}(\overline{B_1(0)})$.

This simple example leads us to the conjecture that it may hold for "reasonable" domains Ω that

$$W^{1,p}(\Omega) \hookrightarrow \begin{cases} L^{\frac{dp}{d-p}}(\Omega) & \text{for } p \in [1, d), \\ \mathcal{C}^{0,1-\frac{d}{p}}(\overline{\Omega}) & \text{for } p \in (d, \infty). \end{cases}$$

We show in the following text that this conjecture is correct. We moreover show that the above stated embedding can be strengthen to compact embeddings, i.e., we show for sufficiently regular (i.e., Lipschitz) domains

$$W^{1,p}(\Omega) \hookrightarrow\hookrightarrow \begin{cases} L^q(\Omega) & \text{for all } q \in [1, \frac{dp}{d-p}) \text{ if } p < d, \\ \mathcal{C}^{0,\mu}(\overline{\Omega}) & \text{for all } \mu \in [0, 1 - \frac{d}{p}) \text{ if } p \in (d, \infty). \end{cases}$$

Let us now summarize the main results of this section. We have for $\Omega \in \mathcal{C}^{0,1}$ the following results.

1. If $p \in [1, d)$, then $\forall q \in [1, \frac{dp}{d-p}]$ it holds $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$; the embedding is compact for $q \in [1, \frac{dp}{d-p})$.

2. If $p = d$, then $\forall q \in [1, \infty)$ it holds $W^{1,d}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ (but $W^{1,d}(\Omega) \not\hookrightarrow L^\infty(\Omega)$).

3. If $p \in (d, \infty)$, then $\forall \mu \in [0, 1 - \frac{d}{p}]$ it holds $W^{1,p}(\Omega) \hookrightarrow \mathcal{C}^{0,\mu}(\Omega)$; the embedding is compact for $\mu \in [0, 1 - \frac{d}{p})$.

4. If $p = \infty$, then $W^{1,\infty}(\Omega) \hookrightarrow \mathcal{C}^{0,1}(\Omega)$ and thus $W^{1,\infty}(\Omega) \hookrightarrow\hookrightarrow \mathcal{C}^{0,\mu}(\Omega)$ for any $\mu \in [0, 1)$.

[8] The Calderón method is based on integral representation of functions. More details can be found (pro $p = 2$) in Nečas (1967) or Ženíšek (2001), the generalization for $p \neq 2$ can the reader due himself; it is based on the theory of Fourier multipliers. The restriction $p \neq 1$ and $p \neq \infty$ is connected with the fact that $L^p - L^p$ estimates of certain types of singular integral operators generally do not hold true for $p = 1$ and $p = \infty$.

6.5.1 Theorems on continuous embedding

As was indicated in the introduction of this section, theorem on continuous (and compact) embedding will differ depending on whether $p < d$, $p = d$ or $p > d$. To present all results in a compact form, we first formulate the corresponding theorem (it will correspond to the conjecture formulated in the introduction of this section). Let us first consider $p < d$. We define

$$p^\star := \frac{dp}{d-p}. \tag{6.24}$$

Then we have

Theorem 6.5.1 — Embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $p < d$. Let $\Omega \in \mathcal{C}^{0,1}$ and $p \in [1, d)$. Then it holds for any $q \in [1, p^\star]$

$$W^{1,p}(\Omega) \hookrightarrow L^q(\Omega).$$

More precisely, for any $q \in [1, p^\star]$ there exists C which depends only on p, d, q and Ω such that for any $u \in W^{1,p}(\Omega)$ it holds

$$\|u\|_{L^q(\Omega)} \le C \|u\|_{W^{1,p}(\Omega)}. \tag{6.25}$$

The theorem above does not say anything about the case $p = d$, however, as a direct consequence of Theorem 6.5.1 we easily get the following.

Theorem 6.5.2 — Embedding $W^{1,d}(\Omega) \hookrightarrow L^q(\Omega)$. Let $\Omega \in \mathcal{C}^{0,1}$. Then it holds for any $q \in [1, \infty)$

$$W^{1,d}(\Omega) \hookrightarrow L^q(\Omega).$$

Let us now consider the case $p > d$. Define

$$\mu^\star := 1 - \frac{d}{p}, \tag{6.26}$$

where we set for $p = \infty$ the value $\mu^\star := 1$. The main result in this situation is the following.

Theorem 6.5.3 — Embedding $W^{1,p}(\Omega) \hookrightarrow \mathcal{C}^{0,\alpha}(\overline{\Omega})$ for $p > d$. Let $\Omega \in \mathcal{C}^{0,1}$ and $p \in (d, \infty]$. Then it holds for any $\alpha \in [0, \mu^\star]$

$$W^{1,p}(\Omega) \hookrightarrow \mathcal{C}^{0,\alpha}(\overline{\Omega}).$$

More precisely, for any $\alpha \in [0, \mu^\star]$ there exists a constant C which depends only on α, p, d and Ω such that for any $u \in W^{1,p}(\Omega)$ there exists a representative $u^\star \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$, i.e., $u^\star \in [u]$, satisfying

$$\|u^\star\|_{\mathcal{C}^{0,\alpha}(\overline{\Omega})} \le C \|u\|_{W^{1,p}(\Omega)}.$$

It follows from the above stated theorems that the case $p = d$ is particular and the embedding from Theorem 6.5.2 is not "sharp". Nonetheless, as shown in the following exercise, better result on the scale of Lebesgue spaces cannot be expected.[9]

Exercise 6.5.4. Show that the function $f(x) := \ln\left(\ln\left(1 + \frac{1}{|x|}\right)\right)$ belongs for $d \ge 2$ to $W^{1,d}(B_1(0))$, but does not belong to $L^\infty(B_1(0))$. Whence $W^{1,d}(\Omega) \not\hookrightarrow L^\infty(\Omega)$.

[9] We show in Subsection 6.7 that if Ω is a bounded set, then

$$\left(\int_\Omega \left|u - \frac{1}{|\Omega|}\int_\Omega u \, \mathrm{d}y\right|^p \mathrm{d}x\right)^{\frac{1}{p}} \le C \|\nabla u\|_{L^p(\Omega)}.$$

Choosing $\Omega = B_1(0)$ and applying the change of variables $z = x + ry$, $y \in B_1(0)$ we get (perform carefully!)

$$\left(\int_{B_r(x)} \left|u - \frac{1}{|B_r(x)|}\int_{B_r(x)} u \, \mathrm{d}y\right|^p \mathrm{d}z\right)^{\frac{1}{p}} \le Cr \|\nabla u\|_{L^p(B_r(x))}.$$

Particularly for $p = 1$ we have

$$\begin{aligned}\frac{1}{|B_r(x)|}\int_{B_r(x)} \left|u - \frac{1}{|B_r(x)|}\int_{B_r(x)} u \, \mathrm{d}y\right| \mathrm{d}z &\le Cr \frac{1}{|B_r(x)|}\int_{B_r(x)} |\nabla u| \, \mathrm{d}z \\ &\le Cr \frac{1}{|B_r(x)|}\left(\int_{B_r(x)} |\nabla u|^d \, \mathrm{d}z\right)^{\frac{1}{d}} |B_r(x)|^{1-\frac{1}{d}} \le C \|\nabla u\|_{L^d(\mathbb{R}^d)}.\end{aligned}$$

The left-hand side characterizes the space $BMO(\mathbb{R}^d)$ (bounded mean oscillations). This space is a Banach space and

$$[u]_{BMO(\mathbb{R}^d)} = \sup_{B_r(x) \subset \mathbb{R}^d} \frac{1}{|B_r(x)|}\int_{B_r(x)} \left|u - \frac{1}{|B_r(x)|}\int_{B_r(x)} u \, \mathrm{d}y\right| \mathrm{d}z$$

is the seminorm in this space. It plays an important role in the harmonic analysis, where it is often used as a replacement of the space $L^\infty(\Omega)$, see, e.g., Stein (1993).

Proof of the embedding for $p \leq d$

Let us first consider the case $p < d$. Our aim is thus to prove Theorems 6.5.1–6.5.2, where Ω is a domain with Lipschitz boundary (and therefore also bounded). Nonetheless, several results stated below remain true even if Ω is unbounded. Let us first note if $|\Omega| < \infty$, then it follows directly from Hölder's inequality that $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $q \in [1, p]$. The aim is to show that q can be larger than p.

Recall the notation (cf. Remark 6.1.8)

$$\nabla u := \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_d} \right).$$

Then the Euclidean norm of ∇u is

$$|\nabla u| := \sqrt{\sum_{i=1}^{d} \left(\frac{\partial u}{\partial x_i} \right)^2}$$

and furthermore

$$\|\nabla u\|_{L^p(\Omega;\mathbb{R}^d)} := \||\nabla u|\|_{L^p(\Omega)}.$$

Then it is not difficult to see that

$$\|u\|_p + \|\nabla u\|_{L^p(\Omega;\mathbb{R}^d)}$$

is an equivalent norm in $W^{1,p}(\Omega)$. To simplify the notation, if no confusion may arise, we always work with ∇u instead of carefully writing all partial derivatives.

Example 6.5.5. Let us try to find a condition for q to satisfy: let $p \in [1, d)$, then there exists $C > 0$ such that for any $u \in W^{1,p}(\mathbb{R}^d)$ it holds

$$\|u\|_{L^q(\mathbb{R}^d)} \leq C \|\nabla u\|_{L^p(\mathbb{R}^d;\mathbb{R}^d)}. \tag{6.27}$$

We take $u \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ for which we assume (6.27) to hold and define the functions

$$u_\lambda(x) := u(\lambda x), \quad \text{ with arbitrary } \lambda \in \mathbb{R}^+.$$

Inequality (6.27) should hold for all functions from $W^{1,p}(\mathbb{R}^d)$ with the constant C independent of u. In particular, it must hold for all functions u_λ. Using the change of variables, we easily get

$$\|u_\lambda\|_{L^q(\mathbb{R}^d)} = \lambda^{-\frac{d}{q}} \|u\|_{L^q(\mathbb{R}^d)}$$
$$\|\nabla u_\lambda\|_{L^p(\mathbb{R}^d;\mathbb{R}^d)} = \lambda^{1-\frac{d}{p}} \|\nabla u\|_{L^p(\mathbb{R}^d;\mathbb{R}^d)}$$

and from inequality (6.27) applied on u_λ it follows

$$\lambda^{-\frac{d}{q}} \|u\|_{L^q(\mathbb{R}^d)} \leq C\lambda^{1-\frac{d}{p}} \|\nabla u\|_{L^p(\mathbb{R}^d;\mathbb{R}^d)}.$$

Since the parameter λ is arbitrary, for the inequality above to hold it is necessary that

$$1 + \frac{d}{q} - \frac{d}{p} = 0, \text{ or } q = \frac{dp}{d-p}.$$

Indeed, we now prove that inequality (6.27) really holds for $u \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, provided d, p and q satisfy $1 + \frac{d}{q} - \frac{d}{p} = 0$.

Lemma 6.5.6 — Gagliardo. Let $i = 1, \dots, d$ and $\hat{u}_i \in L^\infty(\mathbb{R}^{d-1})$ be arbitrary functions with a compact support. We define for any i the function $u_i : \mathbb{R}^d \to \mathbb{R}$ by means of

$$u_i(x) := \hat{u}_i(\hat{x}_i), \quad \text{ where } \quad \hat{x}_i := (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_d).$$

Then it holds for $d \geq 2$

$$\int_{\mathbb{R}^d} \prod_{i=1}^{d} |u_i(x)| \,\mathrm{d}x \leq \prod_{i=1}^{d} \left(\int_{\mathbb{R}^{d-1}} |\hat{u}_i(\hat{x}_i)|^{d-1} \widehat{\mathrm{d}x_i} \right)^{\frac{1}{d-1}} = \prod_{i=1}^{d} \|\hat{u}_i\|_{L^{d-1}(\mathbb{R}^{d-1})},$$

where

$$\widehat{\mathrm{d}x_i} := \mathrm{d}x_1 \cdots \mathrm{d}x_{i-1} \,\mathrm{d}x_{i+1} \cdots \mathrm{d}x_d.$$

Proof. The proof will be performed by means of the induction on the dimension d.

Step 1: The case $d = 2$

The validity of the lemma for $d = 2$ is a simple consequence of the Fubini Theorem

$$\int_{\mathbb{R}^2} |u_1(x)||u_2(x)| \,\mathrm{d}x = \int_{\mathbb{R}^2} |\hat{u}_1(x_2)||\hat{u}_2(x_1)| \,\mathrm{d}x_1 \,\mathrm{d}x_2$$
$$= \int_{\mathbb{R}} |\hat{u}_1(x_2)| \,\mathrm{d}x_2 \int_{\mathbb{R}} |\hat{u}_2(x_1)| \,\mathrm{d}x_1.$$

Step 2: Induction step on the dimension d:
Assume that the claim holds for $d-1$; let us show that it also holds for d. Due to the fact that u_d is independent of x_d, we get the identity

$$\int_{\mathbb{R}^d} |u_1| \dots |u_d| \, \mathrm{d}x_1 \dots \mathrm{d}x_d = \int_{\mathbb{R}^{d-1}} |\hat{u}_d| \left(\int_{\mathbb{R}} |u_1| \dots |u_{d-1}| \, \mathrm{d}x_d \right) \widehat{\mathrm{d}x_d}.$$

We estimate the interior integral over $\mathbb{R}$ by virtue of the generalized Hölder inequality (see Lemma A.3.13) with all $p_i = d-1$ as follows

$$\int_{\mathbb{R}^{d-1}} |\hat{u}_d| \left(\int_{\mathbb{R}} |u_1| \dots |u_{d-1}| \, \mathrm{d}x_d \right) \widehat{\mathrm{d}x_d} \leq \int_{\mathbb{R}^{d-1}} |\hat{u}_d| \prod_{i=1}^{d-1} \left(\int_{\mathbb{R}} |u_i|^{d-1} \, \mathrm{d}x_d \right)^{\frac{1}{d-1}} \widehat{\mathrm{d}x_d}.$$

We now apply Hölder's inequality on the integral over $\mathbb{R}^{d-1}$ and get ($p = d-1$, $p' = \frac{d-1}{d-2}$)

$$\begin{aligned}
&\int_{\mathbb{R}^{d-1}} |\hat{u}_d| \prod_{i=1}^{d-1} \left(\int_{\mathbb{R}} |u_i|^{d-1} \, \mathrm{d}x_d \right)^{\frac{1}{d-1}} \widehat{\mathrm{d}x_d} \\
&\leq \left(\int_{\mathbb{R}^{d-1}} |\hat{u}_d|^{d-1} \widehat{\mathrm{d}x_d} \right)^{\frac{1}{d-1}} \left(\int_{\mathbb{R}^{d-1}} \prod_{i=1}^{d-1} \left(\int_{\mathbb{R}} |u_i|^{d-1} \, \mathrm{d}x_d \right)^{\frac{1}{d-2}} \widehat{\mathrm{d}x_d} \right)^{\frac{d-2}{d-1}} \\
&= \|\hat{u}_d\|_{L^{d-1}(\mathbb{R}^{d-1})} \left(\int_{\mathbb{R}^{d-1}} \prod_{i=1}^{d-1} g_i(x_1, \dots, x_{d-1}) \, \mathrm{d}x_1 \dots \mathrm{d}x_{d-1} \right)^{\frac{d-2}{d-1}},
\end{aligned}$$

where

$$g_i(x_1, \dots, x_{d-1}) := \left(\int_{\mathbb{R}} |\hat{u}_i(x_1, \dots, x_d)|^{d-1} \, \mathrm{d}x_d \right)^{\frac{1}{d-2}}$$

are functions independent of x_i and the related $\hat{g}_i \in L^\infty(\mathbb{R}^{d-2})$ have compact support. It holds for these g_i, due to the induction assumption

$$\begin{aligned}
\int_{\mathbb{R}^{d-1}} \prod_{i=1}^{d-1} g_i \, \mathrm{d}x_1 \dots \mathrm{d}x_{d-1} &\leq \prod_{i=1}^{d-1} \left(\int_{\mathbb{R}^{d-2}} |\hat{g}_i|^{d-2} \, \widehat{\mathrm{d}x_i \, \mathrm{d}x_d} \right)^{\frac{1}{d-2}} \\
&= \prod_{i=1}^{d-1} \left(\int_{\mathbb{R}^{d-1}} |\hat{u}_i|^{d-1} \widehat{\mathrm{d}x_i} \right)^{\frac{1}{d-2}} = \prod_{i=1}^{d-1} \|\hat{u}_i\|_{L^{d-1}(\mathbb{R}^{d-1})}^{\frac{d-1}{d-2}}.
\end{aligned}$$

Altogether we have

$$\begin{aligned}
\int_{\mathbb{R}^d} |u_1| \dots |u_d| \, \mathrm{d}x_1 \dots \mathrm{d}x_d &\leq \|\hat{u}_d\|_{L^{d-1}(\mathbb{R}^{d-1})} \left(\prod_{i=1}^{d-1} \|\hat{u}_i\|_{L^{d-1}(\mathbb{R}^{d-1})}^{\frac{d-1}{d-2}} \right)^{\frac{d-2}{d-1}} \\
&= \prod_{i=1}^{d} \|\hat{u}_i\|_{L^{d-1}(\mathbb{R}^{d-1})}
\end{aligned}$$

which we wanted to prove. ■

We can now come to the proof of the basic inequality for smooth functions with compact support. Let us recall the notation $p^\star$ from (6.24).

Theorem 6.5.7 — Gagliardo–Nirenberg. Let $p \in [1, d)$. Then it hold for any $u \in \mathcal{C}_0^1(\mathbb{R}^d)$

$$\|u\|_{L^{p^\star}(\mathbb{R}^d)} \leq \frac{p(d-1)}{d-p} \|\nabla u\|_{L^p(\mathbb{R}^d)}. \tag{6.28}$$

Proof. **Step 1:** Case $p = 1$
For $p = 1$, the exponent $p^\star = \frac{d}{d-1}$. Further, it clearly holds for sufficiently smooth functions with compact support

$$u(x) = u(x_1, \dots, x_d) = \int_{-\infty}^{x_i} \frac{\partial u}{\partial x_i} (x_1, \dots, x_{i-1}, s, x_{i+1}, \dots, x_d) \, \mathrm{d}s$$

which after increasing the domain of integration yields for any $x \in \mathbb{R}^d$ the estimate

$$|u(x)| \leq \int_{-\infty}^{+\infty} |\nabla u (x_1, \dots, x_{i-1}, s, x_{i+1}, \dots, x_d)| \, \mathrm{d}s.$$

It is easy to deduce from here

$$|u(x)|^{\frac{d}{d-1}} \le \prod_{i=1}^{d} \left(\int_{\mathbb{R}} |\nabla u\,(x_1, \dots, x_{i-1}, s, x_{i+1}, \dots, x_d)|\ \mathrm{d}s \right)^{\frac{1}{d-1}}.$$

We integrate this inequality over $\mathbb{R}^d$ and get

$$\|u\|^{\frac{d}{d-1}}_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} \le \int_{\mathbb{R}^d} \prod_{i=1}^{d} \left(\int_{\mathbb{R}} |\nabla u\,(x_1, \dots, x_{i-1}, s, x_{i+1}, \dots, x_d)|\ \mathrm{d}s \right)^{\frac{1}{d-1}}\ \mathrm{d}x.$$

We apply on the right-hand side the Gagliardo Lemma 6.5.6, where we choose

$$\hat{u}_i\,(\hat{x}_i) := \left(\int_{\mathbb{R}} |\nabla u\,(x_1, \dots, x_{i-1}, s, x_{i+1}, \dots, x_d)|\ \mathrm{d}s \right)^{\frac{1}{d-1}};$$

as a result we get

$$\|u\|^{\frac{d}{d-1}}_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} \le \prod_{i=1}^{d} \left(\int_{\mathbb{R}^d} |\nabla u|\ \mathrm{d}x \right)^{\frac{1}{d-1}} = \|\nabla u\|^{\frac{d}{d-1}}_{L^1(\mathbb{R}^d;\mathbb{R}^d)}$$

which is the desired inequality for $p = 1$.

Step 2: Proof for arbitrary $p \in (1, d)$
Let $v \in \mathcal{C}^1_0(\mathbb{R}^d)$ and define function $u := |v|^\gamma$. For $\gamma > 1$, the function $u \in \mathcal{C}^1_0(\mathbb{R}^d)$. It follows from Step 1 that

$$\||v|^\gamma\|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} = \|u\|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} \le \|\nabla u\|_{L^1(\mathbb{R}^d;\mathbb{R}^d)} = \|\nabla |v|^\gamma\|_{L^1(\mathbb{R}^d;\mathbb{R}^d)} = \int_{\mathbb{R}^d} |\nabla |v|^\gamma|\ \mathrm{d}x.$$

We compute the gradient on the right-hand side and apply Hölder's inequality (our aim is to have $\|\nabla v\|_{L^p(\mathbb{R}^d;\mathbb{R}^d)}$ on the right-hand side)

$$\int_{\mathbb{R}^d} |\nabla |v|^\gamma|\ \mathrm{d}x = \int_{\mathbb{R}^d} \gamma |v|^{\gamma-1} |\nabla v|\ \mathrm{d}x \le \gamma \left(\int_{\mathbb{R}^d} |v|^{(\gamma-1)\frac{p}{p-1}}\ \mathrm{d}x \right)^{\frac{p-1}{p}} \|\nabla v\|_{L^p(\mathbb{R}^d;\mathbb{R}^d)}.$$

We therefore have

$$\left(\int_{\mathbb{R}^d} |v|^{\gamma \frac{d}{d-1}}\ \mathrm{d}x \right)^{\frac{d-1}{d}} = \||v|^\gamma\|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} \le \gamma \left(\int_{\mathbb{R}^d} |v|^{(\gamma-1)\frac{p}{p-1}}\ \mathrm{d}x \right)^{\frac{p-1}{p}} \|\nabla v\|_{L^p(\mathbb{R}^d;\mathbb{R}^d)}.$$

Let us now choose γ so that the powers at $|v|$ are the same on both sides of the inequality. Hence, we require

$$\gamma \frac{d}{d-1} = (\gamma - 1)\, \frac{p}{p-1}$$

which implies the choice

$$\gamma = \frac{p\,(d-1)}{d-p} > 1;$$

the inequality holds due to the assumption $p \in (1, d)$. The final inequality has the form

$$\left(\int_{\mathbb{R}^d} |v|^{\frac{dp}{d-p}}\ \mathrm{d}x \right)^{\frac{d-1}{d}} \le \frac{p\,(d-1)}{d-p} \left(\int_{\mathbb{R}^d} |v|^{\frac{dp}{d-p}}\ \mathrm{d}x \right)^{\frac{p-1}{p}} \|\nabla v\|_{L^p(\mathbb{R}^d;\mathbb{R}^d)}$$

which implies

$$\|v\|_{L^{\frac{dp}{d-p}}(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} |v|^{\frac{dp}{d-p}} \right)^{\frac{d-p}{pd}} \le \frac{p\,(d-1)}{d-p} \|\nabla v\|_{L^p(\mathbb{R}^d;\mathbb{R}^d)}.$$

The proof is complete. ■

We get, based on the density of smooth compactly supported functions.

Corollary 6.5.8 (Embedding of spaces $W^{1,p}(\mathbb{R}^d)$ and $W^{1,p}_0(\Omega)$ for $p \in [1, d)$). Let $p \in [1, d)$. Then it holds.

1. $W^{1,p}(\mathbb{R}^d) \hookrightarrow L^{p^\star}(\mathbb{R}^d)$ and every $u \in W^{1,p}(\mathbb{R}^d)$ satisfies inequality (6.28).
2. Let $\Omega \subset \mathbb{R}^d$ be open. Then $W^{1,p}_0(\Omega) \hookrightarrow L^{p^\star}(\Omega)$ and every $u \in W^{1,p}_0(\Omega)$ satisfies (after defining to be zero outside of Ω) inequality (6.28).

Proof. In both cases, smooth compactly supported functions are dense in the given spaces (it is directly definition of the space $W_0^{1,p}(\Omega)$ and for a similar claim for $W^{1,p}(\mathbb{R}^d)$ see Lemma 6.2.2). The claim is thus a consequence of the completeness of the Lebesgue spaces and inequality (6.28), the details are left as a useful exercise for the kind reader. ■

We can now start with the proof of Theorems 6.5.1–6.5.2.

Proof of Theorem 6.5.1 and of Theorem 6.5.2. Let us first consider the case $p \in [1, d)$, i.e., we prove Theorem 6.5.1. As $\Omega \in \mathcal{C}^{0,1}$, due to the Extension Theorem 6.4.1 there exists extension operator E: $W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^d)$ such that

$$\|Eu\|_{W^{1,p}(\mathbb{R}^d)} \leq c \|u\|_{W^{1,p}(\Omega)},$$

where the constant c is independent of u. Moreover, the support of the extended function Eu is a compact set in $\mathbb{R}^d$.

Corollary 6.5.8 then implies

$$\begin{aligned}\|u\|_{L^{p^\star}(\Omega)} \leq \|Eu\|_{L^{p^\star}(\mathbb{R}^d)} &\leq \frac{p\,(d-1)}{d-p} \|\nabla (Eu)\|_{L^p(\mathbb{R}^d;\mathbb{R}^d)} \\ &\leq \frac{p\,(d-1)}{d-p} \|Eu\|_{W^{1,p}(\mathbb{R}^d)} \leq c \|u\|_{W^{1,p}(\Omega)}.\end{aligned}$$

The claim of the theorem is then consequence of this inequality, of the trivial embedding $L^{p^\star}(\Omega) \hookrightarrow L^q(\Omega)$ for Ω bounded and $q \in [1, p^\star]$ (see Lemma A.3.14) and the transitivity of the continuous embedding.

Let us now consider the case $p = d$. If $q \in [1, d]$, the proof is evident as Ω is bounded; thus $L^d(\Omega) \hookrightarrow L^q(\Omega)$. Let $q > d$. Then there exists $p_q \in [1, d)$ such that $q = p_q^\star = \frac{dp_q}{d-p_q}$ (i.e., $p_q = \frac{dq}{d+q}$) and due to Theorem 6.5.1 it holds

$$W^{1,p_q}(\Omega) \hookrightarrow L^q(\Omega).$$

Further for Ω bounded and $p_q < d$ it holds $W^{1,d}(\Omega) \hookrightarrow W^{1,p_q}(\Omega)$. Altogether,

$$W^{1,d}(\Omega) \hookrightarrow W^{1,p_q}(\Omega) \hookrightarrow L^q(\Omega)$$

and the proof is complete. ■

Remark 6.5.9. It is important to realize that while for $\mathbb{R}^d$ relation (6.28) holds, for a bounded set Ω with Lipschitz boundary we only have (6.25). It means that on the right-hand side of (6.25) we have the full norm in $W^{1,p}(\Omega)$, not only the norm of the gradient, as it is the case for (6.28). Inequality of the type (6.28) cannot hold generally, as the right-hand side does not change if we add a constant to u. On bounded sets, we may expect estimate (6.28) only in special cases, when adding a constant is exluded, as e.g., in the case of $W_0^{1,p}(\Omega)$ (see Corollary 6.5.8).

On the other hand, in the full space, we only have $W^{1,p}(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$ for $q \in [p, p^\star]$. We namely cannot expect in general that $u \in L^q(\mathbb{R}^d)$ for $q < p$.

A natural question arises, whether the assumption $\Omega \in \mathcal{C}^{0,1}$ is really necessary for the optimal embedding exponent. The following example contains counterexamples, showing the necessity.

Example 6.5.10. We show that for $W^{1,p}(\Omega) \hookrightarrow L^{p^\star}(\Omega)$ to hold in the class of sets $\Omega \in \mathcal{C}^{0,\alpha}$ with $\alpha \in [0, 1]$, the necessary condition is $\Omega \in \mathcal{C}^{0,1}$. Assume that $\Omega \subset \mathbb{R}^2$ has a part of the boundary in the neighbourhood of the point $(0, 0)$ described by $|y| = x^\mu$, where $\mu > 1$ and $x \in (0, 1)$, and the rest of the boundary is smooth, cf. Figure 6.5. Check carefully that then $\Omega \in \mathcal{C}^{0,\frac{1}{\mu}}$ (the description of the boundary $x = |y|^{\frac{1}{\mu}}$, $y \in [-1, 1]$, must be considered).

Now, similarly as in Example 6.1.12, we consider functions of the type $u(x, y) := x^{-a}$, where we only deal with the part of the domain close to the origin, i.e. with the set $\Omega := \{(x, y) \,|\, x \in (0, 1),\ y \in (-x^\mu, x^\mu)\}$, since outside the origin, the function is smooth. Then

$$\begin{aligned}\|u\|^q_{L^q(\Omega)} &= \int_0^1 \left(\int_{-x^\mu}^{x^\mu} x^{-aq}\,\mathrm{d}y \right) \mathrm{d}x = 2\int_0^1 x^{\mu-aq}\,\mathrm{d}x \\ \|\nabla u\|^p_{L^p(\Omega)} &= |a|^p \int_0^1 \left(\int_{-x^\mu}^{x^\mu} x^{-(a+1)p}\,\mathrm{d}y \right) \mathrm{d}x = 2|a|^p \int_0^1 x^{\mu-(a+1)p}\,\mathrm{d}x.\end{aligned}$$

From here we immediately get that

$$\begin{aligned}u \in L^q(\Omega) &\iff q < \frac{1+\mu}{a} \\ u \in W^{1,p}(\Omega) &\iff a < \frac{1+\mu-p}{p}.\end{aligned}$$

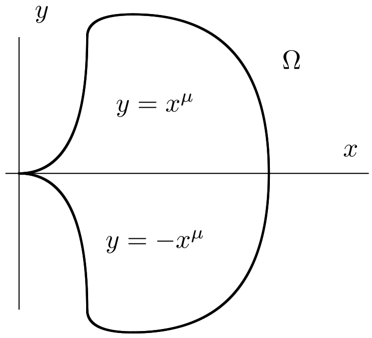

Figure 6.5: Domain Ω from Example 6.5.10.

This example confirms that we have for a two-dimensional domain $\Omega \in \mathcal{C}^{0,\frac{1}{\mu}}$ at most $W^{1,p}(\Omega) \hookrightarrow L^{q_\mu}(\Omega)$, where $q_\mu = \frac{(1+\mu)p}{1+\mu-p}$. This example can be easily transformed to d dimensions, where the correct choice is $q_\mu = \frac{(d-1+\mu)p}{d-1+\mu-p}$. For more details see (Adams, 1975, Theorem 5.35). It is possible to see that q_μ is a decreasing function of μ, having for $\mu = 1$, i.e., for a Lipschitz domain, optimal value $q_1 = p^\star$, and at infinity (domain with continuous boundary) only the value $q_\infty = p$.

We show at the end of this subsection that inequality (6.28) allows to prove interpolation inequalities of the type

$$\|u\|_{L^r(\mathbb{R}^d)} \leq C \|\nabla u\|^{\alpha}_{L^p(\mathbb{R}^d;\mathbb{R}^d)} \|u\|^{1-\alpha}_{L^q(\mathbb{R}^d)},$$

for suitably chosen q, r, p and α.

Example 6.5.11. The following inequalities hold[10].

1. We have for $d = 2$ $\|v\|_{L^4(\mathbb{R}^2)} \leq 2^{\frac{1}{2}} \|\nabla v\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2;\mathbb{R}^2)} \|v\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}$.

2. We have for $d = 3$ $\|v\|_{L^4(\mathbb{R}^3)} \leq \left(\frac{8}{3}\right)^{\frac{3}{4}} \|\nabla v\|^{\frac{3}{4}}_{L^2(\mathbb{R}^3;\mathbb{R}^3)} \|v\|^{\frac{1}{4}}_{L^2(\mathbb{R}^3)}$.

We start with inequality (6.28) for $p = 1$. We take for $d = 2$ the function $u = |v|^2$ and by virtue of Hölder's inequality we compute

$$\int_{\mathbb{R}^2} |v|^4 \,\mathrm{d}x \leq \left(\int_{\mathbb{R}^2} |\nabla |v|^2| \,\mathrm{d}x\right)^2 \leq 4 \left(\int_{\mathbb{R}^2} |\nabla v||v| \,\mathrm{d}x\right)^2 \leq 4 \|\nabla v\|^2_{L^2(\mathbb{R}^2;\mathbb{R}^2)} \|v\|^2_{L^2(\mathbb{R}^2)}.$$

Taking the fourth root we obtain the result.

For $d = 3$ we choose $u = |v|^{\frac{8}{3}}$, then

$$\begin{aligned}\int_{\mathbb{R}^3} |v|^4 \,\mathrm{d}x &\leq \left(\int_{\mathbb{R}^3} |\nabla |v|^{\frac{8}{3}}| \,\mathrm{d}x\right)^{\frac{3}{2}} \leq \left(\frac{8}{3}\right)^{\frac{3}{2}} \left(\int_{\mathbb{R}^3} |\nabla v||v|^{\frac{5}{3}} \,\mathrm{d}x\right)^{\frac{3}{2}} \\ &\leq \left(\frac{8}{3}\right)^{\frac{3}{2}} \left(\int_{\mathbb{R}^3} |\nabla v||v|^{\frac{1}{3}}|v|^{\frac{4}{3}} \,\mathrm{d}x\right)^{\frac{3}{2}} \\ &\leq \left(\frac{8}{3}\right)^{\frac{3}{2}} \|\nabla v\|^{\frac{3}{2}}_{L^2(\mathbb{R}^3;\mathbb{R}^3)} \|v\|^{\frac{1}{2}}_{L^2(\mathbb{R}^3)} \|v\|^2_{L^4(\mathbb{R}^3)}.\end{aligned}$$

We now divide by $\|v\|^2_{L^4(\mathbb{R}^3)}$ and we conclude by taking the square root.

Based on similar considerations, it is possible to show a general claim which is left as a useful exercise for the kind reader.

Exercise 6.5.12 (General interpolation inequality)**.** Show that for any $\alpha \in [0, 1)$ and any $r, p, q \in [1, \infty]$ satisfying

$$\frac{1}{r} = \alpha \left(\frac{1}{p} - \frac{1}{d}\right) + (1 - \alpha) \frac{1}{q}$$

there exists a constant $C = C(p, q, r, d)$ such that it holds for any $u \in \mathcal{C}^\infty_0(\mathbb{R}^d)$

$$\|u\|_{L^r(\mathbb{R}^d)} \leq C \|\nabla u\|^{\alpha}_{L^p(\mathbb{R}^d;\mathbb{R}^d)} \|u\|^{1-\alpha}_{L^q(\mathbb{R}^d)}. \tag{6.29}$$

Moreover, if $p < d$, it is also possible to take $\alpha = 1$. Due to the density argument, these inequalities can be extended for functions $u \in L^r(\mathbb{R}^d)$ for which $\nabla u \in L^p(\mathbb{R}^d; \mathbb{R}^d)$.

[10]The constants obtained above are not optimal and can be improved, see e.g., Temam (2001). These inequalities play an important role in the proof of existence, uniqueness and regularity of weak solutions to the incompressible Navier–Stokes equations and they are "responsible" for a significant difference between the results for two and three space dimensions.

Proof of the embedding theorem for $p > d$

We now deal with the proof of Theorem 6.5.3. We start with the key claim from which we deduce several corollaries.

Lemma 6.5.13 — Morrey I. Let $u \in \mathcal{C}_0^1(\mathbb{R}^d)$. We denote for arbitrary $\mu \in (0,1]$

$$[\nabla u]_{L^{1,\mu}(\mathbb{R}^d)} := \sup_{x\in\mathbb{R}^d} \sup_{\rho>0} \int_{[0,\rho]^d} \frac{|\nabla u(x+y)|}{\rho^{d-1+\mu}} \, \mathrm{d}y. \tag{6.30}$$

Then it holds for any $x_1, x_2 \in \mathbb{R}^d$

$$|u(x_1) - u(x_2)| \leq \frac{2\sqrt{d}}{\mu} |x_1 - x_2|^\mu [\nabla u]_{L^{1,\mu}(\mathbb{R}^d)}. \tag{6.31}$$

Proof. We choose $x_1, x_2 \in \mathbb{R}^d$ arbitrarily, but fixed and denote C_ρ the closed cube with the length of the edge ρ such that x_1 and x_2 lie on the opposite walls of this cube. Then it holds

$$\rho \leq |x_1 - x_2| \leq \sqrt{d}\rho. \tag{6.32}$$

Due to the smoothness of u we further get for all $x \in C_\rho$ and $i = 1, 2$ the estimate

$$\begin{aligned} |u(x) - u(x_i)| &= \left| \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} u(x_i + s(x - x_i)) \, \mathrm{d}s \right| \\ &= \left| \int_0^1 \nabla u(x_i + s(x - x_i)) \cdot (x - x_i) \, \mathrm{d}s \right| \\ &\leq \int_0^1 |\nabla u(x_i + s(x - x_i))| \, |x - x_i| \, \mathrm{d}s \\ &\leq \sqrt{d}\rho \int_0^1 |\nabla u(x_i + s(x - x_i))| \, \mathrm{d}s, \end{aligned} \tag{6.33}$$

where we used in the last inequality estimate (6.32).

We shall now look at the "distance" of the mean value u on the cube C_ρ from the values at the points x_i. We get, using the properties of the Lebesgue integral

$$\left| \int_{C_\rho} \frac{u(x)}{\rho^d} \, \mathrm{d}x - u(x_i) \right| = \left| \int_{C_\rho} \frac{u(x) - u(x_i)}{\rho^d} \, \mathrm{d}x \right| \leq \int_{C_\rho} \frac{|u(x) - u(x_i)|}{\rho^d} \, \mathrm{d}x.$$

Due to estimate (6.33) and Fubini's Theorem we can transform this inequality into the form

$$\left| \int_{C_\rho} \frac{u(x)}{\rho^d} \, \mathrm{d}x - u(x_i) \right| \leq \sqrt{d} \int_0^1 \int_{C_\rho} \frac{|\nabla u(x_i + s(x - x_i))|}{\rho^{d-1}} \, \mathrm{d}x \, \mathrm{d}s. \tag{6.34}$$

We now perform the change of variables

$$z := x_i + s(x - x_i) \qquad C^i_{\rho s} := \left\{ z \in \mathbb{R}^d \,|\, z = x_i + s(x - x_i) \text{ for some } x \in C_\rho \right\},$$

where $C^i_{\rho s}$ is now a cube with the length of the edge ρs. The resulting integral has the form

$$\begin{aligned} \int_0^1 \int_{C_\rho} \frac{|\nabla u(x_i + s(x - x_i))|}{\rho^{d-1}} \, \mathrm{d}x \, \mathrm{d}s &= \int_0^1 \int_{C^i_{\rho s}} \frac{|\nabla u(z)|}{s^d \rho^{d-1}} \, \mathrm{d}z \, \mathrm{d}s \\ &= \rho^\mu \int_0^1 s^{\mu-1} \left(\int_{C^i_{\rho s}} \frac{|\nabla u(z)|}{(\rho s)^{d-1+\mu}} \, \mathrm{d}z \right) \mathrm{d}s. \end{aligned}$$

We plug in this inequality into (6.34) and get, using the assumptions on u

$$\begin{aligned} \left| \int_{C_\rho} \frac{u(x)}{\rho^d} \, \mathrm{d}x - u(x_i) \right| &\leq \sqrt{d}\rho^\mu \int_0^1 s^{\mu-1} \left(\int_{C^i_{\rho s}} \frac{|\nabla u(z)|}{(\rho s)^{d-1+\mu}} \, \mathrm{d}z \right) \mathrm{d}s \\ &\leq \sqrt{d} [\nabla u]_{L^{1,\mu}(\mathbb{R}^d)} \rho^\mu \int_0^1 s^{\mu-1} \, \mathrm{d}s \\ &= \frac{\sqrt{d}}{\mu} [\nabla u]_{L^{1,\mu}(\mathbb{R}^d)} \rho^\mu. \end{aligned} \tag{6.35}$$

The triangle inequality then finishes the whole proof. Indeed,

$$\begin{aligned}|u(x_1) - u(x_2)| &= \left| \int_{C_\rho} \frac{u(x)}{\rho^d} \,\mathrm{d}x - u(x_2) - \int_{C_\rho} \frac{u(x)}{\rho^d} \,\mathrm{d}x + u(x_1) \right| \\ &\leq \left| \int_{C_\rho} \frac{u(x)}{\rho^d} \,\mathrm{d}x - u(x_2) \right| + \left| \int_{C_\rho} \frac{u(x)}{\rho^d} \,\mathrm{d}x - u(x_1) \right| \\ &\leq \frac{2\sqrt{d}}{\mu} [\nabla u]_{L^{1,\mu}(\mathbb{R}^d)} \rho^\mu\end{aligned}$$

and thus (6.31) holds true. ■

It is important to realize that for estimate (6.31) it is not necessary to control the seminorm on the right-hand side on the whole $\mathbb{R}^d$, but it is enough to consider a qualitatively same term taking into account only a certain neighbourhood of x_1 and x_2. We show now this modification.

Lemma 6.5.14 — Morrey II. Let $u \in \mathcal{C}^1((-2R, 2R)^d)$. Then it holds for any $x_1, x_2 \in (-R, R)^d$

$$|u(x_1) - u(x_2)| \leq \frac{2\sqrt{d}}{\mu} |x_1 - x_2|^\mu \sup_{x \in (-R,R)^d} \sup_{\rho \in (0,R)} \int_{(-\rho,\rho)^d} \frac{|\nabla u(x+y)|}{\rho^{d-1+\mu}} \,\mathrm{d}y. \tag{6.36}$$

Proof. The proof follows easily for the previous proof by a slight modification of estimate (6.34) and is left for a kind reader as an exercise. ■

With the help of the Morrey Lemma I 6.5.13 we can now relatively easily prove the following estimate which shows relation between the spaces $W^{1,p}(\mathbb{R}^d)$ and $\mathcal{C}_0^{0,\mu}(\mathbb{R}^d)$.

Theorem 6.5.15 — Morrey I. Let $p \in (d, \infty)$ and $\mu := 1 - \frac{d}{p}$. Then it holds for any $u \in \mathcal{C}_0^1(\mathbb{R}^d)$

$$\|u\|_{\mathcal{C}_0^{0,\mu}(\mathbb{R}^d)} \leq \frac{4\sqrt{d}}{\mu} \|u\|_{W^{1,p}(\mathbb{R}^d)}. \tag{6.37}$$

Proof. Let us recall the definition of the norm in $\mathcal{C}_0^{0,\mu}(\mathbb{R}^d)$ (see (A.4))

$$\|u\|_{\mathcal{C}_0^{0,\mu}(\mathbb{R}^d)} := \|u\|_{L^\infty(\mathbb{R}^d)} + \sup_{\{x,y \in \mathbb{R}^d \,|\, x \neq y\}} \frac{|u(x) - u(y)|}{|x-y|^\mu}.$$

Step 1: Estimate of differences
In this step, we apply the Morrey Lemma I 6.5.13. By virtue of Hölder's inequality and due to our choice of μ we get the estimate

$$\int_{(0,\rho)^d} \frac{|\nabla u(x+y)|}{\rho^{d-1+\mu}} \,\mathrm{d}y \leq \left(\int_{(0,\rho)^d} |\nabla u(x+y)|^p \,\mathrm{d}y \right)^{\frac{1}{p}} \frac{\rho^{\frac{d}{p'}}}{\rho^{d-1+\mu}} \leq \|\nabla u\|_{L^p(\mathbb{R}^d;\mathbb{R}^d)}$$

which immediately leads to the inequality (recall (6.30))

$$[\nabla u]_{L^{1,\mu}(\mathbb{R}^d)} \leq \|\nabla u\|_{L^p(\mathbb{R}^d;\mathbb{R}^d)}.$$

Using (6.31) we therefore have

$$|u(x_1) - u(x_2)| \leq \frac{2\sqrt{d}}{\mu} |x_1 - x_2|^\mu \|\nabla u\|_{L^p(\mathbb{R}^d;\mathbb{R}^d)}$$

or

$$\sup_{\{x_1,x_2 \in \mathbb{R}^d \,|\, x_1 \neq x_2\}} \frac{|u(x_1) - u(x_2)|}{|x_1 - x_2|^\mu} \leq \frac{2\sqrt{d}}{\mu} \|\nabla u\|_{L^p(\mathbb{R}^d;\mathbb{R}^d)}. \tag{6.38}$$

Step 2: Estimate for $\|u\|_\infty$
Let us choose arbitrary $x, y \in \mathbb{R}^d$. Then due to (6.38)

$$|u(x) - u(y)| \leq \frac{2\sqrt{d}}{\mu} \|\nabla u\|_{L^p(\mathbb{R}^d;\mathbb{R}^d)} |x-y|^\mu$$

which after applying the triangle inequality yields

$$|u(x)| \leq |u(y)| + \frac{2\sqrt{d}}{\mu} \|\nabla u\|_{L^p(\mathbb{R}^d;\mathbb{R}^d)} |x-y|^\mu.$$

We integrate this inequality with respect to y over the set $C_\rho := \{y \in \mathbb{R}^d \mid 2|y_i - x_i| \le \rho \,\forall i = 1, 2, \dots, d\}$; it results to

$$|u(x)||C_\rho| \le \int_{C_\rho} |u(y)| \,\mathrm{d}y + |C_\rho| \frac{2\rho^\mu \sqrt{d}\, \|\nabla u\|_{L^p(\mathbb{R}^d;\mathbb{R}^d)}}{\mu}.$$

We estimate the integral on the right-hand side to get the norm of u in $L^p(\mathbb{R}^d)$

$$|u(x)||C_\rho| \le |C_\rho|^{\frac{p-1}{p}} \|u\|_{L^p(C_\rho)} + |C_\rho| \frac{2\rho^\mu \sqrt{d}\, \|\nabla u\|_{L^p(\mathbb{R}^d;\mathbb{R}^d)}}{\mu}, \tag{6.39}$$

from where (we choose for example $\rho = 1$)

$$\sup_{x\in\mathbb{R}^d} |u(x)| \le \|u\|_{L^p(\mathbb{R}^d)} + \frac{2\sqrt{d}}{\mu} \|\nabla u\|_{L^p(\mathbb{R}^d;\mathbb{R}^d)}. \tag{6.40}$$

It follows from inequalities (6.38) and (6.40) for $u \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ that

$$\|u\|_{\mathcal{C}^{0,\mu}(\mathbb{R}^d)} \le \frac{4\sqrt{d}}{\mu} \|u\|_{W^{1,p}(\mathbb{R}^d)} \tag{6.41}$$

and the proof is complete. ∎

Similarly as in the case of Morrey Lemma, the estimate can be localized.

Theorem 6.5.16 — Morrey II. Let $p \in (d, \infty)$ and choose $\mu := 1 - \frac{d}{p}$. Then it holds for any $u \in \mathcal{C}^1((-2R, 2R)^d)$

$$\|u\|_{\mathcal{C}^{0,\mu}([-R,R]^d)} \le \frac{\|u\|_{L^p((-2R,2R)^d)}}{R^{1-\mu}} + (1 + R^\mu)\frac{2\sqrt{d}}{\mu} \|u\|_{W^{1,p}((-2R,2R)^d)}. \tag{6.42}$$

Proof. The proof is a simple modification of the previous proof, in particular of the choice of ρ in (6.39), and is left for the kind reader as an exercise. ∎

Due to the density of $\mathcal{C}_0^\infty(\mathbb{R}^d)$ in $W^{1,p}(\mathbb{R}^d)$ inequality (6.37) also holds for $u \in W^{1,p}(\mathbb{R}^d)$. However, we must be slightly careful here, as $u \in W^{1,p}(\mathbb{R}^d)$ means in fact that a class of equivalent functions belongs to $W^{1,p}(\mathbb{R}^d)$. We therefore must work with a suitable representative of this class..

Corollary 6.5.17. Let $p \in (d, \infty)$. Then it holds for $\mu = 1 - \frac{d}{p}$

$$W^{1,p}(\mathbb{R}^d) \hookrightarrow \mathcal{C}^{0,\mu}(\mathbb{R}^d).$$

More precisely, for any $u \in W^{1,p}(\mathbb{R}^d)$ there exists a representative $u^\star \in \mathcal{C}^{0,\mu}(\mathbb{R}^d)$, $u^\star \in [u]$, such that

$$\|u^\star\|_{\mathcal{C}^{0,\mu}(\mathbb{R}^d)} \le \frac{4\sqrt{d}}{\mu} \|u\|_{W^{1,p}(\mathbb{R}^d)}.$$

Proof. We leave the proof as an exercise for a reader. ∎

The corollary implies that for $u \in W^{1,p}(\mathbb{R}^d)$, $p > d$, all points in $\mathbb{R}^d$ are Lebesgue points (recall Definition A.3.19) of the function $u^\star$ and thus

$$u^\star(x) = \lim_{r\to 0_+} \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y) \,\mathrm{d}y$$

for any $x \in \mathbb{R}^d$.

We can now come to the proof of the main theorem of this part.

Proof of Theorem 6.5.3. Since $\Omega \in \mathcal{C}^{0,1}$, due to Theorem 6.4.1 there exists a continuous linear extension operator

$$E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^d)$$

and there exist $C = C(d, \Omega)$ such that for any $p \in [1, \infty]$ it holds

$$Eu = u \quad \text{almost everywhere in } \Omega, \qquad \|Eu\|_{W^{1,p}(\mathbb{R}^d)} \le C \|u\|_{W^{1,p}(\Omega)}$$

and the support of Eu is a compact set in $\mathbb{R}^d$.

Let us first consider $p \in (d, \infty)$. It follows from Corollary 6.5.17 that ($\mu^\star := 1 - \frac{d}{p}$)

$$\|u^\star\|_{\mathcal{C}^{0,\mu^\star}(\overline{\Omega})} = \|(Eu)^\star\|_{\mathcal{C}^{0,\mu^\star}(\overline{\Omega})} \leq \frac{4\sqrt{d}}{\mu^\star} \|Eu\|_{W^{1,p}(\mathbb{R}^d)} \leq \frac{C(d,\Omega)}{\mu^\star} \|u\|_{W^{1,p}(\Omega)} .$$

The claim of the theorem for $p \in (d, \infty)$ is then a consequence of this inequality, of the embedding $\mathcal{C}^{0,\mu^\star}(\overline{\Omega}) \hookrightarrow \mathcal{C}^{0,\alpha}(\overline{\Omega})$ for Ω bounded and $0 \leq \alpha \leq \mu^\star$ (recall Exercise A.2.15) and of the transitivity of continuous embedding.

We now finish the proof for $p = \infty$. Since Ω is a bounded domain, evidently $W^{1,\infty}(\Omega) \hookrightarrow W^{1,p}(\Omega)$ for arbitrary $p < \infty$. It follows from above that

$$\|u^\star\|_{\mathcal{C}^{0,1-\frac{d}{p}}(\overline{\Omega})} \leq \frac{C(d,\Omega)}{1 - \frac{d}{p}} \|u\|_{W^{1,p}(\Omega)} .$$

Passing to the limit $p \to \infty$ we then easily obtain the claim of the theorem. ■

Corollary 6.5.18 (If $\Omega \in \mathcal{C}^{0,1}$, then $\mathcal{C}^{0,1}(\overline{\Omega}) = W^{1,\infty}(\Omega)$). Let $\Omega \in C^{0,1}$. Then $\mathcal{C}^{0,1}(\overline{\Omega}) = W^{1,\infty}(\Omega)$.

Proof. We already know (cf. Corollary 6.3.2), that for any open set $\mathcal{C}^{0,1}(\overline{\Omega}) \hookrightarrow W^{1,\infty}(\Omega)$. On the other hand, we know for Lipschitz domains that $W^{1,\infty}(\Omega) \hookrightarrow \mathcal{C}^{0,1}(\overline{\Omega})$ and thus also $\mathcal{C}^{0,1}(\overline{\Omega}) = W^{1,\infty}(\Omega)$. ■

The only "weakness" of this approach is the use of the Rademacher Theorem A.2.16 in the proof of the extension theorem. However, this can be removed, since it is possible to show a more general result which includes the Rademacher Theorem as a special case.

As we have already seen in Lemma 6.1.4, the weak derivative of a Sobolev function agrees with the classical one at almost every points, where the classical derivative exists. Moreover, (see Remark 6.3.5) we know that the weak derivative is really a good approximation of the classical derivative. We now generalize these two claims and show existence of the total differential almost everywhere for functions from $W^{1,p}(\Omega)$ with $p > d$. Then we also show the Rademacher Theorem A.2.16 as a particular case. Recall that a function u has at the point x total differential if there exists $a \in \mathbb{R}^d$ such that

$$\lim_{y \to x} \frac{|u(y) - u(x) - a \cdot (y - x)|}{|x - y|} = 0.$$

Recall also that a is equal to the gradient of u at the point x in the classical sense. In the theorem below we deal with the continuous representative $[u]$, since due to Theorem 6.5.3 it holds $W^{1,p}(\Omega) \hookrightarrow \mathcal{C}^0(\overline{\Omega})$.

Theorem 6.5.19 Let $u \in W^{1,p}(\Omega)$ and $p \in (d, \infty]$. Then the function u (more precisely, its continuous representative) has total differential almost everywhere in Ω and the classical derivative is at almost all points, where the total differential exists, equal to the weak derivative.

Proof. **Step 1:** $p \in (d, \infty)$

First, let $p \in (d, \infty)$ and $x \in \Omega$ be arbitrary. Assume further that $r > 0$ is so small that $B_{4r}(x) \subset \Omega$. We know from the density of smooth functions that we may approximate u arbitrarily precisely by smooth functions in $B_{2r}(x)$. We may therefore apply Morrey Lemma II (Lemma 6.5.14) and we have for arbitrary $y \in B_r(x)$ and $v \in W^{1,p}(B_{2r}(x))$ the estimate[11]

$$|v(x) - v(y)| \leq C(d,p)|x - y|^\mu \sup_{z \in B_r(x)} \sup_{\rho \in (0,r)} \int_{B_\rho(z)} \frac{|\nabla v(z')|}{\rho^{d-1+\mu}} \, \mathrm{d}z'.$$

We may further estimate the expression on the right-hand side by means of Hölder's inequality

$$\int_{B_\rho(z)} \frac{|\nabla v(z')|}{\rho^{d-1+\mu}} \, \mathrm{d}z' \leq C(d)\rho^{1-\mu-\frac{d}{p}} \left(\int_{B_\rho(z)} |\nabla v(z')|^p \, \mathrm{d}z' \right)^{\frac{1}{p}} ;$$

hence

$$|v(x) - v(y)| \leq C(d,p) r^{1-\frac{d}{p}} \left(\int_{B_{2r}(x)} |\nabla v(z)|^p \, \mathrm{d}z \right)^{\frac{1}{p}} . \tag{6.43}$$

In what follows, we reduce ourselves on the set of Lebesgue points $\nabla u(x)$ and $|\nabla u(x)|^p$, i.e., on the set of $x \in \Omega$ for which it holds

$$\lim_{r \to 0_+} \frac{1}{|B_r(x)|} \int_{B_r(x)} (|\nabla u(x) - \nabla u(z)| + |\nabla u(x) - \nabla u(z)|^p) \, dz = 0.$$

The Theorem on Lebesgue points A.3.20 and Exercise A.3.24 imply that the above stated relation holds for almost every $x \in \Omega$. For arbitrary $x \in \Omega$ for which the above stated equality holds we set

$$v(y) := u(y) - u(x) - \nabla u(x) \cdot (y - x).$$

[11] Even though the Morrey Lemma is formulated for cubes, we can easily rewrite it by rescaling for balls.

Then clearly $\nabla v(y) = \nabla u(y) - \nabla u(x)$ and $v \in W^{1,p}(B_{4r}(x))$ for sufficiently small $r > 0$. Moreover, we know that v is continuous and $v(x) = 0$. We thus have for each $y \in B_r(x)$

$$|u(y) - u(x) - \nabla u(x) \cdot (y - x)| = |v(y)| = |v(y) - v(x)|.$$

Let us choose now arbitrary $y \in \partial B_r(x)$ (and thus $r = |x - y|$). We get from inequality (6.43) that

$$\begin{aligned} \frac{|u(y) - u(x) - \nabla u(x) \cdot (y - x)|}{|x - y|} &= \frac{|v(x) - v(y)|}{r} \\ &\le C(p,d) \left(\int_{B_{2r}(x)} \frac{|\nabla v(z)|^p}{r^d} \, \mathrm{d}z \right)^{\frac{1}{p}} \\ &= C(p,d) \left(\int_{B_{2r}(x)} \frac{|\nabla u(x) - \nabla u(z)|^p}{r^d} \, \mathrm{d}z \right)^{\frac{1}{p}} \\ &\le C(p,d) \left(\frac{1}{|B_{2r}(x)|} \int_{B_{2r}(x)} |\nabla u(x) - \nabla u(z)|^p \, \mathrm{d}z \right)^{\frac{1}{p}}. \end{aligned}$$

We consider now $y \to x$, thus $r \to 0_+$ in the above stated inequality. Since we consider only such x which are Lebesgue points of ∇u, we see that

$$\begin{aligned} &\lim_{y \to x} \frac{|u(y) - u(x) - \nabla u(x) \cdot (y - x)|}{|x - y|} \\ &\le C(p,d) \lim_{r \to 0_+} \left(\frac{1}{|B_{2r}(x)|} \int_{B_{2r}(x)} |\nabla u(x) - \nabla u(z)|^p \, \mathrm{d}z \right)^{\frac{1}{p}} = 0. \end{aligned}$$

The function u thus has classical total differential at the point x. Moreover, the *classical* gradient is equal to the *weak* gradient $\nabla u(x)$. The remaining claims for $p \in (d, \infty)$ are evident.

Step 2: $p = \infty$

If $u \in W^{1,\infty}(\Omega)$, then also $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ for all $p < \infty$ and we may thus use the first part of the proof. ■

6.5.2 Theorems on compact embedding

We now strengthen the results of the previous subsection and state that the corresponding compact embeddings for Sobolev spaces defined on Lipschitz domains hold true. Furthermore, we also claim that some results remain true even for less smooth domains, in particular for only domains with continuous boundary. We again consider separately the cases $p < d$, $p = d$ and $p > d$. Recall also the notation $p^\star = \frac{dp}{d-p}$ and $\mu^\star = 1 - \frac{d}{p}$. The main result for $p < d$ is the following.

Theorem 6.5.20 — On compact embedding of $W^{1,p}(\Omega)$ for $p < d$. Let $\Omega \in \mathcal{C}^{0,1}$ and $p \in [1, d)$. Then it holds for any $q \in [1, p^\star)$

$$W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega).$$

For $p \ge d$ we have the following result.

Theorem 6.5.21 — On compact embedding of $W^{1,p}(\Omega)$ for $p \ge d$. Let $\Omega \in \mathcal{C}^{0,1}$ and $p \in [d, \infty]$. Then it holds for any $q \in [1, \infty)$

$$W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega).$$

Moreover, if $p > d$, then it holds for any $\alpha \in [0, \mu^\star)$

$$W^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathcal{C}^{0,\alpha}(\overline{\Omega}),$$

and thus also $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^\infty(\Omega)$.

As indicated in Example 6.5.10, the assumption on the Lipschitz continuity of the boundary is a key one, otherwise the embedding (continuous) up to the limit spaces cannot be true. On the other hand, the same example indicates that for domains with Hölder continuous boundary it is possible to expect some improvement in the integrability and we may hope for some compact embedding. The following theorem gives the claim which will be also fundamental for the proof of Theorem 6.5.20.

Theorem 6.5.22 — On compact embedding for domains with $\mathcal{C}^0$-boundary. Let $\Omega \in \mathcal{C}^0$ and $p \in [1, \infty)$.

Then it holds
$$W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega).$$

As a trivial corollary we also have.

Corollary 6.5.23. Let $\Omega \in \mathcal{C}^0$ and $p \in [1, \infty)$. Then it holds for any $q \in [1, p]$
$$W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega).$$

We shall now deal with the proofs of the above stated theorems.

Proof of Theorem 6.5.22. It follows from the definition that $W^{1,p}(\Omega)$ is continuously embedded to $L^p(\Omega)$. To prove the compact embedding, it remains to show that bounded sets in $W^{1,p}(\Omega)$ are totally bounded in $L^p(\Omega)$. Let $C^* > 0$ and $A \subset W^{1,p}(\Omega)$ be such that for any $u \in A$ it holds $\|u\|_{1,p} \le C^*$. Our goal is to show that this set is totally bounded in $L^p(\Omega)$. To this aim we apply the equivalent characterization by the Kolmogorov Theorem (Theorem A.3.40). We define $u \in A$ to be zero outside of Ω and verify the assumptions of the Kolmogorov Theorem. Assumption 1., i.e., the uniform boundedness, is trivially satisfied, as $\|u\|_p \le \|u\|_{1,p}$. Assumption 3., i.e., the uniform equal decay at infinity, is again satisfied trivially, as Ω is a bounded domain. Let us deal with Assumption 2., i.e., with the uniform equal p-mean continuity.

Let $\varepsilon > 0$ be arbitrary, but fixed. We define $\delta_0 := \frac{\varepsilon}{2C^*}$ and without loss of generality we assume that $\delta_0 < 1$. We may find for arbitrary $\mathbf{h} \in \mathbb{R}^d$ a unit vector $\mathbf{e} \in \mathbb{R}^d$ such that $\mathbf{h} = |\mathbf{h}|\mathbf{e}$. In what follows we always consider such $\mathbf{h}$. We now apply the Lemma on the connection between the difference quotient and weak derivative III 6.3.7 and if $|\mathbf{h}| \le \delta_0$, we get (recall that $|\mathbf{e}| = 1$ and $\|u\|_{1,p} \le C^*$ as well as recall that $\Omega_a = \{x \in \Omega \,|\, \mathrm{dist}\,(x, \partial\Omega) > a\}$)
$$\|r_{\mathbf{h}}u - u\|_{L^p(\Omega_{2|\mathbf{h}|})} = |\mathbf{h}|\|\Delta_{\mathbf{e}}^{|\mathbf{h}|}u\|_{L^p(\Omega_{2|\mathbf{h}|})} \le |\mathbf{h}|\,\|\nabla u \cdot \mathbf{e}\|_{L^p(\Omega)} \le \delta_0\,\|\nabla u\|_{L^p(\Omega)} \le \frac{\varepsilon}{2}. \tag{6.44}$$

Therefore we have, due to the triangle inequality and properties of the shift (recall that u is equal to zero outside of Ω)
$$\|r_{\mathbf{h}}u - u\|_{L^p(\mathbb{R}^d)} \le \|r_{\mathbf{h}}u - u\|_{L^p(\Omega_{2|\mathbf{h}|})} + \|r_{\mathbf{h}}u - u\|_{L^p(\mathbb{R}^d\setminus\Omega_{2|\mathbf{h}|})} \le \frac{\varepsilon}{2} + 2\|u\|_{L^p(\Omega\setminus\Omega_{4|\mathbf{h}|})}. \tag{6.45}$$

We now concentrate on the estimate of the second term which deals with the behaviour near the boundary. We keep the notation from Definition 6.2.11. Due to the continuity of the boundary Ω we know that there exist M cartesian coordinate systems and mappings T_r from the local coordinate system (x'_r, x_{r_d}) to (x', x_d) and sets
$$V_r^+ = \left\{(x'_r, x_{r_d}) \in \mathbb{R}^d |\ x'_r \in \Delta_r,\ a_r(x'_r) < x_{rd} < a_r(x'_r) + \beta\right\},$$
where a_r are continuous functions in Δ_r. Due to the continuity of a_r there exists $\delta_1 > 0$ such that whenever $|\mathbf{h}| \le \delta_1$, then $\Omega \setminus \Omega_{4|\mathbf{h}|} \subset \bigcup_{r=1}^M T_r(V_r^+)$. Further, let $\{\phi_r\}_{r=1}^{M+1}$ be the partition of unity from Theorem 6.2.15, where the functions $\phi_r \in \mathcal{C}_0^\infty(T_r(V_r))$ for all $r = 1, \dots, M$ and $\phi_{M+1} \in \mathcal{C}_0^\infty(\Omega)$. We set $u_r := u\phi_r$ (and thus $u = \sum_{r=1}^{M+1} u_r$). Then surely there exists a number $\delta_2 > 0$ such that it holds for all $|\mathbf{h}| \le \delta_2$ that $\operatorname{supp} u_{M+1} \cap (\Omega \setminus \Omega_{4|\mathbf{h}|}) = \emptyset$. Moreover, there exists a constant $C(\Omega)$ such that for all $r = 1, \dots, M$
$$\|u_r\|_{W^{1,p}(T_r(V_r^+))} \le C(\Omega)\|u\|_{1,p} \le C(\Omega)C^*. \tag{6.46}$$

It follows from the above stated that the last term on the right-hand side of (6.45) can be estimated by
$$\|u\|_{L^p(\Omega\setminus\Omega_{4|\mathbf{h}|})} \le \sum_{r=1}^M \|u_r\|_{L^p(T_r(V_r^+)\setminus\Omega_{4|\mathbf{h}|})}.$$

Finally, let us denote for arbitrary $\gamma \le \beta$ the following sets V_r^+
$$V_{r,\gamma}^+ := \left\{(x'_r, x_{r_d}) \in \mathbb{R}^d |\ x'_r \in \Delta_r,\ a_r(x'_r) < x_{r_d} < a_r(x'_r) + \gamma\right\}.$$

We use again the continuity of a_r. It is not difficult to show that there exists a continuous increasing function γ such that $\gamma(0) = 0$ and
$$\Omega \setminus \Omega_{4|\mathbf{h}|} \subset \bigcup_{r=1}^M T_r(V_{r,\gamma(4|\mathbf{h}|)}^+)$$
for each $|\mathbf{h}| < \min(\delta_1, \delta_2)$. From here and from the Theorem on the change of variables we get
$$\|u\|_{L^p(\Omega\setminus\Omega_{4|\mathbf{h}|})} \le \sum_{r=1}^M \|u_r\|_{L^p(T_r(V_{r,\gamma(4|\mathbf{h}|)}^+))} = \sum_{r=1}^M \|\tilde{u}_r\|_{L^p(V_{r,\gamma(4|\mathbf{h}|)}^+)}, \tag{6.47}$$
where $\tilde{u}_r(x'_r, x_{r_d}) := u_r(T_r(x'_r, x_{r_d}))$.

It is therefore enough to estimate the norms on the right-hand side of the above stated inequality. As Ω has continuous boundary, the smooth functions are dense in the given Sobolev space $W^{1,p}(\Omega)$ and we may formally proceed[12]. For arbitrary, but fixed $r \in \{1,\dots,M\}$, and arbitrary $x_r \in V_r^+$ we have (recall that $\tilde{u}_r(x_r', a_r(x_r')+\beta) = 0$) due to Hölder's inequality

$$|\tilde{u}_r(x_r', x_{r_d})|^p = \left| \int_{x_{r_d}}^{a_r(x_r')+\beta} \frac{\mathrm{d}}{\mathrm{d}s} \tilde{u}_r(x_r', s)\,\mathrm{d}s \right|^p \le \beta^{p-1} \int_{a_r(x_r')}^{a_r(x_r')+\beta} |\nabla \tilde{u}_r(x_r', s)|^p\,\mathrm{d}s.$$

We now integrate this inequality over $V^+_{r,\gamma(4|\mathbf{h}|)}$ and get

$$\begin{aligned}
\|\tilde{u}_r\|^p_{L^p(V^+_{r,\gamma(4|\mathbf{h}|)})} &= \int_{V^+_{r,\gamma(4|\mathbf{h}|)}} |\tilde{u}_r(x_r', x_{r_d})|^p\,\mathrm{d}x_r'\,\mathrm{d}x_{r_d} \\
&\le \beta^{p-1} \int_{\Delta_r} \int_{a(x_r')}^{a(x_r')+\gamma(4|\mathbf{h}|)} \int_{a_r(x_r')}^{a_r(x_r')+\beta} |\nabla \tilde{u}_r(x_r', s)|^p\,\mathrm{d}s\,\mathrm{d}x_{r_d}\,\mathrm{d}x_r' \\
&= \beta^{p-1}\gamma(4|\mathbf{h}|) \int_{V_r^+} |\nabla \tilde{u}_r(x_r)|^p\,\mathrm{d}x_r \\
&= \beta^{p-1}\gamma(4|\mathbf{h}|) \int_{T_r(V_r^+)} |\nabla u(x)|^p\,\mathrm{d}x \le \beta^{p-1}\gamma(4|\mathbf{h}|)(C^*)^p,
\end{aligned}$$

where we used in the last inequality (6.46). Plugging into (6.45) and using (6.47) we thus get (recall that u is defined by zero outside of Ω)

$$\|r_{\mathbf{h}}u - u\|_{L^p(\mathbb{R}^d)} \le \frac{\varepsilon}{2} + 2\beta^{\frac{1}{p'}} C^* (\gamma(4|\mathbf{h}|))^{\frac{1}{p}}.$$

We finally define

$$\delta_3 := \frac{1}{4}\gamma^{-1}\left(\frac{\varepsilon^p}{4^p M^p \beta^{p-1} (C^*)^p} \right) > 0.$$

Then it holds for any $|\mathbf{h}| < \min(\delta_0, \delta_1, \delta_2, \delta_3)$

$$\|r_{\mathbf{h}}u - u\|_{L^p(\mathbb{R}^d)} < \varepsilon$$

and the proof is complete. ■

Remark 6.5.24. The most difficult part of the proof was the estimate of the term near the boundary. If we strengthen the assumptions or weaken the claim, the proof would become simpler. If we assumed that Ω has Lipschitz boundary, then we knew from the theorem on continuous embedding that $\|u\|_q \le C$ for some $q > p$, thus from Hölder's inequality

$$\|u\|_{L^p(\Omega\setminus\Omega_{4|\mathbf{h}|})} \le \|u\|_q |\Omega \setminus \Omega_{4|\mathbf{h}|}|^{\frac{q-p}{qp}}.$$

We see that the term on the right-hand side can be arbitrarily small in dependence on $|\mathbf{h}|$. In a similar way we could prove "easier" the compact embedding $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ for $\Omega \in C^0$, whenever $1 \le q < p$.

We can finally perform the proofs of the main theorems of this section.

Proof of Theorem 6.5.20. Let $p < d$. If moreover $q < p$, the proof is simple, since from Theorem 6.5.22 we know that

$$W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega) \hookrightarrow L^q(\Omega) \implies W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega).$$

Let us consider the case $q \in (p, p^\star)$. Since $\Omega \in \mathcal{C}^{0,1}$, from the Theorem on continuous embedding for $p < d$ 6.5.1 we known that $W^{1,p}(\Omega) \hookrightarrow L^{p^\star}(\Omega)$. Using the interpolation Hölder inequality (Lemma A.3.15) we get for any $u \in W^{1,p}(\Omega)$

$$\|u\|_{L^q(\Omega)} \le \|u\|^\alpha_{L^p(\Omega)} \|u\|^{1-\alpha}_{L^{p^\star}(\Omega)} \le C(p,\Omega) \|u\|^\alpha_{L^p(\Omega)} \|u\|^{1-\alpha}_{W^{1,p}(\Omega)},$$

where $\alpha \in (0,1)$ is taken so that $\frac{1}{q} = \frac{\alpha}{p} + \frac{1-\alpha}{p^\star}$. Let now $A \subset W^{1,p}(\Omega)$ be an arbitrary bounded closed set. Due to Theorem 6.5.22 we know that this set is compact in $L^p(\Omega)$ and thus for any $\varepsilon > 0$ there exists an ε-net $\{u_i\}_{i=1}^k \subset A$; it means that for any $u \in A$ it holds

$$\min_i \|u - u_i\|_p \le \varepsilon.$$

Due to the above stated interpolation we get that

$$\min_i \|u - u_i\|_q \le C(p,\Omega) \min_i \|u - u_i\|_p^\alpha \|u - u_i\|_{1,p}^{1-\alpha} \le C(p,q,\Omega,A)\varepsilon^\alpha.$$

Thus we see that for given A and $q < p^\star$ we can construct a finite covering by arbitrarily small balls and the proof is complete. ■

Proof of Theorem 6.5.21. The first claim is a direct consequence of Theorems 6.5.2 and 6.5.20. The second claims then follows from Theorem 6.5.3 and from the compact embedding among the Hölder continuous functions (see Theorem A.2.14). ■

[12] More precisely, we may approximate u_r by a sequence of smooth functions for which we show rigorously the required estimates; they remain true after the limit passage to the functions u_r, too.

6.5.3 General Sobolev embeddings

The previous subsections dealt with $W^{1,p}(\Omega)$. By induction, the results above can be generalized.

Theorem 6.5.25 — General Sobolev embeddings. Let $\Omega \in \mathcal{C}^{0,1}$, $k \in \mathbb{N}$ and $p \in [1,\infty]$. Let further $j \in \{0,1,\dots,k-1\}$ be arbitrary. Denote

$$m_0 := \frac{1}{p} - \frac{k-j}{d} \qquad \text{and if } m_0 \neq 0, \qquad m := \frac{1}{m_0}.$$

Then we have the following.

1. If $m_0 > 0$, then
   (a) $W^{k,p}(\Omega) \hookrightarrow W^{j,m}(\Omega)$
   (b) for any $m_1 \in [1,m)$ it holds $W^{k,p}(\Omega) \hookrightarrow\hookrightarrow W^{j,m_1}(\Omega)$.
2. If $m_0 = 0$, then it holds for any $q \in [1,\infty)$ that $W^{k,p}(\Omega) \hookrightarrow\hookrightarrow W^{j,q}(\Omega)$.
3. If $m_0 < 0$, we set $\mu := -dm_0$ and it holds
   (a) if $\mu \in (0,1)$, then $W^{k,p}(\Omega) \hookrightarrow \mathcal{C}^{j,\mu}(\overline{\Omega})$ and for any $\alpha \in [0,\mu)$ we have $W^{k,p}(\Omega) \hookrightarrow\hookrightarrow \mathcal{C}^{j,\alpha}(\overline{\Omega})$
   (b) if $\mu = 1$, then
   $$\begin{cases} p \neq \infty : \ \forall \alpha \in [0,1) : \ W^{k,p}(\Omega) \hookrightarrow\hookrightarrow \mathcal{C}^{j,\alpha}(\overline{\Omega}) \\ p = \infty : \ W^{k,\infty}(\Omega) \hookrightarrow \mathcal{C}^{k-1,1}(\overline{\Omega}) \end{cases}$$
   (c) if $\mu > 1$, then it holds for any $\alpha \in [0,1]$ that $W^{k,p}(\Omega) \hookrightarrow\hookrightarrow \mathcal{C}^{j,\alpha}(\overline{\Omega})$.

Proof. The proof is based on mathematical induction and embedding theorems for the spaces $W^{1,p}(\Omega)$. It is therefore left as an easy exercise for the kind reader. ∎

As already said, the above mentioned theorem is based on mathematical induction and results on embeddings for $W^{1,p}(\Omega)$. As we showed, these results are in fact optimal and cannot be improved[13]. It may therefore seem that results of Theorem 6.5.25 are optimal. It is generally true, except one case of $W^{d,1}(\Omega)$. By induction we can show that $W^{d,1}(\Omega) \hookrightarrow W^{1,d}(\Omega)$. Note that this embedding is indeed optimal and cannot be improved on the level of Sobolev spaces. We know that the space $W^{1,d}(\Omega)$ is not embedded into continuous functions; even not to bounded ones. On the other hand, at each induction step we did an "insignificant" error, "not seen" for Sobolev spaces. We clarify everything by the last theorem on Sobolev embeddings.

Theorem 6.5.26 — On embedding of $W^{d,1}(\Omega)$. Let $\Omega \in \mathcal{C}^{0,1}$. Then $W^{d,1}(\Omega) \hookrightarrow \mathcal{C}^0_B(\Omega)$, where

$$\mathcal{C}^0_B(\Omega) = \{u \in \mathcal{C}(\Omega) \,|\, \sup_{x\in\Omega} |u(x)| =: \|u\|_{\mathcal{C}^0_B(\Omega)} < \infty\}.$$

Proof. The proof is based on the following lemma, where the embedding is shown for the cubes. Using the fact that a domain with Lipschitz boundary can be written as an at most countable union of open cubes (up to a set of measure zero), whose edges have positive length bounded away from zero, the claim follows for Lipschitz domains. ∎

It remains to prove the above mentioned lemma. Denote the d-dimensional cube as R, i.e.,

$$R = \{x \in \mathbb{R}^d \,|\, a_i < x_i < b_i, i = 1,2,\dots,d\}$$

and a $d-1$-dimensional cube as R', i.e.,

$$R' = \{x \in \mathbb{R}^{d-1} \,|\, a_i < x_i < b_i, i = 1,2,\dots,d-1\}.$$

We can then show the following.

Lemma 6.5.27 It holds
$$W^{d,1}(R) \hookrightarrow \mathcal{C}(\overline{R}).$$

Proof. Since the smooth functions up to the boundary are dense in $W^{d,1}(R)$, it remains to show this embedding for these functions. By virtue of the mean value theorem we have

$$\|u\|_{L^1(R)} = \int_{a_d}^{b_d} \Big(\int_{R'} |u(x',x_d)| \,\mathrm{d}x' \Big) \,\mathrm{d}x_d = (b_d - a_d) \int_{R'} |u(x',\sigma)| \,\mathrm{d}x'$$

[13] On the scale of of Lebesgue spaces; for finer scales like Besov spaces is the situation different, see, e.g., Bahouri et al. (2011).

for some $\sigma \in (a_d, b_d)$. Then for arbitrary $x_d \in (a_d, b_d)$

$$|u(x', x_d)| \le |u(x', \sigma)| + \int_\sigma^{x_d} |D^{(0,\dots,0,1)} u(x', t)| \, \mathrm{d}t.$$

Integrating over R' and then over (a_d, b_d) we thus have

$$\begin{aligned} \|u(\cdot, x_d)\|_{L^1(R')} \le & \|u(\cdot, \sigma)\|_{L^1(R')} + \|D^{(0,\dots,0,1)} u\|_{L^1(R)} \\ \le & (b_d - a_d)^{-1} \|u\|_{L^1(R)} + \|D^{(0,\dots,0,1)} u\|_{L^1(R)}. \end{aligned}$$

Therefore we have for any x_d as above

$$\|u(\cdot, x_d)\|_{W^{d-1,1}(R')} \le C \|u\|_{W^{d,1}(R)},$$

where C depends only on $(b_d - a_d)$. By induction we thus get

$$\|u(\cdot, x_2, x_3, \dots, x_d)\|_{W^{1,1}((a_1,b_1))} \le C \|u\|_{W^{d,1}(R)}.$$

The mean value theorem gives existence of $\sigma \in (a_1, b_1)$ such that

$$\|u(\cdot, x_2, x_3, \dots, x_d)\|_{L^1((a_1,b_1))} = (b_1 - a_1) |u(\sigma, x_2, \dots, x_d)|.$$

Therefore

$$\begin{aligned} |u(x)| &\le |u(\sigma, x_2, \dots, x_d)| + \int_\sigma^{x_1} |D^{(1,0\dots,0)} u(t, x_2, \dots, x_d)| \, \mathrm{d}t \\ &\le \frac{1}{(b_1 - a_1)} \|u(\cdot, x_2, x_3, \dots, x_d)\|_{L^1((a_1,b_1))} + \|D^{(1,0\dots,0)} u(\cdot, x_2, \dots, x_d)\|_{L^1((a_1,b_1))} \\ &\le C \|u\|_{W^{d,1}(R)}. \end{aligned}$$

The continuity of u follows from the fact that by similar argument as above we may get

$$|u(x) - u(y)| \le \|u\|_{W^{d,1}(C_{x,y})},$$

where $C_{x,y}$ is arbitrary cube which contains the points x and y (thus, e.g., the smallest one). The proof is complete. ■

6.6 Traces of Sobolev functions

The Sobolev functions are used to formulate the boundary value problems for partial differential equations. It is therefore necessary to have a possibility to speak about values of Sobolev functions on the boundary of the domain Ω. There are two approaches how to solve this problem. One is based on the notion of "capacity" of a set and can be found, e.g., in Mazja (1985). The other approach which is more commonly used is based on extension of a certain linear operator and will be presented here.

If $u \in W^{1,p}(\Omega)$, then for $p > d$ there exists a continuous representative (up to the boundary of Ω, provided the domain is sufficiently regular) which equals to the given function u almost everywhere in Ω. Whence in this case the boundary value is well defined (and is even Hölder continuous). Thus, below we discuss only the case $p \le d$.

In this situation it is not any more clear what does the boundary value mean, since the d-dimensional measure of the boundary is zero. On the other hand, under suitable assumption on the regularity of the boundary of Ω (at least for $\Omega \in \mathcal{C}^0$) we know that smooth functions up to the boundary are dense in $W^{1,p}(\Omega)$, the restriction of functions $u \in W^{1,p}(\Omega)$ on $\partial\Omega$ is well defined for a dense subset of $W^{1,p}(\Omega)$. It therefore suffices to study if the restriction operator for functions $u \in \mathcal{C}^\infty(\overline{\Omega})$ on $\partial\Omega$ is well defined as an operator from $W^{1,p}(\Omega)$ into a certain function space defined on the boundary of Ω. Below we explain that it is possible. First, however, we need to define correctly the corresponding spaces.

6.6.1 Surface integral and spaces $L^p(\partial\Omega)$

In the whole subsection we assume that the boundary is of the class $\mathcal{C}^{0,1}$ and we use the notation from Definition 6.2.11. Further, let $\{\phi_r\}_{r=1}^M$ be the partition of unity on a certain small neighbourhood of $\partial\Omega$ covered by the local description of the boundary. We first define sets of zero measure.

Definition 6.6.1 — Set of zero measure on $\partial\Omega$. Let $A \subset \partial\Omega$. We say that A is a set of zero measure, if it holds for any $r \in \{1, \dots, M\}$

$$|\{x'_r \in \Delta_r \,|\, T_r(x'_r, a_r(x'_r)) \in A\}|_{d-1} = 0.$$

Similarly as in the case of the standard Lebesgue measure, in what follows, we use the terminology *almost everywhere in* $\partial\Omega$, if the claim holds for all $x \in \partial\Omega \setminus A$, where A is a set of zero measure.

We continue with the definition of measurable functions.

Definition 6.6.2 — Measurable functions on $\partial\Omega$. Let $f\colon \partial\Omega \to \mathbb{R}$. We say that f is measurable, if for any $r \in \{1, \dots, M\}$, $f \circ T_r$ is measurable on Δ_r with respect to the $d-1$-dimensional measure.

We are now ready to define the surface integral.

Definition 6.6.3 — Integral $\int_{\partial\Omega} \cdot \,\mathrm{d}S$. Let $u\colon \partial\Omega \to \mathbb{R}$ be measurable. Denote $u_r(x) := u(x)\phi_r(x)$, where $\{\phi_r\}_{r=1}^M$ is the partition of unity corresponding to the open covering of a certain neighbourhood of the boundary. Then we define the integral of a function u over $\partial\Omega$ as

$$\int_{\partial\Omega} u \,\mathrm{d}S := \sum_{r=1}^{M} \int_{\Delta_r} u_r(T_r(x_r', a_r(x_r')))\sqrt{1 + |\nabla a_r(x_r')|^2}\,\mathrm{d}x_r', \tag{6.48}$$

provided all the integrals on the right-hand side exist and are finite.

Remark 6.6.4. Note that in the definition above, each integrand on the right-hand side of (6.48) is a measurable function. It follows from the fact that u is measurable and thus, due to the smoothness of ϕ_r, also the function u_r is measurable. Furthermore, since functions a_r are Lipschitz, then the Rademacher Theorem A.2.16 ensures existence of ∇a_r almost everywhere in Δ_r and the gradient is also measurable.

The above defined surface integral seems to depend on the choice of parametrization (description) of the boundary $\partial\Omega$. The following key theorem justifies that, indeed, it is not the case.

Theorem 6.6.5 — Independence of surface integral of parametrization. Let $\Omega \in \mathcal{C}^{0,1}$ be arbitrary. Let $\{a_{r1}, \Delta_{r1}, T_{r1}\}_{r1=1}^{M_1}$ and $\{a_{r2}, \Delta_{r2}, T_{r2}\}_{r2=1}^{M_2}$ be two arbitrary descriptions of the boundary which are in agreement with Definition 6.2.11. Let further $\{\phi_{r1}\}_{r1=1}^{M_1}$ and $\{\phi_{r2}\}_{r2=1}^{M_2}$ be two partitions of unity corresponding to them. Then a function u is measurable with respect to the first parametrization, if and only if it is measurable with respect to the second parametrization. Furthermore, it holds for any measurable u

$$\begin{aligned}
&\sum_{r1=1}^{M_1} \int_{\Delta_{r1}} u_{r1}(T_{r1}(x_{r1}', a_{r1}(x_{r1}')))\sqrt{1 + |\nabla a_{r1}(x_{r1}')|^2}\,\mathrm{d}x_{r1}' \\
&= \sum_{r2=1}^{M_2} \int_{\Delta_{r2}} u_{r2}(T_{r2}(x_{r2}', a_{r2}(x_{r2}')))\sqrt{1 + |\nabla a_{r2}(x_{r2}')|^2}\,\mathrm{d}x_{r2}';
\end{aligned} \tag{6.49}$$

whence the surface integral of u does not depend on the choice of parametrization of $\partial\Omega$.

Proof. The proof of the first part of the theorem about the measurability is left for the kind reader as an exercise. In what follows, we concentrate ourselves on the proof of equality (6.49). Since $\{\phi_{r1}\}$ and $\{\phi_{r2}\}$ are partition of unity in the vicinity of the boundary $\partial\Omega$, we get the identity

$$\begin{aligned}
&\sum_{r2=1}^{M_2} \int_{\Delta_{r2}} u_{r2}(T_{r2}(x_{r2}', a_{r2}(x_{r2}')))\sqrt{1 + |\nabla a_{r2}(x_{r2}')|^2}\,\mathrm{d}x_{r2}' \\
&= \sum_{r2=1}^{M_2} \int_{\Delta_{r2}} u\phi_{r2}(T_{r2}(x_{r2}', a_{r2}(x_{r2}')))\sqrt{1 + |\nabla a_{r2}(x_{r2}')|^2}\,\mathrm{d}x_{r2}' \\
&= \sum_{r2=1}^{M_2} \sum_{r1=1}^{M_1} \int_{\Delta_{r2}} u\phi_{r1}\phi_{r2}(T_{r2}(x_{r2}', a_{r2}(x_{r2}')))\sqrt{1 + |\nabla a_{r2}(x_{r2}')|^2}\,\mathrm{d}x_{r2}' \\
&= \sum_{r2=1}^{M_2} \sum_{r1=1}^{M_1} \int_{\Delta_{r2}} u_{r1}\phi_{r2}(T_{r2}(x_{r2}', a_{r2}(x_{r2}')))\sqrt{1 + |\nabla a_{r2}(x_{r2}')|^2}\,\mathrm{d}x_{r2}'.
\end{aligned} \tag{6.50}$$

Let us now concentrate on the last integral. We denote

$$(\partial\Omega)^{r1,r2} := \partial\Omega \cap \operatorname{supp}\phi_{r1} \cap \operatorname{supp}\phi_{r2}.$$

In what follows we consider only cases, when $(\partial\Omega)^{r1,r2}$ is a set of non-zero measure (otherwise, the corresponding integral in (6.50) is identically zero). We then define for these sets the corresponding local description of the boundary

$$\begin{aligned}
\Lambda_{r1}^{r2} &:= (T_{r1})^{-1}(\partial\Omega)^{r1,r2}, \quad \Delta_{r1}^{r2} := \{x_{r1}' \in \Delta_{r1} : x_{r1}' = a_{r1}(x_{r1}')\}, \\
\Lambda_{r2}^{r1} &:= (T_{r2})^{-1}(\partial\Omega)^{r1,r2}, \quad \Delta_{r2}^{r1} := \{x_{r2}' \in \Delta_{r2} : x_{r2}' = a_{r2}(x_{r2}')\}.
\end{aligned}$$

We now define the mapping $\vec{\varphi}\colon \Delta_{r2}^{r1} \to \Delta_{r1}^{r2}$ by the formula

$$\vec{\varphi}(x_{r2}') := \left((T_{r1})^{-1}(T_{r2}(x_{r2}', a_{r2}(x_{r2}')))\right)'.$$

It is easy to verify that this mapping is bijective (the proof of this claim is left to the kind reader). Using this mapping we may rewrite the last integral in (6.50) as

$$\begin{aligned}&\int_{\Delta_{r2}} u_{r1}\phi_{r2}(T_{r2}(x'_{r2}, a_{r2}(x'_{r2})))\sqrt{1+|\nabla a_{r2}(x'_{r2})|^2}\,\mathrm{d}x'_{r2}\\ &= \int_{\Delta_{r2}} u_{r1}\phi_{r2}(T_{r1}(\vec{\varphi}(x'_{r2}), a_{r1}(\vec{\varphi}(x'_{r2}))))\sqrt{1+|\nabla a_{r2}(x'_{r2})|^2}\,\mathrm{d}x'_{r2}.\end{aligned} \tag{6.51}$$

We apply for this integral the change of variables. It remains to express the jacobian of $\vec{\varphi}$ as well as ∇a_{r_2}. For notational simplicity we now define two auxiliary mappings

$$\mathbf{\Sigma}_{r1}(x'_{r1}) := T_{r1}((x'_{r1}, a_{r1}(x'_{r1}))), \qquad \mathbf{\Sigma}_{r2}(x'_{r2}) := T_{r2}((x'_{r2}, a_{r2}(x'_{r2})))$$

and two matrices of derivatives

$$\mathbb{A}_{r1}(x'_{r1}) := \frac{\partial(x'_{r1}, a_{r1}(x'_{r1}))}{\partial x'_{r1}}, \qquad \mathbb{A}_{r2}(x'_{r2}) := \frac{\partial(x'_{r2}, a_{r2}(x'_{r2}))}{\partial x'_{r2}},$$

i.e.,

$$A^{ij}_{r1}(x'_{r1}) := \begin{cases} \delta_{ij} & i<d,\\ \dfrac{\partial a_{r1}(x'_{r1})}{\partial(x'_{r1})_j} & i=d, \end{cases} \qquad A^{ij}_{r2}(x'_{r2}) := \begin{cases} \delta_{ij} & i<d,\\ \dfrac{\partial a_{r2}(x'_{r2})}{\partial(x'_{r2})_j} & i=d. \end{cases}$$

Moreover, since T_{r1} and T_{r2} are only rotations and shifts, there exists matrices $\mathbb{Q}_{r1}$ and $\mathbb{Q}_{r2}$ such that

$$\begin{aligned}\nabla_{r1}\mathbf{\Sigma}_{r1}(x'_{r1}) &:= \frac{\partial \mathbf{\Sigma}_{r1}(x'_{r1})}{\partial x'_{r1}} = \mathbb{Q}_{r1}\mathbb{A}_{r1}(x'_{r1}),\\ \nabla_{r2}\mathbf{\Sigma}_{r2}(x'_{r2}) &:= \frac{\partial \mathbf{\Sigma}_{r2}(x'_{r2})}{\partial x'_{r2}} = \mathbb{Q}_{r2}\mathbb{A}_{r2}(x'_{r2}).\end{aligned}$$

We now apply the Sylvester rule for computation of the determinant of a matrix as well as the definition of the matrix $\mathbb{A}_{r2}$ and get (the upper index T denote the transposed matrix and $\mathbb{I}_i$ denotes the i-dimensional unit matrix)

$$\begin{aligned}1+|\nabla_{r2}a_{r2}(x'_{r2})|^2 &= \det\left(\mathbb{I}_{d-1} + \nabla_{r2}a_{r2}(x'_{r2})(\nabla_{r2}a_{r2}(x'_{r2}))^T\right)\\ &= \det\left(\mathbb{I}_{d-1} + (\nabla_{r2}a_{r2}(x'_{r2}))^T\nabla_{r2}a_{r2}(x'_{r2})\right)\\ &= \det\left((\mathbb{A}_{r2}(x'_{r2}))^T\mathbb{A}_{r2}(x'_{r2})\right)\\ &= \det\left((\mathbb{A}_{r2}(x'_{r2}))^T(\mathbb{Q}_{r2})^T\mathbb{Q}_{r2}\mathbb{A}_{r2}(x'_{r2})\right),\end{aligned}$$

where the last equality follows from the fact that $\mathbb{Q}_{r2}$ is orthogonal. Using now the definitions of $\mathbf{\Sigma}_{r1}$, $\mathbf{\Sigma}_{r2}$ and $\vec{\varphi}$, and due to the chain rule (it can be applied since a_{r1} and a_{r2} are Lipschitz functions) we obtain the identity

$$\begin{aligned}1+&|\nabla_{r2}a_{r2}(x'_{r2})|^2 = \det\left((\mathbb{A}_{r2}(x'_{r2}))^T(\mathbb{Q}_{r2})^T\mathbb{Q}_{r2}\mathbb{A}_{r2}(x'_{r2})\right)\\ &= \det\left((\mathbb{Q}_{r2}\mathbb{A}_{r2}(x'_{r2}))^T\mathbb{Q}_{r2}\mathbb{A}_{r2}(x'_{r2})\right)\\ &= \det\left((\nabla_{r2}\mathbf{\Sigma}_{r2}(x'_{r2}))^T\nabla_{r2}\mathbf{\Sigma}_{r2}(x'_{r2})\right)\\ &= \det\left((\nabla_{r2}\mathbf{\Sigma}_{r1}(\vec{\varphi}(x'_{r2})))^T\nabla_{r2}\mathbf{\Sigma}_{r1}(\vec{\varphi}(x'_{r2}))\right)\\ &= \det\left((\nabla_{r1}\mathbf{\Sigma}_{r1}(\vec{\varphi}(x'_{r2}))\nabla_{r2}\vec{\varphi}(x'_{r2}))^T\nabla_{r1}\mathbf{\Sigma}_{r1}(\vec{\varphi}(x'_{r2}))\nabla_{r2}\vec{\varphi}(x'_{r2})\right)\\ &= |\det\nabla_{r2}\vec{\varphi}(x'_{r2})|^2\det\left((\nabla_{r1}\mathbf{\Sigma}_{r1}(\vec{\varphi}(x'_{r2})))^T\nabla_{r1}\mathbf{\Sigma}_{r1}(\vec{\varphi}(x'_{r2}))\right)\\ &= |\det\nabla_{r2}\vec{\varphi}(x'_{r2})|^2\det\left(\mathbb{Q}_{r1}\mathbb{A}_{r1}(\vec{\varphi}(x'_{r2})))^T\mathbb{Q}_{r1}\mathbb{A}_{r1}(\vec{\varphi}(x'_{r2}))\right)\\ &= |\det\nabla_{r2}\vec{\varphi}(x'_{r2})|^2\det\left(\mathbb{A}_{r1}(\vec{\varphi}(x'_{r2})))^T\mathbb{A}_{r1}(\vec{\varphi}(x'_{r2}))\right)\\ &= |\det\nabla_{r2}\vec{\varphi}(x'_{r2})|^2\left(1+|\nabla_{r1}a_{r1}(\vec{\varphi}(x'_{r2}))|^2\right).\end{aligned}$$

We now apply this identity in (6.51) and get, due to the Theorem on the change of variables,

$$\begin{aligned}&\int_{\Delta_{r2}} u_{r1}\phi_{r2}(T_{r2}(x'_{r2}, a_{r2}(x'_{r2})))\sqrt{1+|\nabla a_{r2}(x'_{r2})|^2}\,\mathrm{d}x'_{r2}\\ &= \int_{\Delta_{r2}} u_{r1}\phi_{r2}(\mathbf{\Sigma}_{r1}(\vec{\varphi}(x'_{r2}))\sqrt{1+|\nabla a_{r1}(\vec{\varphi}(x'_{r2}))|^2}\,|\det\nabla_{r2}\vec{\varphi}(x'_{r2})|^2\,\mathrm{d}x'_{r2}\\ &= \int_{\Delta_{r1}} u_{r1}\phi_{r2}(\mathbf{\Sigma}_{r1}(x'_{r1}))\sqrt{1+|\nabla a_{r1}(x'_{r1})|^2}\,\mathrm{d}x'_{r1}\\ &= \int_{\Delta_{r1}} u_{r1}\phi_{r2}(T_{r1}(x'_{r1}, a_{r1}(x'_{r1})))\sqrt{1+|\nabla a_{r1}(x'_{r1})|^2}\,\mathrm{d}x'_{r1}.\end{aligned} \tag{6.52}$$

Finally, applying (6.52) in (6.50) and due to the fact that ϕ_{r2} is a partition of unity, we get

$$\begin{aligned}&\sum_{r2=1}^{M_2}\int_{\Delta_{r2}} u_{r2}(T_{r2}(x'_{r2},a_{r2}(x'_{r2})))\sqrt{1+|\nabla a_{r2}(x'_{r2})|^2}\,\mathrm{d}x'_{r2}\\&=\sum_{r2=1}^{M_2}\sum_{r1=1}^{M_1}\int_{\Delta_{r1}} u_{r1}\phi_{r2}(T_{r1}(x'_{r1},a_{r1}(x'_{r1})))\sqrt{1+|\nabla a_{r1}(x'_{r1})|^2}\,\mathrm{d}x'_{r1}\\&=\sum_{r1=1}^{M_1}\int_{\Delta_{r1}} u_{r1}(T_{r1}(x'_{r1},a_{r1}(x'_{r1})))\sqrt{1+|\nabla a_{r1}(x'_{r1})|^2}\,\mathrm{d}x'_{r1}\end{aligned} \tag{6.53}$$

which is the desired identity (6.49). The proof is finished. ■

The surface integral is thus well (and uniquely) defined and we can present the definition of the Lebesgue spaces on $\partial\Omega$. This definition is in fact a straightforward modification of the spaces $L^p(\Omega)$.

Definition 6.6.6 — Spaces $L^p(\partial\Omega)$. Let $\Omega\in\mathcal{C}^{0,1}$ and let u be a function defined almost everywhere in $\partial\Omega$. We say that the function u is an element of $L^p(\partial\Omega)$, $p\in[1,\infty)$, if

$$\int_{\partial\Omega}|u|^p\,\mathrm{d}S<\infty.$$

We further say that the function u is an element of $L^\infty(\partial\Omega)$, if (the essential supremum, more precisely, the sets of measure zero, should be understood in the sense of Definition 6.6.1

$$\operatorname*{ess\,sup}_{x\in\partial\Omega}|u(x)|<\infty.$$

We consider the following norms in the spaces $L^p(\partial\Omega)$

$$\|u\|_{L^p(\partial\Omega)}:=\begin{cases}\left(\int_{\partial\Omega}|u|^p\,\mathrm{d}S\right)^{\frac1p} & p\in[1,\infty)\\ \operatorname*{ess\,sup}_{x\in\partial\Omega}|u(x)| & p=\infty.\end{cases} \tag{6.54}$$

Similarly as for the classical Lebesgue spaces we understand elements of $L^p(\partial\Omega)$ as classes of equivalent functions, i.e, we say that $u\sim v$, if $u=v$ almost everywhere on $\partial\Omega$. Based on this convention we have the following result.

Theorem 6.6.7 — Properties of spaces $L^p(\partial\Omega)$. Let $\Omega\in\mathcal{C}^{0,1}$. Then the space $L^p(\partial\Omega)$ with the norm defined in (6.54) is a Banach space which is separable for $p\in[1,\infty)$ and reflexive for $p\in(1,\infty)$.

Proof. The proof can be performed similarly as for the standard Lebesgue spaces and can be found, e.g., in Kufner et al. (1977) or Nečas (1967). ■

Remark 6.6.8. Recall that if $\Omega\in\mathcal{C}^{0,1}$, then due to the Rademacher Theorem A.2.16 the functions a_r are differentiable in Δ_r and there exists a constant $C>0$ such that $\left|\frac{\partial a_r}{\partial x'_{r_i}}\right|\le C<\infty$ almost everywhere in Δ_r for any $r\in\{1,2,\dots,M\}$. Thus we may consider, for $p\in[1,\infty)$, instead of norm (6.54) an equivalent one,

$$\mathcal{N}(u)_{L^p(\partial\Omega)}:=\left(\sum_{r=1}^{M}\int_{\Delta_r}|u_r(T_r(x'_r,a_r(x'_r)))|^p\,\mathrm{d}x'_r\right)^{\frac1p} \tag{6.55}$$

which is sometimes easier to work with.

The following lemma claims that yet another equivalent norm can be defined without the functions $\{\phi_r\}_{r=1}^M$ from the partition of unity.

Lemma 6.6.9 Let $\Omega\in\mathcal{C}^{0,1}$, a function u defined almost everywhere in $\partial\Omega$, be measurable such that it is non-zero only in $T_r(\Lambda_r)$ for some $r\in\{1,\dots,M\}$ fixed. Let it hold $\int_{\Delta_r}|u(T_r(x'_r,a_r(x'_r)))|^p\,\mathrm{d}x'_r<\infty$ for a certain $p\in[1,\infty)$. Then $u\in L^p(\partial\Omega)$ and there exists a constant $C=C(\partial\Omega)$ such that

$$\|u\|_{L^p(\partial\Omega)}\le C\left(\int_{\Delta_r}|u(T_r(x'_r,a_r(x'_r)))|^p\,\mathrm{d}x'_r\right)^{\frac1p}.$$

Proof. The proof is a simple consequence of the definitions and can be found, e.g., in the book Kufner et al. (1977) or Nečas (1967). ■

6.6.2 Theorems on traces for $W^{1,p}(\Omega)$

Before we introduce the main result of this subsection which allows to speak about boundary values for Sobolev functions, let us motivate the optimality of the result.

Example 6.6.10. Let $u(x) := |x|^{-\alpha}$. We already know from Example 6.1.12 that this function belongs to $W^{1,p}(B_1(0))$ provided $\alpha < \frac{d-p}{p}$. The same holds if we replace $B_1(0)$ by the half-ball $\{x \in \mathbb{R}^d \,|\, |x| < 1, x_d > 0\}$. Since this function is smooth outside of the origin, we may expect that its trace will coincide with the function outside the origin. Computing its trace, the only interesting part is the set $x_d = 0$ (otherwise the function is smooth). We therefore have

$$\int_{\{B_1(0)\subset\mathbb{R}^{d-1}\,|\,x_d=0\}} |x|^{-\alpha q}\,\mathrm{d}S = C(d)\int_0^1 r^{-\alpha q+d-2}\,\mathrm{d}r.$$

This integral is finite provided $-\alpha q + d - 2 > -1$, i.e., for $q < \frac{d-1}{\alpha}$. This indicates that the best result we may expect is that if $u \in W^{1,p}(\Omega)$, $p < d$, then $u \in L^q(\partial\Omega)$ for $q \in [1, \frac{(d-1)p}{d-p}]$.

We are now ready to formulate the promised main result.

Theorem 6.6.11 — On trace operator for $W^{1,p}(\Omega)$ with $p \in [1, d)$. Let $\Omega \in \mathcal{C}^{0,1}$. Define the linear continuous operator $T\colon \mathcal{C}^\infty(\overline{\Omega}) \to \mathcal{C}(\partial\Omega)$ by

$$Tu := u|_{\partial\Omega}.$$

Denote for arbitrary $p \in [1, d)$

$$p^\sharp := \frac{dp-p}{d-p}, \qquad \text{i.e.,} \qquad \frac{1}{p^\sharp} = \frac{1}{p} - \frac{p-1}{p(d-1)}.$$

Then there exists a uniquely defined extension of T such that the mapping is linear,

$$T\colon W^{1,p}(\Omega) \to L^q(\partial\Omega),$$

and bounded (thus continuous) for all $q \in [1, p^\sharp]$.

Proof. **Step 1:** Reduction to smooth functions
Assume that we show the following claim

$$\forall v \in \mathcal{C}^\infty(\overline{\Omega}):\ \|Tv\|_{L^{p^\sharp}(\partial\Omega)} \le C(p,\partial\Omega)\,\|v\|_{W^{1,p}(\Omega)}. \tag{6.56}$$

Note that for smooth functions, the operator T is well defined. By virtue of Theorem 6.2.15 we know that for any $u \in W^{1,p}(\Omega)$ there exists a sequence $\{u_n\}_{n=1}^\infty \subset \mathcal{C}^\infty(\overline{\Omega})$ such that $u_n \to u$ in $W^{1,p}(\Omega)$. Due to the definition of the operator T and due to (6.56) the sequence $\{Tu_n\}_{n=1}^\infty$ is a Cauchy sequence in $L^{p^\sharp}(\partial\Omega)$. The space $L^{p^\sharp}(\partial\Omega)$ is complete (recall Theorem 6.6.7); thus there exists a limit of the sequence $\{Tu_n\}_{n=1}^\infty$ in $L^{p^\sharp}(\partial\Omega)$. We can therefore define

$$Tu := \lim_{n\to\infty} Tu_n.$$

It clearly holds that the limit is independent of the approximate sequence and depends only on the element u (the proof of this claim is left an exercise) and thus the operator T has all required properties for $q = p^\sharp$ including inequality (6.56) for Sobolev functions only. It remains to deal with the existence of the trace in $L^q(\partial\Omega)$ for $q < p^\sharp$. The $(d-1)$ dimensional measure of the boundary of Ω is finite, then Hölder's inequality implies that we have for any $q \in [1, p^\sharp]$ that $\|v\|_{L^q(\partial\Omega)} \le C(q, p^\sharp, \partial\Omega)\,\|v\|_{L^{p^\sharp}(\partial\Omega)}$. Hence, from (6.56) we get

$$\forall v \in W^{1,p}(\Omega):\ \|v\|_{L^q(\partial\Omega)} \le C\,\|v\|_{L^{p^\sharp}(\partial\Omega)} \le C\,\|v\|_{W^{1,p}(\Omega)}. \tag{6.57}$$

Step 2: Localisation and partition of unity
We again follow the notation in Definition 6.2.11 and let $\{\phi_r\}_{r=1}^M \in \mathcal{C}_0^\infty(T_r(V_r))$ be a partition of unity in a vicinity of $\partial\Omega$. Denote $u_r := u\phi_r$, then $\operatorname{supp} u_r \subset T_r(V_r)$. Assume now that it holds for any $r \in \{1, 2, \dots, M\}$

$$\left(\int_{\Delta_r} |u_r(T_r(x_r', a_r(x_r')))|^{p^\sharp}\,\mathrm{d}x_r'\right)^{\frac{p}{p^\sharp}} \le C\int_{\Delta_r}\int_{a_r(x_r')}^{a_r(x_r')+\beta} \left(|\nabla_{x_r} u_r(T_r(x_r))|^p + |u_r(T_r(x_r))|^p\right)\mathrm{d}x_r'\,\mathrm{d}x_{r_d}. \tag{6.58}$$

Applying the equivalence of norms (see (6.55)) and the standard change of variables (recall that T_r are orthogonal transformations) we get

$$\|u\|_{L^{p^\sharp}(\partial\Omega)} \le C\sum_{r=1}^M \|u_r\|_{L^{p^\sharp}(\partial\Omega)} \le C\sum_{r=1}^M \|u_r \circ T_r\|_{W^{1,p}(V_r)} = C\sum_{r=1}^M \|u_r\|_{W^{1,p}(T_r(V_r))} \le C\,\|u\|_{W^{1,p}(\Omega)},$$

where we used in the last inequality that ϕ_r are smooth functions. It remains to verify the validity of (6.58) for smooth functions.

Step 3: Proof of inequality (6.58)
Assume without loss of generality that T_r is the identity (if it is not the case, we just apply the corresponding rotation and shift of the coordinate system); thus $T_r(V_r^+) = V_r^+$. Let $u \in \mathcal{C}^\infty(\overline{\Omega})$. From the definition of u_r (ϕ_r has a compact support in V_r) it follows that it holds for every $x' \in \Delta_r$ that $u_r(x', a_r(x') + \beta) = 0$. Assume first the case $p > 1$ and thus also $p^\sharp > 1$. Due to the smoothness of u_r we immediately get that $|u_r|^{p^\sharp} \in \mathcal{C}^1(\overline{V_r^+})$. Denoting

$$v(x') := |u_r(x', a_r(x'))|^{\frac{dp-p}{d-p}},$$

we get

$$v(x') = |u_r(x', a_r(x'))|^{\frac{dp-p}{d-p}} - |u_r(x', a_r(x') + \beta)|^{\frac{dp-p}{d-p}} = -\int_{a_r(x')}^{a_r(x')+\beta} \frac{\partial}{\partial s}\left(|u_r(x', s)|^{\frac{dp-p}{d-p}}\right) \mathrm{d}s.$$

A straightforward computation yields

$$v(x') \le p^\sharp \int_{a_r(x')}^{a_r(x')+\beta} |\nabla u_r(x', s)|\, |u_r(x', s)|^{\frac{d(p-1)}{d-p}} \, \mathrm{d}s.$$

We integrate this inequality over Δ_r. By virtue of Hölder's inequality (Theorem A.3.12) we get

$$\begin{aligned}\int_{\Delta_r} |v(x')| \, \mathrm{d}x' &\le p^\sharp \int_{\Delta_r} \int_{a_r(x')}^{a_r(x')+\beta} |\nabla u_r(x', s)|\, |u_r(x', s)|^{\frac{d(p-1)}{d-p}} \, \mathrm{d}s \, \mathrm{d}x' \\ &= p^\sharp \int_{V_r^+} |\nabla u_r(x)|\, |u_r(x)|^{\frac{d(p-1)}{d-p}} \, \mathrm{d}x \le p^\sharp \, \|\nabla u_r\|_{L^p(V_r^+;\mathbb{R}^d)} \, \|u_r\|_{L^{\frac{dp}{d-p}}(V_r^+)}^{\frac{d(p-1)}{d-p}}.\end{aligned}$$

Since $V_r^+ \in \mathcal{C}^{0,1}$ (we leave the proof of this property to the kind reader), we may use the Theorem on continuous embedding for $p < d$ (Theorem 6.5.1) ans estimate the last term by the $W^{1,p}$-norm. Altogether, we get

$$\|u_r\|_{L^{p^\sharp}(\Lambda_r)} \le C(d, p) \, \|u_r\|_{W^{1,p}(V_r^+)}, \tag{6.59}$$

where the constant $C(d, p)$ remains bounded for $p \to 1_+$. By the limit passage we obtain the validity of (6.59) also for $p = 1$. Inequality (6.58) follows directly from (6.59) and the proof is finished. ■

Remark 6.6.12. Since $C(\partial\Omega) \hookrightarrow L^q(\partial\Omega)$, the operator T defined as $Tu := u|_{\partial\Omega}$ is in fact defined as an operator from $\mathcal{C}^\infty(\overline{\Omega}) \to L^q(\partial\Omega)$ for any $q \in [1, \infty]$. However, it is possible to extend it to $W^{1,p}(\Omega)$ only for $q \le p^\sharp$.

The situation is much simpler for $p \ge d$ and the following claim holds.

Theorem 6.6.13 — On trace operator for $W^{1,p}(\Omega)$ with $p \ge d$. Let $\Omega \in \mathcal{C}^{0,1}$ and T be the trace operator defined in Theorem 6.6.11. Then the operator is continuous from $W^{1,d}(\Omega)$ to $L^q(\partial\Omega)$ for each $q \in [1, \infty)$. Moreover, the operator T is continuous from $W^{1,p}(\Omega)$ to $L^q(\partial\Omega)$ for each $q \in [1, \infty]$ if $p \in (d, \infty]$.

Proof. The situation $p = d$ is similar to the situation in Theorem 6.5.2, the proof is thus left to the kind reader as a useful exercise.

Due to Theorem 6.5.3 we know for $p > d$ that $W^{1,p}(\Omega)$ is continuously embedded into $\mathcal{C}^0(\overline{\Omega})$. The proof of the theorem is thus trivial and follows directly from Hölder's inequality. ■

Next we strengthen the trace theorem. In fact, except for $p = 1$, the operator is compact as an operator from $W^{1,p}(\Omega)$ to certain Lebesgue spaces on $\partial\Omega$.

Theorem 6.6.14 — On compactness of trace operator. Let $\Omega \in \mathcal{C}^{0,1}$ and T be the operator defined in Theorem 6.6.11. Then it holds.

1. If $p \in (1, d)$, then the operator T is compact from $W^{1,p}(\Omega)$ to $L^q(\partial\Omega)$ for any $q \in [1, p^\sharp)$.
2. If $p > d$, then the operator T is compact from $W^{1,p}(\Omega)$ to $L^q(\partial\Omega)$ for each $q \in [1, \infty]$.
3. If $p = d$, then the operator T is compact from $W^{1,d}(\Omega)$ to $L^q(\partial\Omega)$ for each $q \in [1, \infty)$.

Proof. We have for $p > d$ from Theorem 6.5.3 the compact embedding $W^{1,p}(\Omega)$ to $\mathcal{C}^0(\overline{\Omega})$ and the proof is thus trivial. In what follows we deal with the case $p < d$.

We first show a variant of the inequality from the Step 3 of the proof to Theorem 6.6.11. It is the following interpolation theorem

$$\|u\|_{L^q(\partial\Omega)} \le C(q, \Omega) \, \|u\|_{W^{1,q}(\partial\Omega)}^{\frac{1}{q}} \, \|u\|_{L^q(\Omega)}^{1-\frac{1}{q}} \tag{6.60}$$

which holds for any $q \in [1,\infty)$. We use the structure of the proof of Theorem 6.6.11 and only change inequality (6.58). For notational simplicity we skip the index r and reduce everything to the case when T_r is the identity.

$$\begin{aligned}\int_\Delta |u(x',a(x'))|^q \,\mathrm{d}x' &\le \left|\int_\Delta \int_{a(x')}^{a(x')+\beta} \frac{\partial}{\partial s} |u(x',s)|^q \,\mathrm{d}s\,\mathrm{d}x'\right| \\ &\le q\int_\Delta \int_{a(x')}^{a(x')+\beta} |\nabla u(x',s)|\,|u(x',s)|^{q-1} \,\mathrm{d}s\,\mathrm{d}x' \\ &= q\int_{V^+} |\nabla u(x)|\,|u(x)|^{q-1}\,\mathrm{d}x \le q\,\|\nabla u\|_{L^q(V^+;\mathbb{R}^d)}\,\|u\|_{L^q(V^+)}^{q-1}.\end{aligned}$$

We get the required inequality for Ω by applying Steps 1 and 2 from the proof of the Theorem on traces (Theorem 6.6.11).

Let now $A \subset W^{1,p}(\Omega)$ be an arbitrary bounded set. Our goal is to show that $T(A)$ is a totally bounded set in suitable spaces $L^q(\partial\Omega)$. Denote $C^* := \sup_{u\in A}\|u\|_{1,p}$. Due to the Theorem on continuous embedding (Theorem 6.5.1) we can find $C^{**} < \infty$ such that for any $u \in A$ and any $q \in [1,p^\star]$ it holds $\|u\|_q \le C^{**}$. We can now come to the construction of the ε-net. Let $\varepsilon > 0$ be fixed. Due to the Theorem on compact embedding (Theorem 6.5.20) we know that for arbitrary $\delta > 0$ we can find a δ-net $\{u_i\}_{i=1}^k \subset A$ in the space $L^p(\Omega)$ which covers A. Let us choose now

$$\delta := \varepsilon^{\frac{p}{p-1}}\left(C(p,\Omega)(2C^*)^{\frac1p}\right)^{-\frac{p}{p-1}}.$$

Let $u \in A$ be an arbitrary element and let u_i be such that $\|u-u_i\|_p \le \delta$. Applying inequality (6.60) we get

$$\begin{aligned}\|u-u_i\|_{L^p(\partial\Omega)} &\le C(p,\Omega)\,\|u-u_i\|_{W^{1,p}(\Omega)}^{\frac1p}\,\|u-u_i\|_{L^p(\Omega)}^{1-\frac1p} \\ &\le C(p,\Omega)(2C^*)^{\frac1p}\delta^{1-\frac1p} = \varepsilon\end{aligned}$$

and thus $\{Tu_i\}_{i=1}^k$ forms the ε-net in $L^p(\partial\Omega)$; thus the proof is finished for $q = p$. The validity of the theorem for $q \in [1,p)$ is then a consequence of the continuous embedding $L^p(\partial\Omega) \hookrightarrow L^q(\partial\Omega)$.

Let us now consider the case $q \in (p,p^\sharp)$. Similarly as inequality (A.3.15) we can show the interpolation inequality

$$\|u\|_{L^q(\partial\Omega)} \le \|u\|_{L^p(\partial\Omega)}^{\alpha}\,\|u\|_{L^{p^\sharp}(\partial\Omega)}^{1-\alpha}, \qquad \text{where} \qquad \frac1q = \frac{\alpha}{p} + \frac{1-\alpha}{p^\sharp};$$

we leave its proof as an exercise to the kind reader. Combining this inequality and (6.60) we get

$$\begin{aligned}\|u-u_i\|_{L^q(\partial\Omega)} &\le \left(C(p,\Omega)\,\|u-u_i\|_{W^{1,p}(\Omega)}^{\frac1p}\,\|u-u_i\|_{L^p(\Omega)}^{1-\frac1p}\right)^{\alpha}\,\|u-u_i\|_{L^{p^\sharp}(\Omega)}^{1-\alpha} \\ &\le \left(C(p,\Omega)(2C^*)^{\frac1p}\delta^{1-\frac1p}\right)^{\alpha}(2C^{**})^{1-\alpha}.\end{aligned}$$

Clearly, by a suitable choice of δ in dependence on ε we may construct the ε-net in $L^q(\partial\Omega)$ which finishes the proof for $p \ne d$. Nonetheless, for $p = d$ we can proceed similarly, only instead of Theorem 6.6.11 we use Theorem 6.6.13. ■

We saw in the section devoted to continuous and compact embeddings that we may hope for weakening the assumptions on the regularity of the boundary of Ω, from Lipschitz to (Hölder) continuous, for the price to get "non-optimal" results. The following example shows that any weakening of assumptions on the regularity of the boundary leads to problems with the definition of the traces.

Example 6.6.15. Let us consider a domain $\Omega \subset \mathbb{R}^2$ such that part of its boundary is formed by the curve (cf. Example 6.5.10)

$$|y| = x^\mu, \quad x \in [0,1],\ \mu > 1.$$

We showed in Example 6.5.10 that if the rest of the boundary is smooth, then the function $u(x,y) = x^{-a}$ belongs to $W^{1,2}(\Omega)$ if $a < \frac{1+\mu-2}{2} = \frac{\mu-1}{2}$.

If we compute the integral over the boundary, we get[14]

$$a < 1 \Longleftrightarrow \int_{\partial\Omega} |u|\,\mathrm{d}S < \infty.$$

Therefore, for $\mu > 3$, there exist functions from $W^{1,2}(\Omega)$ which do not belong to any $L^q(\partial\Omega)$ for arbitrary $q \ge 1$. It illustrates that the condition $\Omega \in \mathcal{C}^{0,1}$ cannot be weaken.

[14]If we consider the parametric description of the boundary $x = t$, $y = t^\mu$, $t \in [0,1]$, we thus compute

$$\int_0^1 t^{-a}\sqrt{1+(\mu t^{\mu-1})^2}\,\mathrm{d}t$$

which leads to the result presented above.

A question appears, whether $L^{p^\sharp}(\partial\Omega)$ is the range of the trace operator from $W^{1,p}(\Omega)$ (if $p \in [1,d)$). This question is important in connection to the boundary value problems for partial differential equations. The answer is negative. It only holds that the range of the corresponding trace operator is dense in $L^{p^\sharp}(\partial\Omega)$. The precise characterization requires to build the theory for spaces with fractional derivative. In fact, the range of the trace operator is the space $W^{1-\frac{1}{p},p}(\partial\Omega)$, $1 < p \le \infty$. For $p = 1$ the range of the trace operator is $L^1(\partial\Omega)$, but the situation is rather complex here. More precise information about these spaces will be given in Subsection 6.8.1.

6.6.3 Characterization of $W_0^{1,p}(\Omega)$ and integration by parts

The theorem on traces has many important corollaries. It not only allows us to speak about boundary values for Sobolev functions at the boundary $\partial\Omega$, but it also makes possible to generalize the classical theorem on integration by parts for Sobolev functions and to characterize precisely the space $W_0^{1,p}(\Omega)$. Let us start with the integration by parts. It is usually formulated for smooth Ω (piecewise smooth Ω) and functions with continuous derivatives up to the boundary. We saw in Theorem 6.1.22 that the assumption on smoothness up to the boundary can be weaken, provided we consider functions from $W_0^{1,p}(\Omega)$. We now generalize this result for Lipschitz domains and Sobolev functions (generally nonzero on the boundary).

Theorem 6.6.16 — On integration by parts II. Let $\Omega \in \mathcal{C}^{0,1}$. Then the outer normal $\boldsymbol{\nu}$ exists almost everywhere on $\partial\Omega$. Let further $p, q \in [1,\infty)$ be such that one of the possibilities holds:

1. $p \in [1,d)$ and $q \in [1,d)$ such that $\frac{1}{p} + \frac{1}{q} \le \frac{d+1}{d}$
2. $p = d$ and $q > 1$ (or $q = d$ and $p > 1$, respectively)
3. $p > d$ and $q \ge 1$ (or $q > d$ and $p \ge 1$, respectively).

Then it holds for any $u \in W^{1,p}(\Omega)$ and $v \in W^{1,q}(\Omega)$ ($\boldsymbol{\nu}$ is the unit outer normal vector to $\partial\Omega$)

$$\int_\Omega u\frac{\partial v}{\partial x_i}\,\mathrm{d}x = \int_{\partial\Omega} uv\nu_i\,\mathrm{d}S - \int_\Omega \frac{\partial u}{\partial x_i}v\,\mathrm{d}x. \tag{6.61}$$

Proof. The existence of the normal almost everywhere at $\partial\Omega$ is left for a kind reader (it can be found, e.g., in Nečas (1967)).

Let us now deal with the second part of the theorem. We perform the proof slightly formally, we skip the partition of unity. The formula of the Green Theorem (6.61) holds, if $\partial\Omega$ is a C^1 domain and $u, v \in \mathcal{C}^\infty(\overline{\Omega})$. Since every Lipschitz domain can be approximated from inside by smooth domains (see Nečas (1962)), it is possible to verify that (6.61) holds true also in the case of $\Omega \in \mathcal{C}^{0,1}$, $u, v \in \mathcal{C}^\infty(\overline{\Omega})$.

Let u and v be Sobolev functions, as in the formulation of the theorem. Then there exist sequences of smooth functions $\{u_n\}_{n=1}^\infty$, $\{v_n\}_{n=1}^\infty \in \mathcal{C}^\infty(\overline{\Omega})$ which approximate functions u and v in the $\|\cdot\|_{W^{1,p}(\Omega)}$- or in the $\|\cdot\|_{W^{1,q}(\Omega)}$-norm, respectively. Evidently, it holds

$$\int_\Omega u_n\frac{\partial v_n}{\partial x_i}\,\mathrm{d}x = \int_{\partial\Omega} u_n v_n\nu_i\,\mathrm{d}S - \int_\Omega \frac{\partial u_n}{\partial x_i}v_n\,\mathrm{d}x \tag{6.62}$$

and it suffices to verify that under our assumptions we can pass to the limit $n \to \infty$ in the given equality. Let us first look at the first set of conditions, i.e., $p \in [1,d)$, $q \in [1,d)$, the verification of the remaining two sets is left as a useful exercise to the kind reader.

We first consider the integral on the left-hand side of equality (6.62). It converges for $n \to \infty$ to $\int_\Omega u\frac{\partial v}{\partial x_i}\,\mathrm{d}x$, provided

$$\frac{d-p}{dp} + \frac{1}{q} \le 1$$

which can be easily transformed to $\frac{1}{p} + \frac{1}{q} \le \frac{d+1}{d}$ due to the assumptions of the theorem.

Similarly, we can proceed for the volume integral on the right-hand side of (6.62). Let us now consider the surface integral. Here we require (compare with Theorem 6.6.11)

$$\frac{d-p}{dp-p} + \frac{d-q}{dq-q} \le 1$$

which can be transformed to the form $d\left(\frac{1}{p} + \frac{1}{q}\right) \le d+1$; due to the previous condition $\frac{1}{p} + \frac{1}{q} \le \frac{d+1}{d}$ is the other condition fulfilled automatically. ∎

Another important consequence of the theorem on traces is the following characterization of the space $W_0^{1,p}(\Omega)$.

Theorem 6.6.17 — Characterization of $W_0^{1,p}(\Omega)$. Let $\Omega \in \mathcal{C}^{0,1}$. Then

$$W_0^{1,p}(\Omega) = \left\{u \in W^{1,p}(\Omega) \,|\, Tu = 0 \text{ almost everywhere on } \partial\Omega\right\}.$$

Proof. We first show the inclusion "$\subset$". Let $u \in W_0^{1,p}(\Omega)$ be arbitrary. The definition of the space $W_0^{1,p}(\Omega)$ yields a sequence $\{u_n\}_{n=1}^{\infty} \subset \mathcal{C}_0^{\infty}(\Omega)$ such that $u_n \to u$ in $W^{1,p}(\Omega)$. Clearly $Tu_n = 0$ and from the continuity of the trace operator it also follows $Tu = 0$ and thus $W_0^{1,p}(\Omega) \subset \left\{u \in W^{1,p}(\Omega);\ Tu = 0 \text{ almost everywhere on } \partial\Omega\right\}$.

In the second part we concentrate on the more difficult inclusion "$\supset$", i.e., for a given function $u \in W_0^{1,p}(\Omega)$ we have to find a sequence $\{u^n\}_{n=1}^{\infty} \subset \mathcal{C}_0^{\infty}(\Omega)$ such that $\|u^n - u\|_{1,p} \to 0$ for $n \to \infty$. We first use the partition of unity $\{\phi_r\}_{r=1}^{M+1}$ and see that it is enough to consider only functions of the type $u_r := u\phi_r$. For u_{M+1} the situation is clear and it is enough to apply the standard convolution mollification. Let us deal with the case $r = 1, \dots, M$. For simplicity we skip the indeces r and, moreover, we shall not consider the mapping between the local and the global coordinate system (we assume, it is just identity). Recall the notation from Definition 6.2.11

$$V^+ = \left\{x \in \mathbb{R}^d |\ |x_i| < \alpha, i = 1, \dots, d-1,\ a(x') < x_d < a(x') + \beta\right\}.$$

Thus, it is enough to show that if $u \in W^{1,p}(V^+)$ satisfying $Tu = 0$ on Λ and $\operatorname{supp} u \cap \{\partial V^+ \setminus \Lambda\} = \emptyset$, then there exists a sequence such that $\{u_n\}_{n=1}^{\infty} \subset \mathcal{C}_0^{\infty}(V^+)$ and $u_n \to u$ in $W^{1,p}(V^+)$.

We first extend the function u by zero outside of V^+, i.e., we define

$$\tilde{u}(x', x_d) = \begin{cases} u(x', x_d) & (x', x_d) \in V^+ \\ 0 & (x', x_d) \in V^-. \end{cases}$$

We show below that this function belongs to $W^{1,p}(V)$ and its weak derivative is

$$\nabla\tilde{u}(x', x_d) = \begin{cases} \nabla u(x', x_d) & (x', x_d) \in V^+ \\ 0 & (x', x_d) \in V^-. \end{cases}$$

To prove this claim we apply the Theorem on integration by parts II (Theorem 6.6.16). We have for arbitrary $\varphi \in \mathcal{C}_0^{\infty}(V)$ that

$$\begin{aligned}\int_V \tilde{u}\frac{\partial\varphi}{\partial x_i}\,\mathrm{d}x &= \int_{V^+} u\frac{\partial\varphi}{\partial x_i}\,\mathrm{d}x = -\int_{V^+}\frac{\partial u}{\partial x_i}\varphi\,\mathrm{d}x + \int_{\Lambda} u\varphi\nu_i\,\mathrm{d}S \\ &= -\int_{V^+}\frac{\partial u}{\partial x_i}\varphi\,\mathrm{d}x = -\int_V \frac{\partial\tilde{u}}{\partial x_i}\varphi\,\mathrm{d}x,\end{aligned}$$

where we used the assumption $u = 0$ on ∂V^+.

We now mollify this extended function. We proceed as in the proof of Theorem 6.2.15, only instead of sliding this function u "outside" of Ω, we slide it "inside". We define $\tilde{u}^n(x', x_d) := \tilde{u}(x', x_d - \frac{1}{n})$. The function $\tilde{u}^n$ belongs for a sufficiently large n to $W^{1,p}(V^+)$ and, moreover, $\operatorname{supp}\tilde{u}_n \subset V^+$. Further $\lim_{n\to\infty}\|\tilde{u}^n - u\|_{W^{1,p}(V^+)} = 0$. It is now enough to define $u^n = \eta_{h_n} \star \tilde{u}^n$, where η_{h_n} is the mollifier and h_n is a suitably chosen number less than $\frac{1}{n}$ so that $u^n \in \mathcal{C}_0^{\infty}(V^+)$. Due to the properties of the mollifier (Theorem A.3.33) it is not difficult to verify that $u^n \to u$ in $W^{1,p}(V^+)$. The proof is complete. ■

6.7 Poincaré inequalities and equivalent norms

We show in this section that in some cases it is possible to replace the standard norm on the space $W^{k,p}(\Omega)$ by different functionals which define there equivalent norms. These equivalent norms play an important role in the theory of partial differential equations, for example if we prescribe the boundary value on the full boundary or its parts, or if we consider functions with prescribed mean value. We start with one general lemma.

Lemma 6.7.1 Let $\Omega \in \mathcal{C}^0$, $k \in \mathbb{N}$ and $p \in [1, \infty)$. Denote by P_k polynomials of the degree at most k. Let $\{f_i\}_{i=1}^{l}$ be continuous bounded functionals (not necessarily linear) on $W^{k,p}(\Omega)$ which fulfil for any $u \in P_{k-1}$

$$\sum_{i=1}^{l} |f_i(u)| = 0 \quad \Longleftrightarrow \quad u = 0 \text{ almost everywhere in } \Omega.$$

Let it further hold for any $u \in W^{k,p}(\Omega)$, $\lambda \in \mathbb{R}$ and $i = 1, \dots, l$

$$|f_i(\lambda u)| \le |\lambda|\,|f_i(u)|.$$

Then there exist positive constants c_1 and c_2 such that we have for any $u \in W^{k,p}(\Omega)$

$$c_1 \|u\|_{W^{k,p}(\Omega)} \leq \left(\sum_{|\alpha|=k} \|D^\alpha u\|_{L^p(\Omega)}^p + \sum_{i=1}^{l} |f_i(u)|^p \right)^{\frac{1}{p}} \leq c_2 \|u\|_{W^{k,p}(\Omega)} . \tag{6.63}$$

Proof. The second inequality in (6.63) is trivial, let us therefore consider only the first one.

For contradiction, let us assume that there exists a sequence of functions $\{\tilde{u}_n\}_{n=1}^{\infty} \subset W^{k,p}(\Omega)$ such that it holds

$$\left(\sum_{|\alpha|=k} \|D^\alpha \tilde{u}_n\|_{L^p(\Omega)}^p + \sum_{i=1}^{l} |f_i(\tilde{u}_n)|^p \right)^{\frac{1}{p}} < \frac{\|\tilde{u}_n\|_{W^{k,p}(\Omega)}}{n}.$$

Evidently $\tilde{u}_n \neq 0$, therefore $u_n := \tilde{u}_n / \|\tilde{u}_n\|_{W^{k,p}(\Omega)}$ is well defined. Dividing the above stated inequality by $\|\tilde{u}_n\|_{W^{k,p}(\Omega)}$ and using the assumptions on f_i we get

$$\left(\sum_{|\alpha|=k} \|D^\alpha u_n\|_{L^p(\Omega)}^p + \sum_{i=1}^{l} |f_i(u_n)|^p \right)^{\frac{1}{p}} < \frac{1}{n} \tag{6.64}$$

as well as $\|u_n\|_{W^{k,p}(\Omega)} = 1$. Since the sequence $\{u_n\}_{n=1}^{\infty}$ is bounded in $W^{k,p}(\Omega)$, due to the compact embedding $W^{k,p}(\Omega) \hookrightarrow\hookrightarrow W^{k-1,p}(\Omega)$ (Theorem 6.5.22 or 6.5.25) we may choose subsequence (relabeled) and find $u \in W^{k-1,p}(\Omega)$ such that $u_n \to u$ in $W^{k-1,p}(\Omega)$. Furthermore, it follows immediately from (6.64) that for $|\alpha| = k$ we have $D^\alpha u_n \to 0$ in $L^p(\Omega)$. This implies that $u_n \to u$ in $W^{k,p}(\Omega)$, too.

The strong convergence implies for the limit function u that $\|u\|_{W^{k,p}(\Omega)} = 1$ and $D^\alpha u = 0$ for any $|\alpha| = k$. Applying Lemma 6.2.3 and a simple induction it is not difficult to see that $u \in P_{k-1}$. Since the functionals f_i are continuous, we see that $\sum_{i=1}^{l} |f_i(u)|^p = 0$ and since $u \in P_{k-1}$, we also get $u = 0$ which contradicts to $\|u\|_{W^{k,p}(\Omega)} = 1$. ■

Remark 6.7.2. Functionals satisfying assumptions of Lemma 6.7.1 always exist, it is enough to take

$$f_\alpha(u) = \int_{\Omega^\star} x^\alpha u(x)\,\mathrm{d}x, \qquad \text{for all } |\alpha| \leq k-1,$$

or

$$\tilde{f}_\alpha = \int_{\Omega^\star} D^\alpha u(x)\,\mathrm{d}x, \qquad \text{for all } |\alpha| \leq k-1,$$

where $\Omega^\star$ is an arbitrary non-empty subdomain of Ω.

Lemma 6.7.1 has a number of applications. Let us present the most important ones.

Theorem 6.7.3 — On equivalent norms in $W^{1,p}(\Omega)$. Let $\Omega \in \mathcal{C}^{0,1}$. Let $\Omega^\star \subset \Omega$ be such that $|\Omega^\star|_d > 0$ and $\Gamma \subset \partial\Omega$ such that $|\Gamma|_{d-1} > 0$. Let further $p \in [1, \infty)$ and α_i, $i = 1, \dots, 4$, be non-negative numbers such that $\sum_{i=1}^{4} \alpha_i > 0$. Then there exist positive constants c_1 a c_2 such that it holds for any $u \in W^{1,p}(\Omega)$

$$c_1 \|u\|_{W^{1,p}(\Omega)} \leq \left(\|\nabla u\|_{L^p(\Omega)}^p + \alpha_1 \int_\Gamma |u|^p \,\mathrm{d}S + \alpha_2 \left| \int_\Gamma u \,\mathrm{d}S \right|^p + \alpha_3 \int_{\Omega^\star} |u|^p \,\mathrm{d}x + \alpha_4 \left| \int_{\Omega^\star} u \,\mathrm{d}x \right|^p \right)^{\frac{1}{p}} \leq c_2 \|u\|_{W^{1,p}(\Omega)} .$$

Proof. Denote $f_1(u) = \left(\int_\Gamma |u|^p \,\mathrm{d}S\right)^{\frac{1}{p}}$, $f_2(u) = \left|\int_\Gamma u \,\mathrm{d}S\right|$, $f_3(u) = \left(\int_{\Omega^\star} |u|^p \,\mathrm{d}x\right)^{\frac{1}{p}}$ and $f_4(u) = \left|\int_{\Omega^\star} u \,\mathrm{d}x\right|$. All four functionals are clearly positive homogeneous (in the sense as presented in Lemma 6.7.1), bounded and continuous on $W^{1,p}(\Omega)$ (here we use the Theorem on traces, i.e., Theorem 6.6.11 or 6.6.13 and the assumption $\Omega \in \mathcal{C}^{0,1}$). We finally use Lemma 6.7.1 and it is enough to verify that if $u = \textit{const}$ almost everywhere in Ω, then for arbitrary $i \in \{1, \dots, 4\}$

$$f_i(u) = 0 \iff u = 0.$$

This equivalence is, however, evident. ■

Note that unlike in Lemma 6.7.1, we consider in Theorem 6.7.3 domains with Lipschitz boundary. This is connected with the fact that we speak about values of u on the boundary and use the Theorem on traces (Theorem 6.6.11). If we do not consider integral over (a part of) the boundary, it would be enough to consider domains with continuous boundaries only.

Some inequalities (equivalent norms) have traditional names. Below we present their short list.

Remark 6.7.4 (Important inequalities). Let $\Omega \in \mathcal{C}^{0,1}$. Using Theorem 6.7.3 we can easily obtain the following inequalities.

1) Inequality $c_1 \|u\|_{W^{1,p}(\Omega)} \leq \left(\|\nabla u\|^p_{L^p(\Omega)} + \int_\Gamma |u|^p \,\mathrm{d}S \right)^{\frac{1}{p}}$ is called Poincaré–Friedrichs inequality. Evidently, we can replace $\int_\Gamma |u|^p \,\mathrm{d}S$ by $\left(\int_\Gamma |u|^q \,\mathrm{d}S\right)^{\frac{p}{q}}$, where $q \in [1, \frac{dp-p}{d-p}]$ for $p \in [1,d)$ and $q \in [1,\infty)$ for $p \geq d$.

2) Inequality $c_1 \|u\|_{W^{1,p}(\Omega)} \leq \left(\|\nabla u\|^p_{L^p(\Omega)} + \left|\int_\Omega u \,\mathrm{d}x\right|^p \right)^{\frac{1}{p}}$ is called Poincaré inequality. Its generalization for the space $W^{k,p}(\Omega)$ can be found below.

3) It is possible to replace $\int_{\Omega^\star} |u|^p \,\mathrm{d}x$ by $\left(\int_{\Omega^\star} |u|^q \,\mathrm{d}x\right)^{\frac{p}{q}}$, where $q \in [1, \frac{dp}{d-p}]$ for $p \in [1,d)$ and $q \in [1,\infty)$ for $p \geq d$.

Let us finally mention several important inequalities for Sobolev spaces of higher order.

Theorem 6.7.5 — On equivalent norms on $W^{k,p}(\Omega)$. Let $\Omega \in \mathcal{C}^0$, $\Omega^\star \subset \Omega$ be such that $|\Omega^\star|_d > 0$, $p \in [1,\infty)$ and $k \in \mathbb{N}$. Let α_1 and α_2 be non-negative numbers satisfying $\alpha_1 + \alpha_2 > 0$. Then there exist positive numbers c_1 and c_2 such that it holds for any $u \in W^{k,p}(\Omega)$

$$c_1 \|u\|_{W^{k,p}(\Omega)} \leq \left(\sum_{|\alpha|=k} \|D^\alpha u\|^p_{L^p(\Omega)} + \alpha_1 \left(\int_{\Omega^\star} |u| \,\mathrm{d}x \right)^p + \alpha_2 \sum_{|\alpha|\leq k-1} \left| \int_\Omega D^\alpha u \,\mathrm{d}x \right|^p \right)^{\frac{1}{p}} \leq c_2 \|u\|_{W^{k,p}(\Omega)}.$$

Proof. The proof is left for a kind reader as a useful exercise. ■

We did not consider integrals over boundary in the theorem above. There were two reasons for it. The above presented result holds for domains with only smooth boundary, since it is based on compact embedding and we do not need the Theorems on traces and the Lipschitz boundary. Second reason is that unlike Theorem 6.7.3, it is not enough to consider for Sobolev spaces of higher order only integrals over the boundary; we need also certain qualitative assumptions on a part of the boundary. We shall illustrate it on the case of $W^{2,p}(\Omega)$ and leave the general situation for a kind reader.

Theorem 6.7.6 Let $\Omega \in \mathcal{C}^{0,1}$ and $p \in [1,\infty)$. Let $\Gamma \subset \partial\Omega$ be such that Γ is not a hyperplane and satisfies $|\Gamma|_{d-1} > 0$. Then there exist positive numbers c_1 and c_2 such that it holds for any $u \in W^{2,p}(\Omega)$

$$c_1 \|u\|_{W^{2,p}(\Omega)} \leq \left(\sum_{|\alpha|=2} \|D^\alpha u\|^p_{L^p(\Omega)} + \int_\Gamma |u|^p \,\mathrm{d}S \right)^{\frac{1}{p}} \leq c_2 \|u\|_{W^{2,p}(\Omega)}.$$

Since $\partial\Omega$ cannot be a hyperplane, the above presented inequality holds always for $\Gamma = \partial\Omega$.

Proof. Due to Lemma 6.7.1 it is enough to check that if $u \in P_1$ and $\int_\Gamma |u|^p \,\mathrm{d}S = 0$, then $u = 0$. Let u be a linear function and $u = 0$ almost everywhere on Γ. This, however, may happen only in the case when Γ is a hyperplane. By our assumptions, this case is excluded. ■

We considered up to now only the questions concerning equivalent norms. We have seen that it is important to exclude the possibility that the function u is a non-zero polynomial of the $(k-1)$–th order. These functions cannot always be excluded, as for example in the weak formulation for the Poisson equation with the Neumann boundary condition. We therefore introduce subspaces of Sobolev functions which are equivalent up to polynomials of a certain order.

Definition 6.7.7 — Factor space $W^{k,p}(\Omega)/P$. Let $\Omega \subset \mathbb{R}^d$, $k \in \mathbb{N}$ and $p \in [1,\infty]$. Let $P \subset P_{k-1}$ be a subspace of polynomials of the $(k-1)$–th order. Denote by $W^{k,p}(\Omega)/P$ the factor space, i.e., we say that it holds $u_1 \sim u_2$ for $u_1, u_2 \in W^{k,p}(\Omega)$, if $u_1 - u_2 \in P$. We endow this space by the norm

$$\|u\|_{W^{k,p}(\Omega)/P} := \inf_{\tilde{u} \in W^{k,p}(\Omega):\, \tilde{u} \sim u} \|\tilde{u}\|_{W^{k,p}(\Omega)}.$$

This space is evidently a Banach space which is for $p \in [1,\infty)$ separable and for $p \in (1,\infty)$ reflexive. The proof, as well as the proof of the following theorem, is left as an easy exercise for a kind reader.

Theorem 6.7.8 — Poincaré inequality for factor spaces. Let $\Omega \in \mathcal{C}^0$ and $k \in \mathbb{N}$. Then there exist positive constants c_1 and c_2 such that it holds for any $u \in W^{k,p}(\Omega)$

$$c_1 \|u\|_{W^{k,p}(\Omega)/P_{k-1}} \leq \left(\sum_{|\alpha|=k} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}} \leq c_2 \|u\|_{W^{k,p}(\Omega)/P_{k-1}}.$$

6.8 Several further properties of functions from Sobolev spaces

6.8.1 Spaces with fractional derivative, the range of the trace operator and inverse trace theorem

As an analogy of Hölder continuous functions which lie between the space space of continuous and continuously differentiable functions we introduce the space of functions with fractional derivative.

We first consider $\Omega \subsetneq \mathbb{R}^d$.

Definition 6.8.1 — Sobolev spaces with fractional derivative. Let $s \in \mathbb{R}^+$, $p \in [1, \infty)$. Let $[s]$ be the integer part of s. Then $W^{s,p}(\Omega)$ is the space of all functions from $W^{[s],p}(\Omega)$ ($W^{0,p}(\Omega) = L^p(\Omega)$) which satisfy

$$\forall \alpha,\ |\alpha| = [s]\colon\ I_\alpha(u) := \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{d+p(s-[s])}} \,\mathrm{d}x \,\mathrm{d}y < \infty.$$

Denote further

$$\|u\|_{W^{s,p}(\Omega)} = \left(\|u\|^p_{W^{[s],p}(\Omega)} + \sum_{|\alpha|=[s]} I_\alpha(u) \right)^{\frac{1}{p}}.$$

Then it holds

Theorem 6.8.2 — Sobolev spaces with fractional derivative – basic properties. The space $W^{s,p}(\Omega)$ is a Banach space with the norm $\|\cdot\|_{W^{s,p}(\Omega)}$. This space is separable for $p \in [1, \infty)$ and reflexive for $p \in (1, \infty)$.

Proof. The proof can be found, e.g., in Kufner et al. (1977). ■

Remark 6.8.3. Let $p \in [1, \infty)$ and $0 < s < \beta \le 1$, $|\Omega|_d < \infty$. Then

$$\mathcal{C}^{0,\beta}(\overline{\Omega}) \hookrightarrow W^{s,p}(\Omega),$$

since

$$\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{d+ps}} \,\mathrm{d}x \,\mathrm{d}y \le (H_{0,\beta}(u))^p \int_\Omega \int_\Omega \frac{1}{|x-y|^{d+(s-\beta)p}} \,\mathrm{d}x \,\mathrm{d}y \le K \left(H_{0,\beta}(u)\right)^p.$$

Another properties of these function spaces can be found in specialized monographs, see also Di Nezza et al. (2012). Note, however, that

$$\lim_{s \to 0_+} s \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^p}{|x-y|^{d+ps}} \,\mathrm{d}x \,\mathrm{d}y = C \int_{\mathbb{R}^d} |u|^p \,\mathrm{d}x$$

for any $u \in \cap_{s \in (0,1)} W^{s,p}(\mathbb{R}^d)$, where $C = C(p, d)$. Similarly, for $u \in W^{1,p}(\Omega)$

$$\lim_{s \to 1_-} (1-s) \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{d+ps}} \,\mathrm{d}x \,\mathrm{d}y = C \int_{\mathbb{R}^d} |\nabla u|^p \,\mathrm{d}x,$$

where again $C = C(d, p)$.

We are rather interested in the analogous spaces on $\partial\Omega$.

Definition 6.8.4 — Spaces $W^{s,p}(\partial\Omega)$. Let $s \in \mathbb{R}^+$, $p \in [1, \infty)$, $\Omega \in \mathcal{C}^{[s],1}$. Denote for $r = 1, \dots, M$ functions $v_r(x'_r) = u_r \circ T_r(x'_r, a_r(x'_r))$. Then $W^{s,p}(\partial\Omega)$ is the subspace of all functions from $L^p(\partial\Omega)$ which satisfy

$$\forall r \in \{1, \dots, M\}\colon\ v_r \in W^{s,p}(\Delta_r).$$

Denote further[a]

$$\|u\|_{W^{s,p}(\partial\Omega)} = \left(\sum_{r=1}^M \|v_r\|^p_{W^{s,p}(\Delta_r)} \right)^{\frac{1}{p}}.$$

[a]Recall that $\Delta_r \subset \mathbb{R}^{d-1}$.

Similarly as above, it holds (for the proof see, e.g., Kufner et al. (1977))

Theorem 6.8.5 — Sobolev spaces with fractional derivative on the boundary – basic properties. The space $W^{s,p}(\partial\Omega)$ is a Banach space with the norm $\|\cdot\|_{W^{s,p}(\partial\Omega)}$. This space is separable for $p \in [1, \infty)$ and reflexive for $p \in (1, \infty)$.

It appears that spaces $W^{1-\frac{1}{p},p}(\partial\Omega)$, $p \in (1, \infty)$ are exactly the spaces which characterize the range of the trace operator. It holds

Theorem 6.8.6 — Inverse trace theorem. Let $\Omega \in \mathcal{C}^{0,1}$, $p \in (1,\infty)$. Then there exists uniquely defined continuous linear operator

$$T\colon W^{1,p}(\Omega) \to W^{1-\frac{1}{p},p}(\partial\Omega)$$

such that

$$\forall u \in \mathcal{C}^\infty(\overline{\Omega}) : Tu = u|_{\partial\Omega}.$$

Let $\Omega \in \mathcal{C}^{0,1}$, $p \in (1,\infty)$. Then there exists a continuous linear operator

$$P\colon W^{1-\frac{1}{p},p}(\partial\Omega) \to W^{1,p}(\Omega),$$

such that for $v = Pu$ it holds $u = Tv$.

Proof. The proof is rather technical and can be found in all details in Kufner et al. (1977) or in Nečas (1967). ■

Note that the proof of the first part of the above stated theorem is based on the Hardy inequality which is also important in applications in partial differential equations. Let us present here its several formulations. More details can be found in the book Kufner and Opic (1990).

Theorem 6.8.7 — Hardy inequalities. Let $a, b \in \mathbb{R}^d$, $a < b$, $u \in L^p((a,b))$, $p \in (1,\infty)$. Then it holds

$$\int_a^b \left(\frac{1}{x-a} \int_a^x |u(y)|\,\mathrm{d}y \right)^p \mathrm{d}x \le \left(\frac{p}{p-1} \right)^p \int_a^b |u(x)|^p \,\mathrm{d}x,$$
$$\int_a^b \left(\frac{1}{b-x} \int_x^b |u(y)|\,\mathrm{d}y \right)^p \mathrm{d}x \le \left(\frac{p}{p-1} \right)^p \int_a^b |u(x)|^p \,\mathrm{d}x,$$

and

$$\int_0^\infty |u(t)|^p t^{\varepsilon-p}\,\mathrm{d}t \le \left(\frac{p}{|\varepsilon - p + 1|} \right)^p \int_0^\infty |u(t)|^p t^\varepsilon \,\mathrm{d}t,$$

where the inequality holds for $\varepsilon > p-1$ for $u(\infty) = 0$ and $\varepsilon < p-1$ for $u(0) = 0$.

Let $u \in W_0^{1,p}(\Omega)$, $p \in (1,\infty)$ and denote $d(x) = \operatorname{dist}(x, \partial\Omega)$. Then

$$\int_\Omega \left| \frac{u}{d} \right|^p \mathrm{d}x \le C \int_\Omega |\nabla u|^p \,\mathrm{d}x. \tag{6.65}$$

The following example due to Hadamard is interesting for the theory of weak solutions to partial differential equations as it shows that there might be some problems to define a weak solution for an elliptic problem in the situation when the classical solution exists.

Example 6.8.8 (Hadamard)**.** Let $d = 2$ and $\Omega = B_1(0)$. We define

$$u(x,y) = \sum_{n=1}^\infty 2^{-n} \rho^{2^{2n}} \cos(2^{2n}\varphi) \quad \text{in } B_1(0) \setminus \{(0,0)\}$$
$$u(0,0) = 0,$$

where (ρ, φ) are the standard polar coordinates (i.e., $\rho \in (0,1]$, $\varphi \in [0, 2\pi)$). Then the series converges uniformly in $B_1(0)$ and thus $u \in \mathcal{C}(\overline{B_1(0)})$. Moreover, evidently $u \in \mathcal{C}^\infty(B_1(0))$ as the series including all formally taken derivatives converges locally uniformly in $B_1(0)$. In particular $u \in \mathcal{C}(\partial B_1(0))$, but by direct computation it can be shown that $u \notin W^{1,2}(B_1(0))$. Moreover, it can be shown that $u \notin W^{\frac{1}{2},2}(\partial B_1(0))$ and thus there is no $u \in W^{1,2}(B_1(0))$ such that it trace would be given by $u|_{\partial B_1(0)}$. Therefore there does not exist a weak solution $u \in W^{1,2}(B_1(0))$ of the problem $\Delta v = 0$ in $B_1(0)$ with the boundary condition $v = u|_{\partial B_1(0)} \in \mathcal{C}(\partial B_1(0))$.

6.8.2 Sobolev spaces and Fourier transform

In this part we look at another possibility how to introduce the Sobolev spaces, via the Fourier transform. This approach is, however, limited to the case $p = 2$ (i.e., the Hilbert space case).

Theorem 6.8.9 — Equivalent definition of $W^{k,2}(\mathbb{R}^d)$. Let $k \in \mathbb{N}$. Then

1. A function $u \in L^2(\mathbb{R}^d)$ belongs to $W^{k,2}(\mathbb{R}^d)$, if and only if

   $$\left(1 + |\xi|^k\right) \mathcal{F}(u)(\xi) \in L^2(\mathbb{R}^d),$$

   where $\mathcal{F}(u)$ denotes the Fourier transform[a] of the function $u \in L^2(\mathbb{R}^d)$.

2. There exist positive constants c_1 a c_2 such that

$$\forall u \in W^{k,2}(\mathbb{R}^d):\ c_1 \|u\|_{W^{k,2}(\mathbb{R}^d)} \leq \left\| \left(1+|\xi|^k\right) \mathcal{F}(u) \right\|_{L^2(\mathbb{R}^d)} \leq c_2 \|u\|_{W^{k,2}(\mathbb{R}^d)}.$$

[a] We use the definition

$$\mathcal{F}(u)(\xi) = \int_{\mathbb{R}^d} u(x) e^{-\mathrm{i}2\pi(x,\xi)}\,\mathrm{d}x$$

for functions from $L^1(\mathbb{R}^d)$; $(x,\xi) = \sum_{i=1}^d x_i\xi_i$.

Proof. **Step 1:** Proof of the first implication in 1. and the second inequality in 2.
Let $u \in W^{k,2}(\mathbb{R}^d)$, i.e., $\forall |\alpha| \leq k$: $D^\alpha u \in L^2(\mathbb{R}^d)$. In particular, for $u \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ it holds $\mathcal{F}(D^\alpha u)(\xi) = (\mathrm{i}2\pi\xi)^\alpha \mathcal{F}(u)(\xi)$. As $\mathcal{C}_0^\infty(\mathbb{R}^d)$ is dense in $W^{k,2}(\mathbb{R}^d)$, it is not difficult to see that it holds for $u \in W^{k,2}(\mathbb{R}^d)$

$$\mathcal{F}(D^\alpha u)(\xi) = (\mathrm{i}2\pi\xi)^\alpha \mathcal{F}(u)(\xi) \text{ almost everywhere in } \mathbb{R}^d.$$

But then $(\xi)^\alpha \mathcal{F}(u)(\xi) \in L^2(\mathbb{R}^d)$ and choosing $\alpha = (0,\dots,k,\dots,0)$ (k is on the i-th position) we get

$$\int_{\mathbb{R}^d} |\xi|^{2k} |\mathcal{F}(u)(\xi)|^2\,\mathrm{d}\xi \leq C \int_{\mathbb{R}^d} |\nabla^k u|^2\,\mathrm{d}x;$$

this implies

$$\left(\int_{\mathbb{R}^d} \left(1+|\xi|^k\right)^2 |\mathcal{F}(u)(\xi)|^2\,\mathrm{d}\xi \right)^{\frac{1}{2}} \leq C \|u\|_{W^{k,2}(\mathbb{R}^d)}$$

which is the second inequality in 2.

Step 2: Proof of the second implication in 1. and the first inequality in 2.
On the other hand, let $\left(1+|\xi|^k\right)|\mathcal{F}(u)| \in L^2(\mathbb{R}^d)$. Let $|\alpha| \leq k$. Evidently

$$\|(\mathrm{i}2\pi\xi)^\alpha \mathcal{F}(u)(\xi)\|_{L^2(\mathbb{R}^d)}^2 \leq C \left\| \left(1+|\xi|^k\right) \mathcal{F}(u)(\xi) \right\|_{L^2(\mathbb{R}^d)}^2.$$

Denote

$$u_\alpha(x) = \mathcal{F}^{-1}\left[(\mathrm{i}2\pi\xi)^\alpha \mathcal{F}(u)(\xi)\right](x)$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$. Then the Parseval equality implies (we consider real functions)

$$\int_{\mathbb{R}^d} D^\alpha\varphi u\,\mathrm{d}x = \int_{\mathbb{R}^d} D^\alpha\varphi \overline{u}\,\mathrm{d}x = \int_{\mathbb{R}^d} \mathcal{F}(D^\alpha\varphi)\overline{\mathcal{F}(u)}\,\mathrm{d}\xi = \int_{\mathbb{R}^d} (\mathrm{i}2\pi\xi)^\alpha \mathcal{F}(\varphi)(\xi)\overline{\mathcal{F}(u)(\xi)}\,\mathrm{d}\xi$$
$$= (-1)^{|\alpha|} \int_{\mathbb{R}^d} \mathcal{F}(\varphi)(\xi)\overline{(-\mathrm{i}2\pi\xi)^\alpha \mathcal{F}(u)(\xi)}\,\mathrm{d}\xi = (-1)^{|\alpha|} \int_{\mathbb{R}^d} \varphi \overline{u}_\alpha\,\mathrm{d}x,$$

i.e., $u_\alpha = D^\alpha u$ (in the weak sense). Moreover $D^\alpha u \in L^2(\mathbb{R}^d)$ and thus $u \in W^{k,2}(\mathbb{R}^d)$. Clearly

$$\|D^\alpha u\|_{L^2(\mathbb{R}^d)} = \|u_\alpha\|_{L^2(\mathbb{R}^d)} = \|(\mathrm{i}2\pi\xi)^\alpha \mathcal{F}(u)(\xi)\|_{L^2(\mathbb{R}^d)} \leq c \left\| \left(1+|\xi|^k\right) \mathcal{F}(u)(\xi) \right\|_{L^2(\mathbb{R}^d)}.$$

■

Using the Fourier transform we may also define Sobolev spaces with fractional derivative (again, only in the Hilbert case).

Definition 6.8.10 — Definition of $H^s(\mathbb{R}^d)$. Let $s \in (0,\infty)$ and $u \in L^2(\mathbb{R}^d)$. We say that the function $u \in H^s(\mathbb{R}^d)$ if

$$(1+|\xi|^s)\,\mathcal{F}(u)(\xi) \in L^2(\mathbb{R}^d).$$

For s real positive we define

$$\|u\|_{H^s(\mathbb{R}^d)} = \|(1+|\xi|^s)\,\mathcal{F}(u)(\xi)\|_{L^2(\mathbb{R}^d)}.$$

It is possible to show that $H^s(\mathbb{R}^d) = W^{s,2}(\mathbb{R}^d)$ for $0 < s < 1$, where $W^{s,2}(\mathbb{R}^d)$ denotes the Sobolev–Slobodetskii space defined in 6.8.1. The norms $\|\cdot\|_{H^s(\mathbb{R}^d)}$ and $\|\cdot\|_{W^{s,2}(\mathbb{R}^d)}$ are equivalent, see (Di Nezza et al., 2012, Proposition 3.4).

6.9 Dual spaces

Let $k \in \mathbb{N}$ and $p \in (1,\infty)$. Denote

$$\left(W_0^{k,p'}(\Omega)\right)^* = W^{-k,p}(\Omega),$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

The restriction $F \in W^{-k,p}(\Omega)$, $k \in \mathbb{N}$ onto $\mathcal{C}_0^\infty(\Omega)$ clearly defines a distribution. In what follows, we shall characterize these distributions more precisely.

Theorem 6.9.1 — Equivalent characterization of dual spaces to $W_0^{k,p'}(\Omega)$ I. Let $p \in (1,\infty)$, $k \in \mathbb{N}$. Then $F \in W^{-k,p}(\Omega)$, if and only if there exist functions $\{f_\alpha\}_{|\alpha|\le k} \subset L^p(\Omega)$ such that

$$F = \sum_{|\alpha|\le k} (-1)^\alpha D^\alpha f_\alpha,$$

where $D^\alpha f_\alpha$ denotes the distributional derivatives, i.e., it holds for $u \in W_0^{k,p'}(\Omega)$

$$\langle F, u\rangle_{W_0^{k,p'}(\Omega)} = \sum_{|\alpha|\le k} \int_\Omega f_\alpha D^\alpha u \,\mathrm{d}x. \tag{6.66}$$

Moreover,

$$\|F\|_{W^{-k,p}(\Omega)} = \inf \left(\sum_{|\alpha|\le k} \|f_\alpha\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}},$$

where the infimum is taken over all sets of functions $\{f_\alpha\}_{|\alpha|\le k}$ which fulfil (6.66).

Proof. For the general case, see, e.g., (Kufner et al., 1977, Theorem 5.9.2), the special case $k = 1$ and $p = 2$ can be found, e.g., in (Evans, 1998, Section 5.9 Theorem 1). ■

Another possible characterization is as follows.

Theorem 6.9.2 — Equivalent characterization of dual spaces to $W_0^{k,p'}(\Omega)$ II. Let $p \in (1,\infty)$, $k \in \mathbb{N}$ and $\Omega \in \mathcal{C}^{k,0}(\Omega)$. Then for any $g \in W_0^{k,p'}(\Omega)$ the formula

$$\langle \phi_g, f\rangle_{W_0^{k,p}(\Omega)} = \sum_{|\alpha|\le k} \int_\Omega D^\alpha g D^\alpha f \,\mathrm{d}x, \;\; f \in W_0^{k,p}(\Omega)$$

defines a continuous linear functional on $W_0^{k,p}(\Omega)$.

On the contrary, for any $\phi \in \left(W_0^{k,p}(\Omega)\right)^*$ there exists exactly one $g \in W_0^{k,p'}(\Omega)$ such that

$$\forall f \in W_0^{k,p}(\Omega): \;\langle \phi, f\rangle_{W_0^{k,p}(\Omega)} = \sum_{|\alpha|\le k} \int_\Omega D^\alpha g D^\alpha f \,\mathrm{d}x.$$

Furthermore, there exists a positive constant $K = K(d,k,p,\Omega)$ such that

$$K \|g\|_{W^{k,p'}(\Omega)} \le \|\phi\|_{\left(W_0^{k,p}(\Omega)\right)^*} \le \|g\|_{W^{k,p'}(\Omega)}.$$

Proof. The proof can be found in Simader (1972). ■

6.10 Equivalent definition of Sobolev spaces. Beppo Levi spaces

Next spaces which are in fact identical with the Sobolev spaces are the Beppo Levi spaces. Their introduction is slightly more complex, but it is then easier to prove their properties and thus also properties of spaces defined by the standard Definition 6.1.6.

Let us denote $P^{\mathbf{a},\mathbf{b}} := \{x = t\mathbf{a} + (1-t)\mathbf{b} \,|\, t \in \mathbb{R},\, \mathbf{a},\mathbf{b} \in \mathbb{R}^d\}$. Let $\Omega \subset \mathbb{R}^d$ be a domain. Then there exists a sequence of open intervals J_i (finite or infinite) such that

1. $\forall i \neq j : J_i \bigcap J_j = \emptyset$
2. $\Omega \bigcap P^{\mathbf{a},\mathbf{b}} = \bigcup_j \{x = t\mathbf{a} + (1-t)\mathbf{b} |\; t \in J_j\}$.

Let u be a function defined almost everywhere in Ω. We set for $t \in \cup_j J_j$

$$\varphi(t) = u(t\mathbf{a} + (1-t)\mathbf{b}).$$

Definition 6.10.1 — Set $AC(\Omega)$. We say that a function u is absolutely continuous on the line $P^{\mathbf{a},\mathbf{b}}$, if it is absolutely continuous on all compact subintervals J_j.

Let $i \in \mathbb{N},\; i = 1,\dots,d$. We denote $AC_i(\Omega)$ the set of all functions defined on Ω which satisfy the following.

If M is a set of points $(x_1,\dots,x_{i-1},x_{i+1},\dots,x_d)\subset\mathbb{R}^{d-1}$ such that for all parallel lines with the axes x_i, i.e., for

$$P_{(x_1,\dots,x_{i-1},x_{i+1},\dots,x_d)} := \{(x_1,\dots,x_{i-1},\xi,x_{i+1},\dots,x_d)\mid \xi\in\mathbb{R}\}$$

it holds

$$\Omega\bigcap P_{(x_1,\dots,x_{i-1},x_{i+1},\dots,x_d)}\neq\emptyset$$

and u is not absolutely continuous on this line $P_{(x_1,\dots,x_{i-1},x_{i+1},\dots,x_d)}$, then $|M|_{d-1}=0$.

We denote $AC(\Omega)=\bigcap\limits_{i=1}^{d} AC_i(\Omega)$.

Definition 6.10.2 — Beppo Levi space. Let $p\in[1,\infty]$ and $\Omega\subset\mathbb{R}^d$ be a domain. Then $BL^p(\Omega)$ – the Beppo Levi space – is the set of all functions $u\in L^p(\Omega)$ for which there exists $\tilde u\in AC(\Omega)$ such that it holds

1. $\tilde u = u$ almost everywhere in Ω
2. $\forall i=1,\dots,d:\ \left[\frac{\partial\tilde u}{\partial x_i}\right]\in L^p(\Omega)$, where $\left[\frac{\partial\tilde u}{\partial x_i}\right]$ denotes the classical[a] partial derivative of the function $\tilde u$.

[a] Due to the absolute continuity of $\tilde u$ on almost every parallel lines to the axis x_i, $i=1,2,\dots,d$ the derivative $\left[\frac{\partial\tilde u}{\partial x_i}\right]$ exists almost everywhere on almost every parallel line to the axes x_i, $i=1,2,\dots,d$.

In other words, $u\in BL^p(\Omega)$, if after a change of the function u on a set of measure zero we get a function $\tilde u$ which is absolutely continuous on almost all parallel lines to all axes and additionally, u and its all classical derivatives $\left[\frac{\partial\tilde u}{\partial x_i}\right]$ belong to $L^p(\Omega)$.

Exercise 6.10.3. Show that

$$\|u\|_{BL^p(\Omega)} = \|u\|_{L^p(\Omega)} + \left(\sum_{i=1}^{d}\left\|\left[\frac{\partial\tilde u}{\partial x_i}\right]\right\|_{L^p(\Omega)}^{p}\right)^{\frac{1}{p}}$$

defines a norm on the vector space $BL^p(\Omega)$.

Theorem 6.10.4 — Equivalence of Beppo Levi and Sobolev spaces. Let $p\in[1,\infty]$. Then it holds $BL^p(\Omega)=W^{1,p}(\Omega)$ (i.e., the Beppo Levi spaces are isometrically isomorphic to the corresponding Sobolev spaces).

The proof follows from the following two lemmata.

Lemma 6.10.5 Let $u\in L^1_{\text{loc}}(\Omega)\bigcap AC_i(\Omega)$. If $\left[\frac{\partial u}{\partial x_i}\right]\in L^1_{\text{loc}}(\Omega)$, then $\left[\frac{\partial u}{\partial x_i}\right]$ coincides with the weak derivative, i.e.,

$$\left[\frac{\partial u}{\partial x_i}\right] = D_{x_i}u$$

almost everywhere in Ω.

Proof. Take arbitrary $\varphi\in\mathcal{C}_0^\infty(\Omega)$ and extend u and φ by zero outside of Ω. Then

$$\begin{aligned}\int_\Omega u\frac{\partial\varphi}{\partial x_i}\,\mathrm{d}x &= \int_{\mathbb{R}^{d-1}}\left(\int_{P_{(x_1,\dots,x_{i-1},x_{i+1},\dots,x_d)}} u\frac{\partial\varphi}{\partial x_i}\,\mathrm{d}x_i\right)\mathrm{d}x_1\dots\mathrm{d}x_{i-1}\,\mathrm{d}x_{i+1}\dots\mathrm{d}x_d\\ &= -\int_{\mathbb{R}^{d-1}}\left(\int_{P_{(x_1,\dots,x_{i-1},x_{i+1},\dots,x_d)}}\left[\frac{\partial u}{\partial x_i}\right]\varphi\right)\mathrm{d}x_1\dots\mathrm{d}x_{i-1}\,\mathrm{d}x_{i+1}\dots\mathrm{d}x_d\\ &= -\int_\Omega\left[\frac{\partial u}{\partial x_i}\right]\varphi\,\mathrm{d}x,\end{aligned}$$

where we used twice the Fubini theorem and properties of absolutely continuous functions. ■

Lemma 6.10.6 Let u, $D_{x_i}u\in L^1_{\text{loc}}(\Omega)$. Then there exists $\tilde u\in AC_i(\Omega)$ such that $u=\tilde u$ almost everywhere in Ω and, moreover,

$$\left[\frac{\partial\tilde u}{\partial x_i}\right] = D_{x_i}u$$

almost everywhere in Ω.

Proof. Let $\{K_n\}_{n=1}^\infty$ be a sequence of compact sets such that $K_n \subset K_{n+1}$ a $\bigcup_{n=1}^\infty K_n = \Omega$. Let $\varphi_n \in \mathcal{C}_0^\infty(\Omega)$ be such that $\varphi_n = 1$ in K_n and $\varphi_n = 0$ outside of K_{n+1}. We set

$$\overline{u}_n = u\varphi_n,\ \overline{w}_n = D_{x_i}\overline{u}_n.$$

Clearly, $\overline{w}_n = D_{x_i}u\varphi_n + u\frac{\partial\varphi_n}{\partial x_i}$ and $\overline{u}_n, \overline{w}_n \in L^1(\Omega)$. Further $\overline{u}_n = u$ and $\overline{w}_n = D_{x_i}u$ in K_n. We define

$$\overline{u}_n^\star(x) = \overline{u}_n^\star(x_1,\dots,x_{i-1},x_i,x_{i+1},\dots,x_d) := \int_{-\infty}^{x_i} \overline{w}_n(x_1,\dots,x_{i-1},y,x_{i+1},\dots,x_d)\,\mathrm{d}y.$$

The function $\overline{u}_n^\star$ is defined for such $(x_1,\dots,x_{i-1},x_{i+1},\dots,x_d) \in \mathbb{R}^{d-1}$ that

$$\int_{-\infty}^{+\infty} \overline{w}_n(x_1,\dots,x_{i-1},y,x_{i+1},\dots,x_d)\,\mathrm{d}y < \infty$$

which is the case for almost every $(x_1,\dots,x_{i-1},x_{i+1},\dots,x_d) \in \mathbb{R}^{d-1}$ (in the sense of the $(d-1)$-dimensional Lebesgue measure). Evidently, $\overline{u}_n^\star \in AC_i(\Omega)$.

If we succeed to show that

$$\overline{u}_n^\star = \overline{u}_n \text{ almost everywhere in } \Omega, \tag{6.67}$$

we may set for $x \in K_n$

$$\tilde{u}(x) = \overline{u}_n^\star(x),\ n = 1,2,\dots.$$

Then $\tilde{u}(x) \in AC_i(\Omega)$ and it follows from previous Lemma 6.10.5 that

$$\left[\frac{\partial\tilde{u}}{\partial x_i}\right] = D_{x_i}u \qquad \text{almost everywhere in } \Omega$$

which we wanted to show.

Let us return to (6.67). Since $\overline{u}_n$ has compact support in Ω, there exists a sequence $\{u_n^k\}_{k=1}^\infty \subset \mathcal{C}_0^\infty(\Omega)$ such that[15]

$$\lim_{k\to\infty} \left\|u_n^k - \overline{u}_n\right\|_{L^p(\Omega)} = 0$$
$$\lim_{k\to\infty} \left\|\frac{\partial u_n^k}{\partial x_i} - \overline{w}_n\right\|_{L^p(\Omega)} = 0.$$

We therefore have

$$\begin{aligned}\lim_{k\to\infty} \|\overline{u}_n - \overline{u}_n^\star\|_{L^1(\Omega)} &\le \lim_{k\to\infty} \left\|\overline{u}_n - u_n^k\right\|_{L^1(\Omega)} + \lim_{k\to\infty} \left\|u_n^k - \overline{u}_n^\star\right\|_{L^1(\Omega)} \\ &= \lim_{k\to\infty} \left\|u_n^k - \overline{u}_n^\star\right\|_{L^1(\Omega)}\end{aligned}$$

and

$$\begin{aligned}\left\|u_n^k - \overline{u}_n^\star\right\|_{L^1(\Omega)} &= \int_\Omega |u_n^k - \overline{u}_n^\star|\,\mathrm{d}x = \int_\Omega \left|u_n^k - \int_{-\infty}^{x_i} \overline{w}_n(x_1,\dots,x_{i-1},y,x_{i+1},\dots,x_d)\,\mathrm{d}y\right|\,\mathrm{d}x \\ &= \int_\Omega \left|\int_{-\infty}^{x_i} \left(\frac{\partial u_n^k}{\partial x_i} - \overline{w}_n\right)\,\mathrm{d}y\right|\,\mathrm{d}x \\ &\le \int_\Omega \int_{K_{n+2}\bigcap P_{(x_1,\dots,x_{i-1},x_{i+1},\dots,x_d)}} \left|\frac{\partial u_n^k}{\partial x_i} - \overline{w}_n\right|\,\mathrm{d}y\,\mathrm{d}x \\ &\le 2\,\mathrm{diam}(K_{n+2}) \left\|\frac{\partial u_n^k}{\partial x_i} - \overline{w}_n\right\|_{L^1(\Omega)} \to 0,\end{aligned}$$

since $\operatorname{supp} w_n \subset K_{n+1}$ and for a suitable choice of u_n^k we may achieve that $\operatorname{supp} u_n^k \subset K_{n+2}$. Thus $\overline{u}_n^\star = \overline{u}_n$ almost everywhere in Ω and the proof is complete. ■

Due to Theorem 6.10.4 we easily prove the following properties of the spaces $W^{1,p}(\Omega)$.

Corollary 6.10.7 (Several properties of Sobolev spaces). It holds.

1. Let $\Omega = I = (a,b)$, $a,b \in \mathbb{R}$ and $u \in W^{1,p}(\Omega)$, $p \in [1,\infty]$. Then there exists a representative $u^\star = u$ almost everywhere in (a,b) such that $u^\star \in \mathcal{C}([a,b])$.
2. Let $u \in W^{1,p}(\Omega)$, $p \in [1,\infty]$ and let $\nabla u = 0$ almost everywhere in Ω. Then $u = \mathit{const}$ almost everywhere in Ω.

[15] Here we in fact use Theorem 6.2.1 which was shown independently of the results from this section. It is not difficult to see that we may take $u_n^k = \eta_{\frac{1}{k}} \star u_n$, where $\eta_{\frac{1}{k}}$ is a mollifier, see Definition A.3.28.

3. Denote $u^+ = \max(0, u)$ and $u^- = \max(0, -u)$. If $u \in W^{1,p}(\Omega)$, $p \in [1, \infty]$ then also u^+, u^- and $|u| \in W^{1,p}(\Omega)$.

Proof. Claim 1. follows directly from the definition of $BL^p(\Omega)$.
Claim 2. follows from the fact that if $\tilde{u} \in AC(\Omega)$ and $\left[\frac{\partial \tilde{u}}{\partial x_i}\right] = 0$ for $i = 1, \ldots, d$, then necessarily $\tilde{u} = const$.
Claim 3. is a consequence of the following. If u is an absolutely continuous function, then also u^+, u^- and $|u|$ are absolutely continuous. We may use the equality of the spaces $BL^p(\Omega)$ and $W^{1,p}(\Omega)$. Moreover, clearly

$$D_{x_i} u^+ = \begin{cases} D_{x_i} u & \text{almost everywhere in } u > 0 \\ 0 & \text{almost everywhere in } u \leq 0 \end{cases}$$

$$D_{x_i} u^- = \begin{cases} -D_{x_i} u & \text{almost everywhere in } u < 0 \\ 0 & \text{almost everywhere in } u \geq 0 \end{cases}$$

$$D_{x_i} |u| = \begin{cases} D_{x_i} u & \text{almost everywhere in } u > 0 \\ -D_{x_i} u & \text{almost everywhere in } u < 0 \\ 0 & \text{almost everywhere in } u = 0. \end{cases}$$

■

Chapter 7

Nonlinear elliptic equations and introduction to calculus of variations

In this chapter we shall present several techniques which can be used to solve nonlinear problems. First, we show a slight generalization of the Lax–Milgram Lemma which will allow us to treat a class of nonlinear problems, however, with only sublinear growth. Next, we extend the class of operators under the assumption of monotonicity of the operator. Further, we consider problems in which we directly apply the compactness of the embeddings and trace theorems. Another techniques will be connected with different kinds of fixed-point theorems. Finally, we introduce the basics techniques connected with calculus of variations and connections of existence of minimizers and solvability of the corresponding Euler–Lagrange equations, in particular for functionals convex in the gradient of the unknown function.

7.1 Nonlinear version of the Lax–Milgram theorem

Let us consider a more general second order differential equation with the right-hand side $f\colon \Omega \to \mathbb{R}$

$$\begin{aligned} -\sum_{i=1}^{d} \frac{\partial}{\partial x_i}(a_i(x, u(x), \nabla u(x))) + a_0(x, u(x), \nabla u(x)) &= f(x) \quad \text{in } \Omega \\ u &= u_0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{7.1}$$

Such an equation describes, e.g., the thermal equilibrium, when the velocity of the continuum is zero. We have

$$-\operatorname{div} \boldsymbol{q} + r = 0, \tag{7.2}$$

where $\boldsymbol{q} = (q_1, \dots, q_d)$ represents the heat flux and r is the source of the heat. We assume a general dependence of q_i and r on the position $x \in \Omega \subset \mathbb{R}^d$, the temperature u and its gradient, i.e.,

$$\begin{aligned} q_i(x) &= a_i(x, u(x), \nabla u(x)), \qquad i = 1, 2, \dots, d, \\ r(x) &= a_0(x, u(x), \nabla u(x)). \end{aligned}$$

Plugging these constitutive relations into (7.2) we get (7.1) with the right-hand side $f = 0$. Let us assume:

for $i = 0, 1, \dots, d$ the functions $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$
are Carathéodory functions, i.e., (A1)

- for all $z \in \mathbb{R}$ and $\mathbf{p} \in \mathbb{R}^d$ the functions $a_i(\cdot, z, \mathbf{p})$ are measurable in Ω,
- for almost every $x \in \Omega$ the functions $a_i(x, \cdot, \cdot)$ are continuous in $\mathbb{R} \times \mathbb{R}^d$

and

there exist $C_i > 0, i = 0, 1, \dots, d,$ such that for almost every $x \in \Omega$
and for all $z \in \mathbb{R}, \mathbf{p} \in \mathbb{R}^d$ it holds $|a_i(x, z, \mathbf{p})| \le C_i(1 + |z| + |\mathbf{p}|)$. (A2)

While the assumption (A1) is quite general, assumption (A2) significantly limited the considered class of nonlinearities: they all have sublinear growth in u as well as in ∇u. Assumption (A2) thus includes nonlinear functions, they have, however, at most the same growth as operators studied in the chapter devoted to linear elliptic problems. The reader can easily construct nonlinearities with polynomial, logarithmic or exponential growths.

Definition 7.1.1 — Weak solution to problem (7.1). We say that $u \in W^{1,2}(\Omega)$ is a weak solution to problem (7.1), if

- $u - U_0 \in W_0^{1,2}(\Omega)$
- $$\int_\Omega \left(\sum_{i=1}^d a_i(\cdot, u, \nabla u) \frac{\partial \varphi}{\partial x_i} + a_0(\cdot, u, \nabla u)\varphi \right) \mathrm{d}x = \langle f, \varphi \rangle_{W_0^{1,2}(\Omega)} \quad \forall \varphi \in W_0^{1,2}(\Omega), \tag{7.3}$$

where $U_0 \in W^{1,2}(\Omega)$ is such that $TU_0 = u_0$ on $\partial\Omega$. We assume that (A1) and (A2) hold and $f \in W^{-1,2}(\Omega) = (W_0^{1,2}(\Omega))^*$.

Exercise 7.1.2. Show that under the assumptions (A1) and (A2) we have

$$\int_\Omega \left[\sum_{i=1}^d a_i(\cdot, u, \nabla u) \frac{\partial \varphi}{\partial x_i} + a_0(\cdot, u, \nabla u)\varphi \right] \mathrm{d}x < \infty \quad \text{for } u, \varphi \in W^{1,2}(\Omega).$$

It is now natural to introduce (nonlinear!) $\widetilde{T} : W_0^{1,2}(\Omega) \to (W_0^{1,2}(\Omega))^*$ by the formula (we assume for a moment that $U_0 = 0$; the general case of possibly nonzero boundary condition will be considered later)

$$\langle \widetilde{T}(u), \varphi \rangle_{W_0^{1,2}(\Omega)} := \int_\Omega \Big[\sum_{i=1}^d a_i(\cdot, u, \nabla u) \frac{\partial \varphi}{\partial x_i} + a_0(\cdot, u, \nabla u)\varphi \Big] \mathrm{d}x. \tag{7.4}$$

Exercise 7.1.2 yields that $\widetilde{T} : W_0^{1,2}(\Omega) \to \left(W_0^{1,2}(\Omega)\right)^*$ is bounded.

Before we formulate a nonlinear version of the Lax–Milgram theorem, we recall the Banach fixed point Theorem which is the main tool of the proof below.

Theorem 7.1.3 — Banach fixed point Theorem. Let $T\colon X \to X$, where X is a Banach space, such that

$$\|T(u_1) - T(u_2)\|_X \le \alpha \|u_1 - u_2\|_X,$$

where $\alpha \in [0, 1)$. Then there exists a unique fixed point u_0 of T in X.

Theorem 7.1.4 — Nonlinear version of Lax–Milgram theorem. Let X be a real Hilbert space and $T\colon X \to X$ be Lipschitz continuous, i.e.,

$$(\exists M > 0)(\forall u, v \in X) \quad \|T(u) - T(v)\|_X \le M \|u - v\|_X, \tag{7.5}$$

and strongly monotone, i.e.,

$$(\exists m > 0)(\forall u, v \in X) \quad (T(u) - T(v), u - v)_X \ge m \|u - v\|_X^2. \tag{7.6}$$

Then for any $F \in X$ there exists unique $u \in X$ such that

$$T(u) = F. \tag{7.7}$$

Proof. Let us first note that

$$m \|u - v\|_X^2 \le (T(u) - T(v), u - v)_X \le \|T(u) - T(v)\|_X \|u - v\|_X \le M \|u - v\|_X^2$$

and thus $m \le M$. We may assume without loss of generality that $m < M$, otherwise we can increase M. For $\varepsilon > 0$ and $F \in X$ arbitrary we define the operator $A\colon X \to X$ by the formula

$$A(u) := u - \varepsilon(T(u) - F). \tag{7.8}$$

Let us first show that for a suitable choice of ε the operator A is a contraction in X, then the proof follows by the Banach fixed point theorem, as $u = A(u) = u - \varepsilon(T(u) - F)$, i.e., $T(u) = F$. We have for every $u, v \in X$ that

$$\begin{aligned} \|Au - Av\|_X^2 =& (u - v - \varepsilon(T(u) - T(v)), u - v - \varepsilon(T(u) - T(v)))_X \\ =& (u - v, u - v)_X - 2\varepsilon(u - v, T(u) - T(v))_X \\ &+ \varepsilon^2 (T(u) - T(v), T(u) - T(v))_X \\ =& \|u - v\|_X^2 - 2\varepsilon(T(u) - T(v), u - v)_X + \varepsilon^2 \|T(u) - T(v)\|_X^2. \end{aligned}$$

Using the Lipschitz continuity and the strong monotonicity of T, we get

$$\|A(u) - A(v)\|_X^2 \le (1 - 2\varepsilon m + \varepsilon^2 M^2) \|u - v\|_X^2.$$

The function $g(\varepsilon) := 1 - 2\varepsilon m + \varepsilon^2 M^2$ attains its minimum for $\varepsilon = \frac{m}{M^2}$ and $g\left(\frac{m}{M^2}\right) = 1 - \frac{m^2}{M^2} < 1$. The mapping A is hence contractive and the claim of Theorem 7.1.4 is proved. ■

Let us return to our problem (7.1). Theorem 7.1.4 is not possible to apply directly on the operator $\widetilde{T}$, since it does not map $W_0^{1,2}(\Omega)$ onto itself. We may, however, use the Riesz representation Theorem and assign to $\widetilde{T}(u) \in \left(W_0^{1,2}(\Omega)\right)^*$ the function $T(u) \in W_0^{1,2}(\Omega)$ so that

$$\langle \widetilde{T}(u), \varphi \rangle_{W_0^{1,2}(\Omega)} = (T(u), \varphi)_{W_0^{1,2}(\Omega)}, \tag{7.9}$$

and

$$\|T(u)\|_{1,2} = \|\widetilde{T}(u)\|_{(W_0^{1,2}(\Omega))^*}. \tag{7.10}$$

To verify that $T\colon W_0^{1,2}(\Omega) \to W_0^{1,2}(\Omega)$ is Lipschitz continuous and strongly monotone, it is enough to show that $\widetilde{T} : W_0^{1,2}(\Omega) \to (W_0^{1,2}(\Omega))^*$ is Lipschitz continuous and strongly monotone, i.e.,

$$(\exists \widetilde{M} > 0)(\forall u, v \in W_0^{1,2}(\Omega)) \quad \left\|\widetilde{T}(u) - \widetilde{T}(v)\right\|_{(W_0^{1,2}(\Omega))^*} \leq \widetilde{M}\, \|u - v\|_{W^{1,2}(\Omega)} \tag{7.11}$$

and

$$(\exists \widetilde{m} > 0)(\forall u, v \in W_0^{1,2}(\Omega)) \quad \langle \widetilde{T}(u) - \widetilde{T}(v, u - v\rangle_{W_0^{1,2}(\Omega)} \geq \widetilde{m}\, \|u - v\|^2_{W^{1,2}(\Omega)}. \tag{7.12}$$

The following lemma contains sufficient conditions, when (7.11) and (7.12) hold true.

Lemma 7.1.5 1. If

$$\frac{\partial a_i}{\partial z} \text{ and } \frac{\partial a_i}{\partial p_j} \text{ are bounded in } \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^d \text{ for } i = 0, 1, \ldots, d,\ j = 1, 2, \ldots, d, \tag{A3}$$

then $\widetilde{T}$ is Lipschitz continuous.

2. If there exists $\alpha > 0$ such that for almost every $x \in \Omega$ and all $\vec{\xi} = (\xi_0, \ldots, \xi_d) \in \mathbb{R}^{d+1}$ it holds

$$\sum_{\substack{i=0\\j=1}}^{d} \left(\frac{\partial a_i}{\partial p_j}\xi_i\xi_j + \frac{\partial a_i}{\partial z}\xi_i\xi_0\right) \geq \alpha|\vec{\xi}|^2, \tag{A4}$$

then $\widetilde{T}$ is strongly monotone.

Proof. The proof is based on the following identity

$$\begin{aligned}
\langle \widetilde{T}(u) - \widetilde{T}(v), \varphi\rangle_{W_0^{1,2}(\Omega)} &= \int_\Omega \Big[\sum_{i=1}^{d} \big(a_i(\cdot, u, \nabla u) - a_i(\cdot, v, \nabla v)\big)\frac{\partial \varphi}{\partial x_i} + \big(a_0(\cdot, u, \nabla u) - a_0(\cdot, v, \nabla v)\big)\varphi\Big]\,\mathrm{d}x \\
&= \int_\Omega \Bigg[\sum_{i=1}^{d} \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} a_i(\cdot, v + s(u - v), \nabla v + s\nabla(u - v))\,\mathrm{d}s\, \frac{\partial \varphi}{\partial x_i} \\
&\quad + \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} a_0(\cdot, v + s(u - v), \nabla v + s\nabla(u - v))\,\mathrm{d}s\, \varphi\Bigg]\,\mathrm{d}x \\
&= \int_\Omega \Bigg[\sum_{i=1}^{d} \int_0^1 \frac{\partial a_i}{\partial z}(\cdot, v + s(u - v), \nabla v + s\nabla(u - v))(u - v)\,\mathrm{d}s\, \frac{\partial \varphi}{\partial x_i} \\
&\quad + \sum_{i,j=1}^{d} \int_0^1 \frac{\partial a_i}{\partial p_j}(\cdot, v + s(u - v), \nabla v + s\nabla(u - v))\frac{\partial (u - v)}{\partial x_j}\,\mathrm{d}s\, \frac{\partial \varphi}{\partial x_i}\Bigg]\,\mathrm{d}x \\
&\quad + \int_\Omega \varphi \int_0^1 \Big[\frac{\partial a_0}{\partial z}(\cdot, v + s(u - v), \nabla v + s\nabla(u - v))(u - v) \\
&\quad + \sum_{j=1}^{d} \frac{\partial a_0}{\partial p_j}(\cdot, v + s(u - v), \nabla v + s\nabla(u - v))\frac{\partial (u - v)}{\partial x_j}\Big]\,\mathrm{d}s\,\mathrm{d}x.
\end{aligned}$$

The rest of the proof of this lemma is rather straightforward and is thus left to the kind reader as a useful exercise. ■

Theorem 7.1.6 — Existence, uniqueness and continuous dependence on the data. Let $f \in (W_0^{1,2}(\Omega))^*$ and let the extension U_0 of the boundary value u_0 fulfil

$$U_0 \in W^{1,2}(\Omega) \text{ and the trace of } U_0 \text{ at } \partial\Omega \text{ equals to } u_0.$$

Let the coefficients $a_i\colon \Omega\times\mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$ satisfy (A1)–(A4). Then there exists unique solution $u \in W^{1,2}(\Omega)$ of problem (7.1) in the sense of Definition 7.1.1. Moreover, there exists $C > 0$ such that

$$\|u\|_{W^{1,2}(\Omega)} \le C\left[\|f\|_{(W_0^{1,2}(\Omega))^*} + \|U_0\|_{W^{1,2}(\Omega)} + 1\right]. \tag{7.13}$$

Proof. Let us first introduce the operator $\widetilde{T}\colon W^{1,2}(\Omega) \to (W_0^{1,2}(\Omega))^*$ by formula (7.4). The aim is to show existence of $u \in W^{1,2}(\Omega)$ such that

$$\begin{aligned} u - U_0 &\in W_0^{1,2}(\Omega) \\ \langle \widetilde{T}(u), \varphi\rangle_{W_0^{1,2}(\Omega)} &= \langle f, \varphi\rangle_{W_0^{1,2}(\Omega)} \quad \forall \varphi \in W_0^{1,2}(\Omega). \end{aligned} \tag{7.14}$$

Let us search u in the form $u = U_0 + w$, where $w \in W_0^{1,2}(\Omega)$. We define an operator $\widetilde{S}\colon W_0^{1,2}(\Omega) \to (W_0^{1,2}(\Omega))^*$ by the formula

$$\widetilde{S}(w) = \widetilde{T}(U_0 + w).$$

We now identify the operator $\widetilde{S}$ with the operator $S\colon W_0^{1,2}(\Omega) \to W_0^{1,2}(\Omega)$ and $f \in (W^{1,2}(\Omega))^*$ with $g \in W_0^{1,2}(\Omega)$ by virtue of the Riesz representation Theorem (see (7.9)–(7.10)) and apply Lemma 7.1.5 and Theorem 7.1.4. We get existence and uniqueness of the solution to the problem

$$S(w) = g \quad \text{in } W^{1,2}(\Omega)$$

which gives the existence part of the first claim. To get the uniqueness, assume we have two different solution for the same data, i.e. $u^1 = w^1 + U_0^1$ and $u^2 = w^2 + U_0^2$, where $U_0^1 - U_0^2 \in W_0^{1,2}(\Omega)$. Due to the monotonicity property of S which is inherited by the properties of $\widetilde{T}$ we get

$$0 = \langle \widetilde{T}(w^1 + U_0^1) - \widetilde{T}(w^2 + U_0^2), w^1 + U_0^1 - (w^1 + U_0^2)\rangle_{W_0^{1,2}(\Omega)} \ge \|w^1 + U_0^1 - (w^1 + U_0^2)\|^2_{W^{1,2}(\Omega)}$$

which yields the uniqueness part.

To show (7.13), we first use in (7.14) the function $\varphi = u - U_0$ and then add to this equality $-\langle \widetilde{T}(U_0), u - U_0\rangle_{W_0^{1,2}(\Omega)}$. We get

$$\langle \widetilde{T}(u) - \widetilde{T}(U_0), u - U_0\rangle_{W_0^{1,2}(\Omega)} = \langle f, u - U_0\rangle_{W_0^{1,2}(\Omega)} - \langle \widetilde{T}(U_0), u - U_0\rangle_{W_0^{1,2}(\Omega)}.$$

Using the strong monotonicity of $\widetilde{T}$ and the linear growth of a_i, $i = 0, \ldots, d$, we get

$$\|u - U_0\|^2_{W^{1,2}(\Omega)} \le C\left[\|f\|_{(W_0^{1,2}(\Omega))^*} + \|U_0\|_{W^{1,2}(\Omega)} + 1\right]\|u - U_0\|_{W^{1,2}(\Omega)}.$$

Since $u = u - U_0 + U_0$, estimate (7.13) is proved. ■

Example 7.1.7. Let us consider the problem

$$\begin{aligned} -\Delta u + (\operatorname{arctg} u + \pi)u &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{7.15}$$

We show that for any $f \in W^{-1,2}(\Omega)$ there exists unique weak solution to problem (7.15).

Solution. The function $a_0(x, z, \mathbf{p}) := (\operatorname{arctg} z + \pi)z$ has clearly linear growth, it is therefore sufficient to verify assumption of Lemma 7.1.5. Further we have $a_i(x, z, \mathbf{p}) := p_i$. Then the conditions from the first part of the lemma are fulfilled and it is enough to verify that $\frac{\partial a_0}{\partial z} \ge \alpha_0 > 0$. Let us compute

$$\frac{\partial a_0(x, z, \mathbf{p})}{\partial z} = \operatorname{arctg} z + \pi + \frac{z}{z^2 + 1} \ge \frac{\pi}{2} - \frac{1}{2} > 1.$$

Hence all necessary assumptions are fulfilled and problem (7.15) possesses for arbitrary right-hand side $f \in (W^{1,2}(\Omega))^*$ a unique weak solution. Easily, we could also consider the inhomogeneous Dirichlet boundary condition. □

7.2 Application of the theory of monotone operators

We consider a similar problem as in the previous section, hence we look for a function $u\colon \Omega \to \mathbb{R}$ solving the problem

$$
\begin{aligned}
-\sum_{i=1}^{d} \frac{\partial}{\partial x_i} a_i(x, u(x), \nabla u(x)) + a_0(x, u(x), \nabla u(x)) = f(x) \qquad &\text{in } \Omega,\\
u = u_0 \qquad &\text{on } \partial\Omega,
\end{aligned}
\tag{7.16}
$$

where $\Omega \subset \mathbb{R}^d$, functions $a_i\colon \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ are Carathéodory, similarly as in the previous section, thus for any $z \in \mathbb{R}$ and $\mathbf{p} \in \mathbb{R}^d$ are the functions $a_i(\cdot, z, \mathbf{p})$ measurable on Ω and for a.e. $x \in \Omega$ the functions $a_i(x, \cdot, \cdot)$ are continuous in $\mathbb{R} \times \mathbb{R}^d$, $i = 0, 1, \ldots, d$, and

$$
|a_i(x, z, \mathbf{p})| \le C_i(|z|^{r-1} + |\mathbf{p}|^{r-1}) + h_i(x), \qquad i = 0, 1, \ldots, d, \tag{7.17}
$$

where $h_i \in L^{\frac{r}{r-1}}(\Omega)$, $1 < r < \infty$.

Recall that due to Theorem A.3.42 we know that the mappings

$$
u \mapsto a_i(\cdot, u, \nabla u), \qquad i = 0, 1, \ldots, d
$$

are continuous from $W^{1,r}(\Omega)$ to $L^{\frac{r}{r-1}}(\Omega)$, $1 < r < \infty$. We further assume that the following two conditions are fulfilled. First, let the coercivity condition hold,

$$
\sum_{i=1}^{d} a_i(x, z, \mathbf{p}) p_i + a_0(x, z, \mathbf{p}) z \ge C_1 |\mathbf{p}|^r + C_2(x)|z|^r - C_3(x), \tag{7.18}
$$

where $C_1 > 0$, $C_2(x) \ge 0$ almost everywhere in Ω and $C_3 \in L^1(\Omega)$. Next, let the monotonicity condition hold,

$$
\sum_{i=1}^{d} \Big(a_i(x, z^1, \mathbf{p}^1) - a_i(x, z^2, \mathbf{p}^2)\Big)(p_i^1 - p_i^2) + \Big(a_0(x, z^1, \mathbf{p}^1) - a_0(x, z^2, \mathbf{p}^2)\Big)(z^1 - z^2) \ge 0 \tag{7.19}
$$

for arbitrary z^1, $z^2 \in \mathbb{R}$ and arbitrary $\mathbf{p}^1$, $\mathbf{p}^2 \in \mathbb{R}^d$; the latter will be in some situations replaced by the strict monotonicity condition

$$
\sum_{i=1}^{d} \Big(a_i(x, z^1, \mathbf{p}^1) - a_i(x, z^2, \mathbf{p}^2)\Big)(p_i^1 - p_i^2) + \Big(a_0(x, z^1, \mathbf{p}^1) - a_0(x, z^2, \mathbf{p}^2)\Big)(z^1 - z^2) > 0 \tag{7.20}
$$

for all z^1, $z^2 \in \mathbb{R}$ and all $\mathbf{p}^1$, $\mathbf{p}^2 \in \mathbb{R}^d$ such that $(z^1, \mathbf{p}^1) \neq (z^2, \mathbf{p}^2)$.

The definition of a weak weak solution is similar to the case of the nonlinear version of the Lax–Milgram lemma

Definition 7.2.1 — Weak solution for problems with a monotone operator. A function $u \in W^{1,r}(\Omega)$ is called a weak solution to (7.16), if $u - U_0 \in W_0^{1,r}(\Omega)$, $U_0 \in W^{1,r}(\Omega)$, the trace of U_0 is u_0 on $\partial\Omega$, and the following identity holds

$$
\int_\Omega \Big(\sum_{i=1}^{d} a_i(\cdot, u, \nabla u) \frac{\partial \varphi}{\partial x_i} + a_0(\cdot, u, \nabla u)\varphi\Big)\, \mathrm{d}x = \langle f, \varphi \rangle_{W_0^{1,r}(\Omega)} \tag{7.21}
$$

for any $\varphi \in W_0^{1,r}(\Omega)$.

Note that all integrals in the weak formulation of (7.21) are due to our assumptions finite. We look for the solution in the form $u = v + U_0$. We define the operator $T\colon W_0^{1,r}(\Omega) \to (W_0^{1,r}(\Omega))^* =: W^{-1,r'}(\Omega)$ as follows:

$$
\langle T(v), \varphi \rangle_{W_0^{1,r}(\Omega)} := \int_\Omega \Big(\sum_{i=1}^{d} a_i(\cdot, v + U_0, \nabla v + \nabla U_0) \frac{\partial \varphi}{\partial x_i} + a_0(\cdot, v + U_0, \nabla v + \nabla U_0)\varphi\Big)\, \mathrm{d}x. \tag{7.22}
$$

Then the weak formulation can be rewritten as operator equation

$$
T(v) = f \qquad \text{in } W^{-1,r'}(\Omega). \tag{7.23}
$$

Assumptions formulated on a_i ensure the following properties of T.

Lemma 7.2.2 Let (7.17)–(7.19) or (7.20) hold. Then

1. the operator T is continuous from $W_0^{1,r}(\Omega)$ to $W^{-1,r'}(\Omega)$
2. the operator T is coercive, i.e.,

$$
\lim_{\|v\|_{W^{1,r}(\Omega)} \to \infty} \frac{\langle T(v), v \rangle_{W_0^{1,r}(\Omega)}}{\|v\|_{W^{1,r}(\Omega)}} = \infty
$$

3. the operator T is monotone, i.e., for $v^1, v^2 \in W_0^{1,r}(\Omega)$

$$\langle T(v^1) - T(v^2), v^1 - v^2 \rangle_{W_0^{1,r}(\Omega)} \geq 0$$

or strictly monotone, i.e., for $v^1, v^2 \in W_0^{1,r}(\Omega)$, $v^1 \neq v^2$

$$\langle T(v^1) - T(v^2), v^1 - v^2 \rangle_{W_0^{1,r}(\Omega)} > 0.$$

Proof. **Step 1:** Continuity of T
The continuity follows from the Theorem on Nemytskii operator (Theorem A.3.42), due to which the mappings $u \mapsto a_i(\cdot, u, \nabla u)$ are continuous from $W^{1,r}(\Omega)$ to $L^{\frac{r}{r-1}}(\Omega)$.
Step 2: Coercivity of T
Let us compute

$$\begin{aligned}
\langle T(v), v \rangle_{W_0^{1,r}(\Omega)} = & \int_\Omega \Big(\sum_{i=1}^d a_i(\cdot, v + U_0, \nabla v + \nabla U_0) \frac{\partial (v + U_0)}{\partial x_i} \\
& + a_0(\cdot, v + U_0, \nabla v + \nabla U_0)(v + U_0) \Big) \, \mathrm{d}x \\
& - \int_\Omega \Big(\sum_{i=1}^d a_i(\cdot, v + U_0, \nabla v + \nabla U_0) \frac{\partial U_0}{\partial x_i} \\
& + a_0(\cdot, v + U_0, \nabla v + \nabla U_0) U_0 \Big) \, \mathrm{d}x \\
& \geq C_1 \|\nabla (v + U_0)\|_{L^r(\Omega)}^r - C_2 \|v + U_0\|_{W^{1,r}(\Omega)}^{r-1} \|U_0\|_{W^{1,r}(\Omega)} - C_3 \\
& \geq C \|\nabla (v + U_0)\|_{L^r(\Omega)}^r - C_4,
\end{aligned}$$

which yields Claim 2.
Step 3: Monotonicity
Finally, due to the fact that $v^1 - v^2 = (v^1 + U_0) - (v^2 + U_0)$, the monotonicity (the strict monotonicity) follows directly from property (7.18). ■

We will also need one corollary of the Brouwer fixed point Theorem proved later:

Theorem 7.2.3 — Brouwer fixed point Theorem. Let $\vec{u}$: $\overline{B_R(0)} \to \overline{B_R(0)} \subset \mathbb{R}^N$ be a continuous mapping, $N \geq 1$. Then there exists at least one fixed point of the mapping $\vec{u}$ in $\overline{B_R(0)}$, i.e., there exists $\vec{x} \in \overline{B_R(0)}$ such that $\vec{u}(\vec{x}) = \vec{x}$.

The proof will be given later, in Section 4.5.

We in fact need its corollary

Corollary 7.2.4 (Variant of the Brouwer theorem). Let $\vec{g}$: $\mathbb{R}^N \to \mathbb{R}^N$ be continuous. Assume there exists $R > 0$ such that

$$(\vec{g}(\vec{v}), \vec{v})_{\mathbb{R}^N} \geq 0$$

for all $\vec{v} \in \mathbb{R}^N$, $|\vec{v}| = R$. Then there exists at least one $\vec{z} \in \overline{B_R(0)}$ such that $\vec{g}(\vec{z}) = \vec{0}$.

Proof. Assume that such $\vec{z}$ does not exist, hence

$$\vec{g}(\vec{x}) \neq \vec{0} \qquad \forall \vec{x} \in \overline{B_R(0)}.$$

Due to the continuity, $\min_{\vec{x} \in \overline{B_R(0)}} |\vec{g}(\vec{x})| > 0$. Denote

$$\vec{F}(\vec{x}) = -R \frac{\vec{g}(\vec{x})}{|\vec{g}(\vec{x})|}.$$

Then $\vec{F}$ is a continuous mapping which maps $\overline{B_R(0)}$ into itself. Due to the Brouwer fixed point theorem (Theorem 7.2.3) there exists $\vec{x}_1 \in \overline{B_R(0)}$, a fixed point of the mapping $\vec{F}$. Then

$$\vec{x}_1 = -R \frac{\vec{g}(\vec{x}_1)}{|\vec{g}(\vec{x}_1)|}.$$

Hence $|\vec{x}_1| = R$ and

$$0 \leq (\vec{g}(\vec{x}_1), \vec{x}_1)_{\mathbb{R}^N} = \Big(\vec{g}(\vec{x}_1), -R \frac{\vec{g}(\vec{x}_1)}{|\vec{g}(\vec{x}_1)|} \Big)_{\mathbb{R}^N} = -R |\vec{g}(\vec{x}_1)| < 0$$

which yields a contradiction. ■

We can now formulate and prove the main result from this section.

Theorem 7.2.5 — Existence result for the problem with monotone operator. Assume that the Carathéodory functions $\{a_i\}_{i=0}^d$ fulfil (7.17)–(7.19), where $1 < r < \infty$. Let $f \in W^{-1,r'}(\Omega)$, $U_0 \in W^{1,r}(\Omega)$, where the trace $U_0 = u_0$ on $\partial\Omega$. Then there exists a weak solution to (7.16) in the sense of Definition 7.2.1. If the condition of strict monotonicity (7.20) holds, the solution is unique.

Proof. Let the operator T be as above. We look for a solution to the problem

$$T(v) = f \qquad \text{in } W^{-1,r'}(\Omega).$$

Let $\{h_i\}$, $i \in \mathbb{N}$ be a countable dense subset of $W_0^{1,r}(\Omega)$ and we denote by V_n the linear hull of the first n such functions. We define

$$A\colon (a_1, a_2, \dots, a_n) \in \mathbb{R}^n \mapsto \sum_{i=1}^n a_i h_i =: v_{\vec{a}} \in V_n.$$

We also define g: $\mathbb{R}^n \to \mathbb{R}^n$ by

$$\vec{g}(\vec{a}) = \big\{\langle T(A\vec{a}) - f, h_i\rangle_{W_0^{1,r}(\Omega)}\big\}_{i=1}^n.$$

As the operator T is continuous (a weaker notion of continuity would be enough at this moment), the mapping $\vec{g}$ is also continuous. Moreover, the mapping T is coercive, therefore

$$\begin{aligned}(\vec{g}(\vec{a}), \vec{a})_{\mathbb{R}^n} &= \langle T(v_{\vec{a}}), v_{\vec{a}}\rangle_{W_0^{1,r}(\Omega)} - \langle f, v_{\vec{a}}\rangle_{W_0^{1,r}(\Omega)} \\ &\geq \Big(\frac{\langle T(v_{\vec{a}}), v_{\vec{a}}\rangle_{W_0^{1,r}(\Omega)}}{\|v_{\vec{a}}\|_{W^{1,r}(\Omega)}} - \|f\|_{W^{-1,r'}(\Omega)}\Big)\|v_{\vec{a}}\|_{W^{1,r}(\Omega)} \geq 0,\end{aligned}$$

if $\|v_{\vec{a}}\|_{W^{1,r}(\Omega)} \geq R$ for a sufficiently large R (independent of n!). Therefore due to Corollary 7.2.4 for any $n \in \mathbb{N}$ there exists at least one $\vec{a}^{\,n} \in \mathbb{R}^n$ such that

$$\vec{g}(\vec{a}^{\,n}) = \vec{0},$$

hence

$$\langle T(A\vec{a}^{\,n}) - f, h\rangle_{W_0^{1,r}(\Omega)} = 0, \qquad h \in \operatorname{Lin}\{h_i\}_{i=1}^n$$

and moreover, for $v^n := \sum_{i=1}^n a_i^n h_i$ the estimate $\|v^n\|_{W^{1,r}(\Omega)} \leq R$ holds for R independent of n.

Therefore due to Theorem B.2.7 we know that there exists $v \in W_0^{1,r}(\Omega)$ (note in particular that the limit function has also zero trace) and a subsequence $\{v^{n_k}\}_{k\in\mathbb{N}} \subset \{v^n\}_{n\in\mathbb{N}}$ such that

$$v^{n_k} \rightharpoonup v \qquad \text{in } W^{1,r}(\Omega).$$

Moreover, as $T(v^n)$ is a bounded sequence in $W^{-1,r'}(\Omega)$ (recall that T: $W_0^{1,r}(\Omega) \to W^{-1,r'}(\Omega)$ is bounded) and $\{h_i\}_{i\in\mathbb{N}}$ is a dense subset of $W^{1,r}(\Omega)$, we also have

$$T(v^{n_k}) \overset{*}{\rightharpoonup} f$$

in $W^{-1,r'}(\Omega)$ (and due to the reflexivity the sequence converges also weakly). Furthermore,

$$\lim_{k\to\infty} \langle T(v^{n_k}), v^{n_k}\rangle_{W_0^{1,r}(\Omega)} = \lim_{k\to\infty} \langle f, v^{n_k}\rangle_{W_0^{1,r}(\Omega)} = \langle f, v\rangle_{W_0^{1,r}(\Omega)}. \tag{7.24}$$

Using the monotonicity of T, we have for arbitrary $\varphi \in W_0^{1,r}(\Omega)$

$$\langle T(v^{n_k}) - T(\varphi), v^{n_k} - \varphi\rangle_{W_0^{1,r}(\Omega)} \geq 0. \tag{7.25}$$

By virtue of (7.24) and letting $k \to \infty$ in (7.25) we get

$$\langle f - T(\varphi), v - \varphi\rangle_{W_0^{1,r}(\Omega)} \geq 0 \qquad \text{for any } \varphi \in W_0^{1,r}(\Omega). \tag{7.26}$$

We now perform Minty's trick. We set $\varphi := v - \lambda w$ in inequality (7.26), where $w \in W_0^{1,r}(\Omega)$ is arbitrary and $\lambda > 0$. This yields

$$\langle f - T(v - \lambda w), \lambda w\rangle_{W_0^{1,r}(\Omega)} \geq 0 \qquad \text{for any } w \in W_0^{1,r}(\Omega). \tag{7.27}$$

Dividing (7.27) by λ, we have

$$\langle f - T(v - \lambda w), w\rangle_{W_0^{1,r}(\Omega)} \geq 0 \qquad \text{for any } w \in W_0^{1,r}(\Omega). \tag{7.28}$$

Due to the continuity of T, letting $\lambda \to 0_+$, we have

$$\langle f - T(v), w\rangle_{W_0^{1,r}(\Omega)} \geq 0 \qquad \text{for any } w \in W_0^{1,r}(\Omega). \tag{7.29}$$

Finally, it is enough to recall that (7.29) holds for arbitrary $w \in W_0^{1,r}(\Omega)$ (and thus also for $-w$). We get equality

$$\langle f - T(v), w\rangle_{W_0^{1,r}(\Omega)} = 0 \qquad \text{for any } w \in W_0^{1,r}(\Omega) \tag{7.30}$$

and thus $u := v + U_0$ is a weak solution to our problem.

Assuming additionally the strict monotonicity condition (7.20) (hence the operator T is strictly monotone, see Lemma (7.2.2), item (iii)), we can show the uniqueness of solutions. Assume that u^1 and u^2 are two distinct solutions to our problem with the same data. As $u^1 - u^2 \in W_0^{1,r}(\Omega)$, we have

$$0 < \langle T(u^1) - T(u^2), u^1 - u^2 \rangle_{W_0^{1,r}(\Omega)} = \langle f - f, u^1 - u^2 \rangle_{W_0^{1,r}(\Omega)} = 0$$

which yields a contradiction. Therefore $u^1 = u^2$. ■

Example 7.2.6. Consider problem (7.16), where

$$\begin{aligned} a_i(x, u, \nabla u) &= |\nabla u|^{r-2} \frac{\partial u}{\partial x_i}, \qquad i = 1, 2, \dots d, \\ a_0(x, u, \nabla u) &= |u|^{r-2} u. \end{aligned}$$

The coercivity condition (7.18) and the growth condition (7.17) are indeed fulfilled. Let us verify that the strict monotonicity condition (7.20) holds. We compute

$$\begin{aligned} &\big(|\nabla u^1|^{r-2}\nabla u^1 - |\nabla u^2|^{r-2}\nabla u^2\big)\cdot(\nabla u^1 - \nabla u^2) + \big(|u^1|^{r-2}u^1 - |u^2|^{r-2}u^2\big)(u^1 - u^2) \\ &\quad = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\Big(|\nabla(u^2 + t(u^1 - u^2))|^{r-2}\nabla(u^2 + t(u^1 - u^2))\Big)\,\mathrm{d}t \cdot \nabla(u^1 - u^2) \\ &\qquad + \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\Big(|u^2 + t(u^1 - u^2)|^{r-2}(u^2 + t(u^1 - u^2))\Big)\,\mathrm{d}t(u^1 - u^2) \\ &\quad = (r-1)\int_0^1 \nabla(u^2 + t(u^1 - u^2))|^{r-2}\,\mathrm{d}t|\nabla(u^1 - u^2)|^2 \\ &\qquad\qquad + (r-1)\int_0^1 |u^2 + t(u^1 - u^2)|^{r-2}\,\mathrm{d}t|u^1 - u^2|^2 \ge 0 \end{aligned}$$

and even, for $u^1 \neq u^2$, the inequality is sharp. Therefore, the strict monotonicity condition (7.20) holds and thus for any $f \in W^{-1,r'}(\Omega)$ and any $u_0 \in W^{1-\frac{1}{r},r}(\Omega)$ there exists a unique weak solution to

$$\begin{aligned} -\sum_{i=1}^d \frac{\partial}{\partial x_i}\Big(|\nabla u|^{r-2}\frac{\partial u}{\partial x_i}\Big) + |u|^{r-2}u &= f \qquad \text{in } \Omega \\ u &= u_0 \qquad \text{on } \partial\Omega \end{aligned} \tag{7.31}$$

in the sense of Definition 7.2.1.

Note that the previous example is a particular situation of a more general case. Assume that there exists

$$U\colon \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$$

which is Carathéodory and satisfies the following. We assume the growth condition

$$U(x, z, \mathbf{p}| \ge C(|z|^r + |\mathbf{p}|^r) - h_1(x),$$

where $h_1 \in L^r(\Omega)$, we have

$$\Big|\sum_{i=1}^d \frac{\partial U(x, z, \mathbf{p})}{\partial p_i}\Big| + \Big|\frac{\partial U(x, z, \mathbf{p})}{\partial z}\Big| \le C(1 + |z|^{r-1} + |\mathbf{p}|^{r-1}) + h_2(x),$$

where $h_2 \in L^{\frac{r}{r-1}}(\Omega)$ and we have the monotonicity condition

$$\sum_{i,j=1}^d \frac{\partial^2 U(x, z, \mathbf{p})}{\partial p_i \partial p_j}\Big)\xi_i\xi_j + \sum_{i=1}^d \frac{\partial^2 U(x, z, \mathbf{p})}{\partial p_i \partial z}\xi_i\xi_0 + \frac{\partial^2 U(x, z, \mathbf{p})}{\partial z^2}\xi_0^2 \ge C(x, z, \mathbf{p})|\vec{\xi}|^2$$

for any $\vec{\xi} \in \mathbb{R}^{d+1}$, $C(x, z, \mathbf{p}) \ge 0$ with the $r-2$th growth. We set

$$\begin{aligned} a_i(x, z, \mathbf{p}) &= \frac{\partial U(x, z, \mathbf{p})}{\partial p_i}, \qquad i = 1, 2, \dots, d \\ a_0(x, z, \mathbf{p}) &= \frac{\partial U(x, z, \mathbf{p})}{\partial z}. \end{aligned}$$

Then all assumptions of Theorem 7.2.5 are satisfied and the corresponding problem possesses at least one weak solution.

Exercise 7.2.7. Show the claim above in more details.

7.3 Compact perturbations

The aim of this section is to show that some nonlinear problems can be solved using the compact embedding theorems together with the compactness of the trace operator. We will demonstrate this on the following example. We consider

$$\begin{aligned} -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big) + f(u) &= F \qquad && \text{in } \Omega \\ \frac{\partial u}{\partial n_{\mathbb{A}}} + g(u) &= G && \text{on } \Gamma_1 \\ u &= 0 && \text{on } \Gamma_2, \end{aligned} \tag{7.32}$$

where $\Omega \subset \mathbb{R}^d$, $\Omega \in C^{0,1}$, $\frac{\partial u}{\partial n_{\mathbb{A}}} = \sum_{i,j=1}^d a_{ij}\frac{\partial u}{\partial x_i}\nu_i$ is the derivative with respect to the conormal, $\boldsymbol{\nu}$ is the unit outer normal vector to $\partial\Omega$, and $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup M$, where $|M|_{d-1} = 0$ and Γ_1, Γ_2 are open subsets of $\partial\Omega$.

We will assume the following

$$\begin{gathered} a_{ij} \in L^\infty(\Omega),\ i,j = 1,2,\dots,d \\ \sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq C_1|\xi|^2, \qquad \text{almost everywhere in } \Omega,\ C_1 > 0, \boldsymbol{\xi} \in \mathbb{R}^d, \text{ arbitrary} \\ \int_\Omega f(v)v\,\mathrm{d}x + \int_{\Gamma_1} g(v)v\,\mathrm{d}S \geq 0, \qquad \forall v \in V = \{u \in W^{1,2}(\Omega)\,|\,Tu = 0 \text{ on } \Gamma_2\} \\ f:\mathbb{R}\to\mathbb{R} \text{ is continuous}, \qquad |f(v)| \leq C(1+|v|^p), 0 \leq p \leq \frac{d+2}{d-2}\ (p<\infty \text{ if } d=2) \\ g:\mathbb{R}\to\mathbb{R} \text{ is continuous}, \qquad |g(v)| \leq C(1+|v|^q), 0 \leq q \leq \frac{d}{d-2}\ (q<\infty \text{ if } d=2) \\ F \in V^*,\ G \in W^{-\frac12,2}(\Gamma_1) = \big(W^{\frac12,2}(\Gamma_1)\big)^*. \end{gathered} \tag{7.33}$$

We aim at proving the following result.

Theorem 7.3.1 Under the assumptions (7.33) there exists a weak solution to (7.32), i.e., there exists a function $u \in V$ such that

$$\int_\Omega \Big(\sum_{i,j=1}^d a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial \varphi}{\partial x_i} + f(u)\varphi\Big)\,\mathrm{d}x + \int_{\Gamma_1} g(u)\varphi\,\mathrm{d}S = \langle F,\varphi\rangle_V + \langle G,\varphi\rangle_{W^{\frac12,2}(\Gamma_1)}$$

for all $\varphi \in V$.

Proof. Before starting, note that all terms in the weak formulation in Theorem 7.3.1 are finite. We proceed very similarly as in the proof of the main result from the previous section, i.e., for the monotone operator case. We only modify the arguments in order to pass to the limit from the Galerkin approximation to the continuous one.

Step 1: Approximation

We take $\{w_i\}_{i=1}^\infty$ a dense subset of V. Note that we may take in particular such functions that $w_i \in C^\infty(\overline{\Omega})$ and $w_i = 0$ on Γ_2 for all $i \in \mathbb{N}$. Another possibility is to work with the basis formed by eigenfunctions of the Laplace equation with the homogeneous Neumann condition on Γ_1 and homogeneous Dirichlet condition on Γ_2. Let us fix $n \in \mathbb{N}$ and look for

$$u^n(x) := \sum_{i=1}^n c_i^n w_i(x)$$

such that

$$\int_\Omega \Big(\sum_{i,j=1}^d a_{ij}\frac{\partial u^n}{\partial x_j}\frac{\partial w_k}{\partial x_i} + f(u^n)w_k\Big)\,\mathrm{d}x + \int_{\Gamma_1} g(u^n)w_k\,\mathrm{d}S = \langle F,w_k\rangle_V + \langle G,w_k\rangle_{W^{\frac12,2}(\Gamma_1)}, \tag{7.34}$$

$k = 1,2,\dots,n$.

Step 2: Existence of a solution for the approximation

We use as before Corollary 7.2.4 (corollary of the Brouwer fixed point theorem). We consider the mapping $\vec{\Phi}$:

$$\begin{aligned} \{d_i\}_{i=1}^n \mapsto \Big\{ &\int_\Omega \Big(\sum_{i,j=1}^d a_{ij}\frac{\partial(\sum_{l=1}^n d_l w_l)}{\partial x_j}\frac{\partial w_k}{\partial x_i} + f\Big(\sum_{l=1}^n d_l w_l\Big)w_k\Big)\,\mathrm{d}x \\ &+ \int_{\Gamma_1} g\Big(\sum_{l=1}^n d_l w_l\Big)w_k\,\mathrm{d}S - \langle F,w_k\rangle_V - \langle G,w_k\rangle_{W^{\frac12,2}(\Gamma_1)}\Big\}_{k=1}^n. \end{aligned}$$

Indeed, the mapping $\vec{\Phi}$: $\mathbb{R}^n \to \mathbb{R}^n$ is continuous. Moreover (we multiply the k-th equation by d_k and sum up, for $k = 1$ to n)

$$\int_\Omega \Big(\sum_{i,j=1}^{d} a_{ij} \frac{\partial\big(\sum_{l=1}^n d_l w_l\big)}{\partial x_j} \frac{\partial\big(\sum_{l=1}^n d_l w_l\big)}{\partial x_i} + f\Big(\sum_{l=1}^n d_l w_l\Big) \sum_{l=1}^n d_l w_l \Big) \,\mathrm{d}x$$
$$+ \int_{\Gamma_1} g\Big(\sum_{l=1}^n d_l w_l\Big) \sum_{l=1}^n d_l w_l \,\mathrm{d}S - \Big\langle F, \sum_{l=1}^n d_l w_l \Big\rangle_V - \Big\langle G, \sum_{l=1}^n d_l w_l \Big\rangle_{W^{\frac{1}{2},2}(\Gamma_1)}$$
$$\geq C_1 \int_\Omega \Big| \nabla (\sum_{l=1}^n d_l w_l) \Big|^2 \,\mathrm{d}x - \Big\langle F, \sum_{l=1}^n d_l w_l \Big\rangle_V - \Big\langle G, \sum_{l=1}^n d_l w_l \Big\rangle_{W^{\frac{1}{2},2}(\Gamma_1)} \geq 0$$

provided $\| \sum_{l=1}^n \nabla(d_l w_l)\|_{L^2(\Omega)} \geq \widetilde{R}$, where $\widetilde{R}$ depends on C_1, F, G, Ω, but in particular, is independent of n. Due to the Poincaré–Friedrichs inequality there exists $R > 0$ such that if

$$\Big\| \sum_{l=1}^n d_l w_l \Big\|_{W^{1,2}(\Omega)} \geq R \Longrightarrow \big(\vec{\Phi}(\vec{d}), \vec{d}\big)_{\mathbb{R}^n} \geq 0.$$

Hence using the corollary of the Brouwer fixed point theorem (Corollary 7.2.4) there exists $\{c_i^n\}_{i=1}^n$ such that

$$\vec{\Phi}(\vec{c}^{\,n}) = \vec{0}.$$

Moreover, for $u^n := \sum_{i=1}^n c_i^n w_i$ we have

$$\|u^n\|_{W^{1,2}(\Omega)} \leq R,$$

where R is independent of n.

Step 3: Limit passage

We consider the limit $n \to \infty$. As $W^{1,2}(\Omega)$ is a Hilbert space, there exists $\{u^{n_k}\}_{k=1}^\infty \subset \{u_n\}_{n=1}^\infty$ such that

$$u^{n_k} \rightharpoonup u \qquad \text{in } W^{1,2}(\Omega),$$

where the limit function $u \in V$. Moreover, for another subsequence (however, we do not relabel it)

$$u^{n_k} \to u \qquad \text{in } L^{\tilde{p}}(\Omega), \quad 1 \leq \tilde{p} < \frac{2d}{d-2} \quad \text{(compact embedding)},$$
$$u^{n_k} \to u \qquad \text{in } L^{\tilde{q}}(\partial\Omega), \quad 1 \leq \tilde{q} < \frac{2d-2}{d-2} \quad \text{(compactness of the trace operator)}.$$

We now easily have for $k \to \infty$ (the weak convergence suffices here)

$$\int_\Omega \sum_{i,j=1}^d a_{ij} \frac{\partial u^{n_k}}{\partial x_j} \frac{\partial w_l}{\partial x_i} \,\mathrm{d}x \to \int_\Omega \sum_{i,j=1}^d a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial w_l}{\partial x_i} \,\mathrm{d}x$$

for any $l \in \mathbb{N}$. Next, we want to show that as $k \to \infty$,

$$\int_\Omega f(u^{n_k}) w_l \,\mathrm{d}x \to \int_\Omega f(u) w_l \,\mathrm{d}x$$

for any $l \in \mathbb{N}$. To this aim, we apply the Vitali's convergence theorem. Recall that for a suitable subsequence (we do not relabel it) we have

$$u^{n_k} \to u \quad \text{almost everywhere in } \Omega,$$

hence

$$f(u^{n_k}) \to f(u) \text{ almost everywhere in } \Omega$$

due to the continuity of f. As the functions w_l are bounded for any $l \in \mathbb{N}$ and the growth of f is subcritical, we know that due to the Hölder inequality

$$\Big| \int_E f(u^{n_k}) w_l \,\mathrm{d}x \Big|$$

are uniformly small provided the measure of E is sufficiently small. The argument for $w_l \in V$ only is slightly more involved, we need to use the approximation of functions from V by smooth functions.

More or less similarly we may show that for $k \to \infty$

$$\int_{\Gamma_1} g(u^{n_k}) w_l \,\mathrm{d}S \to \int_{\Gamma_1} g(u) w_l \,\mathrm{d}S$$

for any $l \in \mathbb{N}$. Hence we have

$$\int_\Omega \Big(\sum_{i,j=1}^d a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial w_l}{\partial x_i} + f(u) w_l \Big) \,\mathrm{d}x + \int_{\Gamma_1} g(u) w_l \,\mathrm{d}S = \langle F, w_l \rangle_V + \langle G, w_l \rangle_{W^{\frac{1}{2},2}(\Gamma_1)} \tag{7.35}$$

for any $l \in \mathbb{N}$ and therefore, for any test function φ from the linear hull of $\{w_i\}_{i=1}^\infty$.

Step 4: General test function
Using that for arbitrary $v \in V$ there exists $v^n \to v$ in $W^{1,2}(\Omega)$, where v^n belongs to the linear hull of $\{w_i\}_{i=1}^\infty$, we easily verify that the weak formulation (7.35) holds for any test function $v \in V$. Theorem 7.3.1 is proved. ■

Remark 7.3.2. (i) It is not difficult to see that if we relax condition on the test function in the weak formulation, i.e. we require that the test functions are continuously differentiable functions in $\overline{\Omega}$ which are equal to zero on Γ_2, then we may relax the conditions on f and g in (7.33). More precisely, we may assume that

$$f: \mathbb{R} \to \mathbb{R} \text{ is continuous,} \qquad |f(v)| \le C(1 + |v|^p),\ 0 \le p < \frac{2d}{d-2} \ (p < \infty \text{ if } d = 2)$$
$$g: \mathbb{R} \to \mathbb{R} \text{ is continuous,} \qquad |g(v)| \le C(1 + |v|^q), 0 \le q < \frac{2d-2}{d-2} \ (q < \infty \text{ if } d = 2).$$

(ii) Assumptions (7.33) for f and g are fulfilled e.g. if $f(v) = \widetilde{f}(v)v$, where $\widetilde{f} \ge 0$ satisfies the growth condition above for $p \le \frac{4}{d-2}$ and $g(v) = \widetilde{g}(v)v$, where $\widetilde{g} \ge 0$ satisfies the growth condition above for $q \le \frac{2}{d-2}$. As in part (i), this condition can be further relaxed if we work with smooth test functions only.
(iii) If we assume that

$$f(v)v \ge C|v|^p - C_1, \qquad C_1 \in \mathbb{R}$$
$$g(v)v \ge C|v|^q - C_2, \qquad C_2 \in \mathbb{R},$$

then we can get the existence of a solution for arbitrary p, q finite, provided we consider (assume that $p, q \ge 1$)

$$V = \{u \in W^{1,2}(\Omega) \cap L^p(\Omega) \cap L^q(\partial\Omega) \,|\, Tu = 0 \text{ on } \Gamma_2\}.$$

The details are left as an exercise.

7.4 Methods based on fixed point theorems

We list several fixed point theorems used in the analysis of partial differential equations. All Banach spaces are assumed real. While the Banach fixed point Theorem can be found in most Textbooks on Analysis (see, e.g., (Černý and Pokorný, 2021, Theorem 11.8.2)), the other theorems will be proved in the following text.

Theorem 7.4.1 — Banach fixed point Theorem. Let $T\colon X \to X$, where X is a complete (metric or normed) space, such that

$$\varrho(T(u_1), T(u_2))_X \le \alpha \varrho(u_1, u_2)_X$$

for the metric space, or

$$\|T(u_1) - T(u_2)\|_X \le \alpha \|u_1 - u_2\|_X$$

for the Banach space, where $\alpha \in [0, 1)$. Then there exists a unique fixed point u_0 of T in X.

Theorem 7.4.2 — Brouwer fixed point Theorem. Let $T\colon B \subset \mathbb{R}^N \to B$ be continuous, where B is a convex, closed and bounded set. Then there exists a (generally non-unique) fixed point of T in B.

Theorem 7.4.3 — Schauder fixed point Theorem. Let $T\colon K \subset X \to K$ be continuous, where K is compact and convex and X is a Banach space. Then there exists a fixed point of T in K.

The following version of the Schauder fixed point theorem is more suitable for direct applications in the analysis of PDEs.

Theorem 7.4.4 — Schaeffer or a version of the Schauder fixed point Theorem. Let $T\colon X \to X$ be continuous and compact, X Banach space and let the set

$$\{u \in X \,|\, u = \lambda T(u) \quad \text{for some } 0 \le \lambda \le 1\}$$

be bounded. Then there exists a fixed point of T in X.

We start with an application of Theorem 7.4.1.

7.4.1 Application of the Banach fixed point Theorem

We have already used this theorem in the proof of the nonlinear version of the Lax–Milgram theorem (see Theorem 7.1.4). Another typical application are local existence theorems for nonlinear problems (local in time existence of nonlinear problems or global in time existence for small data as well as existence of solutions to nonlinear elliptic problems for data closed to a known solution, typically a zero one). We will consider the latter.

We consider the following problem

$$\begin{aligned} -\Delta u + f(u, \nabla u) &= g && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{7.36}$$

where $\Omega \subset \mathbb{R}^3$ is a $C^{1,1}$ domain and $g \in L^2(\Omega)$ is a given function such that $\|g\|_{L^2(\Omega)} \ll 1$. Moreover, we assume that f is a Lipschitz function in $\mathbb{R} \times \mathbb{R}^3$ such that

$$\begin{aligned} |f(v, \vec{z})| &\le C(|\vec{z}|^q + 1)|v|^r \\ |f(v_1, \vec{z}) - f(v_2, \vec{z})| &\le C|v_1 - v_2|(|\vec{z}|^q + 1)(|v_1|^{r-1} + |v_2|^{r-1}) \\ |f(v, \vec{z}_1) - f(v, \vec{z}_2)| &\le C|\vec{z}_1 - \vec{z}_2||v|^r(|\vec{z}_1|^{q-1} + |\vec{z}_2|^{q-1}), \end{aligned} \tag{7.37}$$

$1 \le q \le 3$, $1 < r < \infty$, for any v, v_1 and $v_2 \in \mathbb{R}$ and $\vec{z}$, $\vec{z}_1$ and $\vec{z}_2 \in \mathbb{R}^3$.

We aim at showing the following

Theorem 7.4.5 There exists a $\delta_0 > 0$ such that if $\|g\|_{L^2(\Omega)} \le \delta_0$, then there exists a unique solution (weak and strong) of (7.36) in the ball

$$B_\delta = \{u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega) \,|\, \|u\|_{W^{2,2}(\Omega)} \le \delta\}$$

for a suitable $\delta > 0$.

Proof. Take $v \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$, $\|u\|_{W^{2,2}(\Omega)} \le \delta$, where $\delta < 1$ will be fixed later. Assume $\delta_0 > 0$ sufficiently small (will be fixed later) and consider a mapping

$$T\colon W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega) \to W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$$

such that $T(v) = u$, where

$$\begin{aligned} -\Delta u &= g - f(v, \nabla v) && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Since due to our assumptions $g - f(v, \nabla v) \in L^2(\Omega)$, there exists unique $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ (cf. Theorem 3.7.8), a solution to (7.36), and

$$\|u\|_{W^{2,2}(\Omega)} \le C(\|g\|_{L^2(\Omega)} + \|f(v, \nabla v\|_{L^2(\Omega)}) \le C(\|g\|_{L^2(\Omega)} + \|v\|^\alpha_{W^{2,2}(\Omega)})$$

for some $\alpha > 1$ (here we used that $\delta < 1$). Therefore, in order to justify that T maps B_δ into itself, we need to verify that

$$C(\delta_0 + \delta^\alpha) < \delta.$$

Indeed, taking δ sufficiently small so that $C\delta^\alpha < \frac{\delta}{2}$ and then δ_0 small so that $C\delta_0 < \frac{\delta}{2}$ we verify our required property of T. Note that there exists sufficiently small $\overline{\delta}$ such that for any $\delta \le \overline{\delta}$, and for $\delta_0 \le \frac{\delta}{2C}$ the inequality above holds.

Furthermore, for v_1, $v_2 \in B_\delta$ we have

$$\begin{aligned} -\Delta(u_1 - u_2) &= f(v_2, \nabla v_2) - f(v_1, \nabla v_1) \\ &= f(v_2, \nabla v_2) - f(v_2, \nabla v_1) + f(v_2, \nabla v_1) - f(v_1, \nabla v_1) && \text{in } \Omega \\ u_1 - u_2 &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Using our assumptions we have

$$\begin{aligned} \|u_1 - u_2\|_{W^{2,2}(\Omega)} &\le C\|(v_1 - v_2)(|\nabla v_1|^q + 1)(|v_1|^{r-1} + |v_2|^{r-1})\|_{L^2(\Omega)} \\ &\quad + C\||\nabla v_1 - \nabla v_2||v_2|^r(|\nabla v_1|^{q-1} + |\nabla v_2|^{q-1})\|_{L^2(\Omega)} \\ &\le C\delta^\beta\|v_1 - v_2\|_{W^{2,2}(\Omega)} \end{aligned}$$

for some $\beta > 0$. Choosing δ sufficiently small (possibly smaller than in the previous considerations) we may conclude that $C\delta^\beta < 1$, hence T is a contraction in B_δ. The theorem is proved. ■

Example 7.4.6. We may take, e.g., $f(u, \nabla u) := |\nabla u|^2|u|^r$ for some $r > 1$ (even $r = 1$ is enough) so that assumptions (7.37) are verified (for $r = 1$ a slight modification is needed).

Exercise 7.4.7. A following generalization of the Banach fixed point Theorem holds:

Theorem 7.4.8 Let X, Y be Banach spaces, X is reflexive and $X \hookrightarrow Y$. Let H be a non-empty, closed, convex and bounded subset of X and let $T\colon H \to H$ be a mapping such that

$$\|T(u) - T(v)\|_Y \leq \kappa \|u - v\|_Y$$

for any u, $v \in H$ and $0 \leq \kappa < 1$. Then T possesses a unique fixed point in H.

Prove it!

7.4.2 Proof of the Schauder fixed point Theorem. Its applications

First, using the Brouwer fixed point Theorem 7.4.2 we show the Schauder fixed point Theorem 7.4.3.

Proof. (of Theorem 7.4.3.) **Step 1:** Construction of an operator in a finite dimensional space
Choose $\varepsilon > 0$ and $\{u_i\}_{i=1}^{N_\varepsilon} \subset K$ such that

$$\bigcup_{i=1}^{N_\varepsilon} B_\varepsilon(u_i) \supset K.$$

This follows due to the fact that K is compact. Denote by K_ε the convex hull of points $\{u_i\}_{i=1}^{N_\varepsilon}$, i.e.,

$$K_\varepsilon := \Big\{ \sum_{i=1}^{N_\varepsilon} \lambda_i u_i \,|\, 0 \leq \lambda_i \leq 1, i = 1, 2, \ldots, N_\varepsilon, \sum_{i=1}^{N_\varepsilon} \lambda_i = 1 \Big\}.$$

Since K is convex, we know that $K_\varepsilon \subset K$. Define $P_\varepsilon\colon K \to K_\varepsilon$ as

$$P_\varepsilon(u) := \frac{\sum_{i=1}^{N_\varepsilon} \operatorname{dist}\big(u, K \setminus B_\varepsilon(u_i)\big) u_i}{\sum_{i=1}^{N_\varepsilon} \operatorname{dist}\big(u, K \setminus B_\varepsilon(u_i)\big)}.$$

As K is covered by the union of the balls, the denominator is non zero for any $u \in K$. Clearly, P_ε is continuous, and for any $u \in K$

$$\|P_\varepsilon(u) - u\|_X \leq \frac{\sum_{i=1}^{N_\varepsilon} \operatorname{dist}\big(u, K \setminus B_\varepsilon(u_i)\big) \|u_i - u\|_X}{\sum_{i=1}^{N_\varepsilon} \operatorname{dist}\big(u, K \setminus B_\varepsilon(u_i)\big)} \leq \varepsilon.$$

(Recall that either $\|u_i - u\|_X \leq \varepsilon$ or $\operatorname{dist}\big(u, K \setminus B_\varepsilon(u_i)\big) = 0$.)
Step 2: Application of the Brouwer fixed point Theorem
Consider $T_\varepsilon := P_\varepsilon(T(u))$, $u \in K_\varepsilon$. Now K_ε is finite dimensional and homeomorphic to $\mathbb{R}^M$, $M \leq N_\varepsilon$. Then the Brouwer fixed point Theorem (Theorem 7.4.2) yields existence of at least one $u_\varepsilon \in K_\varepsilon$ such that

$$T_\varepsilon(u_\varepsilon) = u_\varepsilon \qquad \forall \varepsilon > 0.$$

Step 3: Fixed point of T
Since K is compact, there exists a sequence $\varepsilon_j \to 0_+$ and $u \in K$ such that $u_{\varepsilon_j} \to u$ in X. Let us show that u is a fixed point of T in K. We have

$$\|u_{\varepsilon_j} - T(u_{\varepsilon_j})\|_X = \|T_{\varepsilon_j}(u_{\varepsilon_j}) - T(u_{\varepsilon_j})\|_X = \|P_{\varepsilon_j}(T(u_{\varepsilon_j})) - T(u_{\varepsilon_j})\|_X \leq \varepsilon_j.$$

As T is continuous, $\|u_{\varepsilon_j} - T(u_{\varepsilon_j})\|_X \to \|u - T(u)\|_X$ which is due to the computation above equal to zero. The theorem is proved. ∎

Next, we show Theorem 7.4.4.

Proof. (of Theorem 7.4.4.) **Step 1:** Candidate for the fixed point
Choose $M > 0$ such that if $u = \lambda T(u)$ for some $0 \leq \lambda \leq 1$, then $\|u\|_X < M$. Define

$$\widetilde{T}(u) := \begin{cases} T(u) & \text{if } \|T(u)\|_X \leq M \\ \dfrac{MT(u)}{\|T(u)\|_X} & \text{if } \|T(u)\|_X > M. \end{cases}$$

Now, $\widetilde{T}\colon B_M(0) \to B_M(0)$. Define K as convex hull of $\widetilde{T}(B_M(0))$. As T is compact, then also $\widetilde{T}$ is compact. Hence, as K is a convex, compact subset of X and $\widetilde{T}$ maps K into K, $\widetilde{T}$ possesses a fixed point $u \in K$.
Step 2: Fixed point of T
Let us now show that u is in fact also a fixed point of T. If not, then necessarily, $\|T(u)\|_X > M$ and $u = \lambda T(u)$ for $\lambda = \frac{M}{\|T(u)\|_X} \in [0, 1)$. Then $\|u\|_X = \|\widetilde{T}(u)\|_X = M$ which contradicts to the choice of M. ∎

Exercise 7.4.9. Prove the following variant of the Schaeffer fixed point Theorem; the proof follows the same idea as the proof of the Schaeffer fixed point Theorem.

Corollary 7.4.10. Let $K \subset X$ be convex, $0 \in K$ and $T\colon K \to K$ be continuous, compact such that

$$\{u \in K \,|\, u = \lambda T(u),\, 0 \le \lambda \le 1\}$$

is bounded. Then T has a fixed point in K.

Next we consider the following example

$$\begin{aligned} -\Delta u + F(\nabla u) + \mu u &= f && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{7.38}$$

where $\Omega \subset \mathbb{R}^d$ is of class $C^{1,1}$, $F\colon \mathbb{R}^d \to \mathbb{R}$ is continuous with linear growth

$$|F(\mathbf{p})| \le C(1 + |\mathbf{p}|) \qquad \forall \mathbf{p} \in \mathbb{R}^d,\ C > 0. \tag{7.39}$$

We aim at proving the following result

Theorem 7.4.11 Let $f \in L^2(\Omega)$ and $\mu > 0$ be sufficiently large. Then under the assumptions on Ω and F above, there exists a strong (and weak) solution to (7.38) such that $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. If F is additionally globally Lipschitz continuous, the solution is unique.

Proof. **Step 1:** Operator T
We recall that for a given $u \in W_0^{1,2}(\Omega)$

$$f - F(\nabla u) \in L^2(\Omega).$$

Next we denote by $w \in W_0^{1,2}(\Omega)$ the unique weak solution to

$$\begin{aligned} -\Delta w + \mu w &= f - F(\nabla u) && \text{in } \Omega \\ w &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Moreover, we know that $w \in W^{2,2}(\Omega)$ and

$$\|w\|_{2,2} \le C\|f - F(\nabla u)\|_2 \le C(1 + \|f\|_2 + \|\nabla u\|_2).$$

Hence we define the operator

$$T : W_0^{1,2}(\Omega) \to W_0^{1,2}(\Omega)$$

so that $T(u) = w$. We have

$$\|T(u)\|_{2,2} \le C(1 + \|f\|_2 + \|u\|_{1,2}).$$

Step 2: Continuity and compactness of T
We show that $T\colon W_0^{1,2}(\Omega) \to W_0^{1,2}(\Omega)$ is continuous and compact. Let $u_k \to u$ in $W_0^{1,2}(\Omega)$. Then

$$\sup_{k\in\mathbb{N}} \|w_k\|_{2,2} < +\infty,$$

where $w_k = T(u_k)$. Thus there exists a subsequence $\{w_{k_j}\}_{j=1}^{\infty} \subset \{w_k\}_{k=1}^{\infty}$ and $w \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ such that

$$w_{k_j} \to w \qquad \text{in } W_0^{1,2}(\Omega)$$

(and $w_{k_j} \rightharpoonup w$ in $W^{2,2}(\Omega)$). As

$$\int_\Omega \big(\nabla w_{k_j} \cdot \nabla v + \mu w_{k_j} v\big)\,\mathrm{d}x = \int_\Omega (f - F(\nabla u_{k_j}))v\,\mathrm{d}x,$$

we get letting $j \to \infty$

$$\int_\Omega (\nabla w \cdot \nabla v + \mu w v)\,\mathrm{d}x = \int_\Omega (f - F(\nabla u))v\,\mathrm{d}x$$

for all $v \in W_0^{1,2}(\Omega)$. To pass to the limit in the nonlinear term we use again the continuity of F, its linear growth and the Vitali convergence theorem. Hence $w = T(u)$. Due to the uniqueness of the solution in fact the whole sequence $T(u_k) \to T(u)$ in $W^{1,2}(\Omega)$ and T is continuous.

The compactness is immediate. As we have that $\sup_{k\in\mathbb{N}} \|u_k\|_{1,2} < +\infty$, it follows that $\sup_{k\in\mathbb{N}} \|T(u_k)\|_{2,2} < +\infty$ and the compact embedding $W^{2,2}(\Omega) \hookrightarrow\hookrightarrow W^{1,2}(\Omega)$ yields that a bounded sequence in $W^{1,2}(\Omega)$ provides a strongly convergent subsequence of $T(u_k)$ in the same space.
Step 3: Bounds for possible fixed points
We have to verify that the set of possible fixed points

$$\{u \in W_0^{1,2}(\Omega) \,|\, u = \lambda T(u) \quad \text{for some } 0 \le \lambda \le 1\}$$

is bounded provided μ is sufficiently large. Since the case $\lambda = 0$ is trivial, let

$$u = \lambda T(u), \qquad 0 < \lambda \le 1,$$

i.e., $\frac{u}{\lambda} = T(u)$. Therefore,

$$-\Delta u + \mu u = \lambda(f - F(\nabla u)).$$

Multiplying the equality by u and integrating it over Ω we obtain

$$\|\nabla u\|_2^2 + \mu\|u\|_2^2 \le \lambda C \int_\Omega (|f| + |\nabla u| + 1)|u| \,\mathrm{d}x \le \frac{1}{2}\|\nabla u\|_2^2 + \frac{\mu}{2}\|u\|_2^2 + C(\mu)\|f\|_2^2 + \frac{C^2}{2}\|u\|_2^2.$$

Assuming $C^2 \le \mu$ we get

$$\|u\|_{1,2} \le C,$$

where the constant depends only on the data of the problem.
Step 4: Fixed point
Schaeffer's fixed point Theorem 7.4.4 implies the existence of a fixed point of T, i.e., existence of a solution to (7.38) which belongs to $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$.
Step 5: Uniqueness
Let w_1 and w_2 be two distinct solutions. We now employ the Lipschitz continuity of F. Denoting $W = w_1 - w_2$, we have

$$\begin{aligned} -\Delta W + \mu W &= F(\nabla w_2) - F(\nabla w_1) && \text{in } \Omega \\ W &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Multiplying the equation by W we get using the Lipschitz continuity of F

$$\begin{aligned} \|\nabla W\|_{L^2(\Omega;\mathbb{R}^d)}^2 + \mu\|W\|_{L^2(\Omega)}^2 &\le C\|\nabla W\|_{L^2(\Omega;\mathbb{R}^d)}\|W\|_{L^2(\Omega)} \\ &\le \frac{1}{2}\|\nabla W\|_{L^2(\Omega;\mathbb{R}^d)}^2 + C_1\|W\|_{L^2(\Omega)}^2. \end{aligned}$$

The uniqueness follows provided $C_1 \le \mu$. ■

7.5 Introduction to the calculus of variations

7.5.1 Basic ideas

We consider the problem

$$A(u) = 0 \tag{7.40}$$

for a special class of operators A, namely such that A is a "gradient" of a potential, i.e.,

$$A(u) = I'[u] = 0. \tag{7.41}$$

It means (at least formally) that a solution to (7.40) is a critical point of the functional I. Under additional assumptions on I we even show that the critical point is a point of (local) minimum of I. The advantage is that to prove existence of a minimum for certain class of functionals is easier than to prove existence of a solution to a nonlinear equation.

Definition 7.5.1 Let L: $\mathbb{R}^d \times \mathbb{R} \times \Omega \to \mathbb{R}$. Then we call L the Lagrangian and we define

$$I[u] := \int_\Omega L(\nabla u, u, \cdot) \,\mathrm{d}x.$$

We also use the following notation:

$$\begin{aligned} L(\mathbf{p}, z, x) &:= L(p_1, p_2, \dots, p_d, z, x_1, x_2, \dots, x_d) \\ \nabla_{\mathbf{p}} L &:= \Big(\frac{\partial L}{\partial p_1}, \frac{\partial L}{\partial p_2}, \dots, \frac{\partial L}{\partial p_d}\Big) \\ \nabla_z L &:= \frac{\partial L}{\partial z} \\ \nabla_x L &:= \Big(\frac{\partial L}{\partial x_1}, \frac{\partial L}{\partial x_2}, \dots, \frac{\partial L}{\partial x_d}\Big). \end{aligned}$$

We consider the following problem. We look for

$$u := \operatorname*{argmin}_{\{w \in X \,|\, w = g \text{ on } \partial\Omega\}} I[w],$$

where X is a certain function space (typically a Sobolev space). The fact that we fix possibly an inhomogeneous boundary condition leads to the problem that the set $\{w \in X \mid w = g \text{ on } \partial\Omega\}$ does not have a linear structure. We resolve this problem as usually, we look for the solution in the form $u = w_0 + U$, where w_0 is a fixed function with the given boundary condition and U has zero boundary condition, i.e., we look for U from a linear space. We show that if the function L is smooth and u is sufficiently smooth, then u is a (classical) solution to a certain partial differential equation.

We set $i(\tau) := I[u + \tau v]$, where $\tau \in \mathbb{R}$, $u = g$ on $\partial\Omega$ and $v \in C_0^\infty(\Omega)$. Then $u + \tau v = g$ on $\partial\Omega$. Assume that u is a minimizer of I. Then $i(\tau)$ has a minimum at $\tau = 0$. Since the function i is according to our assumptions smooth, we have

$$i'(0) = 0.$$

(This derivative is in the context of the functional analysis called the Gateaux derivative or differential of the functional I, however, we need to work with u in the form presented above.) Formally, we get

$$i'(\tau) = \int_\Omega \Big(\nabla_{\mathbf{p}} L(\nabla(u + \tau v), u + \tau v, \cdot) \cdot \nabla v + \nabla_z L(\nabla(u + \tau v), u + \tau v, \cdot) v \Big) \, \mathrm{d}x.$$

Therefore

$$0 = i'(0) = \int_\Omega \Big(\nabla_{\mathbf{p}} L(\nabla u, u, \cdot) \cdot \nabla v + \nabla_z L(\nabla u, u, \cdot) v \Big) \, \mathrm{d}x$$

for all $v \in C_0^\infty(\Omega)$. This equality is in fact a weak formulation of the problem below. We "integrate by parts" and get

$$0 = \int_\Omega \Big(- \sum_{i=1}^d \frac{\partial}{\partial x_i} \Big(\frac{\partial L}{\partial p_i}(\nabla u, u, \cdot) \Big) + \frac{\partial L}{\partial z}(\nabla u, u, \cdot) \Big) v \, \mathrm{d}x$$

for all $v \in C_0^\infty(\Omega)$. The Fundamental Lemma of calculus of variations yields finally

$$\begin{aligned} - \sum_{i=1}^d \frac{\partial}{\partial x_i} \Big(\frac{\partial L}{\partial p_i}(\nabla u(x), u(x), x) \Big) + \frac{\partial L}{\partial z}(\nabla u(x), u(x), x) &= 0 && \text{in } \Omega, \\ u &= g && \text{on } \partial\Omega. \end{aligned}$$

The equation above is called the Euler–Lagrange equation corresponding to the functional I.

Example 7.5.2. a) Let

$$L(\mathbf{p}, z, x) = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(x) p_i p_j - z f(x),$$

where $a_{ij} = a_{ji}$ for $i, j = 1, \ldots, d$ in Ω. Then

$$I[w] = \int_\Omega \Big(\frac{1}{2} \sum_{i,j=1}^d a_{ij}(x) \frac{\partial w}{\partial x_j} \frac{\partial w}{\partial x_i} - w f \Big) \, \mathrm{d}x.$$

The corresponding Euler–Lagrange equation is then

$$- \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \Big(a_{ij}(x) \frac{\partial u}{\partial x_j} \Big) = f.$$

b) Let

$$L(\mathbf{p}, z, x) = \frac{1}{2} |\mathbf{p}|^2 - F(z),$$

where $F'(z) = f(z)$. Then

$$I[w] = \int_\Omega \Big(\frac{1}{2} |\nabla w|^2 - F(w) \Big) \, \mathrm{d}x,$$

and the Euler–Lagrange equation is

$$-\Delta u = f(u).$$

c) Let

$$L(\mathbf{p}, z, x) = \big(1 + |\mathbf{p}|^2 \big)^{\frac{1}{2}}$$

(minimal surface problem). Then

$$I[w] = \int_\Omega \big(1 + |\nabla w|^2 \big)^{\frac{1}{2}} \, \mathrm{d}x,$$

and the Euler–Lagrange equation is

$$\sum_{i=1}^{d} \frac{\partial}{\partial x_i}\Big(\frac{\frac{\partial u}{\partial x_i}}{\sqrt{1+|\nabla u|^2}}\Big) = 0$$

in Ω. Adding the boundary condition $u = g$ on $\partial\Omega$ we get a problem whose solution solves the following problem. Let $\Omega \subset \mathbb{R}^d$ be given. We look for a graph of function $u = u(x)$ defined in Ω with fixed value $u = g$ on $\partial\Omega$ such that the area of the surface $(x, u(x))$, $x \in \Omega$, is minimal.

We next consider the second derivative of our function $i(\tau)$. Recall that under the smoothness assumption the necessary condition for having a minimum of i at $\tau = 0$ is $i''(0) \geq 0$. We compute

$$\begin{aligned} i''(\tau) = \int_\Omega \Big[&\sum_{i,j=1}^{d} \frac{\partial^2 L}{\partial p_i \partial p_j}(\nabla u + \tau\nabla v, u + \tau v, \cdot)\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j} \\ &+ 2\sum_{i=1}^{d} \frac{\partial^2 L}{\partial p_i \partial z}(\nabla u + \tau\nabla v, u+\tau v, \cdot) v\frac{\partial v}{\partial x_i} + \frac{\partial^2 L}{\partial z^2}(\nabla u + \tau \nabla v, u + \tau v, \cdot) v^2\Big]\, \mathrm{d}x. \end{aligned}$$

Hence

$$i''(0) = \int_\Omega \Big[\sum_{i,j=1}^{d} \frac{\partial^2 L}{\partial p_i\partial p_j}(\nabla u, u, \cdot)\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j} + 2\sum_{i=1}^{d}\frac{\partial^2 L}{\partial p_i \partial z}(\nabla u, u, \cdot) v \frac{\partial v}{\partial x_i} + \frac{\partial^2 L}{\partial z^2}(\nabla u, u, \cdot)v^2\Big]\, \mathrm{d}x \tag{7.42}$$

for all $v \in C_0^\infty(\Omega)$. By approximation, we can in fact use functions which are only Lipschitz continuous, however, still compactly supported in Ω. We now aim at proving the necessary condition for the minimum. We fix $\boldsymbol{\xi} \in \mathbb{R}^d$, take $\varepsilon > 0$ and a function $\eta \in C_0^\infty(\Omega)$ and define

$$v(x) := \varepsilon\varrho\Big(\frac{x\cdot\boldsymbol{\xi}}{\varepsilon}\Big)\eta(x),$$

where the function

$$\varrho(z) = \begin{cases} z & 0 \leq z \leq \frac{1}{2} \\ 1 - z & \frac{1}{2} < z \leq 1 \end{cases}$$

and extend the function as 1-periodic to $\mathbb{R}$, i.e., $\varrho(z) = \varrho(z+1)$ for all $z \in \mathbb{R}$. Therefore $|\varrho'(z)| = 1$ a.e. in $\mathbb{R}$, and

$$\frac{\partial v}{\partial x_i}(x) = \varrho'\Big(\frac{x\cdot\boldsymbol{\xi}}{\varepsilon}\Big)\xi_i(x)\eta(x) + O(\varepsilon) \qquad \text{for } \varepsilon \to 0_+.$$

Using this function in (7.42) and passing with $\varepsilon \to 0_+$ we get due to the fact that $i''(0) \geq 0$

$$0 \leq \int_\Omega \sum_{i,j=1}^{d} \frac{\partial^2 L}{\partial p_i \partial p_j}(\nabla u, u, \cdot)\xi_i\xi_j\eta^2\, \mathrm{d}x.$$

As this inequality holds for arbitrary $\eta \in C_0^\infty(\Omega)$, we get

$$\sum_{i,j=1}^{d} \frac{\partial^2 L}{\partial p_i\partial p_j}(\nabla u(x), u(x), x)\xi_i\xi_j \geq 0$$

for all $\boldsymbol{\xi} \in \mathbb{R}^d$ and all $x \in \Omega$. This indicates (but does not prove!) that the existence of a minimum for our class of functionals is connected with the convexity of L in the first variable.

7.5.2 Systems

We now assume that the Lagrangian depends on vector-valued function, i.e., its gradient is a tensor (matrix); hence

$$L\colon \mathbb{R}^{N\times d} \times \mathbb{R}^d \times \Omega \to \mathbb{R}.$$

We use the notation

$$L = L(\vec{\mathbf{P}}, \vec{z}, x),$$

where

$$\vec{\mathbf{P}} = \begin{pmatrix} p_1^1 & \dots & p_d^1 \\ \cdot & & \cdot \\ \cdot & & \cdot \\ p_1^N & \dots & p_d^N \end{pmatrix}$$

and define

$$I[\vec{w}] = \int_\Omega L(\nabla\vec{w}, \vec{w}, \cdot)\, \mathrm{d}x,$$

where $\vec{w}$: $\Omega \to \mathbb{R}^N$,

$$\nabla \vec{w} = \begin{pmatrix} \frac{\partial w_1}{\partial x_1} & \cdots & \frac{\partial w_1}{\partial x_d} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \frac{\partial w_N}{\partial x_1} & \cdots & \frac{\partial w_N}{\partial x_d} \end{pmatrix}$$

and we fix again the boundary condition $\vec{w} = \vec{g}$. We now deduce the Euler–Lagrange equations (or rather the system of Euler–Lagrange equations) for the minimizer of I. We set $i(\tau) = I[\vec{u} + \tau\vec{v}]$, where $\vec{v} \in C_0^\infty(\Omega; \mathbb{R}^N)$ and compute $i'(0)$ which we set to be equal to 0. We get

$$0 = i'(0) = \int_\Omega \Big(\sum_{i=1}^d \sum_{k=1}^N \frac{\partial L}{\partial p_i^k}(\nabla \vec{u}, \vec{u}, \cdot) \frac{\partial v^k}{\partial x_i} + \sum_{k=1}^N \frac{\partial L}{\partial z^k}(\nabla \vec{u}, \vec{u}, \cdot) v^k \Big) \, \mathrm{d}x,$$

i.e.,

$$-\sum_{i=1}^d \frac{\partial}{\partial x_i} \Big(\frac{\partial L}{\partial p_i^k}(\nabla \vec{u}(x), \vec{u}(x), x) \Big) + \frac{\partial L}{\partial z^k}(\nabla \vec{u}(x), \vec{u}(x), x) = 0 \tag{7.43}$$

in Ω, $k = 1, 2, \ldots, N$, and $\vec{u} = \vec{g}$ on $\partial\Omega$.

7.5.3 Null Lagrangians. Proof of Brouwer fixed point Theorem

There exist special Lagrangians such that any smooth function is a solution to the corresponding Euler–Lagrange equations.

Definition 7.5.3 The function L: $\mathbb{R}^{N \times d} \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ is called a null Lagrangian, if the corresponding Euler–Lagrange equations

$$-\sum_{i=1}^d \frac{\partial}{\partial x_k} \Big(\frac{\partial L}{\partial p_i^k}(\nabla \vec{u}(x), \vec{u}(x), x) \Big) + \frac{\partial L}{\partial z^k}(\nabla \vec{u}(x), \vec{u}(x), x) = 0,$$

$k = 1, 2, \ldots, d$, are fulfilled by any smooth function $\vec{u}$: $\Omega \to \mathbb{R}^N$.

In the scalar case ($N = 1$) only the linear functions in $\vec{\mathbf{P}}$ (with constant coefficients) are null Lagrangians. A more interesting situation is in the vectoriel case, especially if $d = N$, as we shall see later.

We start with one special property of the null Lagrangians which says that the corresponding energy depends only on the boundary conditions of the functions.

Theorem 7.5.4 — Properties of null Lagrangians. Let L be a null Lagrangian. Let $\vec{u}_1$ and $\vec{u}_2$ be two functions from $C^2(\Omega; \mathbb{R}^N)$ such that

$$\vec{u}_1 = \vec{u}_2 \qquad \text{on } \partial\Omega.$$

Then

$$I[\vec{u}_1] = I[\vec{u}_2].$$

Proof. Define

$$i(\tau) = I(\tau\vec{u}_1 + (1-\tau)\vec{u}_2], \qquad 0 \le \tau \le 1.$$

Then

$$\begin{aligned} i'(\tau) = \int_\Omega \Big[& \sum_{i=1}^d \sum_{k=1}^N \frac{\partial L}{\partial p_i^k}(\tau\nabla\vec{u}_1 + (1-\tau)\nabla\vec{u}_2, \tau\vec{u}_1 + (1-\tau)\vec{u}_2, \cdot) \Big(\frac{\partial u_1^k}{\partial x_i} - \frac{\partial u_2^k}{\partial x_i} \Big) \\ & + \sum_{k=1}^d \frac{\partial L}{\partial z^k}(\tau\nabla\vec{u}_1 + (1-\tau)\nabla\vec{u}_2, \tau\vec{u}_1 + (1-\tau)\vec{u}_2, \cdot)(u_1^k - u_2^k) \Big] \, \mathrm{d}x \\ = \sum_{k=1}^N \int_\Omega \Big[& -\sum_{i=1}^d \frac{\partial}{\partial x_i} \Big(\frac{\partial L}{\partial p_i^k}(\tau\nabla\vec{u}_1 + (1-\tau)\nabla\vec{u}_2, \tau\vec{u}_1 + (1-\tau)\vec{u}_2, \cdot) \Big) \\ & + \frac{\partial L}{\partial z^k}(\tau\nabla\vec{u}_1 + (1-\tau)\nabla\vec{u}_2, \tau\vec{u}_1 + (1-\tau)\vec{u}_2, \cdot)(u_1^k - u_2^k) \Big] \, \mathrm{d}x = 0, \end{aligned}$$

where we used in the application of the Gauss theorem that $\vec{u}_1 - \vec{u}_2 = \vec{0}$ on $\partial\Omega$ as well as the fact that L is a null Lagrangian. ■

Definition 7.5.5 Let $\mathbb{A}$ be a $d \times d$ matrix. Then we define $\operatorname{cof} \mathbb{A}$ as a matrix, whose (k, i) entries are as follows:

$$(\operatorname{cof} \mathbb{A})_i^k = (-1)^{k+i} \det(\mathbb{A}_i^k),$$

where the matrix $(\mathbb{A}_i^k)$ is a matrix obtained from the matrix $\mathbb{A}$ by deleting the k-th row and the i-th column.

Lemma 7.5.6 Let $\mathbf{u}\colon \mathbb{R}^d \to \mathbb{R}^d$ be a smooth function. Then

$$\sum_{i=1}^{d} \frac{\partial}{\partial x_i}\Big(\mathrm{cof}(\nabla \mathbf{u})\Big)_i^k = 0, \qquad k = 1, 2, \ldots, d.$$

Proof. **Step 1:** Derivative of the determinant
We recall the following linear algebra identity

$$(\det \mathbb{P})\mathbb{I} = \mathbb{P}^T \mathrm{cof}\, \mathbb{P}$$

which holds for any $\mathbb{P}$, matrix of the type $d \times d$. We may rewrite the identity above component-wise

$$(\det \mathbb{P})\delta_{ij} = \sum_{k=1}^{d} p_i^k (\mathrm{cof}\, \mathbb{P})_j^k, \tag{7.44}$$

$i, j = 1, 2, \ldots, d$. Therefore

$$\frac{\partial(\det \mathbb{P})}{\partial p_m^k} = (\mathrm{cof}\, \mathbb{P})_m^k$$

for $k, m = 1, 2, \ldots, d$ (recall that $(\mathrm{cof}\, \mathbb{P})_m^k$ does not contain the entry p_m^k).
Step 2: Identity for the derivative of the cofactor
Take $\mathbb{P} = \nabla \mathbf{u}$ in (7.44), differentiate the identity with respect to x_j and sum the identity over j. It yields

$$\sum_{j=1}^{d} \delta_{ij} \frac{\partial}{\partial x_j} \det \mathbb{P} = \sum_{j,k}^{d} \delta_{ij} (\mathrm{cof}(\nabla \mathbf{u}))_m^k \frac{\partial^2 u^k}{\partial x_m \partial x_j} = \sum_{j,k,m=1}^{d} \Big(\frac{\partial^2 u^k}{\partial x_i \partial x_j} (\mathrm{cof}(\nabla \mathbf{u}))_j^k + \frac{\partial u^k}{\partial x_i} \frac{\partial}{\partial x_j} (\mathrm{cof}(\nabla \mathbf{u}))_j^k \Big).$$

Hence, it follows from the last equality

$$\sum_{k=1}^{d} \frac{\partial u^k(x)}{\partial x_i} \Big(\sum_{j=1}^{d} \frac{\partial}{\partial x_j} (\mathrm{cof}\big(\nabla \mathbf{u}(x)\big))_j^k \Big) = 0. \tag{7.45}$$

Step 3: Identity for the cofactor
If $\det(\nabla \mathbf{u})(x_0) \neq 0$, then

$$\sum_{j=1}^{d} \frac{\partial}{\partial x_j} (\mathrm{cof}\, \nabla \mathbf{u}(x_0))_j^k = 0, \qquad k = 1, 2, \ldots, d,$$

where we used that the non-zero determinant implies unique solvability (the trivial one) of (7.45). If the determinant is zero, we choose $\varepsilon > 0$, sufficiently small, such that $\det(\nabla(\mathbf{u})(x_0) + \varepsilon \mathbb{I}) \neq 0$ and repeat Steps 1–3 for $\widetilde{\mathbf{u}}_\varepsilon := \mathbf{u}(x) + \varepsilon x$ and finally let $\varepsilon \to 0_+$. ∎

The previous result allows us to present a non-trivial example of a null Lagrangian.

Lemma 7.5.7 — Determinant function as null Lagrangian. Let $d = N$. Then the determinant function is a null Lagrangian.

Proof. We have to show that

$$\sum_{i=1}^{d} \frac{\partial}{\partial x_i} \Big(\frac{\partial \big(\det(\nabla \mathbf{u}(x))\big)}{\partial p_i^k} \Big) = 0, \qquad k = 1, 2, \ldots, d.$$

Recall that

$$\frac{\partial \big(\det(\nabla \mathbf{u})\big)}{\partial p_i^k} = (\mathrm{cof}(\nabla \vec{u}))_i^k.$$

Then, due to Lemma 7.5.6

$$\sum_{i=1}^{d} \frac{\partial}{\partial x_i} \Big(\frac{\partial(\det(\nabla \mathbf{u}))}{\partial p_i^k} \Big) = \sum_{i=1}^{d} \frac{\partial}{\partial x_i} (\mathrm{cof}\, (\nabla \mathbf{u}))_i^k = 0, \qquad k = 1, 2, \ldots, d.$$

∎

Using this result we may now prove the Brouwer fixed point Theorem

Theorem 7.5.8 — Brouwer fixed point Theorem. Let $\vec{u}$: $\overline{B_1(0)} \to \overline{B_1(0)} \subset \mathbb{R}^N$ be continuous. Then $\vec{u}$ possesses at least one fixed point in $\overline{B_1(0)}$, i.e., there exists at least one $\vec{x}_0 \in \overline{B_1(0)} \subset \mathbb{R}^N$ such that

$$\vec{u}(\vec{x}_0) = \vec{x}_0.$$

Proof. **Step 1:** Non-existence of a smooth function with given boundary values
We first claim that there does not exist a smooth function

$$\vec{w}\colon B := \overline{B_1(0)} \to \partial B$$

such that

$$\vec{w}(\vec{x}) = \vec{x} \qquad \text{on } \partial B. \tag{7.46}$$

Suppose that such a function exists. Let $\vec{I}$ denote the identity mapping, i.e. $\vec{I}(\vec{x}) = \vec{x}$ for all $\vec{x} \in B$. Due to our assumption, $\vec{w} = \vec{I}$ on ∂B. As the determinant is a null Lagrangian, Theorem 7.5.4 implies

$$\int_B \det(\nabla \vec{w})\,\mathrm{d}x = \int_B \det(\nabla \vec{I})\,\mathrm{d}x = |B| \neq 0. \tag{7.47}$$

On the other hand, as $\vec{w}$: $B \to \partial B$, we have $|\vec{w}| = 1$. Therefore ($\frac{\partial}{\partial x_i}|\vec{w}|^2 = 0$, $i = 1, 2, \ldots, N$)

$$(\nabla \vec{w})^T \vec{w} = \vec{0}.$$

As $|\vec{w}| = 1$, the equality says that 0 is an eigenvalue of $(\nabla \vec{w})^T$ for all $\vec{x} \in B$, whence $\det(\nabla \vec{w}) \equiv 0$ in B which contradicts to (7.47) and thus the smooth function considered in Step 1 cannot exist.
Step 2: The function from Step 1 does not exist even if it is only continuous
Let us show that even no such merely continuous function may exist. Let $\vec{w}$ be such a continuous function. We extend it by the identity function ($\vec{w}(\vec{x}) := \vec{x}$) for $\vec{x} \in \mathbb{R}^N \setminus B$. Observe that $\vec{w}(\vec{x}) \neq \vec{0}$ for $\vec{x} \in \mathbb{R}^N$. Fix $\varepsilon > 0$ and mollify the function, i.e., $\vec{w}_\varepsilon := \eta_\varepsilon \star \vec{w}$. Then also $\vec{w}_\varepsilon \neq \vec{0}$ in $\mathbb{R}^N$. As η_ε is a radially symmetric function, we also have $\vec{w}_\varepsilon(\vec{x}) = \vec{x}$ for $\mathbf{x} \in \mathbb{R}^N \setminus B_2(0)$ if $\varepsilon < 1$ (recall that $\int_{\mathbb{R}^N} (\vec{x} - \vec{y})\eta_\varepsilon(\vec{y})\,\mathrm{d}\vec{y} = \vec{x}$).

Then

$$\widetilde{\vec{w}} := \frac{2\vec{w}_\varepsilon}{|\vec{w}_\varepsilon|}$$

would be a smooth function fulfilling assumptions from Step 1 (with $B = \overline{B_2(0)}$). Hence such a function cannot exist.
Step 3: Fixed point by contradiction
Let $\vec{u}$: $B \to B$ be continuous such that $\vec{u}$ does not possess any fixed point in B. Define $\vec{w}$: $B \to \partial B$ so that $\vec{w}(\vec{x}) = \vec{y}$, where $\vec{y}$ is the point at which the ray starting at $\vec{u}(\vec{x})$ and going through $\vec{x}$ hits ∂B. Since $\vec{u}(\vec{x}) \neq \vec{x}$ and $\vec{u}$ is continuous, the mapping $\vec{w}$ is well defined and continuous on B such that $\vec{w}$ maps B to ∂B and $\vec{w}(\vec{x}) = \vec{x}$ on ∂B. This contradicts to Step 2. ■

7.6 Existence of minimizers for some problems of calculus of variations. Convexity

Let us start with the following abstract problem. We look for a function u (provided it exists) such that the functional $I[\cdot]$: $X \to \mathbb{R}$, where X is a Banach space and $\mathcal{A} \subset X$, has its minimum over $\mathcal{A}$ at u, i.e.,

$$u = \mathrm{argmin}_{w \in \mathcal{A}} I[w], \text{ i.e.,} \qquad I[u] = \min_{w \in \mathcal{A}} I[w].$$

Let $u_n \in \mathcal{A}$ be a minimizing sequence, i.e., $\lim_{n\to\infty} I[u_n] = \inf_{w\in\mathcal{A}} I[w]$. We would like to extract at least a weakly convergent subsequence. To this aim, we assume that
(1) $\mathcal{A}$ is non-empty
(2) I is bounded from below on $\mathcal{A}$ ($\inf_{w\in\mathcal{A}} I[w] > -\infty$)
(3) I is coercive on $\mathcal{A}$, i.e., $\lim_{\|w\|_X\to\infty} I[w] = +\infty$.
Then the minimizing sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded. Further
(4) X is a reflexive Banach space
(5) the set $\mathcal{A}$ is closed and convex (hence weakly closed, see Theorem B.2.15).
Then the sequence contains a weakly convergent subsequence whose limit belongs to $\mathcal{A}$. However, we can hardly expect that $I[u_{n_k}] \to I[u]$ for $k \to \infty$, such a condition would restrict the class of functionals considerably. We therefore assume a weaker condition, namely that
(6) I is weakly lower semicontinuous, i.e., $I[u] \le \liminf_{k\to\infty} I[u_k]$ whenever $u_k \rightharpoonup u$ in X.
Then we get that

$$I[u] \le \liminf_{k\to\infty} I[u_{n_k}] = \lim_{k\to\infty} I[u_{n_k}] = \inf_{w\in\mathcal{A}} I[w].$$

Since $u \in \mathcal{A}$, then u is the minimizer. We proved

Theorem 7.6.1 — Main Theorem of the calculus of variations. Let X be a reflexive Banach space, $\mathcal{A} \subset X$ non-empty, closed and convex. Let $I\colon \mathcal{A} \to \mathbb{R}$ be coercive, bounded from below and weakly lower semicontinuous on X. Then there exists $u \in \mathcal{A}$ such that $I[u] = \min_{w\in\mathcal{A}} I[w]$.

We restrict ourselves on functionals which can be represented by means of Lagrangians, i.e.,

$$I[w] := \int_\Omega L(\nabla u, u, \cdot)\,\mathrm{d}x, \qquad u \in \mathcal{A} \tag{7.48}$$

and we aim at formulating suitable conditions on L and $\mathcal{A}$ so that we will be able to apply Theorem 7.6.1. We deal separately with the case when u is a scalar-valued and a vector-valued function.

7.6.1 Scalar case (one Euler–Lagrange equation)

In order to verify that the functional is coercive, we assume that there exist $\alpha > 0$, $q \in (1,\infty)$ and $b \in L^1(\Omega)$ such that

$$L(\mathbf{p}, z, x) \geq \alpha|\mathbf{p}|^q - b(x), \tag{7.49}$$

for any $\mathbf{p} \in \mathbb{R}^d$, $z \in \mathbb{R}$, $x \in \Omega$. Then

$$I[w] = \int_\Omega L(\nabla w, w, \cdot)\,\mathrm{d}x \geq \int_\Omega (\alpha|\nabla w|^q - b)\,\mathrm{d}x = \alpha\|\nabla w\|^q_{L^q(\Omega;\mathbb{R}^d)} - \gamma.$$

Therefore $I[w] \to +\infty$ as $\|\nabla w\|_{L^q(\Omega;\mathbb{R}^d)} \to +\infty$. Additionally, if we assume

$$\mathcal{A} = \Big\{w \in W^{1,q}(\Omega)\,|\, w = g \text{ on } \partial\Omega\Big\} \quad \text{with } \Omega \in C^{0,1}$$

and fix $g \in W^{1-\frac{1}{q},q}(\partial\Omega)$, then even

$$I[w] \to +\infty \qquad \text{as} \qquad \|w\|_{1,q} \to +\infty,$$

as $\|\nabla w\|_{L^q(\Omega);\mathbb{R}^d)} + \|g\|_{L^q(\Omega)}$ is a norm equivalent to the $W^{1,q}$-norm on $\mathcal{A}$. Note also that $\mathcal{A}$ is weakly closed.

The only condition which remains to verify is the weak lower semicontinuity of I on $W^{1,q}(\Omega)$. We already know that a necessary condition for a function u to be a minimizer of a smooth Lagrangian is

$$\sum_{i,j=1}^d \frac{\partial^2 L}{\partial p_i \partial p_j}(\nabla u(x), u(x), x)\xi_i\xi_j \geq 0$$

for all $\boldsymbol{\xi} \in \mathbb{R}^d$ and almost every $x \in \Omega$.

We will show that the convexity is indeed the right condition. We will assume (for simplicity) that $L(\cdot,\cdot,\cdot)$ is a continuous function on $\mathbb{R}^d \times \mathbb{R}$ for almost every $x \in \Omega$ and measurable on Ω on $\mathbb{R}^d \times \mathbb{R}$. Further, we also assume that it is one continuously differentiable with respect to $\mathbf{p}$. We have the following result (for more general situation, see Theorem A.3.43 from the Appendix)

Lemma 7.6.2 Let L be as above, additionally bounded from below and let the mapping

$$\mathbf{p} \mapsto L(\mathbf{p}, z, x)$$

be convex $\forall z \in \mathbb{R}$ and $x \in \Omega$ on $\mathbb{R}^d$. Let $\Omega \in C^0$. Then

$$I[w] := \int_\Omega L(\nabla w, w, \cdot)\,\mathrm{d}x$$

is weakly lower semicontinuous on $W^{1,q}(\Omega)$, $1 < q < \infty$.

Proof. **Step 1:** Formulation
Let $\{u_k\}_{k\in\mathbb{N}} \subset W^{1,q}(\Omega)$ be such that $u_k \rightharpoonup u$ (weakly) in $W^{1,q}(\Omega)$. Let

$$\ell := \liminf_{k\to\infty} I[u_k].$$

We aim at showing that $I[u] \leq \ell$.

Note that we may choose a subsequence (we do not relabel it) such $\ell = \lim_{k\to\infty} I[u_k]$ and $u_k \to u$ in $L^q(\Omega)$. Moreover, $\sup_{k\in\mathbb{N}} \|u_k\|_{W^{1,q}(\Omega)} < +\infty$.

Step 2: Application of Egorov Theorem
We fix $\varepsilon > 0$. Then there exists $E_\varepsilon \subset \Omega$, $|\Omega \setminus E_\varepsilon| < \varepsilon$ such that $u_k \rightrightarrows u$ in E_ε. We set

$$F_\varepsilon = \Big\{x \in \Omega\,|\,|u(x)| + |\nabla u(x)| \leq \frac{1}{\varepsilon}\Big\}.$$

Recall that $|\Omega \setminus F_\varepsilon| \to 0$ as $\varepsilon \to 0$. Finally, we set

$$G_\varepsilon = F_\varepsilon \bigcap E_\varepsilon$$

(note that $|\Omega \setminus G_\varepsilon| \to 0$ as $\varepsilon \to 0$).

Step 3: Application of convexity

By adding a suitable integrable function we may without loss of generality assume that $L(\nabla u_k(x), u_k(x), x)$ is non-negative in Ω. Then, by convexity of L

$$\begin{aligned} I[u_k] = \int_\Omega L(\nabla u_k, u_k, \cdot)\,\mathrm{d}x &\geq \int_{G_\varepsilon} L(\nabla u_k, u_k, \cdot)\,\mathrm{d}x \\ &\geq \int_{G\varepsilon} L(\nabla u, u_k, \cdot)\,\mathrm{d}x + \int_{G_\varepsilon} \sum_{i=1}^d \frac{\partial L}{\partial p_i}(\nabla u, u_k, \cdot)\Big(\frac{\partial u_k}{\partial x_i} - \frac{\partial u}{\partial x_i}\Big)\,\mathrm{d}x. \end{aligned}$$

Now it is easy to verify, due to the definition of G_ε,

$$\int_{G_\varepsilon} L(\nabla u, u_k, \cdot)\,\mathrm{d}x \to \int_{G_\varepsilon} L(\nabla u, u, \cdot)\,\mathrm{d}x$$

and due to the weak convergence of $\nabla u_k \rightharpoonup \nabla u$ in $L^q(\Omega; \mathbb{R}^d)$

$$\int_{G_\varepsilon} \sum_{i=1}^d \frac{\partial L}{\partial p_i}(\nabla u, u_k, \cdot)\Big(\frac{\partial u_k}{\partial x_i} - \frac{\partial u}{\partial x_i}\Big)\,\mathrm{d}x \to 0,$$

both for $k \to \infty$. Whence

$$\ell \geq \int_{G_\varepsilon} L(\nabla u, u, \cdot)\,\mathrm{d}x$$

for any $\varepsilon > 0$. Passing with $\varepsilon \to 0$ and using the Lebesgue monotone convergence Theorem A.3.2 we conclude

$$\ell \geq \int_\Omega L(\nabla u, u, \cdot)\,\mathrm{d}x = I[u]$$

which finishes the proof. ■

Therefore we have

Theorem 7.6.3 — Existence of minimizers for convex Lagrangians. Let $\mathcal{A} = \{u \in W^{1,q}(\Omega) \,|\, u = g \text{ on } \partial\Omega\}$ with $g \in W^{1-\frac{1}{q},q}(\partial\Omega)$, $\Omega \in C^{0,1}$. Let L fulfil the coercivity condition (7.49) and let L be convex in the first variable. Let L be differentiable in the first variable, L and $\nabla_{\mathbf{p}} L$ be continuous in the second variable for almost every x in Ω and measurable in Ω for all $(\mathbf{p}, z) \in \mathbb{R}^d \times \mathbb{R}$.

Then there exists at least one $u \in \mathcal{A}$ such that

$$I[u] = \min_{w \in \mathcal{A}} I[w].$$

Proof. As $\mathcal{A}$ is non-empty, closed and convex, L is coercive and weakly lower semicontinuous, we may apply Theorem 7.6.1. ■

In order to ensure the uniqueness of the minimizer, we have to require more. One possibility is to assume that

$$L = L(\mathbf{p}, x) \qquad (L \text{ does not depend on } z), \tag{7.50}$$

$$\exists \theta > 0\colon \quad \sum_{i,j=1}^d \frac{\partial^2 L}{\partial p_i \partial p_j}(\mathbf{p}, x)\xi_i \xi_j \geq \theta |\boldsymbol{\xi}|^2 \quad \forall \boldsymbol{\xi} \in \mathbb{R}^d \text{ and almost everywhere in } \Omega \qquad \text{(uniform convexity).} \tag{7.51}$$

Then

Theorem 7.6.4 — Uniqueness of minimizers. Under assumptions (7.50)–(7.51), for L twice differentiable in the first variable for almost every $x \in \Omega$ and measurable in x for all $\mathbf{p} \in \mathbb{R}^d$, $\Omega \in C^{0,1}$, the minimizer of $I[\cdot]$ over $\mathcal{A}$ (defined above) is unique, provided it exists.

Proof. Assume that u_1, u_2 are two different minimizers. As $\mathcal{A}$ is convex, we have that $\frac{u_1+u_2}{2} \in \mathcal{A}$. We show that

$$I\Big[\frac{u_1 + u_2}{2}\Big] \leq \frac{1}{2} I[u_1] + \frac{1}{2} I[u_2]$$

with strict inequality provided $u_1 \neq u_2$. Note that

$$L(\mathbf{p}, x) \geq L(\mathbf{q}, x) + \sum_{i=1}^{d} \frac{\partial}{\partial p_i} L(\mathbf{q}, x)(p_i - q_i) + \frac{\theta}{2}|\mathbf{p} - \mathbf{q}|^2.$$

Taking $\mathbf{p} = \nabla u_1$, $\mathbf{q} = \frac{1}{2}(\nabla u_1 + \nabla u_2)$ we get

$$I\Big[\frac{u_1 + u_2}{2}\Big] + \frac{1}{2}\int_\Omega \sum_{i=1}^{d} \frac{\partial}{\partial p_i} L\Big(\frac{\nabla u_1 + \nabla u_2}{2}, \cdot\Big)\Big(\frac{\partial u_1}{\partial x_i} - \frac{\partial u_2}{\partial x_i}\Big)\, \mathrm{d}x + \frac{\theta}{8}\int_\Omega |\nabla u_1 - \nabla u_2|^2 \,\mathrm{d}x \leq I[u_1] \tag{7.52}$$

and for $\mathbf{p} = \nabla u_2$, $\mathbf{q} = \frac{1}{2}(\nabla u_1 + \nabla u_2)$ we get

$$I\Big[\frac{u_1 + u_2}{2}\Big] + \frac{1}{2}\int_\Omega \sum_{i=1}^{d} \frac{\partial}{\partial p_i} L\Big(\frac{\nabla u_1 + \nabla u_2}{2}, \cdot\Big)\Big(\frac{\partial u_2}{\partial x_i} - \frac{\partial u_1}{\partial x_i}\Big)\, \mathrm{d}x + \frac{\theta}{8}\int_\Omega |\nabla u_1 - \nabla u_2|^2 \,\mathrm{d}x \leq I[u_2]. \tag{7.53}$$

Summing up (7.53) with (7.52) and multiplying the resulted inequality by $\frac{1}{2}$ we obtain

$$I\Big[\frac{u_1 + u_2}{2}\Big] + \frac{\theta}{8}\int_\Omega |\nabla u_1 - \nabla u_2|^2 \,\mathrm{d}x \leq \frac{I[u_1] + I[u_2]}{2}.$$

Evidently, if $I[u_1] = I[u_2]$ and u_1 and u_2 are minimizers, then $\nabla u_1 = \nabla u_2$ a.e. in Ω. But since $u_1 = u_2$ a.e. on $\partial\Omega$, then $u_1 = u_2$ a.e. in Ω which finishes the proof. ■

Next, under certain growth conditions on L, we verify that the minimizer solves the corresponding Euler–Lagrange equation. We assume

$$|L(\mathbf{p}, z, x)| \leq C(|\mathbf{p}|^q + |z|^q + 1), \tag{7.54}$$

and

$$|\nabla_{\mathbf{p}} L(\mathbf{p}, z, x)| + |\nabla_z L(\mathbf{p}, z, x)| \leq C(|\mathbf{p}|^{q-1} + |z|^{q-1} + 1) \tag{7.55}$$

for some $1 < q < \infty$.

Theorem 7.6.5 — Minimizer solves Euler–Lagrange equation. Assume conditions (7.54)–(7.55), where L is measurable in x for all $(\mathbf{p}, z) \in \mathbb{R}^d \times \mathbb{R}$ and differentiable in $\mathbf{p}$ and z for almost every $x \in \Omega$. Let u be a minimizer of I over $\mathcal{A} = \{u \in W^{1,q}(\Omega) \,|\, u = g \text{ on } \partial\Omega\}$, $\Omega \in C^{0,1}$ with q as in (7.54)–(7.55). Then u is a weak solution to the corresponding Euler–Lagrange equations, i.e.

$$\int_\Omega \Big(\sum_{i=1}^{d} \frac{\partial L}{\partial p_i}(\nabla u, u, \cdot)\frac{\partial v}{\partial x_i} + \frac{\partial L}{\partial z}(\nabla u, u, \cdot)v\Big)\,\mathrm{d}x = 0$$

for all $v \in W_0^{1,q}(\Omega)$.

Proof. We performed the formal proof at the beginning of Section 7.5. We now present a rigorous proof, based on the verification that we may differentiate the function (i.e., an integral) with respect to a parameter.

We set $i(\tau) := I[u + \tau v]$, where $v \in W^{1,q}(\Omega)$. Due to (7.54), $i(\tau)$ is finite. We study for $\tau \neq 0$ the difference quotients

$$\frac{i(\tau) - i(0)}{\tau} = \int_\Omega \frac{L(\nabla u + \tau\nabla v, u + \tau v, \cdot) - L(\nabla u, u, \cdot)}{\tau}\,\mathrm{d}x =: \int_\Omega L^\tau(\cdot)\,\mathrm{d}x$$

and aim at computing the limit when $\tau \to 0_+$. We see that due to our assumptions

$$\lim_{\tau\to 0} L^\tau(x) = \sum_{i=1}^{d} \frac{\partial L}{\partial p_i}(\nabla u(x), u(x), x)\frac{\partial v}{\partial x_i} + \frac{\partial L}{\partial z}(\nabla u(x), u(x), x)v(x)$$

almost everywhere in Ω. Using the Theorem on integrals dependent on a parameter (consequence of the Lebesgue dominated convergence Theorem) we get the desired equality since estimate (7.55) provides an integrable majorant. Since u is a minimizer of $I[\cdot]$, we know that $i'(0) = 0$. ■

Remark 7.6.6. If L is convex in $\mathbf{p}$ and z and differentiable in both variables almost everywhere in Ω, then each weak solution is a minimizer. In this case namely

$$L(\mathbf{p}, z, x) + \sum_{i=1}^{d} \frac{\partial L}{\partial p_i}(\mathbf{p}, z, x)(q_i - p_i) + \frac{\partial L}{\partial z}(\mathbf{p}, z, x)(r - z) \leq L(\mathbf{q}, r, x).$$

Taking $\mathbf{p} = \nabla u$, $\mathbf{q} = \nabla w$, $z = u$, $r = w$, integrating over Ω yields

$$I[u] + \int_\Omega \Big(\sum_{i=1}^{d} \frac{\partial L}{\partial p_i}(\nabla u, u, \cdot)(\frac{\partial w}{\partial x_i} - \frac{\partial u}{\partial x_i}) + \frac{\partial L}{\partial z}(\nabla u, u, \cdot)(w-u) \Big)\, \mathrm{d}x \leq I[w]. \tag{7.56}$$

Hence

$$I[u] \leq I[w]$$

for all $w \in \mathcal{A}$ due to the fact that $u - w = 0$ on Ω and u is a weak solution to the corresponding Euler–Lagrange equation. Then the integral on the left-hand side of (7.56) vanishes.

7.6.2 Vector case (system of equations)

Since the Fundamental Theorem of calculus of variations 7.6.1 holds also in this case, we can modify the previous conditions to get existence of a minimizer, its uniqueness and connection to the solution of (now system of) the Euler–Lagrange equations under basically the same assumptions as above. As before, we assume that L is measurable in x and sufficiently regular in the first two variables (this will be specified for each result separately). Furthermore, let

$$|L(\vec{\mathbf{P}}, \vec{z}, x)| \geq C_1 |\vec{\mathbf{P}}|^q - b(x), \qquad \text{coercivity, where } b \in L^1(\Omega) \tag{7.57}$$

$$L \text{ is convex in the variable } \vec{\mathbf{P}}, \tag{7.58}$$

and

$$\mathcal{A} = \Big\{ \vec{u} \in W^{1,q}(\Omega; \mathbb{R}^N) \,|\, \vec{u} = \vec{g} \text{ on } \partial\Omega \Big\} \text{ is non-empty}, \quad \Omega \in C^{0,1} \tag{7.59}$$

(i.e., $\vec{g} \in W^{1-\frac{1}{q},q}(\partial\Omega; \mathbb{R}^N)$). Then

Theorem 7.6.7 — Weak lower semicontinuity for convex Lagrangian. Under assumptions (7.57)–(7.59), let L be measurable in x for all $(\vec{\mathbf{P}}, \vec{z}) \in \mathbb{R}^{N\times d} \times \mathbb{R}^N$, differentiable in the first variable for all $\vec{z} \in \mathbb{R}^d$ and almost every $x \in \Omega$ and continuous in the second variable fo r all $\vec{\mathbf{P}} \in \mathbb{R}^{N\times d}$ and almost every $x \in \Omega$. Then there exists a minimizer of $I[\vec{w}] = \int_\Omega L(\nabla \vec{w}, \vec{w}, \cdot)\, \mathrm{d}x$ over the set $\mathcal{A}$, i.e. a function $\vec{u} \in \mathcal{A}$ such that

$$I[\vec{u}] \leq I[\vec{w}] \qquad \forall \vec{w} \in \mathcal{A}.$$

If we assume further that

$$L = L(\vec{\mathbf{P}}, x), \qquad \sum_{i,j=1}^{d} \sum_{k,l=1}^{N} \frac{\partial^2 L}{\partial p_i^k \partial p_j^l}(\vec{\mathbf{P}}, x) \xi_i^k \xi_j^l \geq \theta |\vec{\boldsymbol{\xi}}|^2 \tag{7.60}$$

for all $\vec{\boldsymbol{\xi}} \in \mathbb{R}^{N\times d}$ and some $\theta > 0$ (for all $\vec{\mathbf{P}} \in \mathbb{R}^{N\times d}$ and almost every $x \in \Omega$), then

Theorem 7.6.8 — Uniqueness of minimizers. Under the assumptions (7.60) and (7.59), the minimizer of $I[\mathbf{w}]$ (provided it exists), is unique.

Finally, let

$$L(\vec{\mathbf{P}}, \mathbf{z}, x)| \leq C(|\vec{\mathbf{P}}|^q + |\mathbf{z}|^q + 1), \tag{7.61}$$

and

$$|\nabla_{\vec{\mathbf{P}}} L(\vec{\mathbf{P}}, \mathbf{z}, x)| + |\nabla_{\mathbf{z}} L(\vec{\mathbf{P}}, \mathbf{z}, x)| \leq C(|\vec{\mathbf{P}}|^{q-1} + |\mathbf{z}|^{q-1} + 1) \tag{7.62}$$

for some $1 < q < \infty$. Then

Theorem 7.6.9 — Minimizers fulfil the Euler–Lagrange equations. Let L, continuously differentiable in the first two variables for almost every $x \in \Omega$ and measurable in x for all $(\vec{\mathbf{P}}, \vec{z}) \in \mathbb{R}^{N\times d} \times \mathbb{R}^N$ satisfy (7.61) and (7.62). Let $\vec{u}$ be a minimizer of $I[\cdot]$ over the set $\mathcal{A}$ from (7.59). Then $\vec{u}$ is a solution to the corresponding system of the Euler–Lagrange equations, i.e., $\vec{u} \in \mathcal{A}$ and

$$\sum_{k=1}^{N} \int_\Omega \Big(\sum_{i=1}^{d} \frac{\partial L}{\partial p_i^k}(\nabla \vec{u}, \vec{u}, \cdot) \frac{\partial v^k}{\partial x_i} + \frac{\partial L}{\partial z^k}(\nabla \vec{u}, \vec{u}, \cdot) v^k \Big)\, \mathrm{d}x = 0$$

for any $\vec{v} \in W_0^{1,q}(\Omega; \mathbb{R}^N)$.

In applications (e.g., in the elasticity theory), the convexity of the Lagrangian in $\mathbb{P}$ (here naturally $d = N$) contradicts physical principles. However, we may replace the convexity by so-called polyconvexity, namely that for $d = N$ the Lagrangian is a convex function of $\mathbb{P}$ and of the determinant of $\mathbb{P}$, see below. The fact that the theory works perfectly in this case is based on the following partially counterintuitive result which has close connection to problems of compensated compactness.

Lemma 7.6.10 — Weak compactness of the determinant function. Let $d < q < \infty$ and let

$$\mathbf{u_k} \rightharpoonup \mathbf{u} \qquad \text{(weakly) in } W^{1,q}(\Omega;\mathbb{R}^d),$$

where $\Omega \subset \mathbb{R}^d$ is at least of class $C^{0,1}$. Then

$$\det \nabla \mathbf{u}_k \rightharpoonup \det \nabla \mathbf{u} \qquad \text{(weakly) in } L^{\frac{q}{d}}(\Omega).$$

Proof. Recall that we have $(\det \mathbb{P})\mathbb{I} = \mathbb{P}(\operatorname{cof}\mathbb{P})^T$ (see Lemma 7.5.6). Hence

$$\det \mathbb{P} = \sum_{j=1}^d p_j^i (\operatorname{cof}\mathbb{P})_j^i, \qquad i = 1,2,\dots,d.$$

Step 1: First integration by parts
Let $\mathbf{w} \in C^\infty(\Omega;\mathbb{R}^d)$. Then

$$\det(\nabla \mathbf{w}) = \sum_{j=1}^d \frac{\partial w^i}{\partial x_j}(\operatorname{cof}\nabla\mathbf{w})_j^i, \qquad i = 1,2,\dots,d.$$

Due to Lemma 7.5.6

$$\sum_{j=1}^d \frac{\partial}{\partial x_j}\Big[\Big(\operatorname{cof}\nabla\mathbf{w}\Big)_j^i\Big] = 0, \qquad i = 1,2,\dots,d.$$

Therefore

$$\det(\nabla \mathbf{w}) = \sum_{j=1}^d \frac{\partial}{\partial x_j}\Big(w^i\Big(\operatorname{cof}\nabla\mathbf{w})_j^i\Big), \qquad i = 1,2,\dots,d.$$

Hence for arbitrary $v \in C_0^\infty(\Omega)$

$$\int_\Omega v\det(\nabla\mathbf{w})\,\mathrm{d}x = -\sum_{j=1}^d \int_\Omega w^i\Big(\operatorname{cof}\nabla\mathbf{w}\Big)_j^i \frac{\partial v}{\partial x_j}\,\mathrm{d}x, \qquad i = 1,2,\dots,d.$$

Step 2: Induction in the integration by parts
Due to the last inequality

$$\int_\Omega v\det(\nabla\mathbf{u}_k)\,\mathrm{d}x = -\sum_{j=1}^d \int_\Omega u_k^i\Big(\operatorname{cof}\nabla\mathbf{u}_k\Big)_j^i \frac{\partial v}{\partial x_j}\,\mathrm{d}x, \qquad i = 1,2,\dots,d,\ k\in\mathbb{N}.$$

As $W^{1,q}(\Omega) \hookrightarrow\hookrightarrow C(\overline{\Omega})$, the sequence $\mathbf{u}_k \to \mathbf{u}$ in $C(\overline{\Omega};\mathbb{R}^d)$. (In fact, the compact embedding ensures this only for a subsequence, but as $\mathbf{u}_k \rightharpoonup \vec{u}$ in $W^{1,q}(\Omega;\mathbb{R}^d)$, the convergence holds for the whole sequence). Assume for a moment that we know

$$\int_\Omega \psi\Big(\operatorname{cof}\nabla\mathbf{u}_k)_j^i\,\mathrm{d}x \to \int_\Omega \psi\Big(\operatorname{cof}\nabla\mathbf{u}\Big)_j^i\,\mathrm{d}x, \qquad i,j = 1,2,\dots,d$$

for any $\psi \in C_0^\infty(\Omega)$. Then we have the desired weak convergence

$$\int_\Omega v\det(\nabla\mathbf{u}_k)\,\mathrm{d}x = -\sum_{j=1}^d \int_\Omega u_k^i\Big(\operatorname{cof}\nabla\mathbf{u}_k\Big)_j^i \frac{\partial v}{\partial x_j}\,\mathrm{d}x$$
$$\to -\sum_{j=1}^d \int_\Omega u^i\Big(\operatorname{cof}\nabla\mathbf{u})_j^i \frac{\partial v}{\partial x_j}\,\mathrm{d}x = \int_\Omega v\det(\nabla\mathbf{u})\,\mathrm{d}x.$$

However, $(\operatorname{cof}\nabla\mathbf{u}_k)$ is a matrix whose entries are determinants of matrices of the type $(d-1)\times(d-1)$. Therefore they can be analyzed as above and they can be rewritten as above after integration by parts to a product of a determinant of order $(d-2)\times(d-2)$ of derivatives of the function and the function itself. After finitely many steps we end up with a product of a derivative and the function; the function converges strongly even in the $C(\overline{\Omega})$ and the derivative converges weakly in $L^q(\Omega)$.
Step 3: Weak convergence of the determinant
As

$$\int_\Omega v\det(\nabla\mathbf{u}_k)\,\mathrm{d}x \to \int_\Omega v\det(\nabla\mathbf{u})\,\mathrm{d}x$$

for any $v \in C_0^\infty(\Omega)$ and the sequence $\det(\nabla\mathbf{u}_k)$ is bounded in $L^{\frac{q}{d}}(\Omega)$, by density argument we conclude the weak convergence in the same space. ■

Assume now that $N = d$ and let

$$L(\mathbb{P}, \mathbf{z}, x) = F(\mathbb{P}, \det \mathbb{P}, \mathbf{z}, x).$$

Finally, let

$$\begin{aligned} &\forall \mathbf{z} \in \mathbb{R}^d, \text{ almost every } x \in \Omega \text{ the mapping} \\ &\qquad (\mathbb{P}, r) \mapsto F(\mathbb{P}, r, \mathbf{z}, x) \end{aligned} \tag{7.63}$$

be convex. Additionally, the function F is measurable in x for all $(\mathbb{P}, r, \mathbf{z}) \in \mathbb{R}^{d\times d} \times \mathbb{R} \times \mathbb{R}^d$, continuously differentiable in the first two variables for all $\mathbf{z} \in \mathbb{R}$ and almost all $x \in \Omega$ and continuous in the third variable for all $(\mathbb{P}, r) \in \mathbb{R}^{d\times d} \times \mathbb{R}$ and almost all $x \in \Omega$. We call such Lagrangians polyconvex.

Lemma 7.6.11 — Polyconvexity implies weak lower semicontinuity. Let $d < q < \infty$, L be bounded from below, smooth as stated above and polyconvex, $\Omega \in C^{0,1}$. Then the functional

$$I[\mathbf{w}] = \int_\Omega L(\nabla \mathbf{w}, \mathbf{w}, \cdot)\,\mathrm{d}x = \int_\Omega L(\nabla \mathbf{w}, \det(\nabla \mathbf{w}), \mathbf{w}, \cdot)\,\mathrm{d}x$$

is weakly lower semicontinuous on $W^{1,q}(\Omega; \mathbb{R}^d)$.

Proof. We proceed similarly as in the proof of Lemma 7.6.2. Let $\mathbf{u}_k \rightharpoonup \mathbf{u}$ in $W^{1,q}(\Omega; \mathbb{R}^d)$. Then due to Lemma 7.6.11 $\det(\nabla \mathbf{u}_k) \rightharpoonup \det(\nabla \mathbf{u})$ in $L^{\frac{q}{d}}(\Omega)$. We define the set G_ε similarly as in Lemma 7.6.2 and assume without loss of generality that $I \geq 0$. Then

$$\begin{aligned} I[\mathbf{u}_k] &= \int_\Omega L(\nabla \mathbf{u}_k, \mathbf{u}_k, \cdot)\,\mathrm{d}x \geq \int_{G_\varepsilon} L(\nabla \mathbf{u}_k, \mathbf{u}_k, \cdot)\,\mathrm{d}x \\ &= \int_{G_\varepsilon} F(\nabla \mathbf{u}_k, \det(\nabla \mathbf{u}_k), \mathbf{u}_k, \cdot)\,\mathrm{d}x \geq \int_{G_\varepsilon} F(\nabla \mathbf{u}, \det(\nabla \mathbf{u}), \mathbf{u}_k, \cdot)\,\mathrm{d}x \\ &\quad + \int_{G_\varepsilon} \sum_{i,j=1}^d \frac{\partial F}{\partial p^i_j}(\nabla \mathbf{u}, \det(\nabla \mathbf{u}), \mathbf{u}_k, \cdot)\Big(\frac{\partial u^i_k}{\partial x_j} - \frac{\partial u_i}{\partial x_j}\Big)\,\mathrm{d}x \\ &\quad + \int_{G_\varepsilon} \frac{\partial F}{\partial r}(\nabla \mathbf{u}, \det(\nabla \mathbf{u}), \mathbf{u}_k, \cdot)\Big(\det(\nabla \mathbf{u}_k) - \det(\nabla \mathbf{u})\Big)\,\mathrm{d}x. \end{aligned}$$

We finish the proof as in Lemma 7.6.2. ■

We can therefore prove, exactly as above

Theorem 7.6.12 — Existence of a minimizer for polyconvex Lagrangians. Let $d < q < \infty$, let L be as in Lemma 7.6.11 and fulfil (7.57) and (7.63). Let $\mathcal{A}$ be non-empty. Then there exists $\mathbf{u} \in \mathcal{A}$ such that

$$I[\mathbf{u}] = \min_{\mathbf{w} \in \mathcal{A}} I[\mathbf{w}].$$

Chapter 8

A more detailed guide to Lebesgue–Bochner and Sobolev–Bochner spaces

In this chapter, we introduce in more details the necessary function spaces and tools needed for the correct formulation of the evolutionary partial differential equations in the weak setting. We present proofs for almost all results, even though part of the text will rather remind a textbook in functional analysis or measure theory; therein, however, such results are not always carefully studied due to the fact that most of the applications of such results are directly connected with the theory of partial differential equations.

We aim at studying mappings

$$f\colon I \subset \mathbb{R} \to X,$$

where X is a Banach space. Most of the results in this section remain true even if we replace the one-dimensional interval by a measurable subset of $\mathbb{R}^d$, however, since we do not need this level of generality, we remain in $\mathbb{R}$. Moreover, instead of the Lebesgue measure on $\mathbb{R}$ we may as well consider a more general measure μ. The space X is always a Banach space; sometimes, however, we will require more properties (separability, reflexivity etc.). For simplicity, we always assume in this chapter that $I \subset \mathbb{R}$ is open and bounded, even though most of the results remain true for arbitrary interval in $\mathbb{R}$; however, the proofs must be slightly modified in this case. The presentation below follows in the first part nicely written thesis Kreuter (2015), the second part is then based on Boyer and Fabrie (2006) or more precisely, on Pokorný (2022).

8.1 Bochner integral

First, we construct a new type of integral, in order to be able to integrate functions with values in general Banach spaces.

Definition 8.1.1 — Simple function. A function $s\colon I \to X$ is called a simple function, if we can write

$$s(t) = \sum_{i=1}^{n} x_i \chi_{E_i}(t), \tag{8.1}$$

where $x_i \in X$, E_i are pairwise disjoint, measurable, and $\sum_{i=1}^{n} \lambda_1(E_i) < +\infty$.

Remark 8.1.2. Assuming $x_i \neq x_j$ for $i \neq j$ and $\cup_{i=1}^{n} E_i = I$ (it also means that we define s to be the zero element on the set of one dimensional Lebesgue measure zero, where the function is possibly not defined), then the representation of the function s in the form (8.1) is unique.

Definition 8.1.3 — Strong and weak measurability.

1. A function $f\colon I \to X$ is (strongly) measurable, if there exists a sequence of simple functions $\{s_n\}_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \|s_n(t) - f(t)\|_X = 0$$

for almost all $t \in I$.

2. A function $f\colon I \to X$ is weakly measurable, if

$$\langle x', f\rangle_X$$

is measurable for all $x' \in X^*$, where X^* denotes the dual space to X.

Definition 8.1.4 — Almost separably valued function. A function $f\colon I \to X$ is called almost separably valued, if there exists a set $N \subset I$, $\lambda_1(N) = 0$ such that $f(I \setminus N)$ is separable. If N is empty, then we say that the function f is separably valued.

We have the following important result

Theorem 8.1.5 — Pettis. A function $f\colon I \to X$ is measurable, if and only if f is weakly measurable and almost separably valued.

Proof. **Step 1: "$\Longrightarrow$"**
If f is measurable, then there exists $\{s_n\}_{n\in\mathbb{N}}$ a sequence of simple functions such that $s_n \to f$ almost everywhere in I (in the norm of X). Then $\langle x', s_n\rangle_X \to \langle x', f\rangle_X$ pointwise almost everywhere in I which implies that f is weakly measurable. Moreover, up to a set of measure zero, f takes values in the closure of values taken by $\{s_n\}_{n\in\mathbb{N}}$, therefore f is almost separably valued.

Step 2: "$\Longleftarrow$"
Let f be weakly measurable and almost separably valued. Let $N \subset I$ be a null set such that $f(I \setminus N)$ is separable. Let $\{x_n\}_{n\in\mathbb{N}}$ be dense in $f(I \setminus N)$. Due to the Hahn–Banach Theorem there exists a sequence $\{x'_n\}_{n\in\mathbb{N}} \subset X^*$ such that $\|x'_n\|_{X^*} = 1$ and $\langle x'_n, x_n\rangle = \|x_n\|_X$. Let $t \in I \setminus N$ and let $x_{n_k} \to f(t)$. Then for every $\varepsilon > 0$ there exists $k \in \mathbb{N}$ (sufficiently large) such that

$$\begin{aligned}\langle x'_{n_k}, f(t)\rangle_X &\le \|f(t)\|_X \le \|x_{n_k}\|_X + \varepsilon = \langle x'_{n_k}, x_{n_k}\rangle_X + \varepsilon \\ &= \langle x'_{n_k}, x_{n_k} - f(t)\rangle_X + \langle x'_{n_k}, f(t)\rangle_X + \varepsilon \le \langle x'_{n_k}, f(t)\rangle_X + 2\varepsilon.\end{aligned}$$

Letting $\varepsilon \to 0_+$ and recalling that $\|x'_{n_k}\|_{X^*} = 1$ we get

$$\|f(t)\|_X = \sup_{k\in\mathbb{N}} \langle x'_{n_k}, f(t)\rangle_X,$$

therefore, as $\|f(t)\|_X$ is a pointwise supremum of countably many measurable functions, it is measurable.

Define

$$f_n(t) := \|f(t) - x_n\|_X,$$

then as above, f_n is measurable. Let $\varepsilon > 0$ and $E_n = \{t \in I \,|\, f_n(t) \le \varepsilon\}$. Then E_n is measurable. Define $g\colon I \to X$ as follows

$$g(t) := \begin{cases} x_n & \text{if } t \in E_n \setminus \bigcup_{m<n} E_m \\ 0 & \text{otherwise.} \end{cases}$$

Then $\|f - g\|_X \le \varepsilon$ almost everywhere (as $\{x_n\}_{n\in\mathbb{N}}$ is dense in $f(I \setminus N)$). Let $\varepsilon_m = 2^{-m}$, $m \in \mathbb{N}$. We thus construct a sequence $g_m = \sum_{n=1}^{\infty} x_{n,m}\chi_{E_{n,m}}$, where $x_{n,m} \in X$ and $\cup_{i\in\mathbb{N}} E_{n,m} = I$, of countably valued functions which converge to f almost everywhere. For each $m \in \mathbb{N}$, let $F_m = \cup_{n=1}^{k_m} E_{n,m}$, where k_m is chosen so large that $\lambda_1(I \setminus F_m) < 2^{-m}$.

Let $s_m := g_m \chi_{F_m}$. Then $\{s_m\}_{m\in\mathbb{N}}$ are simple functions, $s_m \to f$ almost everywhere in I. To see this, let $t \in \cap_{m=k}^{\infty} F_m$ for some $k \in \mathbb{N}$. Then for all $m > k$ we have $s_m(t) = g_m(t)$, thus $\|f(t) - s_m(t)\|_X < 2^{-m}$ and therefore $s_m(t) \to f(t)$ for all $t \in \cup_{k=1}^{\infty} \cap_{m=k}^{\infty} F_m$. But for each $k \in \mathbb{N}$ we have

$$\lambda_1(I \setminus \bigcap_{m=k}^{\infty} F_m) \le \sum_{m=k}^{\infty} \lambda_1(I \setminus F_m) < 2^{-k+1},$$

whence $I \setminus \cup_{k=1}^{\infty} \cap_{m=k}^{\infty} F_m$ is a null set. Thus $s_m \to f$ outside a null set. ∎

We have the following corollaries

Corollary 8.1.6. Let $f\colon I \to X$ be continuous. Then f is measurable.

Proof. As f is continuous, then f is separably valued (recall that rational numbers from I are dense in I). Moreover, for any $x' \in X^*$, the duality $\langle x', f\rangle_X$ is continuous and thus measurable. We may therefore use the Pettis Theorem 8.1.5. ∎

Corollary 8.1.7. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of measurable functions from I to X such that $f_n(t) \rightharpoonup f(t)$ in X for almost every $t \in I$. Then f is measurable.

Proof. For any $x' \in X^*$ and t outside a null set $N \subset I$ we have $\langle x', f_n(t)\rangle_X \to \langle x', f(t)\rangle_X$. Thus $\langle x', f\rangle_X$ as a pointwise almost everywhere limit of measurable functions is measurable. Let us show that f is almost separably valued. For each $n \in \mathbb{N}$ choose E_n, a null set such that $f_n(I \setminus E_n)$ lies in a separable subspace X_n. Let $E = \cup_{n\in\mathbb{N}} E_n \cup N$. Then $\lambda_1(E) = 0$ and $f|_{I\setminus E}$ takes values in the weak closure of the span of $\cup_{n\in\mathbb{N}} X_n$. As this set is convex, then this set is also closed by the corollary of the Mazur theorem B.2.15 and thus $f|_{I\setminus E}$ is separable. We may therefore use Theorem 8.1.5. ∎

We are now ready to define the Bochner integral. For a simple function s, we may evidently set

$$\int_I s \, \mathrm{d}\lambda_1 := \sum_{i=1}^n x_i \lambda_1(E_i),$$

where $s = \sum_{i=1}^n x_i \chi_{E_i}$. It can be easily shown, similarly as for the Lebesgue integral, that this definition is independent of the representative of s and that

$$\Big\| \int_I s \, \mathrm{d}\lambda_1 \Big\|_X \leq \int_I \|s\|_X \, \mathrm{d}t.$$

We now extend the definition to measurable functions.

Definition 8.1.8 — Bochner integral. A measurable function $f \colon I \to X$ is Bochner integrable, if there exists a sequence $\{s_n\}_{n\in\mathbb{N}}$ of simple functions converging to f almost everywhere such that

$$\lim_{n\to\infty} \int_I \|f - s_n\|_X \, \mathrm{d}t = 0.$$

Then for such a function f we define

$$\int_I f \, \mathrm{d}\lambda_1 := \lim_{n\to\infty} \int_I s_n \, \mathrm{d}\lambda_1.$$

Note that the definition is correct in the sense that

$$\Big\| \int_I s_n \, \mathrm{d}\lambda_1 - \int_I s_m \, \mathrm{d}\lambda_1 \Big\|_X \leq \int_I \|s_n - s_m\|_X \, \mathrm{d}t \leq \int_I \|s_n - f\|_X \, \mathrm{d}t + \int_I \|s_m - f\|_X \, \mathrm{d}t \to 0 \qquad \text{as } n, m \to \infty.$$

Hence $\int_I s_n \, \mathrm{d}\lambda_1$ is a Cauchy sequence and due to the completeness of X it is convergent. If $\{\widetilde{s}_n\}_{n\in\mathbb{N}}$ is another sequence of simple functions approximating f in X almost everywhere together with $\lim_{n\to\infty} \int_I \|f - \widetilde{s}_n\|_X \, \mathrm{d}t = 0$, it is easy to see that the limits of sequences $\Big\{ \int_I s_n \, \mathrm{d}\lambda_1 \Big\}_{n\in\mathbb{N}}$ and $\Big\{ \int_I \widetilde{s}_n \, \mathrm{d}\lambda_1 \Big\}_{n\in\mathbb{N}}$ are the same which shows that the definition of the integral is independent of the approximating sequence of the simple functions. The linearity of the Bochner integral is a consequence of the linearity of the integral for simple functions.

The following fundamental theorem yields the correspondence between the Bochner and Lebesgue integrals.

Theorem 8.1.9 — Bochner. Let $f \colon I \to X$ be a measurable function. Then f is Bochner integrable over I, if and only if $\|f\|_X$ is Lebesgue integrable over I. Moreover,

$$\Big\| \int_I f \, \mathrm{d}\lambda_1 \Big\|_X \leq \int_I \|f\|_X \, \mathrm{d}t. \tag{8.2}$$

Proof. **Step 1: "$\Longrightarrow$"**

Let f be Bochner integrable, $\{s_n\}_{n\in\mathbb{N}}$ be the corresponding sequence of simple functions. As $\Big| \|f\|_X - \|s_n\|_X \Big| \leq \|f - s_n\|_X$, we see that $\|f\|_X$ is the almost everywhere limit of $\{\|s_n\|_X\}_{n\in\mathbb{N}}$ and thus $\|f\|_X$ is Lebesgue measurable. Moreover, we have

$$\int_I \|f\|_X \, \mathrm{d}t \leq \int_I \|f - s_n\|_X \, \mathrm{d}t + \int_I \|s_n\|_X \, \mathrm{d}t.$$

The second integral on the right-hand side is finite due to the properties of the simple functions, while the first integral is finite due to the definition of the Bochner integral, at least for n sufficiently large. Therefore $\|f\|_X$ is Lebesgue integrable over I. Furthermore,

$$\Big\| \int_I f \, \mathrm{d}\lambda_1 \Big\|_X = \lim_{n\to\infty} \Big\| \int_I s_n \, \mathrm{d}\lambda_1 \Big\|_X \leq \liminf_{n\to\infty} \int_I \|s_n\|_X \, \mathrm{d}t,$$

as $\int_I s_n \, \mathrm{d}\lambda_1 \to \int_I f \, \mathrm{d}\lambda_1$ in X. Since $\int_I \|f - s_n\|_X \, \mathrm{d}t \to 0$ and

$$\int_I \|s_n\|_X \, \mathrm{d}t \leq \int_I \|f - s_n\|_X \, \mathrm{d}t + \int_I \|f\|_X \, \mathrm{d}t,$$

we have that

$$\limsup_{n\to\infty} \int_I \|s_n\|_X \, \mathrm{d}t \leq \int_I \|f\|_X \, \mathrm{d}t.$$

which proves the second claim of the theorem as $\lim_{n\to\infty} \int_I \|s_n\|_X \, \mathrm{d}t = \int_I \|f\|_X \, \mathrm{d}t$.

Step 2: "$\Longleftarrow$"
Let $\|f\|_X$ be measurable and $\int_I \|f\|_X \,\mathrm{d}t < \infty$. Let $\{s_n\}_{n\in\mathbb{N}}$ be a sequence of simple functions converging to f almost everywhere. We define new simple functions

$$\widetilde{s}_n(t) := \begin{cases} s_n(t) & \text{if } \|s_n(t)\|_X \leq 2\|f(t)\|_X, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\|\widetilde{s}_n - f\|_X \to 0$ almost everywhere as well as $\|\widetilde{s}_n - f\|_X$ are measurable for any $n \in \mathbb{N}$. Thus

$$\|(\widetilde{s}_n - f)(t)\|_X \leq \|\widetilde{s}_n(t)\|_X + \|f(t)\|_X \leq 3\|f(t)\|_X.$$

As $\|f\|_X$ is integrable, the Lebesgue dominated convergence Theorem A.3.4 yields

$$\int_I \|\widetilde{s}_n - f\|_X \,\mathrm{d}t \to 0.$$

Therefore f is Bochner integrable. ∎

Corollary 8.1.10 (Dominated convergence Theorem). Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of Bochner integrable functions and let f be such that $f_n \to f$ almost everywhere in I. Let $g \in L^1(I;\mathbb{R})$ be such that $\|f_n\|_X \leq g$ for all $n \in \mathbb{N}$ almost everywhere in I. Then f is Bochner integrable and

$$\int_I f \,\mathrm{d}\lambda_1 = \lim_{n\to\infty} \int_I f_n \,\mathrm{d}\lambda_1.$$

Proof. By the Lebesgue dominated convergence Theorem A.3.4 we know that $\|f\|_X$ is integrable, by Corollary 8.1.7 f is measurable and thus f is Bochner integrable. Now

$$\|f - f_n\|_X \leq \|f\|_X + \|f_n\|_X \leq \|f\|_X + g,$$

hence we may apply again the Lebesgue dominated convergence Theorem to compute

$$\Big\| \int_I f_n \,\mathrm{d}\lambda_1 - \int_I f \,\mathrm{d}\lambda_1 \Big\|_X \leq \int_I \|f_n - f\|_X \,\mathrm{d}t \to 0,$$

i.e., $\int_I f_n \,\mathrm{d}\lambda_1 \to \int_I f \,\mathrm{d}\lambda_1$. ∎

Corollary 8.1.11. Let f be Bochner integrable. Then the sequence s_n of simple functions converging to f can be chosen so that $\|s_n(t)\|_X \leq 2\|f(t)\|_X$ holds for almost every $t \in I$.

Proof. We can use the construction from Theorem 8.1.9. ∎

Corollary 8.1.12. Let $x' \in X^*$ and let f be Bochner integrable over I. Then

$$\int_I \langle x', f\rangle_X \,\mathrm{d}t = \Big\langle x', \int_I f \,\mathrm{d}t \Big\rangle_X.$$

Proof. By definition, for every simple function s and every $x' \in X^*$,

$$\int_I \langle x', s\rangle_X \,\mathrm{d}t = \Big\langle x', \int_I s \,\mathrm{d}\lambda_1 \Big\rangle_X.$$

Let $\{s_n\}_{n\in\mathbb{N}}$ be a sequence of simple function from the definition of the Bochner integral of f, possibly modified by Corollary 8.1.11, i.e., $\|s_n(t)\|_X \leq 2\|f(t)\|_X$ almost everywhere in I. Then $\langle x', s_n\rangle_X \to \langle x', f\rangle_X$ almost everywhere in I and $|\langle x', s_n(t)\rangle_X| \leq 2\|x'\|_{X^*}\|f(t)\|_X$ almost everywhere in I. Thus by the Lebesgue dominated convergence Theorem A.3.4

$$\int_I \langle x', f\rangle_X \,\mathrm{d}t = \lim_{n\to\infty} \int_I \langle x', s_n\rangle_X \,\mathrm{d}t = \lim_{n\to\infty} \Big\langle x', \int_I s_n \,\mathrm{d}\lambda_1 \Big\rangle_X = \Big\langle x', \int_I f \,\mathrm{d}\lambda_1 \Big\rangle_X,$$

where the last equality follows from the continuity of x' and the definition of the Bochner integral. ∎

Corollary 8.1.13. Let f be Bochner integrable over I. Then

$$\lim_{\lambda_1(J)\to 0^+, J\subset I} \int_J f \,\mathrm{d}\lambda_1 = 0 \in X.$$

Proof. As $\|f\|_X$ is Lebesgue integrable by the Bochner Theorem 8.1.9, we have

$$\Big\| \int_J f \,\mathrm{d}\lambda_1 \Big\|_X \leq \int_J \|f\|_X \,\mathrm{d}t \to 0$$

as $J \subset I$, $\lambda_1(J) \to 0$, hence $\int_J f \,\mathrm{d}\lambda_1 \to 0 \in X$. ∎

We finish by a generalization of the Fubini theorem for the Bochner integral.

Theorem 8.1.14 — Fubini. Let $J = I_1 \times I_2$ be a product measure space with respect to the measure $\lambda_1 \otimes \lambda_1 = \lambda_2$ and let $f\colon J \to X$ be measurable. Assume that the integral

$$\int_{I_1} \Big(\int_{I_2} \|f(t_1,t_2)\|_X \,\mathrm{d}t_2 \Big) \,\mathrm{d}t_1 \tag{8.3}$$

exists and is finite. Then f is Bochner integrable over $I_1 \times I_2$ and we have

$$\int_{I_1\times I_2} f \,\mathrm{d}\lambda_2 = \int_{I_1} \Big(\int_{I_2} f(t_1,t_2) \,\mathrm{d}\lambda_1(t_2) \Big) \,\mathrm{d}\lambda_1(t_1) = \int_{I_2} \Big(\int_{I_1} f(t_1,t_2) \,\mathrm{d}\lambda_1(t_1) \Big) \,\mathrm{d}\lambda_1(t_2). \tag{8.4}$$

Conversely, if f is Bochner integrable over $I_1 \times I_2$, then the integrals above exist and the equalities in (8.4) hold.

Proof. If integral (8.3) exists, then the Fubini–Tonelli Theorem for the scalar case (real valued functions) implies that $\|f\|_X$ is integrable and therefore also f is integrable over J by virtue of the Bochner Theorem 8.1.9. Further, the integrals $\int_{I_i} \|f(t_1,t_2)\|_X \,\mathrm{d}t_i$, $i = 1,2$ also exist almost everywhere on I_j, $i \cdot j = 2$, and the same is true for $\int_{I_i} f(t_1,t_2) \,\mathrm{d}\lambda_1(t_i)$, $i = 1,2$. Since f is almost separably valued, then the functions $t_j \mapsto \int_{I_i} f(t_1,t_2) \,\mathrm{d}\lambda_1(t_i)$ are almost separably valued as well. For arbitrary $x' \in X^*$ the duality pairing $\langle x', f\rangle_X$ is measurable and integrable as $|\langle x', f\rangle_X| \le \|x'\|_{X^*}\|f\|_X$.

By virtue of the scalar version of the Fubini Theorem the functions

$$t_j \mapsto \int_{I_i} \langle x', f(t_1,t_2)\rangle \,\mathrm{d}t_i,$$

$i \cdot j = 2$, are measurable and integrable. By Corollary 8.1.12, these functions are equal to

$$\Big\langle x', \int_{I_i} f(t_1,t_2) \,\mathrm{d}\lambda_1(t_i) \Big\rangle_X.$$

The Pettis Theorem 8.1.5 yields that $t_j \mapsto \int_{I_i} f(t_1,t_2) \,\mathrm{d}t_i$, $i \cdot j = 2$ are measurable and thus by the Bochner Theorem 8.1.9 integrable. For $x' \in X^*$ the Fubini Theorem for scalar functions yields that

$$\int_{I_1\times I_2} \langle x', f(t_1,t_2)\rangle_X \,\mathrm{d}t_1 \,\mathrm{d}t_2 = \int_{I_1} \Big(\int_{I_2} \langle x', f(t_1,t_2)\rangle_X \,\mathrm{d}t_2 \Big) \,\mathrm{d}t_1 = \int_{I_2} \Big(\int_{I_1} \langle x', f(t_1,t_2)\rangle_X \,\mathrm{d}t_1 \Big) \,\mathrm{d}t_2.$$

Therefore we may interchange the integration and the duality pairing with x' in the line above and the claim holds.

Conversely, if f is Bochner integrable, then $\|f\|_X$ is integrable due to the Bochner Theorem. The Fubini–Tonelli Theorem for the scalar case yields that

$$\int_{I_1\times I_2} \|f(t_1,t_2)\|_X \,\mathrm{d}t_1 \,\mathrm{d}t_2 = \int_{I_1} \Big(\int_{I_2} \|f(t_1,t_2)\|_X \,\mathrm{d}t_2 \Big) \,\mathrm{d}t_1 = \int_{I_2} \Big(\int_{I_1} \|f(t_1,t_2)\|_X \,\mathrm{d}t_1 \Big) \,\mathrm{d}t_2$$

exists and we may use the first part of the proof. ■

8.2 The spaces $L^p(I;X)$ (Lebesgue–Bochner spaces)

We now introduce an analogue of the Lebesgue spaces.

Definition 8.2.1 — Lebesgue–Bochner spaces. A measurable function $f \in L^p(I;X)$, $1 \le p \le \infty$, if for $1 \le p < \infty$

$$\int_I \|f\|_X^p \,\mathrm{d}t < \infty,$$

and for $p = \infty$

$$\operatorname*{ess\,sup}_I \|f\|_X < \infty.$$

Remark 8.2.2. Note that a function from $L^p(I;X)$ is (recall that I is bounded) Bochner integrable. In case we would accept the possibility that I is unbounded, then the function is at least locally Bochner integrable over I. We also introduce the notation

$$\|f\|_{L^p(I;X)} := \begin{cases} \Big(\int_I \|f\|_X^p \,\mathrm{d}t \Big)^{\frac{1}{p}} & \text{if } 1 \le p < \infty \\ \operatorname{ess\,sup}_I \|f\|_X & \text{if } p = \infty. \end{cases}$$

We now prove several results which are more or less similar to the standard results for Lebesgue spaces and which will finally lead to the result that the spaces defined above (often called Lebesgue–Bochner spaces) are complete normed spaces.

Lemma 8.2.3 Let $1 \le p \le \infty$, $f \in L^p(I;X)$ and $g \in L^{p'}(I;\mathbb{R})$, $\frac{1}{p}+\frac{1}{p'}=1$. Then $f\cdot g \in L^1(I;X)$ and

$$\Big\| \int_I fg \,\mathrm{d}\lambda_1 \Big\|_X \le \int_I \|f\|_X |g| \,\mathrm{d}t \le \|f\|_{L^p(I;X)} \|g\|_{L^{p'}(I)}.$$

Proof. Evidently, fg is measurable as both functions are limits of simple functions. Using now the Hölder inequality,

$$\int_I \|fg\|_X \,\mathrm{d}t = \int_I \|f\|_X |g| \,\mathrm{d}t \le \|f\|_{L^p(I;X)} \|g\|_{L^{p'}(I)}.$$

By virtue of the Bochner Theorem 8.1.9 we conclude $fg \in L^1(I;X)$ and the second part of the estimate holds. The first part is trivial. ■

Lemma 8.2.4 Let $1 \le p \le \infty$, $f \in L^p(I;X)$ and $g \in L^{p'}(I;X^*)$, $\frac{1}{p}+\frac{1}{p'}=1$ and X^* is the dual space to X. Then $\langle g,f\rangle_X \in L^1(I;\mathbb{R})$ and

$$\Big| \int_I \langle g,f\rangle_X \,\mathrm{d}t \Big| \le \|f\|_{L^p(I;X)} \|g\|_{L^{p'}(I;X^*)}.$$

Proof. The proof is similar to Lemma 8.2.3 as $\langle g,f\rangle_X$ is measurable due to the fact that it is a limit of $\langle s_n', s_n\rangle_X$ which is a sequence of simple functions with values in $\mathbb{R}$. The estimate follows then by virtue of the standard Hölder inequality. ■

Corollary 8.2.5. We have for $1 \le p \le q \le \infty$ that $L^q(I;X) \hookrightarrow L^p(I;X)$ and

$$\|f\|_{L^p(I;X)} \le \lambda_1(I)^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^q(I;X)}.$$

In general, for I possibly unbounded, we have at least that any $f \in L^q(I;X)$ is locally integrable over I and belongs to $L^1_{\mathrm{loc}}(I;X)$.

Proof. The proof is a trivial consequence of the results above. ■

Lemma 8.2.6 Let $1 \le p \le \infty$ and let $f_n \to f$ in $L^p(I;X)$. Then there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ which converges to f almost everywhere in I (in the norm of X).

Proof. The sequence $\|f_n - f\|_X$ converges to zero in $L^p(I;\mathbb{R})$, hence there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ such that $\|f_{n_k} - f\|_X$ converges to zero pointwise almost everywhere in I. Therefore $f_{n_k} \to f$ pointwise almost everywhere in I (in the norm of X). ■

As in the scalar case we therefore have

Theorem 8.2.7 — Properties of Lebesgue–Bochner spaces. The spaces $L^p(I;X)$ are Banach spaces with respect to the norms $\|f\|_{L^p(I;X)}$ introduced above, where we consider two functions identical, $f_1 = f_2$, provided $f_1(t) = f_2(t)$ for almost every $t \in I$ (in the sense of equality in X).

If $p=2$ and X is a Hilbert space, then $L^2(I;X)$ is a Hilbert space with respect to the scalar product

$$(f,g)_{L^2(I;X)} := \int_I (f,g)_X \,\mathrm{d}t.$$

Proof. The proof is similar to the scalar case $X=\mathbb{R}$. ■

We now study the question of the density for different classes of functions in the spaces $L^p(I;X)$ and the related question of the separability of these function spaces. We denote (recall, I is a bounded, open interval in $\mathbb{R}$)

$$\begin{aligned}
\mathcal{C}(I;X) &= \{f\colon I\to X \mid \text{ continuous in } I \text{ with values in } X\}\\
\mathcal{C}^k(I;X) &= \{f\colon I\to X \mid f, f', \dots, f^{(k)} \in \mathcal{C}(I;X)\}\\
\mathcal{C}(\overline{I};X) &= \{f\colon I\to X \mid \text{ continuous in } I \text{ with values in } X \text{ up to the endpoints}\}\\
\mathcal{C}^k(\overline{I};X) &= \{f\colon I\to X \mid f, f', \dots, f^{(k)} \in \mathcal{C}(\overline{I};X)\}\\
\mathcal{C}_0(I;X) &= \{f\colon I\to X \mid f\in\mathcal{C}(I;X), f \text{ is compactly supported in } I\},\\
\mathcal{C}^\infty(\overline{I};X) &= \bigcap_{k\in\mathbb{N}} \mathcal{C}^k(\overline{I};X)\\
\mathcal{C}_0^\infty(I;X) &= \mathcal{C}^\infty(I;X) \bigcap \mathcal{C}_0(I;X).
\end{aligned}$$

Theorem 8.2.8 — Dense subsets of Lebesgue–Bochner spaces. Let $1 \le p < \infty$. Then

1. simple functions are dense in $L^p(I;X)$
2. functions of the form $s(t) = \sum_{j=1}^n \varphi_j(t)x_j$, $\varphi_j \in \mathcal{C}_0^\infty(I;\mathbb{R})$, $x_j \in X$, are dense in $L^p(I;X)$
3. if the set Y is dense in X, then $\mathcal{C}_0^\infty(I;Y)$ is dense in $L^p(I;X)$
4. let η_ε be the regularizing kernel in $\mathbb{R}$ and suppose f is extended by zero outside I; then $f \star \eta_\varepsilon \to f$ in $L^p(I;X)$ for $\varepsilon \to 0_+$.

Proof. **Step 1:** Claim 1.
Let $f \in L^p(I;X)$ be given. Then there exists a sequence of simple functions $\{\widetilde{s}_n\}_{n\in\mathbb{N}}$ such that $\widetilde{s}_n \to f$ almost everywhere in I. Define

$$s_n(t) := \begin{cases} \widetilde{s}_n(t) & \text{if } \|\widetilde{s}_n(t)\|_X \le 1 + \|f(t)\|_X \\ 0 & \text{otherwise.} \end{cases}$$

Then $s_n(t) \to f(t)$ for almost every $t \in I$, but also $\|s_n(t)\|_X \le 1 + \|f(t)\|_X$ for all $t \in I$. Hence

$$\|s_n - f\|_X^p \le (1 + 2\|f\|_X)^p \in L^1(I).$$

We may therefore use the Lebesgue dominated convergence Theorem A.3.4 to conclude.
Step 2: Claim 2.
We know that functions of the type

$$\sum_{j=1}^n \chi_{E_j}(t)x_j, \qquad x_j \in X$$

are dense in $L^p(I;X)$. We may approximate the characteristic functions of a measurable set in $\mathbb{R}$ (in the norm of $L^p(I;\mathbb{R})$) by smooth compactly supported functions in I exactly as for the standard theory of the Lebesgue spaces, which yields the result.
Step 3: Claim 3.
Functions

$$s_n(t) = \sum_{j=1}^n \varphi_j(t)x_j, \qquad \varphi_j \in \mathcal{C}_0^\infty(I;\mathbb{R}),\ x_j \in X$$

can be approximated in $L^p(I;X)$ by

$$\widetilde{s}_n(t) = \sum_{j=1}^n \varphi_j(t)y_j, \qquad \varphi_j \in \mathcal{C}_0^\infty(I;\mathbb{R}),\ y_j \in Y$$

which yields the result.
Step 4: Claim 4.
Let $\delta > 0$. We find $v \in \mathcal{C}_0^\infty(I;X)$ such that $\|f - v\|_{L^p(I;X)} < \delta$. Then we write

$$f - \eta_\varepsilon \star f = f - v + v - \eta_\varepsilon \star v + \eta_\varepsilon \star v - \eta_\varepsilon \star f.$$

Since

$$\eta_\varepsilon \star v - v \rightrightarrows 0 \quad \text{in } I$$

by standard argument for the mollification, we have for ε sufficiently small using the triangle and Young inequalities

$$\begin{aligned} \|f - \eta_\varepsilon \star f\|_{L^p(I;X)} &\le \|f - v\|_{L^p(I;X)} + \|v - \eta_\varepsilon \star v\|_{L^p(I;X)} \\ &\quad + \|\eta_\varepsilon\|_{L^1(I;\mathbb{R})}\|f - v\|_{L^p(I;X)} < 3\delta \end{aligned}$$

which finishes the proof. ■

Corollary 8.2.9. Let X be a separable Banach space, then $L^p(I;X)$ is separable for $1 \le p < \infty$.

Proof. It is a consequence of Theorem 8.2.8, Claim 3., combined with the standard approach based on density of polynomials with rational coefficients in $C(\overline{I})$. ■

Theorem 8.2.10 — Lebesgue points. Let $f \in L^1_{\mathrm{loc}}(I;X)$. Then

$$\frac{1}{2h}\int_{-h}^{h} \|f(t+s) - f(t)\|_X \,\mathrm{d}s \to 0$$

for $h \to 0_+$ for almost every $t \in I$. In particular,

$$f(t) = \lim_{h\to 0_+} \frac{1}{2h} \int_{-h}^{h} f(t+s)\,\mathrm{d}\lambda_1(s)$$

almost everywhere in I.

Proof. As f is measurable, then f is almost separably valued and therefore we may without loss of generality assume that X is separable. Let $\{x_n\}_{n\in\mathbb{N}}$ be a countable dense subset of X and consider functions $\|f - x_n\|_X$. By the Theorem on Lebesgue points for scalar-valued functions A.3.20 we know that

$$\|f - x_n\|_X = \lim_{h\to 0_+} \frac{1}{2h} \int_{-h}^{h} \|f(t+s) - x_n\|_X \,\mathrm{d}s$$

for all $t \in I \setminus N_n$, where $\lambda_1(N_n) = 0$. Let $N = \cup_{n\in\mathbb{N}} N_n$. Then $\lambda_1(N) = 0$. Let $\varepsilon > 0$, $t \in I \setminus N$ and let $n \in \mathbb{N}$ be such that $\|f(t) - x_n\|_X < \frac{\varepsilon}{2}$. Then

$$\begin{aligned} 0 &\le \limsup_{h\to 0_+} \frac{1}{2h} \int_{-h}^{h} \|f(t+s) - f(t)\|_X \,\mathrm{d}s \le \limsup_{h\to 0_+} \frac{1}{2h} \int_{-h}^{h} \Big(\|f(t+s) - x_n\|_X + \|f(t) - x_n\|_X\Big)\,\mathrm{d}s \\ &\le 2\|f(t) - x_n\|_X < \varepsilon. \end{aligned}$$

The second claim follows immediately from (8.2) from the Bochner Theorem 8.1.9. ■

Finally, we have

Lemma 8.2.11 Let f_n be a sequence of functions from $L^p(I;X)$, $1 \le p \le \infty$ such that $\|f_n\|_{L^p(I;X)} \le C < +\infty$ for all $n \in \mathbb{N}$. Let there exist $f\colon I \to X$ such that for almost every $t \in I$

$$f_n(t) \rightharpoonup f(t) \quad \text{in } X.$$

Then $f \in L^p(I;X)$ and $\|f\|_{L^p(I;X)} \le C$.

Proof. By Corollary 8.1.7 the function f is measurable. For $t \in I$ we choose $x'(t) \in X^*$ such that $\|x'(t)\|_{X^*} = 1$ and $\langle x'(t), f(t)\rangle_X = \|f(t)\|_X$. By Fatou Lemma for $1 \le p < \infty$

$$\begin{aligned} \|f\|^p_{L^p(I;X)} &= \int_I \|f\|^p_X \,\mathrm{d}t = \int_I |\langle x', f\rangle_X|^p \,\mathrm{d}t = \int_I \lim_{n\to\infty} |\langle x', f_n\rangle_X|^p \,\mathrm{d}t \\ &\le \liminf_{n\to\infty} \int_I |\langle x', f_n\rangle_X| \,\mathrm{d}t \le \liminf_{n\to\infty} \int_I \|f_n\|^p_X \,\mathrm{d}t \le C^p, \end{aligned}$$

hence $f \in L^p(I;X)$. For $p = \infty$ we first proceed for $p < \infty$ as above and then pass with $p \to \infty$. ■

8.2.1 The Radon–Nikodym property and the dual space to $L^p(I;X)$

We might expect that the dual space to $L^p(I;X)$ for $p < \infty$ is $L^{p'}(I;X^*)$. However, this is in general not true for arbitrary Banach space and we have to add some properties of X. On the other hand, using Lemma 8.2.4, we at least know that for arbitrary Banach space X we have

$$L^{p'}(I;X^*) \hookrightarrow \big(L^p(I;X)\big)^*.$$

Moreover, for any $g \in L^{p'}(I;X^*)$ we also have

$$\|g\|_{(L^p(I;X))^*} \le \|g\|_{L^{p'}(I;X)}. \tag{8.5}$$

In fact, we can prove:

Proposition 8.2.12. For $1 \le p < \infty$ the inclusion mapping $I\colon L^{p'}(I;X^*) \to \big(L^p(I;X)\big)^*$ is an isometry, i.e., equality in (8.5) holds.

Proof. First, let $g = \sum_{i=1}^{\infty} x_i' \chi_{E_i}(t) \in L^{p'}(I;X^*)$, where $x_i' \in X^*$, E_i, $i \in \mathbb{N}$ are pairwise disjoint and $\cup_{i=1}^{\infty} E_i = I$. Then $\|g\|_{X^*} \in L^{p'}(I;\mathbb{R})$ and there exists a non-negative function $h \in L^p(I;\mathbb{R})$ such that $\|h\|_{L^p(I;\mathbb{R})} = 1$ and

$$\|g\|_{L^{p'}(I;X^*)} = \|\, \|g\|_{X^*} \,\|_{L^{p'}(I)} \le \int_I h\|g\|_{X^*} \,\mathrm{d}t + \frac{\varepsilon}{2}.$$

Choose $x_i \in X$, $\|x_i\|_X = 1$ such that

$$\|x_i'\|_{X^*} \le \langle x_i', x_i\rangle_X + \frac{\varepsilon}{2\|h\|_{L^1(I;\mathbb{R})}}$$

and define $f := \sum_{i=1}^{\infty} x_i h \chi_{E_i}(t)$. Then

$$\|f\|_{L^p(I;X)}^p = \sum_{i=1}^{\infty} \int_{E_i} \|x_i\|_X^p h^p \,\mathrm{d}t = \int_I h^p \,\mathrm{d}t = \|h\|_{L^p(I;\mathbb{R})}^p = 1$$

and for this f

$$\begin{aligned}\int_I \langle g, f\rangle_X \,\mathrm{d}\lambda_1 &= \int_I \Big\langle g, \sum_{i=1}^{\infty} h(t) x_i \chi_{E_i}(t)\Big\rangle_X \,\mathrm{d}t = \int_I h(t) \sum_{i=1}^{\infty} \big\langle x_i', x_i\big\rangle_X \chi_{E_i}(t)\,\mathrm{d}t \\ &\geq \int_I h(t) \sum_{i=1}^{\infty} \Big(\|x_i'\|_{X^*} - \frac{\varepsilon}{2\|h\|_{L^1(I;\mathbb{R})}}\Big)\chi_{E_i}(t)\,\mathrm{d}t = \int_I h\|g\|_{X^*}\,\mathrm{d}t - \frac{\varepsilon}{2} \geq \|g\|_{L^{p'}(I;X^*)} - \varepsilon.\end{aligned}$$

Therefore, as we may obtain this inequality for arbitrary $\varepsilon > 0$,

$$\|g\|_{(L^p(I;X))^*} \geq \|g\|_{L^{p'}(I;X^*)}$$

and the equality holds.

If $g \in L^{p'}(I;X^*)$ is arbitrary, then there exists a sequence $\{g_n\}_{n\in\mathbb{N}}$ such that g_n are countably valued, $g_n \to g$ in $L^{p'}(I;X^*)$. As $\|g_n - g\|_{(L^p(I;X))^*} \leq \|g_n - g\|_{L^{p'}(I;X^*)}$, the convergence holds also in $\big(L^p(I;X)\big)^*$ and thus

$$\|g\|_{(L^p(I;X))^*} = \lim_{n\to\infty} \|g_n\|_{(L^p(I;X))^*} = \lim_{n\to\infty} \|g_n\|_{L^{p'}(I;X^*)} = \|g\|_{L^{p'}(I;X^*)}.$$

Note that we used that I is bounded. The proof can be extended to unbounded interval by approximation of the unbounded interval by a sequence of bounded intervals. ■

We have the following abstract result the proof of which can be found in (Benyamini and Lindenstrauss, 2000, Theorem 5.21).

Theorem 8.2.13 — Radon–Nikodym–Rademacher property. Let X be a Banach space and $I \subset \mathbb{R}$ an interval. Then the following two assertions are equivalent.

1. For any vector measure $\nu\colon \Sigma \to X$ with bounded variations, where Σ denotes the σ-algebra of measurable subsets of I with respect to the Lebesgue measure λ_1 that is absolutely continuous with respect to λ_1, there exists $f \in L^1(I;X)$ such that
$$\nu(A) = \int_A f \,\mathrm{d}\lambda_1$$
for any $A \in \Sigma$.
2. Every Lipschitz continuous function $f\colon I \to X$ is differentiable almost everywhere in I.

The first property is called the Radon–Nikodym property, the second one the Rademacher property.

We will now present one class of spaces that possesses Property 2.

Proposition 8.2.14 (Modified Dunford–Pettis Theorem). Let X be a separable reflexive (real) Banach space. Then X has the Rademacher property.

Proof. We will show that for X separable reflexive Banach space Claim 2. from Theorem 8.2.13 holds true. Let $F\colon I \to X$ be a Lipschitz continuous function with Lipschitz constant L. Let $a \in I$. We may assume without loss of generality that $F(a) = 0$ and $L = 1$ (we may consider $G(t) := \frac{F(t)-F(a)}{L}$). Therefore for any $y \in X^*$ the function $\langle y, F(\cdot)\rangle_X$ is Lipschitz continuous with the Lipschitz constant bounded by $\|y\|_{X^*}$.

By the scalar version of the Rademacher Theorem A.2.16 there exists g_y, unique up to sets of measure zero, such that $\|g_y\|_{L^\infty(I;\mathbb{R})} \leq \|y\|_{X^*}$, and

$$\langle y, F(t)\rangle_X = \int_a^t g_y \,\mathrm{d}s \qquad \text{almost everywhere in } I.$$

As X is separable and reflexive, then X^* is also separable. Let $D \subset X^*$ be a countable dense subset and consider $y = \sum_{i=1}^{n} \alpha_i y_i$ for some $n \in \mathbb{N}$, $y_i \in D$ and $\alpha_i \in \mathbb{Q}$. Then

$$\langle y, F(t)\rangle_X = \Big\langle \sum_{i=1}^{n} \alpha_i y_i, F(t)\Big\rangle_X = \sum_{i=1}^{n} \alpha_i \big\langle y_i, F(t)\big\rangle_X = \sum_{i=1}^{n} \alpha_i \int_a^t g_{y_i}(s)\,\mathrm{d}s = \int_a^t \sum_{i=1}^{n} \alpha_i g_{y_i}(s)\,\mathrm{d}s,$$

hence

$$g_y = \sum_{i=1}^{n} \alpha_i g_{y_i}.$$

Therefore
$$\Big|\sum_{i=1}^n \alpha_i g_{y_i}(s)\Big| = |g_y(s)| \le \|g_y\|_{L^\infty(I;\mathbb{R})} \le \|y\|_{X^*} = \Big\|\sum_{i=1}^n \alpha_i y_i\Big\|_{X^*} \tag{8.6}$$
for almost every $t \in I$. Thus there exists a null set $E \subset I$ such that (8.6) holds for every $t \in I \setminus E$ and all choices $\{n \in \mathbb{N} \,|\, \{\alpha_i\}_{i=1}^n \subset (\mathbb{Q})^n; \{y_i\}_{i=1}^n \subset D\}$. As $\mathbb{Q}$ is dense in $\mathbb{R}$, the estimate holds for all choices $\{n \in \mathbb{N} \,|\, \{\alpha_i\}_{i=1}^n \subset \mathbb{R}^n; \{y_i\}_{i=1}^n \subset D\}$. Thus $y \mapsto g_y(t)$ is a linear mapping from the span of D to $\mathbb{R}$ with a norm bounded by 1. By density of D it can be uniquely extended to X^*. We obtain an element $f(t) \in X^{**}$, $\|f(t)\|_{X^{**}} \le 1$. We identify this element, via the canonical mapping, with an element from X (we use the same notation) and $\|f(t)\|_X = 1$. Therefore we have for all $y \in D$ and almost every $s \in I$ that
$$g_y(s) = \langle y, f(s)\rangle_X$$
which is measurable and bounded. Let $y \in X^*$ and $\{y_n\}_{n\in\mathbb{N}} \subset D$ such that $y_n \to y$ in X^*. Then $g_{y_n}(s) = \langle y_n, f(\cdot)\rangle_X$ are measurable and bounded by $\|y_n\|_{X^*}$. Therefore $\langle y, f(\cdot)\rangle_X$ is measurable and
$$\begin{aligned}\langle y, F(t)\rangle_X &= \lim_{n\to\infty}\langle y_n, F(t)\rangle_X = \lim_{n\to\infty}\int_a^t g_{y_n}\,\mathrm{d}s\\ &= \lim_{n\to\infty}\int_a^t \langle y_n, f(s)\rangle_X\,\mathrm{d}s = \int_a^t \langle y, f(s)\rangle_X\,\mathrm{d}s.\end{aligned}$$
Now, as f is measurable due to the Pettis Theorem 8.1.5 (it is weakly measurable and X is separable) and bounded, it is Bochner integrable and
$$\langle y, F(t)\rangle_X = \int_a^t \big\langle y, f(s)\big\rangle_X\,\mathrm{d}s = \Big\langle y, \int_a^t f(s)\,\mathrm{d}\lambda_1(s)\Big\rangle_X$$
for all $y \in X^*$. Hence
$$F(t) = \int_a^t f(s)\,\mathrm{d}\lambda_1(s)$$
and by the Theorem on Lebesgue points 8.2.10 the function F is almost everywhere differentiable on I. Whence X has the Rademacher property and by means of Theorem 8.2.13 also the Radon–Nikodym property. ■

Remark 8.2.15. In fact, the previous proposition holds also under less restrictive assumptions. It is enough if either X is reflexive or $X = Y^*$ and X is separable. While $X = Y^*$ with X separable follows due to the same proof as the proposition above, the case X solely reflexive is slightly more complex.

If $f\colon I \to X$ is continuous, then $f(I)$ is separable and we may therefore reduce our considerations to the smallest closed subset of X containing $f(I)$. But any closed subset of reflexive space is reflexive. Hence we may proceed exactly as in the proof of the proposition above, replacing X by the smallest closed subset of X containing $f(I)$.

The previous proposition has the following connection to the characterization of the dual space to $L^p(I;X)$.

Theorem 8.2.16 — Duality for Lebesgue–Bochner spaces. Let X be a Banach space such that X^* has the Radon–Nikodym property. Then $\big(L^p(I;X)\big)^*$ is isometrically isomorphic to the space $L^{p'}(I;X^*)$ for $1 \le p < \infty$.

Proof. We use the Radon–Nikodym characterization. Let Σ denote all measurable subsets of I. Recall that I is bounded. We define for $l \in \big(L^p(I;X)\big)^*$ a vector measure $\nu\colon \Sigma \to X^*$ as
$$\langle \nu(E), x\rangle_X = \langle l, x\chi_E\rangle_{L^p(I;X)}$$
for any $E \in \Sigma$, $x \in X$. Then
$$\big|\langle l, x\chi_E\rangle_{L^p(I;X)}\big| \le \|l\|_{(L^p(I;X))^*}\|x\chi_E\|_{L^p(I;X)} = \|l\|_{(L^p(I;X))^*}\|x\|_X(\lambda_1(E))^{\frac{1}{p}}.$$
Therefore $\nu(E) \in X^*$ and $\|\nu(E)\|_{X^*} \le \|l\|_{(L^p(I;X))^*}(\lambda_1(E))^{\frac{1}{p}}$. Let $\{E_n\}_{n\in\mathbb{N}}$ be pairwise disjoint and measurable. Let $x \in X$. Then
$$\big\langle \nu\big(\cup_{n=1}^\infty E_n\big), x\big\rangle_X = \langle l, x\chi_{\cup_{n=1}^\infty E_n}\rangle_{L^p(I;X)} = \Big\langle l, \sum_{n=1}^\infty x\chi_{E_n}\Big\rangle_{L^p(I;X)} = \sum_{n=1}^\infty \langle l, x\chi_{E_n}\rangle_{L^p(I;X)} = \sum_{n=1}^\infty \langle \nu(E_n), x\rangle_X.$$
Note that all series converge as $\langle l, x\chi_{\cup_{n=1}^\infty E_n}\rangle_{L^p(I;X)}$ converge. Finally, let us show that ν has bounded variation. We have for E_i
$$\|\nu(E_i)\|_{X^*} = \sup_{\|x_i\|_X=1}\langle \nu(E_i), x_i\rangle_X = \sup_{\|x_i\|_X=1}\langle l, x_i\chi_{E_i}\rangle_{L^p(I;X)}.$$
Therefore for any partition π of I
$$\sum_{E_i\in\pi}\|\nu(E_i)\|_{X^*} = \sum_{E_i\in\pi}\sup_{\|x_i\|_X=1}\langle l, x_i\chi_{E_i}\rangle_{L^p(I;X)} = \sup_{\|x_i\|_X=1}\sum_{E_i\in\pi}\langle l, x_i\chi_{E_i}\rangle_{L^p(I;X)}.$$

As

$$\sum_{E_i\in\pi} \big\langle l, x_i\chi_{E_i}\big\rangle_{L^p(I;X)} = \Big\langle l, \sum_{E_i\in\pi} x_i\chi_{E_i}\Big\rangle_{L^p(I;X)} \le \|l\|_{(L^p(I;X))^*}\Big\|\sum_{E_i\in\pi} x_i\chi_{E_i}\Big\|_{L^p(I;X)}$$
$$\le \|l\|_{(L^p(I;X))^*}\Big(\sum_{E_i\in\pi}\|x_i\|_X^p\lambda_1(E_i)\Big)^{\frac1p} \le \|l\|_{(L^p(I;X))^*}\Big(\lambda_1(I)\Big)^{\frac1p},$$

we have

$$|\nu|(I) \le \|l\|_{(L^p(I;X))^*}\Big(\lambda_1(I)\Big)^{\frac1p} < \infty,$$

and ν is of bounded variation. Recall X^* has the Radon–Nikodym property, hence there exists $g \in L^1(I;X^*)$ such that

$$\nu(E) = \int_E g\,\mathrm{d}\lambda_1, \qquad E \in \Sigma.$$

Let $s = \sum_{i=1}^n x_i\chi_{E_i}(t)$ be a simple function. Then

$$\langle l, s\rangle_{L^p(I;X)} = \sum_{i=1}^n \langle l, x_i\chi_{E_i}\rangle_{L^p(I;X)} = \sum_{i=1}^n \langle \nu(E_i), x_i\rangle_{L^p(I;X)} = \sum_{i=1}^n \Big\langle \int_{E_i} g\,\mathrm{d}\lambda_1, x_i\Big\rangle_X$$
$$= \sum_{i=1}^n \int_{E_i}\langle g, x_i\rangle_X\,\mathrm{d}t = \sum_{i=1}^n \int_I \langle g, x_i\chi_{E_i}\rangle_X\,\mathrm{d}t = \int_I \langle g, s\rangle_X\,\mathrm{d}t.$$

We need to extend the formula for an arbitrary L^p-function. We define the set $E_n := \{t \in I; \|g(t)\|_{X^*} \le n\}$. Recall that $\cup_{n=1}^\infty E_n = I \setminus E$, where $\lambda_1(E) = 0$. For a fixed n we have $g\chi_{E_n} \in L^\infty(I;X^*) \subset L^{p'}(I;X^*)$ and thus

$$f \mapsto \int_{E_n} \langle g, f\rangle_X\,\mathrm{d}t$$

is a bounded functional in $\big(L^p(I;X)\big)^*$ which is equal to $\langle l, f\rangle_{L^p(I;X)}$ if f is a simple function supported in E_n. For any $f \in L^p(I;X)$ there exists a sequence $\{s_k\}_{k\in\mathbb{N}}$ supported in E_n which converge to $f\chi_{E_n}$ almost everywhere in I and in the $L^p(I;X)$ norm. Thus

$$\langle l, f\chi_{E_n}\rangle_{L^p(I;X)} = \lim_{k\to\infty}\langle l, s_k\rangle_{L^p(I;X)} = \lim_{k\to\infty}\int_{E_n}\langle g, s_k\rangle_X\,\mathrm{d}t = \int_{E_n}\langle g, f\rangle_X\,\mathrm{d}t,$$

where the last equality is a consequence of $g\chi_{E_n} \in L^{p'}(I;X^*) \subset \big(L^p(I;X)\big)^*$. Then we can compute

$$\|g\chi_{E_n}\|_{L^{p'}(I;X^*)} = \|g\chi_{E_n}\|_{(L^p(I;X))^*} = \sup_{\|f\|_{L^p(I;X)}\le 1}\int_I \langle g\chi_{E_n}, f\rangle_X\,\mathrm{d}t$$
$$= \sup_{\|f\|_{L^p(I;X)}\le 1}\int_{E_n}\langle g, f\rangle_X\,\mathrm{d}t = \sup_{\|f\|_{L^p(I;X)}\le 1}\langle l, f\chi_{E_n}\rangle_{L^p(I;X)} \le \|l\|_{(L^p(I;X))^*}.$$

We have that $\|g\chi_n\|_{X^*} \to \|g\|_{X^*}$ for $n\to\infty$ for almost every $t \in I$ and the convergence is monotonically increasing. By virtue of the Lebesgue monotone convergence Theorem A.3.4 then $g \in L^{p'}(I;X^*)$, $\|g\|_{L^{p'}(I;X^*)} \le \|l\|_{(L^p(I;X))^*}$. We now have

$$\langle l, f\rangle_{L^p(I;X)} = \lim_{n\to\infty}\langle l, f\chi_{E_n}\rangle_{L^p(I;X)} = \lim_{n\to\infty}\int_{E_n}\langle g, f\rangle_X\,\mathrm{d}t = \lim_{n\to\infty}\int_I\langle g\chi_{E_n}, f\rangle_X\,\mathrm{d}t = \int_I \langle g, f\rangle_X\,\mathrm{d}t$$

for all $f \in L^p(I;X)$. Thus $\big(L^p(I;X)\big)^* = L^{p'}(I;X^*)$ in the sense that the identity mapping conserves the norm and is bijective. For unbounded intervals we may proceed by means of an approximation of the interval by bounded ones. ■

Remark 8.2.17. Using the remark before this theorem we see that for $1 < p < \infty$ we can identify $\big(L^p(I;X)\big)^*$ with $L^{p'}(I;X^*)$ if either X is reflexive or if X^* is separable.

Corollary 8.2.18. If X is reflexive and $1 < p < \infty$, then $L^p(I;X)$ is reflexive.

Proof. We know that $\big(L^p(I;X)\big)^{**} \cong L^p(I;X^{**}) \cong L^p(I;X)$. To show the reflexivity, we proceed similarly as in the case of the standard L^p-spaces. ■

8.3 Spaces $W^{1,p}(I;X)$ (Sobolev–Bochner spaces)

Definition 8.3.1 — Weak time derivative. Let $u \in L^1_{\mathrm{loc}}(I;X)$ and $g \in L^1_{\mathrm{loc}}(I;X)$. We say that g is the weak derivative of u with respect to t, i.e., $g = u'$, if

$$\int_I u(t)\varphi'(t)\,\mathrm{d}\lambda_1(t) = -\int_I g(t)\varphi(t)\,\mathrm{d}\lambda_1(t)$$

holds for all $\varphi \in \mathcal{C}_0^\infty(I;\mathbb{R})$.

We have (we extend f by zero outside of I if f is defined only in I).

Proposition 8.3.2. Let $1 \le p < \infty$, $f \in L^p(\mathbb{R};X)$. Define for any $h > 0$

$$M_h(f)(t) := \frac{1}{h}\int_t^{t+h} f(s)\,\mathrm{d}\lambda_1(s). \tag{8.7}$$

Then $M_h(f) \in L^p(\mathbb{R};X) \cap \mathcal{C}(\mathbb{R};X)$ and

$$\lim_{h\to 0^+} M_h(f) = f$$

in $L^p(\mathbb{R};X)$ and almost everywhere in $\mathbb{R}$ (in the sense of convergence in X).

Proof. Let $t_n \to t_+$. Then

$$\|M_h(f)(t_n) - M_h(f)(t)\|_X \le \frac{1}{h}\Big\|\int_t^{t_n} f(s)\,\mathrm{d}\lambda_1(s)\Big\|_X + \frac{1}{h}\Big\|\int_{t+h}^{t_n+h} f(s)\,\mathrm{d}\lambda_1(s)\Big\|_X \to 0$$

as $f \in L^1_{\mathrm{loc}}(\mathbb{R};X)$. Similarly we proceed for $t_n \to t_-$. Thus

$$M_h(f)(t_n) \to M_h(f)(t),$$

and $M_h(f)$ is continuous. By Hölder's inequality

$$\|M_h(f)(t)\|_X \le \frac{1}{h}\int_t^{t+h}\|f\|_X\,\mathrm{d}s \le h^{\frac{1}{p'}-1}\Big(\int_t^{t+h}\|f\|_X^p\,\mathrm{d}s\Big)^{\frac{1}{p}},$$

thus

$$\|M_h(f)(t)\|_X^p \le \frac{1}{h}\int_t^{t+h}\|f\|_X^p\,\mathrm{d}s.$$

By the standard Fubini–Tonelli Theorem

$$\begin{aligned}\|M_h(f)\|^p_{L^p(\mathbb{R};X)} &\le \frac{1}{h}\int_{\mathbb{R}}\Big(\int_t^{t+h}\|f\|_X^p\,\mathrm{d}s\Big)\,\mathrm{d}t = \frac{1}{h}\int_{\mathbb{R}}\Big(\int_0^h\|f(t+\cdot)\|_X^p\,\mathrm{d}s\Big)\,\mathrm{d}t\\ &= \frac{1}{h}\int_0^h\Big(\int_{\mathbb{R}}\|f(t+\cdot)\|_X^p\,\mathrm{d}s\Big)\,\mathrm{d}t = \|f\|^p_{L^p(\mathbb{R};X)},\end{aligned}$$

thus $M_h(f) \in L^p(\mathbb{R};X)$ and $M_h \in \mathcal{L}(L^p(\mathbb{R};X))$ with $\|M_h\|_{\mathcal{L}(L^p(\mathbb{R};X))} \le 1$.

We already know (see the Theorem on Lebesgue points 8.2.10) that for $h \to 0_+$ we have $M_h(f) \to f$ almost everywhere. To show the convergence in the $L^p(\mathbb{R};X)$ norm, we compute

$$\|f - M_h(f)\|_{L^p(\mathbb{R};X)} \le \|f-\varphi_n\|_{L^p(\mathbb{R};X)} + \|M_h(f-\varphi_n)\|_{L^p(\mathbb{R};X)} + \|\varphi_n - M_h\varphi_n\|_{L^p(\mathbb{R};X)},$$

where $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathcal{C}_0^\infty(\mathbb{R};X)$ is such that $\varphi_n \to f$ in $L^p(\mathbb{R};X)$. Then

$$\|f - M_h(f)\|_{L^p(\mathbb{R};X)} \le 2\|f-\varphi_n\|_{L^p(\mathbb{R};X)} + \|\varphi_n - M_h\varphi_n\|_{L^p(\mathbb{R};X)}$$

and as $\|\varphi_n - M_h\varphi_n\|_X$ converges to zero uniformly in $\mathbb{R}$ and both functions have bounded support, we proved that

$$\|f - M_h(f)\|_{L^p(\mathbb{R};X)} \to 0$$

for $h \to 0_+$. ∎

The previous proposition yields

Corollary 8.3.3. Let $g \in L^1_{\mathrm{loc}}(I;X)$, $t_0 \in I$ and let I be an open interval. Let

$$f(t) := \int_{t_0}^{t} g(s)\,\mathrm{d}\lambda_1(s).$$

Then $f \in \mathcal{C}(I;X)$ and

$$f' = g$$

in the weak sense and almost everywhere in I.

Proof. In what follows we consider sufficiently small h so that for $t \in I$ we have $t+h$ and $t \in I$. We know that $M_h(g) = \frac{f(t+h)-f(t)}{h}$, hence the continuity of f and differentiability almost everywhere follows from Proposition 8.3.2. We prove the weak differentiability. Let $\varphi \in C_0^\infty(I;\mathbb{R})$. Then

$$\begin{aligned} -\int_I f\varphi'\,\mathrm{d}\lambda_1(t) &= -\int_I f(t) \lim_{h\to 0_+} \frac{\varphi(t+h)-\varphi(t)}{h}\,\mathrm{d}\lambda_1(t) = -\lim_{h\to 0_+} \int_I f(t)\frac{\varphi(t+h)-\varphi(t)}{h}\,\mathrm{d}\lambda_1(t) \\ &= \lim_{h\to 0_+} \int_I \frac{f(t-h)-f(t)}{-h}\varphi(t)\,\mathrm{d}\lambda_1(t) = \lim_{h\to 0_+} \int_I M_{-h}(g)(t)\varphi(t)\,\mathrm{d}\lambda_1(t). \end{aligned}$$

By a direct modification of Proposition 8.3.2 we know that $M_{-h}g \to g$ almost everywhere and in $L^1_{\mathrm{loc}}(I;X)$ which implies that the last limit equals to $\int_I g\varphi\,\mathrm{d}\lambda_1(t)$. ■

We further have

Corollary 8.3.4. Let $f \in L^1_{\mathrm{loc}}(I;X)$ be such that $f' = 0$ almost everywhere in I. Then there exists $x_0 \in X$ such that $f = x_0$ almost everywhere in I.

Proof. Let $\vartheta \in \mathcal{C}_0^\infty(I;\mathbb{R})$ be such that $\int_I \vartheta\,\mathrm{d}s = 1$. Define

$$x_0 := \int_I f\vartheta\,\mathrm{d}\lambda_1.$$

Let $\operatorname{supp}\vartheta \subset [a,b]$, $t_0 < a$. Then for any $\varphi \in \mathcal{C}_0^\infty(I;\mathbb{R})$ we set

$$\psi(t) := \int_{t_0}^{t} \Big(\varphi(s) - \vartheta(s)\int_I \varphi(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s.$$

Hence $\psi \in \mathcal{C}_0^\infty(I;\mathbb{R})$, and

$$0 = \int_I f\psi'\,\mathrm{d}\lambda_1 = \int_I f(t)\Big(\varphi(t) - \vartheta(t)\int_I \varphi(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}\lambda_1(t) = \int_I f\varphi\,\mathrm{d}\lambda_1 - x_0\int_I \varphi\,\mathrm{d}t = \int_I (f-x_0)\varphi\,\mathrm{d}\lambda_1.$$

Therefore $f = x_0$ almost everywhere in I. ■

Definition 8.3.5 — Sobolev–Bochner spaces. Let $1 \le p \le \infty$ and $u \in L^p(I;X)$ be such that $u' \in L^p(I;X)$. Then we say that u belongs to the Sobolev–Bochner space $W^{1,p}(I;X)$ and we define

$$\|u\|_{W^{1,p}(I;X)} := \|u\|_{L^p(I;X)} + \|u'\|_{L^p(I;X)}.$$

Proposition 8.3.6. If $1 \le p \le \infty$, then $W^{1,p}(I;X)$ is with respect to the norm above a Banach space. If X is a Hilbert space, then $W^{1,2}(I;X)$ is a Hilbert space with respect to the scalar product

$$(u,v)_{W^{1,2}(I;X)} := \int_I (u,v)_X\,\mathrm{d}\lambda_1 + \int_I (u',v')_X\,\mathrm{d}\lambda_1$$

for any u, $v \in W^{1,2}(I;X)$. The associated norm is an equivalent norm on $W^{1,2}(I;X)$.

Proof. Let u_n be a Cauchy sequence in $W^{1,p}(I;X)$. Then there exist u, $v \in L^p(I;X)$ such that $u_n \to u$ in $L^p(I;X)$ and $u_n' \to v$ in $L^p(I;X)$. Since evidently

$$\int_I u_n\varphi'\,\mathrm{d}\lambda_1 \to \int_I u\varphi'\,\mathrm{d}\lambda_1,$$
$$\int_I u_n'\varphi\,\mathrm{d}\lambda_1 \to \int_I v\varphi\,\mathrm{d}\lambda_1$$

(in X), we have that $u' = v$. Hence $W^{1,p}(I;X)$ is complete. The fact that the above defined functional is a norm follows from properties of the norm in $L^p(I;X)$. The claim for the Hilbert spaces is trivial. ■

Proposition 8.3.7. If X is reflexive, then $W^{1,p}(I;X)$ is for $1 < p < \infty$ reflexive. If $1 \le p < \infty$ and X is separable, then $W^{1,p}(I;X)$ is separable.

Proof. We know that under the assumptions of the proposition, the space $L^p(I;X)$ is separable or reflexive, respectively, and so is $L^p(I;X) \times L^p(I;X)$. As $W^{1,p}(I;X)$ is a closed subset of $L^p(I;X) \times L^p(I;X)$, the claim follows. ■

Proposition 8.3.8. Let $1 \le p \le \infty$ and $u \in W^{1,p}(I;X)$. Then there exists $t_0 \in I$ such that for almost all $t \in I$

$$u(t) = u(t_0) + \int_{t_0}^{t} u'(s)\,\mathrm{d}\lambda_1(s).$$

Proof. Let $t_1 \in I$ and let $g(t) := \int_{t_1}^{t} u'(s)\,\mathrm{d}\lambda_1(s)$. Let $w(t) := u(t) - g(t)$. By virtue of Corollary 8.3.3 we see that $w'(t) = 0$ almost everywhere in I, hence Corollary 8.3.4 yields that $w(t) = x_0 \in X$ almost everywhere in I. Thus $u(t) = x_0 + \int_{t_1}^{t} u'(s)\,\mathrm{d}\lambda_1(s)$ almost everywhere in I. We choose $t_0 \in I$ such that the previous equality holds, therefore

$$u(t) - u(t_0) = \int_{t_1}^{t} u'(s)\,\mathrm{d}\lambda_1(s) - \int_{t_1}^{t_0} u'(s)\,\mathrm{d}\lambda_1(s) = \int_{t_0}^{t} u'(s)\,\mathrm{d}\lambda_1(s).$$

■

In fact, Proposition 8.3.8 yields existence of $N \subset I$, a null set, such that $u(t) = u(t_0) + \int_{t_0}^{t} u'(s)\,\mathrm{d}\lambda_1(s)$ for all t, $t_0 \in I\backslash N$. Let $I = (a,b)$. If $g \in L^1(a,b;X)$ and $f(t) = \int_a^t g(s)\,\mathrm{d}\lambda_1(s)$, then also $\|g\|_X \in L^1((a,b))$ and $f \in \mathcal{AC}([a,b];X)$. Recall that $f\colon [a,b] \to X$ is absolutely continuous on $[a,b]$, if $\forall\varepsilon > 0\,\exists\delta > 0$: $\sum_{i=1}^{n} \|f(b_i) - f(a_i)\|_X < \varepsilon$ for all $\{[a_i,b_i]\}_{i=1}^{n}$, disjoint collection of subintervals of I with the total length less than δ. Therefore, if $u \in W^{1,p}(a,b;X)$, then there exists a representative of u, a function $\widetilde{u} \in \mathcal{AC}([a,b];X)$ such that $\widetilde{u}$ is differentiable almost everywhere, $\widetilde{u}' = u'$ and $\widetilde{u} = u$ almost everywhere in $[a,b]$. Unless stated differently, we shall always assume that we work directly with this representative.

We further have

Proposition 8.3.9. Let $1 \le p \le \infty$. Then $W^{1,p}(I;X) \hookrightarrow \mathcal{C}(\overline{I};X)$, i.e., working with the representative from above, we have

$$\|u\|_{C(\overline{I};X)} \le C\|u\|_{W^{1,p}(I;X)}.$$

Proof. Due to Proposition 8.3.8

$$u(t) = u(t_0) + \int_{t_0}^{t} u'(s)\,\mathrm{d}\lambda_1(s)$$

holds for the continuous representative for any t_0, $t \in I$. Therefore

$$\max_{t\in\overline{I}} \|u(t)\|_X \le \|u(t_0)\|_X + \int_I \|u'\|_X\,\mathrm{d}t.$$

Furthermore,

$$\lambda_1(I)\|u(t_0)\|_X \le \int_I \|u\|_X\,\mathrm{d}t + \lambda_1(I)\int_I \|u'\|_X\,\mathrm{d}t.$$

Combining the last two inequalities we get the claim of the proposition. ■

The previous results are summarized in the following theorem.

Theorem 8.3.10 — Properties of weak derivative. Let $u \in L^p(I;X)$, $1 \le p \le \infty$. Then the following assertions are equivalent:

1. $u \in W^{1,p}(I;X)$
2. u is absolutely continuous on $\overline{I}$ and $u' \in L^p(I;X)$
3. there exists a function $v \in L^p(I;X)$ such that for all $x' \in X^*$ the function $\psi(t) := \langle x', u\rangle_X$ is absolutely continuous on I and $\psi'(t) = \langle x', v(t)\rangle_X$ almost everywhere in I.

Proof. **Step 1:** "(i) $\implies$ (ii)"
It follows from Proposition 8.3.8 and the remark below it, and from Corollary 8.3.3.
Step 2: "(ii) $\implies$ (iii)"
It follows from the linearity of x' and from Corollary 8.1.12.
Step 3: "(iii) $\implies$ (i)"
This part is more demanding. We define

$$g(t) := u(t_0) + \int_{t_0}^{t} v(s)\,\mathrm{d}\lambda_1(s), \qquad t_0 \in I.$$

Then $g \in W^{1,p}(I;X)$ by Corollary 8.3.3. For any $x' \in X^*$ the function ψ defined above is absolutely continuous on $\overline{I}$, therefore by the standard properties of absolutely continuous functions

$$\langle x', u(t)\rangle_X = \psi(t) = \psi(t_0) + \int_{t_0}^t \psi'(s)\,\mathrm{d}s = \langle x', u(t_0)\rangle_X + \int_{t_0}^t \langle x', v\rangle_X\,\mathrm{d}s.$$

Thus by Corollary 8.1.12

$$\langle x', g(t)\rangle_X = \langle x', u(t)\rangle_X$$

which yields that

$$u(t) = u(t_0) + \int_{t_0}^t v(s)\,\mathrm{d}\lambda_1(s),$$

hence $u' = v$ and $u \in W^{1,p}(I;X)$. ■

Furthermore, we have

Theorem 8.3.11 — Density of smooth functions in time. Let $1 \le p < \infty$. Then $\mathcal{C}^\infty(\overline{I};X)$ is dense in $W^{1,p}(I;X)$.

Proof. Let $u \in W^{1,p}(I;X)$. Then $u' \in L^p(I;X)$ and thus there exists a sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ of functions from $\mathcal{C}_0^\infty(I;X)$ such that $\varphi_n \to u'$ in $L^p(I;X)$. By Proposition 8.3.8 there exists t_0 such that $u(t) = u(t_0) + \int_{t_0}^t u'(s)\,\mathrm{d}\lambda_1(s)$ almost everywhere in I. Define

$$u_n(t) := u(t_0) + \int_{t_0}^t \varphi_n(s)\,\mathrm{d}\lambda_1(s).$$

Then $u_n \in \mathcal{C}^\infty(\overline{I};X)$, $u_n' = \varphi_n$ in I. It remains to show that $u_n \to u$ in $L^p(I;X)$. We have (recall that $\frac{1}{p'} = 0$ for $p=1$)

$$\|u_n - u\|_X = \left\| \int_{t_0}^t (\varphi_n - u')(s)\,\mathrm{d}\lambda_1(s)\right\|_X \le |t-t_0|^{\frac{1}{p'}} \left(\int_{t_0}^t \|\varphi_n - u'\|_X^p\,\mathrm{d}s\right)^{\frac{1}{p}}.$$

Therefore

$$\|u_n - u\|_{L^p(I;X)}^p \le \int_I |t-t_0|^{\frac{p}{p'}} \int_{t_0}^t \|\varphi_n - u'\|_X^p\,\mathrm{d}s\,\mathrm{d}t \le (\lambda_1(I))^{\frac{p}{p'}+1} \|\varphi_n - u'\|_{L^p(I;X)}^p \to 0$$

as $n \to \infty$. ■

8.4 Gelfand triple and properties of corresponding Bochner spaces

We next consider a more general setting, where the time derivative in general belongs to a different space than the function itself.

Definition 8.4.1 — Gelfand triple. Let X be a separable reflexive Banach space such that there exists a Hilbert space H, where $X \hookrightarrow H$ densely. Then we call the triple X, $H \cong H^*$ and X^* the Gelfand triple.

In what follows we shall prove that

$$X \hookrightarrow H \cong H^* \hookrightarrow X^*,$$

where both embeddings are dense. The identification of H and H^* is through the Riesz representation Theorem. Let $x \in X$ and $Ix \in H$, where $I\colon X \to H$ represents the embedding. Let $\Phi\colon H^* \to H$ be the mapping which represents the Riesz representation Theorem. Then we may define $i\colon X \to X^*$ as follows

$$\langle ix_0, x\rangle_X := (Ix_0, Ix)_H = \langle \Phi^{-1} Ix_0, Ix\rangle_H,$$

where x, $x_0 \in X$. The mapping $i\colon X \to X^*$ is injective and $i(X)$ is dense in X^* (due to the fact that both embeddings are dense).

Example 8.4.2. Let $X = W_0^{1,2}(\Omega)$ and $H = L^2(\Omega)$. As $W_0^{1,2}(\Omega)$ is densely embedded into $L^2(\Omega)$, the spaces $W_0^{1,2}(\Omega)$, $L^2(\Omega)$ and $W^{-1,2}(\Omega) = \left(W_0^{1,2}(\Omega)\right)^*$ form the Gelfand triple. By the Riesz representation Theorem, for any $f \in W^{-1,2}(\Omega)$ there exists unique $u_f \in W_0^{1,2}(\Omega)$ such that

$$\langle f, v\rangle_{W_0^{1,2}(\Omega)} = ((u_f, v))_{W_0^{1,2}(\Omega)} := \int_\Omega \nabla u \cdot \nabla v\,\mathrm{d}x,$$

where the scalar product is equivalent to the standard scalar product on $W^{1,2}(\Omega)$. Now, if $u \in C_0^\infty(\Omega)$ (which is dense in $W_0^{1,2}(\Omega)$), then

$$((u,v))_{W_0^{1,2}(\Omega)} = \int_\Omega \nabla u \cdot \nabla v\,\mathrm{d}x = -\int_\Omega \Delta u v\,\mathrm{d}x$$

for any $v \in W_0^{1,2}(\Omega)$. Hence for any $f \in W^{-1,2}(\Omega)$ there exists $\{u_n\}_{n\in\mathbb{N}}$ such that $u_n \to u_f$ in $W_0^{1,2}(\Omega)$, thus

$$\begin{aligned}\langle f, v\rangle_{W_0^{1,2}(\Omega)} &= \lim_{n\to\infty} \int_\Omega -\Delta u_n v \,\mathrm{d}x = \lim_{n\to\infty} (-\Delta u_n, v)_{L^2(\Omega)} \\ &= \lim_{n\to\infty} \langle i(-\Delta u_n), v\rangle_{W^{1,2}(\Omega)} =: \lim_{n\to\infty} \langle i(f_n), v\rangle_{W_0^{1,2}(\Omega)},\end{aligned}$$

where $f_n \in C_0^\infty(\Omega)$. Clearly, i is injective by definition and $i(W_0^{1,2}(\Omega))$ is dense in $L^2(\Omega)$. Moreover, if $f \in L^2(\Omega)$, then $-\Delta u_n \to f$ in $L^2(\Omega)$ and we have that

$$\langle f, v\rangle_{W_0^{1,2}(\Omega)} = \int_\Omega f v \,\mathrm{d}x$$

for any $v \in W_0^{1,2}(\Omega)$.

In this setting, we can generalize the definition of the time derivative. From now on, we use $\mathrm{d}t$ instead of $\mathrm{d}\lambda_1$.

Definition 8.4.3 — Generalized weak time derivative. Let $u \in L^p(I; X)$, where X, H and X^* form the Gelfand triple. Then we say that $v \in L^q(I; X^*)$ is the time derivative of u, if

$$\int_I \langle v, w\rangle_X \psi \,\mathrm{d}t = -\int_I (u, w)_H \psi' \,\mathrm{d}t$$

holds for all $w \in X$, $\psi \in \mathcal{C}_0^\infty(I; \mathbb{R})$. We denote $u' := v$.

We have the following important result.

Theorem 8.4.4 — Continuous representative for Gelfand triple. Let X, H, X^* form the Gelfand triple, and let $u \in L^p(I; X)$ with $u' \in L^{p'}(I; X^*)$, $1 < p < \infty$. Then $u = \widetilde{u}$ almost everywhere in I, where $\widetilde{u} \in \mathcal{C}(\overline{I}; H)$. Moreover, $t \mapsto \|u(t)\|_H^2$ is weakly differentiable, and

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_H^2 = 2\langle u'(t), u(t)\rangle_X$$

almost everywhere in I. In particular,

$$\|\widetilde{u}(t_2)\|_H^2 = \|\widetilde{u}(t_1)\|_H^2 + \int_{t_1}^{t_2} 2\langle u'(s), u(s)\rangle_X \,\mathrm{d}s.$$

Remark 8.4.5. Similarly as in Theorem 8.4.4 we may also show that if u, $v \in L^p(I; X)$, u', $v' \in L^{p'}(I; X^*)$, then for $\psi \in \mathcal{C}_0^\infty(I; \mathbb{R})$

$$\int_I \Big(\langle u', v\rangle_X + \langle v', u\rangle_X\Big)\psi \,\mathrm{d}t = -\int_I (u, v)_H \psi' \,\mathrm{d}t$$

as well as

$$\int_{t_1}^{t_2} \Big(\langle u', v\rangle_X + \langle v', u\rangle_X\Big)\psi \,\mathrm{d}t = (u(t_2), v(t_2))_H \psi(t_2) - (u(t_1), v(t_1))_H \psi(t_1) - \int_{t_1}^{t_2} (u, v)_H \psi' \,\mathrm{d}t$$

for any $\psi \in C^\infty(\overline{I}; \mathbb{R})$ and any $t_1, t_2 \in \overline{I}$.

Before proving the theorem we need several auxiliary results.

Lemma 8.4.6 Let $u \in L^p(I; X)$ and $u' \in L^q(I; Y)$, $1 \le p, q < \infty$, where X, Y are Banach spaces, $X \hookrightarrow Y$. Then there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{C}^\infty(\overline{I}; X)$ such that $\{u_n'\}_{n\in\mathbb{N}} \subset C^\infty(\overline{I}; Y)$, and

$$u_n \to u \qquad \text{in } L^p(I; X) \qquad \text{and} \qquad u_n' \to u' \qquad \text{in } L^q(I; Y).$$

<u>*Proof.*</u> Let without loss of generality $I = (0, T)$. We define

$$v(t) := \begin{cases} u(-t) & t \in (-T, 0) \\ u(t) & t \in (0, T) \\ u(2T - t) & t \in (T, 2T), \end{cases}$$

and

$$\mu(t) := \begin{cases} 1 & t \in (-\frac{T}{4}, \frac{5}{4}T) \\ 0 & t < -\frac{T}{2} \text{ or } t > \frac{3}{2}T \\ \in [0, 1] & \text{otherwise} \end{cases}$$

such that $\mu \in \mathcal{C}_0^\infty((-T,2T))$. Denote $\mathcal{I} = (-\frac{T}{4}, \frac{5}{4}T)$. Then $v\mu \in L^p(\mathcal{I};X)$ and $(\mu v)' \in L^q(\mathcal{I};Y)$ (recall that $v \in C(\mathcal{I};Y)$). Define

$$u_n := (v\mu) \star \eta_{\frac{1}{n}},$$

where $\eta_{\frac{1}{n}}$ is the standard mollifier. Using Theorem 8.2.8, Claim 4. we finish the proof, as $u_n' = (v\mu)' \star \eta_{\frac{1}{n}}$ in $\mathcal{I}$. ■

Lemma 8.4.7 Let X be a reflexive Banach space, Y a Banach space and $X \hookrightarrow Y$ densely. Then $Y^* \hookrightarrow X^*$ densely.

Proof. Denote $I\colon X \to Y$ the continuous injective mapping (the identity) which represents the embedding of X to Y. We know that $I(X)$ is dense in Y. We define $I^*\colon Y^* \to X^*$ as

$$\langle I^*(y'), x\rangle_X := \langle y', I(x)\rangle_Y.$$

We show that I^* is the (identity) mapping from Y^* to X^* which is injective, continuous and $I^*(Y^*)$ is dense in X^*.

Let $I^*(y') = 0$, i.e., $\langle I^*(y'), x\rangle_X = 0$ for all $x \in X$. Then $\langle y', I(x)\rangle_Y = 0$ for all $x \in X$, i.e., $y' = 0$ as $I(x)$ is dense in Y. Thus I^* is injective. Clearly, from the definition it follows that I^* is bounded. Let us show the density of the embedding.

Recall that X is reflexive and assume that $\overline{Y^*} \neq X^*$. Then there exists $x'' \in X^{**}$ such that for all $y' \in Y^*$ we have $\langle x'', I^*(y')\rangle_{X^*} = 0$, but $x'' \neq 0$. Hence there exists $x \in X$, $x \neq 0$, such that $x'' = \mathcal{J}(x)$ (canonical mapping) such that

$$\langle I^*(y'), x\rangle_X = 0 \qquad \forall y' \in Y^* \implies \langle y', I(x)\rangle_Y = 0 \qquad \forall y' \in Y^*,$$

thus $I(x) = 0$, i.e., $x = 0$ which leads to the contradiction. Hence $\overline{Y^*} = X^*$. ■

Definition 8.4.8 — Weak continuity. Let X be a Banach space. We say that $u\colon I \to X$ is continuous in the weak topology of X (weakly continuous in X), $u \in \mathcal{C}(I;X_w)$, if the mapping

$$t \mapsto \langle x', u(t)\rangle_X$$

is continuous in I for all $x' \in X^*$.

Clearly, if $u \in \mathcal{C}(I;X)$, then $u \in \mathcal{C}(I;X_w)$, the other implication is generally true only if X is finite dimensional.

Lemma 8.4.9 Let X, Y be two Banach spaces, X reflexive, $X \hookrightarrow Y$ densely, $I \subset \mathbb{R}$ be bounded, open. Let $\varphi \in L^\infty(I;X)$ and $\varphi \in \mathcal{C}(\overline{I};Y_w)$. Then $\varphi \in \mathcal{C}(\overline{I};X_w)$.

Proof. Due to Lemma 8.4.7 we know that $Y^* \hookrightarrow X^*$ densely. Further, the mapping

$$t \mapsto \langle \eta, I(\varphi)(t)\rangle_Y$$

is continuous in $\overline{I}$ for all $\eta \in Y^*$, where $I\colon X \to Y$ represents the embedding of X to Y. We aim at showing that

$$t \mapsto \langle \mu, \varphi(t)\rangle_X$$

is continuous for all $\mu \in X^*$. We first show that $\varphi(t) \in X$ is well defined for all values $t \in \overline{I}$. We extend the function to the whole $\mathbb{R}$ (by reflection so that we keep the weak continuity in Y) and fix $t_0 \in \overline{I}$. We set (η is the standard regularization kernel)

$$\varphi_n(t) := (\eta_{\frac{1}{n}} \star \varphi)(t) \in X.$$

We have

$$\|\varphi_n(t_0)\|_X = \|(\eta_{\frac{1}{n}} \star \varphi)(t_0)\|_X \leq \|\varphi\|_{L^\infty(I;X)}.$$

Since X is reflexive, there exists a subsequence $\varphi_{n_k}(t_0)$ which converges weakly to an element of $x \in X$. Since for any $\mu \in Y^*$ we have

$$\langle \mu, (\eta_{\frac{1}{n}} \star \varphi)(t_0) - \varphi(t_0)\rangle_Y = (\eta_{\frac{1}{n}} \langle \mu, \varphi\rangle_Y)(t_0) - \langle \mu, \varphi\rangle_Y(t_0) \to 0$$

for $n \to \infty$ by continuity of $\langle \mu, \varphi\rangle_Y$, by uniqueness of the weak limit it follows that $x = \varphi(t_0)$ and

$$\|\varphi(t)\|_X \leq \|\varphi\|_{L^\infty(I;X)} \qquad \text{for all } t \in \overline{I}.$$

Now, as Y^* is dense in X^*, for any $\mu \in X^*$ and arbitrary $\varepsilon > 0$, there exists $\mu_\varepsilon \in Y^*$ such that

$$\|I^*(\mu_\varepsilon) - \mu\|_{X^*} < \varepsilon.$$

Let us fix $\varepsilon > 0$. Then we write

$$\langle \mu, \varphi(t) - \varphi(t_0)\rangle_X = \langle \mu - I^*\mu_\varepsilon, \varphi(t) - \varphi(t_0)\rangle_X + \langle I^*\mu_\varepsilon, \varphi(t) - \varphi(t_0)\rangle_X.$$

By the choice of μ_ε it holds

$$|\langle \mu - I^*(\mu_\varepsilon), \varphi(t) - \varphi(t_0)\rangle_X| \le \|\mu - I^*(\mu_\varepsilon)\|_{X^*}\|\varphi(t) - \varphi(t_0)\|_X \le 2\varepsilon\|\varphi\|_{L^\infty(I;X)},$$

and

$$|\langle I^*(\mu_\varepsilon), \varphi(t) - \varphi(t_0)\rangle_X| = |\langle \mu_\varepsilon, I(\varphi)(t) - I(\varphi)(t_0)\rangle_Y|.$$

Taking t sufficiently close to t_0, for fixed μ_ε, we can, due to the continuity of $\langle \mu_\varepsilon, I(\varphi)(t)\rangle$, also the second term make smaller than ε. The lemma is proved. ■

We are now prepared to prove Theorem 8.4.4.

Proof. (of Theorem 8.4.4). Using Lemma 8.4.6 we know that there exists $\{u_m\}_{m\in\mathbb{N}} \subset \mathcal{C}^\infty(\overline{I}; X)$ such that $u_m \to u$ in $L^p(I; X)$ and $u'_m \to u'$ in $L^{p'}(I; X^*)$ as $m \to \infty$. Since $X \hookrightarrow H$ and $H^* \hookrightarrow X^*$, both embeddings being dense, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_m\|_H^2 = 2(u'_m, u_m)_H = 2\langle u'_m, u_m\rangle_X$$

(see Example 8.4.2) and integrating the identity we also see that $\|u_m\|_H$ is bounded in I. Moreover, we easily get that the sequence $\|u_m\|_H^2$ is a Cauchy sequence in $\mathcal{C}(\overline{I}; \mathbb{R})$ and as $\{u_m\}_{m\in\mathbb{N}}$ converges in $L^p(I; H)$ to u, the limit is $\|u\|_H^2$. Therefore we have

$$\|u(t_2)\|_H^2 - \|u(t_1)\|_H^2 = 2\int_{t_1}^{t_2} \langle u', u\rangle_X \,\mathrm{d}t,$$

as well as

$$-\int_I \|u\|_H^2 \psi' \,\mathrm{d}t = 2\int_I \langle u', u\rangle_X \psi \,\mathrm{d}t$$

for arbitrary $\psi \in \mathcal{C}_0^\infty(I; \mathbb{R})$.

As $u \in \mathcal{C}(\overline{I}; X^*)$, due to Theorem 8.3.10, we have $u \in \mathcal{C}(\overline{I}; X_w^*)$ and hence by Lemma 8.4.9 $u \in \mathcal{C}(\overline{I}; H_w)$; note that computations above imply the required property that $u \in L^\infty(I; H)$.

Let $t_0 \in \overline{I}$ be arbitrary. As $\lim_{t\to t_0}(u(t), \varphi)_H = (u(t_0), \varphi)_H$ and $\lim_{t\to t_0}\|u(t)\|_H^2 = \|u(t_0)\|_H^2$, we see that

$$\lim_{t\to t_0}\|u(t) - u(t_0)\|_H^2 = 0$$

which yields $u \in \mathcal{C}(\overline{I}; H)$. ■

In what follows, if no confusion may arise, we stop using the notation I, i, I^*, i^* or Φ for the corresponding operators.

8.5 Compact embedding of spaces with time derivative

We close this chapter by showing a result which will replace for the evolutionary equations the result on compact embedding of Sobolev spaces and thus will allow to solve certain class of nonlinear problems. We consider for X_0 and X_1 Banach spaces

$$W = W^{\alpha_0,\alpha_1}_{X_0,X_1} = \{v \in L^{\alpha_0}(I; X_0) \,|\, v' \in L^{\alpha_1}(I; X_1)\},$$

where $I \subset \mathbb{R}$ is an open bounded interval. We define

$$\|u\|_W := \|u\|_{L^{\alpha_0}(I;X_0)} + \|u'\|_{L^{\alpha_1}(I;X_1)}.$$

First we have

Lemma 8.5.1 — Ehrling. Let X_0, X_1 and X be Banach spaces such that $X_0 \hookrightarrow\hookrightarrow X \hookrightarrow X_1$. Then for any $\eta > 0$ there exists $C_\eta > 0$ such that for all $v \in X_0$

$$\|v\|_X \le \eta\|v\|_{X_0} + C_\eta\|v\|_{X_1}. \tag{8.8}$$

Proof. We proceed by contradiction. Let (8.8) be not true, i.e., there exists $\eta > 0$ such that for any $m \in \mathbb{N}$ there exists $w_m \in X_0$, for which

$$\|w_m\|_X > \eta\|w_m\|_{X_0} + m\|w_m\|_{X_1}.$$

We set $v_m := \frac{w_m}{\|w_m\|_{X_0}}$, i.e.,

$$\|v_m\|_X > \eta + m\|v_m\|_{X_1}.$$

As $\|v_m\|_{X_0} = 1$, then v_m is bounded in X and thus $\|v_m\|_{X_1} \to 0$. Furthermore, since $X_0 \hookrightarrow\hookrightarrow X$, there exists $\{v_{m_k}\}_{k\in\mathbb{N}}$ such that $v_{m_k} \to v$ in X. However, necessarily, $v = 0$, but on the other hand,

$$\|v_{m_k}\|_X > \eta$$

which leads to the contradiction. ■

The main result is

Theorem 8.5.2 — Aubin–Lions. Let X_0, X_1 and X be Banach spaces such that $X_0 \hookrightarrow\hookrightarrow X \hookrightarrow X_1$. Let X_0, X_1 be reflexive, $1 < \alpha_0, \alpha_1 < \infty$ and let I be open, bounded. Then $W \hookrightarrow\hookrightarrow L^{\alpha_0}(I; X)$.

Proof. **Step 1:** Weak convergences
Let $\{u_m\}_{m\in\mathbb{N}}$ be a bounded sequence in W. We aim at showing that there exists $\{u_{m_k}\}_{k\in\mathbb{N}}$ a subsequence such that $u_{m_k} \to u$ in $L^{\alpha_0}(I; X)$ as $k \to \infty$. As X_0, X_1 are reflexive, there exists $u \in W$ such that $u_{m_k} \rightharpoonup u$ in W, i.e.

$$\begin{aligned} u_{m_k} &\rightharpoonup u && \text{in } L^{\alpha_0}(I; X_0), \\ u'_{m_k} &\rightharpoonup u' && \text{in } L^{\alpha_1}(I; X_1). \end{aligned}$$

We need to show that $v_{m_k} := u_{m_k} - u \to 0$ (strongly) in $L^{\alpha_0}(I; X)$.

Step 2: Replacing X by X_1
In fact, it is enough to verify that $v_{m_k} \to 0$ in $L^{\alpha_0}(I; X_1)$. Under this assumption, by virtue of Lemma 8.5.1, we have

$$\|v_{m_k}\|_{L^{\alpha_0}(I;X)} \le \eta \|v_{m_k}\|_{L^{\alpha_0}(I;X_0)} + C_\eta \|v_{m_k}\|_{L^{\alpha_0}(I;X_1)}.$$

As $\|v_{m_k}\|_{L^{\alpha_0}(I;X_0)}$ is bounded, we have

$$\|v_{m_k}\|_{L^{\alpha_0}(I;X)} \le C\eta + C_\eta \|v_{m_k}\|_{L^{\alpha_0}(I;X_1)}.$$

Let us fix $\varepsilon > 0$. Then there exists $\eta > 0$ such that $C\eta < \frac{\varepsilon}{2}$ and to this η there exists $k_0 \in \mathbb{N}$ such that for $k \ge k_0$, $C_\eta \|v_{m_k}\|_{L^{\alpha_0}(I;X_1)} < \frac{\varepsilon}{2}$. Hence $v_{m_k} \to 0$ in $L^{\alpha_0}(I; X)$.

Step 3: Continuous embedding
Repeating the proof from Proposition 8.3.9 we get that

$$W \hookrightarrow \mathcal{C}(\overline{I}; X_1),$$

i.e.,

$$\max_{t\in\overline{I}} \|u(t)\|_{X_1} \le C\|u\|_W.$$

Step 4: Pointwise convergence
As $\|v_{m_k}(t)\|_{X_1} \le C$ for all $t \in \overline{I}$, it is enough to show that $v_{m_k}(t) \to 0$ (strongly) in X_1 for arbitrary $t \in \overline{I}$. Let us assume, without loss of generality, that $0 \in \overline{I}$ and let us show that $v_{m_k}(0) \to 0$ in X_1. We have

$$v_{m_k}(0) = v_{m_k}(t) - \int_0^t v'_{m_k}(\tau)\,\mathrm{d}\lambda_1(\tau).$$

We integrate this inequality, from 0 to s. Then

$$\begin{aligned} v_{m_k}(0) &= \frac{1}{s}\Big\{ \int_0^s v_{m_k}(t)\,\mathrm{d}\lambda_1(t) - \int_0^s \Big(\int_0^t v'_{m_k}(\tau)\,\mathrm{d}\lambda_1(\tau) \Big)\,\mathrm{d}\lambda_1(t) \Big\} \\ &= \frac{1}{s}\int_0^s v_{m_k}(t)\,\mathrm{d}\lambda_1(t) - \frac{1}{s}\int_0^s (s-\tau) v'_{m_k}(\tau)\,\mathrm{d}\lambda_1(\tau) =: a_{m_k} + b_{m_k}, \end{aligned}$$

where we used the Fubini Theorem 8.1.14. We choose $\varepsilon > 0$. Then

$$\|b_{m_k}\|_{X_1} \le \int_0^s \|v'_{m_k}\|_{X_1}\,\mathrm{d}t < \frac{\varepsilon}{2}$$

provided s is sufficiently small (recall, $\alpha_1 > 1$). As we know that $v_{m_k} \rightharpoonup 0$ in $L^{\alpha_0}(I; X_0)$, then $\|a_{m_k}\|_{X_0}$ is bounded and hence $\|a_{m_k}\|_{X_1} \to 0$ strongly for a fixed $s > 0$. Therefore, for s fixed from the estimate of b_{m_k}, we choose k_0 sufficiently large so that for any $k \ge k_0$, $\|a_{m_k}\|_{X_1} < \frac{\varepsilon}{2}$. The theorem is proved. ■

Remark 8.5.3. The theorem above also holds for $\alpha_1 = 1$ (time derivative bounded only in L^1 over time) which might be in some situations important. However, this requires some changes in the proof and since we shall not use this result, we do not present the proof in this situation.

Chapter 9

Semigroup theory. Nonlinear parabolic equations

We first return to linear evolutionary partial differential equations and present another method how to construct solutions. The method will be based on the semigroup theory. Next we consider two examples of nonlinear parabolic equations and show two methods of construction of weak solutions.

9.1 Semigroup theory

We will now present another point of view to linear evolutionary PDEs: instead of studying a PDE in $\mathbb{R}^{d+1}$ we consider an ODE in a Banach space. This approach is closer to functional analysis. We restrict ourselves only on the homogeneous equation, the situation with a nontrivial right-hand side will be only briefly mentioned at the end of the section.

Recall that the ODE

$$\frac{\mathrm{d}u}{\mathrm{d}t} = au, \qquad u(0) = u_0$$

has a unique solution

$$u(t) = u_0 \mathrm{e}^{at}.$$

By analogy, considering

$$\partial_t u - \Delta u = 0, \qquad u(0) = u_0$$

(together with some boundary conditions) we expect the solution in the form

$$u(t) = u_0 \mathrm{e}^{\Delta t}$$

for suitably defined exponential of the Laplace operator. More generally, let A: $X \to X$, where X is a Banach space. (Note that in the energy method we considered the Laplace operator as an operator from $W_0^{1,2}$ to $W^{-1,2}$, i.e., this approach is not the same.) We consider the problem

$$\frac{\mathrm{d}u}{\mathrm{d}t} = Au, \qquad u(0) = u_0$$

and hope to get the solution in the form

$$u(t) = u_0 \mathrm{e}^{At}.$$

However, as A is generally an unbounded operator, it is not always true that the series

$$\sum_{n=0}^{\infty} \frac{t^n A^n}{n!}$$

converges, at least, not for arbitrary $t \in \mathbb{R}$.

Before solving this problem, we slightly change our point of view. We consider the solution to our ODE in a Banach space u: $[0, \infty) \to X$. We assume that X is a Banach space, A: $X \to X$ a linear operator with $D(A) \subsetneq X$, a linear subspace of X. The operator A is usually unbounded. We will consider such A that
(a) the ODE has a unique solution for arbitrary $u(0) = u_0 \in X$
(b) the theory covers as many as possible interesting PDEs.
Assume for a moment that (a) and (b) hold true and we write

$$u(t) = S(t)u_0, \qquad t > 0.$$

We expect the following properties:

1. $S(t)\colon X \to X$ is linear
2. $S(0)u = u$ for all $u \in X$ in the sense of the limit in X, i.e., $\lim_{t\to 0_+} S(t)u = u$
3. $S(t+s)u = S(t)S(s)u = S(s)S(t)u \ \forall t, s \geq 0 \ u \in X$.

Definition 9.1.1 — Semigroup. A family $\{S(t)\}_{t\geq 0}$ of bounded linear operators mapping X to X is called a semigroup (a c_0-semigroup) if 1.–3. hold. Moreover, if $\|S(t)\|_{\mathcal{L}(X)} \leq 1$ holds, we call the semigroup a contraction semigroup.

Remark 9.1.2. A contraction semigroup satisfies $\|S(t)u\|_X \leq \|u\|_X$ for all $t \geq 0$ and $u \in X$.

Lemma 9.1.3 Let $S(t)$ be a c_0-semigroup in X. Then the following holds.

1. There exist $M \geq 1$, $\omega \geq 0$ such that
$$\|\mathcal{S}(t)\|_{\mathcal{L}(X)} \leq M\mathrm{e}^{\omega t} \qquad \forall t \geq 0.$$
2. The mapping $t \mapsto S(t)y$ is continuous from $[0,\infty) \to X$, for all $u \in X$.

Proof. **Step 1:** Claim 1.
We proceed by contradiction. We claim that there exists $M \geq 1$ and $\delta > 0$ such that $\|S(t)\|_{\mathcal{L}(X)} \leq M$ for all $t \in [0,\delta]$. If not, then there exists $t_n \to 0_+$ such that $\|S(t_n)\|_{\mathcal{L}(X)} \to \infty$, but $S(t_n)u \to u$ for all $u \in X$. Therefore $\|S(t_n)u\|_X$ is bounded. This contradicts to the principle of uniform boundedness, as $\|S(t_n)\|_{\mathcal{L}(X)}$ are bounded if and only if $\|S(t_n)u\|_X$ are bounded for all $u \in X$.

Now, let $\omega := \frac{1}{\delta}\ln M$, i.e., $M = \mathrm{e}^{\omega\delta}$. Then for $t \geq 0$ arbitrary, $t = n\delta + \varepsilon$, $\varepsilon \in [0,\delta)$, we have
$$\begin{aligned}\|S(t)\|_{\mathcal{L}(X)} &= \|S(\delta+\delta+\cdots+\delta+\varepsilon)\|_{\mathcal{L}(X)} = \|S(\delta)S(\delta)\ldots S(\delta)S(\varepsilon)\|_{\mathcal{L}(X)}\\ &\leq \|S(\delta)\|^n_{\mathcal{L}(X)}\|S(\varepsilon)\|_{\mathcal{L}(X)} \leq M^n \cdot M \leq M\mathrm{e}^{\omega t}.\end{aligned}$$

Step 2: Claim 2.
The continuity at 0_+ is Property 2. from the definition of the semigroup. Continuity from the right at $t > 0$ is a consequence of the following property
$$S(t+h)u = S(t)S(h)u \to S(t)u$$
for $h \to 0_+$ due to the continuity of $S(t)$ in X and the fact that $S(h)u \to u$ in X for $h \to 0_+$. Similarly, we consider the continuity from the left. Let $0 < h < t$. Then
$$S(t-h)u - S(t)u = S(t-h)\big(u - S(h)u\big).$$
As $u - S(h)u \to 0$ in X for $h \to 0_+$ and $\|S(t-h)\|_{\mathcal{L}(X)} \leq M\mathrm{e}^{\omega t}$ independently of h, we have
$$\|S(t-h)u - S(t)u\|_X \leq \|S(t-h)\|_{\mathcal{L}(X)}\|u - S(h)u\|_X \leq M\mathrm{e}^{\omega t}\|u - S(h)u\|_X \to 0$$
for $h \to 0_+$. ■

Definition 9.1.4 — Generator of a semigroup. An unbounded operator $A\colon X \to X$ with the domain $D(A)$ is called a *generator of the semigroup* $\{S(t)\}_{t\geq 0}$, if
$$Au = \lim_{h\to 0_+} \frac{S(h)u - u}{h},$$
where
$$D(A) = \Big\{u \in X \mid \lim_{h\to 0_+} \frac{1}{h}(S(h)u - u) \ \text{ exists in } X\Big\}.$$

Remark 9.1.5. Note that the above defined operator is linear and $D(A) \subset X$ is a linear subspace.

Theorem 9.1.6 — Properties of a c_0-semigroup. Let A with its domain $D(A)$ be the generator of a c_0-semigroup $\{S(t)\}_{t\geq 0}$. Then

1. if $u \in D(A)$, then $S(t)u \in D(A) \ \forall t \geq 0$
2. if $u \in D(A)$, then $AS(t)u = S(t)Au = \frac{\mathrm{d}}{\mathrm{d}t}(S(t)u)$ for all $t \geq 0$ (at $t = 0$, the derivative is only from the right)
3. if $u \in X$, $t > 0$, then $\int_0^t S(s)u \,\mathrm{d}\lambda_1(s) \in D(A)$ and $A\big(\int_0^t S(s)u\,\mathrm{d}\lambda_1(s)\big) = S(t)u - u$.

Proof. **Step 1:** Claim 1.
Let $u \in D(A)$ and $t \geq 0$ be given. We ask whether $\frac{1}{h}\big(S(h)S(t)u - S(t)u\big) \to y$ in X for $h \to 0_+$, where $y \in X$. Then $S(t)u \in D(A)$ and $A(S(t)u) = y$. However,

$$\frac{1}{h}\Big(S(h)S(t)u - S(t)u\Big) = S(t)\Big(\frac{1}{h}\big(S(h)u - u\big)\Big) \to S(t)Au$$

in X as $h \to 0_+$, since $S(t)$ is linear and bounded.
Step 2: Claim 2.
Let $u \in D(A)$. Above in 1. we also proved that $A(S(t)u) = S(t)(Au)$ and $\frac{\mathrm{d}}{\mathrm{d}t}(S(t)u) = S(t)(Au)$ from the right for $t \geq 0$. We need to compute the derivative from the left, for $t > 0$. We have

$$\frac{S(t-h)u - S(t)u}{-h} = S(t-h)\Big(\frac{u - S(h)u}{-h}\Big),$$

therefore

$$\begin{aligned}\frac{S(t-h)u - S(t)u}{-h} - S(t)(Au) &= S(t-h)\Big(\frac{u - S(h)u}{-h}\Big) - S(t-h)S(h)(Au)\\ &= S(t-h)\Big[\frac{u - S(h)u}{-h} - S(h)(Au)\Big] \to 0\end{aligned}$$

for $h \to 0_+$, as $S(t-h)$ is bounded uniformly with respect to h and both terms in the bracket converge to Au.
Step 3: Claim 3.
Denote

$$y = \int_0^t S(s)u \,\mathrm{d}\lambda_1(s), \qquad u \in X, \quad t > 0$$

(as $S(s)$ is bounded, the Bochner integral exists). Then due to the continuity of $t \mapsto S(t)u$

$$\begin{aligned}\frac{1}{h}\big(S(h)y - y\big) &= \frac{1}{h}\Big(S(h)\int_0^t S(s)u\,\mathrm{d}\lambda_1(s) - \int_0^t S(s)u\,\mathrm{d}\lambda_1(s)\Big) = \frac{1}{h}\Big(\int_0^t S(h+s)u\,\mathrm{d}\lambda_1(s) - \frac{1}{h}\int_0^t S(s)u\,\mathrm{d}\lambda_1(s)\Big)\\ &= \frac{1}{h}\Big(\int_h^{t+h} S(s)u\,\mathrm{d}\lambda_1(s) - \int_0^t S(s)u\,\mathrm{d}\lambda_1(s)\Big) = \frac{1}{h}\int_t^{t+h} S(s)u\,\mathrm{d}\lambda_1(s) - \frac{1}{h}\int_0^h S(s)u\,\mathrm{d}\lambda_1(s)\\ &\to S(t)u - S(0)u = S(t)u - u.\end{aligned}$$

Therefore $y \in D(A)$ and $Ay = S(t)u - u$. ■

Remark 9.1.7. 1. Note that for $u \in D(A)$

$$S(t)u - u = S(t)u - S(0)u = \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}S(s)u\,\mathrm{d}\lambda_1(s) = \int_0^t S(s)(Au)\,\mathrm{d}\lambda_1(s).$$

2. We proved that $D(A)$ is invariant with respect to $S(t)$, and $S(t)$ and A commute in $D(A)$. Moreover, $t \mapsto S(t)u_0$ is a classical solution to $\frac{\mathrm{d}u}{\mathrm{d}t} = Au$, $u(0) = u_0$, if $u_0 \in D(A)$. Thus also $S(t)u \in C^1((0,\infty);X)$.

Recall that an (unbounded) operator A with its domain $D(A)$ is closed, if for $u_n \in D(A)$, $u_n \to u$ in X, $Au_n \to v$ in X we have $u \in D(A)$ and $Au = v$. It follows that
A with its domain $D(A)$ is closed $\iff$ $D(A)$ is a Banach space (complete) with respect to the norm $\|u\|_X + \|Au\|_X$ (the graph norm).

Theorem 9.1.8 — Density of the domain and closedness of the generator. Let A with $D(A)$ be a generator of a c_0-semigroup $\{S(t)\}_{t\geq 0}$ in X. Then $D(A)$ is dense in X and A with its domain $D(A)$ is closed.

Proof. **Step 1:** Density
Let $x \in X$ be given. Then $x = \lim_{h\to 0^+} \frac{1}{h}\int_0^h S(s)x\,\mathrm{d}\lambda_1(s)$, due to the continuity of $S(s)$ and standard properties of the Bochner integral. But due to Theorem 9.1.6 Claim 3. we know that $\int_0^h S(s)x\,\mathrm{d}\lambda_1)s) \in D(A)$ and so is $\frac{1}{h}\int_0^h S(s)x\,\mathrm{d}\lambda_1)s) \in D(A)$ for any $h > 0$.
Step 2: Closedness
Let $u_k \in D(A)$ and suppose $u_k \to u$, $Au_k \to v$ in X. We have to verify that $u \in D(A)$ and $v = Au$. According to Remark 9.1.7

$$S(t)u_k - u_k = \int_0^t S(s)(Au_k)\,\mathrm{d}\lambda_1(s).$$

Letting $k \to \infty$ (recall that $\|S(s)(Au_k) - S(s)v\|_X \leq \|S(s)\|_{\mathcal{L}(X)}\|Au_k - v\|_X \to 0$ uniformly with respect to s from a bounded interval, see Lemma 9.1.3)

$$S(t)u - u = \int_0^t S(s)v\,\mathrm{d}\lambda_1(s).$$

Therefore

$$\lim_{t\to 0_+} \frac{S(t)u-u}{t} = \lim_{t\to 0_+} \frac{1}{t}\int_0^t S(s)v \,\mathrm{d}\lambda_1(s) = v.$$

Then $u \in D(A)$ and $v = Au$. ■

We also need the following uniqueness result.

Lemma 9.1.9 Let $\{S(t)\}_{t\geq 0}$, $\{\widetilde{S}(t)\}_{t\geq 0}$ be two c_0-semigroups with the same generator. Then $S(t) = \widetilde{S}(t)$ for all $t \geq 0$.

Proof. Let $u \in D(A)$. Define $y(t) = S(T-t)\widetilde{S}(t)u$. We have

$$\frac{\mathrm{d}}{\mathrm{d}t} y(t) = -AS(T-t)\widetilde{S}(t)u + AS(T-t)\widetilde{S}(t)u = 0,$$

hence $\frac{\mathrm{d}}{\mathrm{d}t}y(t) = 0$ in $(0,T)$ and since $y \in C([0,T];X)$, we have $S(T)u = y(0) = y(T) = \widetilde{S}(T)u$ for any $u \in D(A)$ and any $T > 0$. Finally, due to the density of $D(A)$ in X and boundedness of $S(t)$ and $\widetilde{S}(t)$ as operators from X to X we get that $S(t)u = \widetilde{S}(t)u$ for any $u \in X$ and $t \geq 0$. ■

Definition 9.1.10 — Resolvent of unbounded operator. Let A with its domain $D(A)$ be an unbounded operator. We define

1. the resolvent set $\varrho(A) := \{\lambda \in \mathbb{R} \,|\, \lambda I - A\colon X \to X \text{ is one to one}\}$ (can also be considered as a subset of $\mathbb{C}$)
2. the resolvent $R_\lambda (= R_\lambda(A)) = (\lambda I - A)^{-1}\colon X \to D(A), \lambda \in \varrho(A)$
3. The spectrum $\sigma(A) = \{\lambda \in \mathbb{C} \,|\, \lambda I - A \text{ is not invertible}\}$ (generally, $\sigma(A) = \mathbb{C} \setminus \varrho(A)$).

Remark 9.1.11. The operator $A\colon D(A) \to X$ is continuous, therefore the same holds for $\lambda I - A$. Hence R_λ is by the Open mappings Theorem continuous as an operator from X to $D(A)$, see, e.g., (Edwards, 1995, Theorem 6.4.5).

Lemma 9.1.12
1. Let $\lambda \in \varrho(A)$. Then $AR_\lambda x = \lambda R_\lambda x - x$ for all $x \in X$ and $R_\lambda Ax = \lambda R_\lambda x - x$ for all $x \in D(A)$.
2. Let λ, $\mu \in \varrho(A)$. Then $R_\lambda x - R_\mu x = (\mu - \lambda)R_\lambda R_\mu x$ for all $x \in X$; whence $R_\lambda R_\mu = R_\mu R_\lambda$.
3. Let A with its domain $D(A)$ be a generator of a c_0-semigroup. Let further $\|S(t)\|_{\mathcal{L}(X)} \leq M\mathrm{e}^{\omega t}$. Then $\lambda \in \varrho(A)$ for all $\lambda > \omega$ and
$$R_\lambda x = \int_0^\infty \mathrm{e}^{-\lambda t} S(t)x \,\mathrm{d}\lambda_1(t)$$
for all $x \in X$; thus $\|R_\lambda\|_{\mathcal{L}(X)} \leq \frac{M}{\lambda - \omega}$.

Proof. **Step 1:** Claim 1.
We have for any $x \in X$, due to the definition of R_λ

$$AR_\lambda x = (A - \lambda I + \lambda I)R_\lambda x = -x + \lambda R_\lambda x.$$

Similarly for $x \in D(A)$

$$R_\lambda Ax = R_\lambda(A - \lambda I + \lambda I)x = -x + R_\lambda \lambda x = \lambda R_\lambda x - x.$$

Step 2: Claim 2.
We can directly compute

$$R_\lambda = R_\lambda(\mu I - A)R_\mu = R_\lambda\big((\mu-\lambda)I + \lambda I - A\big)R_\mu = (\mu - \lambda)R\lambda R_\mu + R_\mu$$

which gives directly the first formula.

As $R_\mu - R_\lambda = (\lambda - \mu)R_\mu R_\lambda$, we easily see that

$$R_\mu R_\lambda = R_\lambda R_\mu = \frac{R_\lambda - R_\mu}{\mu - \lambda}.$$

Step 3: Claim 3.
First note that we may assume, without loss of generality, that $\omega = 0$. Indeed, if A with its domain $D(A)$ generates the c_0-semigroup $\{S(t)\}_{t\geq 0}$, then $\widetilde{A} = A - \omega I$ with $D(\widetilde{A}) = D(A)$ generates the semigroup $\widetilde{S}(t) = \mathrm{e}^{-\omega t}S(t)$. Indeed, we have

$$\widetilde{A}x = \lim_{t\to 0_+} \frac{\mathrm{e}^{-\omega t}S(t)x - x}{t} = \lim_{t\to 0_+} \Big(\frac{S(t)x - x}{t}\mathrm{e}^{-\omega t} + \frac{\mathrm{e}^{-\omega t} - 1}{t}x\Big) = Ax - \omega Ix.$$

Moreover, $R_\lambda(\widetilde{A}) = R_{\lambda+\mu}(A)$. To see it, we compute

$$R_\lambda(\widetilde{A}) = (\lambda I - \widetilde{A})^{-1} = \big((\lambda+\mu)I - A\big)^{-1} = R_{\lambda+\mu}(A).$$

Assume therefore that $\|S(t)\|_{\mathcal{L}(X)} \le M$ and $\lambda > 0$. We show that $\lambda \in \varrho(A)$. Denote

$$\widetilde{R}x = \int_0^\infty \mathrm{e}^{-\lambda t} S(t)x \,\mathrm{d}\lambda_1(t),$$

the Laplace transform of the semigroup $\{S(t)\}_{t\ge 0}$. As

$$\|\mathrm{e}^{-\lambda t} S(t)x\|_X \le \mathrm{e}^{-\lambda t} M \|x\|_X \in L^1((0,\infty)),$$

we have

$$\|\widetilde{R}x\|_X \le \int_0^\infty \mathrm{e}^{-\lambda t} M\|x\|_X \,\mathrm{d}t = \frac{M}{\lambda}\|x\|_X.$$

Thus $\widetilde{R} \in \mathcal{L}(X)$ and $\|\widetilde{R}\|_{\mathcal{L}(X)} \le \frac{M}{\lambda}$. Let us show that $\widetilde{R}x \in D(A)$. We compute for $x \in X$

$$\begin{aligned}
\frac{1}{h}(S(h) - I)\widetilde{R}x &= \frac{1}{h}\int_0^\infty \big(\mathrm{e}^{-\lambda t} S(h)S(t)x - \mathrm{e}^{-\lambda t}S(t)x\big)\,\mathrm{d}\lambda_1(t) \\
&= \frac{1}{h}\int_0^\infty \mathrm{e}^{-\lambda t} S(t+h)x \,\mathrm{d}\lambda_1(t) - \frac{1}{h}\int_0^\infty \mathrm{e}^{-\lambda t} S(t)x\,\mathrm{d}\lambda_1(t) \\
&= \frac{1}{h}\int_h^\infty \mathrm{e}^{-\lambda (t-h)} S(t)x \,\mathrm{d}\lambda_1(t) - \frac{1}{h}\int_0^\infty \mathrm{e}^{-\lambda t} S(t)x\,\mathrm{d}\lambda_1(t) \\
&= \frac{\mathrm{e}^{\lambda h} - 1}{h}\int_0^\infty \mathrm{e}^{-\lambda t} S(t)x \,\mathrm{d}\lambda_1(t) - \frac{\mathrm{e}^{\lambda h}}{h}\int_0^h \mathrm{e}^{-\lambda t} S(t)x\,\mathrm{d}\lambda_1(t) \\
&\to \lambda\widetilde{R}x - x \qquad \text{as } h \to 0^+.
\end{aligned}$$

It means that $\widetilde{R}x \in D(A)$, $A\widetilde{R}x = \lambda\widetilde{R}x - x$ for all $x \in X$, i.e., $(\lambda I - A)\widetilde{R}x = x$. Then $\lambda I - A\colon D(A) \to X$ is onto.

We now show that it is also injective. Let $x \in D(A)$ be fixed. Then

$$\begin{aligned}
A\widetilde{R}x = A\Big(\int_0^\infty \mathrm{e}^{-\lambda t} S(t)x\,\mathrm{d}\lambda_1(t)\Big) &= \int_0^\infty A\big(\mathrm{e}^{-\lambda t}S(t)x\big)\,\mathrm{d}\lambda_1(t) \\
&= \int_0^\infty \mathrm{e}^{-\lambda t} S(t)Ax\,\mathrm{d}\lambda_1(t) = \widetilde{R}Ax \qquad \text{due to Theorem 9.1.6.}
\end{aligned}$$

Thus $A\widetilde{R} = \widetilde{R}A$ in $D(A)$ and therefore $\widetilde{R}(\lambda I - A)x = \lambda\widetilde{R}x - \widetilde{R}Ax = \lambda\widetilde{R}x - (\lambda\widetilde{R}x - x) = x$. Hence $(\lambda I - A)x = 0 \Rightarrow x = \widetilde{R}0 = 0$, $\lambda I - A$ is injective, $\lambda \in \varrho(A)$ and $\widetilde{R} = (\lambda I - A)^{-1} = R_\lambda(A)$. ■

We are now prepared to prove the main result of this subsection.

Theorem 9.1.13 — Hille–Yosida. Let A with its domain $D(A) \subset X$ be an unbounded operator. Then A with its domain $D(A)$ is a generator of a contraction semigroup $\{S(t)\}_{t\ge0}$, if and only if A with its domain $D(A)$ is closed, $D(A)$ is dense in X, $(0,\infty) \subset \varrho(A)$ and $\|R_\lambda\|_{\mathcal{L}(X)} \le \frac{1}{\lambda}$ for $\lambda > 0$.

Proof. "$\Longrightarrow$" Density and closedness follow from Theorem 9.1.8. The fact that $(0,\infty) \subset \varrho(A)$ together with the estimate of the resolvent follow from Lemma 9.1.12.

"$\Longleftarrow$" We use the Yosida approximation. We set $A_n := nAR_n(A)$ and aim at showing that $\lim_{n\to\infty} \mathrm{e}^{tA_n} = S(t)$. We also recall that if $F_n\colon X \to X$ is linear, $\|F_n\|_{\mathcal{L}(X)} \le C$ and $F_n y \to y$ for all $y \in S$, S is dense in X, then $F_n x \to x$ for all $x \in X$.

Step 1: Yosida approximation

We show that $A_n = n^2R_n(A) - nI$. Then $A_n \in \mathcal{L}(X)$, $nR_n(A)x \to x$ for all $x \in X$, $A_n x \to Ax$ for all $x \in D(A)$.

We have $AR_n(A) = A(nI - A)^{-1} = (A - nI)(nI - A)^{-1} + nI(nI - A)^{-1} = -I + nR_n(A)$. Therefore $nAR_n(A) = n^2R_n(A) - nI \in \mathcal{L}(X)$, as $R_n(A) \in \mathcal{L}(X)$. Define $F_n := nR_n(A)$. Then $\|F_n\|_{\mathcal{L}(X)} \le 1$ as $\|R_n(A)\|_{\mathcal{L}(X)} \le \frac{1}{n}$ for all $n \in \mathbb{N}$, $S := D(A)$ is dense in X. Thus we have for all $y \in D(A)$ that

$$nR_n(A)y = y + AR_n(A)y = \text{ (due to Lemma 9.1.12) } y + R_n(A)(Ay) \to y$$

for $n \to \infty$ as $\|R_n(A)Ay\|_X \le \frac{1}{n}\|Ay\|_X \to 0$ for $n \to \infty$. Therefore $nR_n(A)x \to x$ for all $x \in X$ and $A_n x = nAR_n(A)x = nR_n(A)Ax \to Ax$ as $n \to \infty$ for all $x \in D(A)$.

Step 2: Approximation of the semigroup

We set $S_n(t) := \mathrm{e}^{tA_n} = \sum_{k=0}^\infty \frac{t^k A_n^k}{k!} \in \mathcal{L}(X)$, as $A_n \in \mathcal{L}(X)$ due to Step 1. It also holds $A_n = n^2R_n(A) - nI$, hence

$$\mathrm{e}^{tA_n} = \mathrm{e}^{tn^2R_n(A) - tnI} = \mathrm{e}^{-nt}\mathrm{e}^{n^2tR_n(A)}.$$

Now

$$\|e^{n^2tR_n(A)}\|_{\mathcal{L}(X)} = \Big\|\sum_{k=0}^{\infty}\frac{(nt)^k(nR_n(A))^k}{k!}\Big\|_{\mathcal{L}(X)} \le \sum_{k=0}^{\infty}\frac{(nt)^k}{k!}\|nR_n(A)\|^k_{\mathcal{L}(X)} \le e^{nt}.$$

Thus

$$\|S_n(t)\|_{\mathcal{L}(X)} \le 1$$

for all $n \in \mathbb{N}$ and $S_n(t)$ are contractions.

Step 3: Limit of the approximative semigroups

We now study $\lim_{n\to\infty} S_n(t)x$ for $x \in X$. First, let $x \in D(A)$. Let t be fixed for a moment. We have

$$\begin{aligned}
S_n(t)x - S_m(t)x &= -[S_n(t-s)S_m(s)x]_{s=0}^t = -\left[e^{(t-s)A_n}e^{sA_m}x\right]_{s=0}^t \\
&= -\int_0^t \frac{d}{ds}\left[e^{(t-s)A_n}e^{sA_m}x\right] d\lambda_1(s) \\
&= \int_0^t \left[A_n e^{(t-s)A_n}e^{sA_m}x - e^{(t-s)A_n}A_m e^{sA_m}x\right] d\lambda_1(s).
\end{aligned}$$

Recall that $A_nA_m = A_mA_n$ (as $R_n(A)$ and $R_m(A)$ commute, see Lemma 9.1.12). Then

$$\begin{aligned}
S_n(t)x - S_m(t)x &= \int_0^t e^{(t-s)A_n}e^{sA_m}(A_nx - A_mx)\, d\lambda_1(s) \\
&= \int_0^t S_n(t-s)S_m(s)(A_nx - A_mx)\, d\lambda_1(s).
\end{aligned}$$

Hence

$$\|S_n(t)x - S_m(t)x\|_X \le t\|A_mx - A_nx\|_X.$$

As both A_nx and $A_mx \to Ax$, then $A_nx - A_mx \to 0$ as $m, n \to \infty$ and $\{S_n(t)x\}_{t\ge 0}$ satisfies the Bolzano–Cauchy condition uniformly with respect to $t \in [0,T]$, $T < \infty$, i.e., $S_n(t)x \rightrightarrows S(t)x$ as $n \to \infty$, for all $x \in D(A)$. Thus, due to the convergence principle above

$$S_n(t)x \rightrightarrows S(t)x \qquad \forall x \in X,$$

where $S(t)$ is a c_0-semigroup of contractions.

Step 4: Generator of the semigroup

Let us show that $S(t)$ is generated by A with the domain $D(A)$. Let $\widetilde{A}$ with the domain $D(\widetilde{A})$ be the generator of $S(t)$. Let $x \in D(A)$. Then

$$\begin{aligned}
S_n(t)x - x &= \int_0^t \frac{d}{ds}S_n(s)x\, d\lambda_1(s) = \int_0^t A_nS_n(s)x\, d\lambda_1(s) \\
&= \int_0^t S_n(s)A_nx\, d\lambda_1(s) \to \int_0^t S(s)Ax\, d\lambda_1(s) \qquad \text{as } n \to \infty,
\end{aligned}$$

due to the fact that $S_n(t) \rightrightarrows S(t)$ in $\mathcal{L}(X)$ and $A_nx \to Ax$ in X. Therefore

$$S(t)x - x = \int_0^t S(s)Ax\, d\lambda_1(s).$$

It implies

$$\frac{1}{h}(S(h)x - x) = \frac{1}{h}\int_0^h S(s)Ax\, d\lambda_1(s) \to Ax,$$

as $h \to 0_+$. Hence $x \in D(\widetilde{A})$, $\widetilde{A}x = Ax$, i.e., $D(A) \subset D(\widetilde{A})$, $\widetilde{A}x = Ax$ for all $x \in D(A)$. Now, let $\lambda > 0$. As $\|S(t)\|_{\mathcal{L}(X)} \le 1$, $\lambda \in \varrho(A)$ (due to our assumptions) and $\lambda \in \varrho(\widetilde{A})$ (due to Lemma 9.1.12), we have $\lambda I - \widetilde{A} = \lambda I - A$ in $D(A)$. Hence $\lambda I - \widetilde{A}|_{D(A)}$ is onto, injective. Then $D(A) = D(\widetilde{A})$, $A = \widetilde{A}$ and A with its domain $D(A)$ is the generator of the semigroup $\{S(t)\}_{t\ge 0}$. ∎

Remark 9.1.14. A semigroup $\{S(t)\}_{t\ge 0}$ is called ω-contractive, if $\|S(t)\|_{\mathcal{L}(X)} \le e^{\omega t}$.

Corollary 9.1.15. Let A with the domain $D(A) \subset X$ be an unbounded operator. Then A with the domain $D(A)$ is a generator of an ω-contractive semigroup $\{S(t)\}_{t\ge 0}$ if and only if A with its domain $D(A) \subset X$ is closed, $D(A)$ is dense in X, $(\omega, \infty) \subset \varrho(A)$ and $\|R_\lambda\|_{\mathcal{L}(X)} \le \frac{1}{\lambda-\omega}$ for $\lambda > \omega$.

Proof. As $\widetilde{S}(t) := e^{-\omega t}S(t)$ is a contraction semigroup, we may proceed as in Theorem 9.1.13. Note that even the case $M > 1$ can be treated similarly, however, we do not need it here. ∎

Remark 9.1.16. If $u_0 \in D(A)$, then $S(t)u_0$ is a classical solution to

$$\frac{\mathrm{d}u}{\mathrm{d}t} = Au, \qquad u(0) = u_0,$$

where $S(t)$ is a c_0-semigroup generated by A with its domain $D(A)$. If $u_0 \in X$ merely, then we may take $S(t)u_0 := \lim_{n\to\infty} S(t)u_n$, where $\{u_n\}_{n\in\mathbb{N}} \subset D(A)$ and $u_n \to u_0$ in X.

Example 9.1.17. Consider the second order parabolic problem

$$\begin{aligned} \partial_t u + Lu &= 0 \qquad \text{in } (0,T)\times\Omega \\ u &= 0 \qquad \text{on } (0,T)\times\partial\Omega \\ u(0) &= g \qquad \text{in } \Omega, \end{aligned}$$

where

$$Lu = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big) + \sum_{i=1}^{d} c_i \frac{\partial u}{\partial x_i} + bu$$

with smooth coefficients $\{a_{ij}\}_{i,j=1}^d$, $\{c_i\}_{i=1}^d$ and b in the x-variable, independent of the time variable t. Moreover, let $\Omega \in C^{1,1}$, $g \in W_0^{1,2}(\Omega)$. We take $X = L^2(\Omega)$, $D(A) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ and $Au = -Lu$. Then A is an unbounded linear operator. Recall that $C_1\|u\|^2_{W_0^{1,2}(\Omega)} \le B[u,u] + \gamma\|u\|^2_{L^2(\Omega)}$ holds for $C_1 > 0$, $\gamma \ge 0$ and $B[\cdot,\cdot]$ is defined via Lu as in Section 5.1.

Theorem 9.1.18 — Semigroup for parabolic problem. The operator A from Example 9.1.17 generates a c_0-semigroup $\{S(t)\}_{t\ge 0}$ on $L^2(\Omega)$ which is γ-contractive.

Proof. We need to verify assumptions of Corollary 9.1.15 with $\omega := \gamma$.
Step 1: Density of $D(A)$
Clearly, $D(A) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ is dense in $L^2(\Omega)$.
Step 2: Closedness of A
The operator A is closed. To this aim, let $\{u_k\}_{k\in\mathbb{N}} \subset D(A)$ be such that

$$\begin{aligned} u_k &\to u \text{ in } L^2(\Omega) \\ Au_k &\to f \text{ in } L^2(\Omega). \end{aligned}$$

Let u_k solve

$$\begin{aligned} Au_k &= G_k \qquad \text{in } \Omega \\ u_k &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

Then, as $Au_k + \gamma u_k = G_k + \gamma u_k$, we have due to the result on the elliptic regularity

$$\|u_k\|_{W^{2,2}(\Omega)} \le C(\|G_k\|_{L^2(\Omega)} + \|u_k\|_{L^2(\Omega)}) = C(\|Au_k\|_{L^2(\Omega)} + \|u_k\|_{L^2(\Omega)}),$$

hence due to the linearity of the operator A we have

$$\|u_k - u_l\|_{W^{2,2}(\Omega)} \le C\big(\|Au_k - Au_l\|_{L^2(\Omega)} + \|u_k - u_l\|_{L^2(\Omega)}\big).$$

Whence $u_k \to u$ in $W^{2,2}(\Omega)$, which implies $Au = f$ (as $Au_k \to Au$ in $L^2(\Omega)$).
Step 3: Spectrum
We know that for each $\lambda \ge \gamma$ the boundary value problem

$$\begin{aligned} Lu + \lambda u &= f \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

has unique solution $u \in W_0^{1,2}(\Omega)$ and due to the elliptic regularity, further, $u \in W^{2,2}(\Omega)$. Thus $u \in D(A)$. Therefore we may write

$$\lambda u - Au = f.$$

Thus $\lambda I - A$: $D(A) \to X$ is one to one, onto, for $\lambda \ge \gamma$. Hence $\varrho(A) \subset (\gamma, \infty)$.
Step 4: Spectral estimate
Therefore, using as test function u in the weak formulation,

$$(\lambda - \gamma)\|u\|^2_{L^2(\Omega)} \le \|f\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}.$$

As $u = R_\lambda f$, we have

$$\|R_\lambda f\|_{L^2(\Omega)} \le \frac{1}{\lambda - \gamma}\|f\|_{L^2(\Omega)}$$

for any $f \in L^2(\Omega)$. Whence

$$\|R_\lambda\|_{\mathcal{L}(X)} \le \frac{1}{\lambda - \gamma}.$$

■

To summarize the difference between the energy method and the semigroup approach for the parabolic problems, by the semigroup theory we immediately get regular solution and the method is elegant, however, it requires more restrictive assumptions on the coefficients of the problem and the regularity can anyway be proved by other methods.

Example 9.1.19. We now consider the hyperbolic problem

$$\begin{aligned} \partial_{tt}u + Lu &= 0 && \text{in } (0,T)\times\Omega \\ u &= 0 && \text{on } (0,T)\times\partial\Omega \\ u(0) &= g && \text{in } \Omega \\ \partial_t u(0) &= h && \text{in } \Omega. \end{aligned}$$

We first rewrite the system as a system of first order equations

$$\begin{aligned} \partial_t u &= v && \text{in } (0,T)\times\Omega \\ \partial_t v + Lu &= 0 && \text{in } (0,T)\times\Omega \\ u &= 0 && \text{on } (0,T)\times\partial\Omega \\ u(0) &= g && \text{in } \Omega \\ v(0) &= h && \text{in } \Omega. \end{aligned}$$

We assume that

$$Lu = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_i}\Big) + bu,$$

where $b \geq 0$ and $a_{ij} = a_{ji}$, $i,j = 1,2,\dots,d$ are sufficiently smooth in x (and independent of time).

We denote $X = W_0^{1,2}(\Omega) \times L^2(\Omega)$ with the norm

$$\|(u,v)\|_X := \Big(B[u,u] + \|v\|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}},$$

where

$$B[u,u] = \int_\Omega \sum_{i,j=1}^{d} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\,\mathrm{d}x + \int_\Omega bu^2\,\mathrm{d}x.$$

We define

$$D(A) := \big(W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)\big)\times W_0^{1,2}(\Omega)$$

and set

$$A(u,v) = (v, -Lu), \qquad u,v \in D(A).$$

We show that A with its domain $D(A)$ fulfills the assumptions of the Hille–Yosida Theorem 9.1.13.

Theorem 9.1.20 — Semigroup for the hyperbolic problem. The operator A from Example 9.1.19 generates a c_0-contraction semigroup $\{S(t)\}_{t\geq 0}$ on $W_0^{1,2}(\Omega)\times L^2(\Omega)$.

Proof. We have to verify the assumptions of the Hille–Yosida Theorem 9.1.13.
Step 1: Density of $D(A)$
Clearly, $\big(W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)\big)\times W_0^{1,2}(\Omega)$ is dense in $W_0^{1,2}(\Omega)\times L^2(\Omega)$.
Step 2: Closedness of A
Let $(u_k,v_k)\subset D(A)$ be such that $(u_k,v_k)\to(u,v)$ and $A(u_k,v_k)\to(F,G)$ in X. As $A(u_k,v_k) = (v_k,-Lu_k)$, we have $F = v$ and $Lu_k \to -G$ in $L^2(\Omega)$. Using the same argument as in the previous theorem (here the situation is even easier) we get

$$\|u_k - u_l\|_{W^{2,2}(\Omega)} \leq \|L(u_k-u_l)\|_{L^2(\Omega)},$$

and we see that u_k is a Cauchy sequence in $W^{2,2}(\Omega)$. Whence $u_k\to u$ in $W^{2,2}(\Omega)$ and $Lu = -G$. Since $v_k\in W_0^{1,2}(\Omega)$ and $v_k\to v$ in $W_0^{1,2}(\Omega)$, we have that $(u,v)\in D(A)$, $A(u,v) = (v,-Lu) = (F,G)$.
Step 3: Spectrum
Let $\lambda>0$, $(F,G)\in X = W_0^{1,2}(\Omega)\times L^2(\Omega)$ and consider

$$\lambda(u,v) - A(u,v) = (F,G). \tag{9.1}$$

This is equivalent to

$$\begin{aligned} \lambda u - v &= F && u\in W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega), \\ \lambda v + Lu &= G && v \in W_0^{1,2}(\Omega). \end{aligned} \tag{9.2}$$

Then $\lambda^2 u + Lu = \lambda F + G$. As $\lambda^2 > 0$, there exists unique solution $u\in W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)$ (due to the regularity of the elliptic problem). Then $v = \lambda u - F \in W_0^{1,2}(\Omega)$, hence (9.1) has a unique solution (u,v). Consequently, $\varrho(A)\subset(0,\infty)$.

Step 4: Resolvent estimate
We write the solution to (9.1) as

$$(u, v) = R_\lambda(F, G).$$

From $(9.2)_2$ we see that

$$\lambda\|v\|^2_{L^2(\Omega)} + B[u, v] = \int_\Omega Gv \, \mathrm{d}x.$$

Since $v = \lambda u - F$, we have (here we need the symmetry of $\{a_{ij}\}_{i,j=1}^d$)

$$\begin{aligned}\lambda\big(\|v\|^2_{L^2(\Omega)} + B[u, u]\big) &= B[u, F] + \int_\Omega Gv \, \mathrm{d}x \\ &\leq \big(\|G\|^2_{L^2(\Omega)} + B[F, F]\big)^{\frac{1}{2}} \big(\|v\|_{L^2(\Omega)} + B[u, u]\big)^{\frac{1}{2}}.\end{aligned}$$

Whence

$$\|(u, v)\|_X \leq \frac{1}{\lambda}\|(F, G)\|_X$$

and

$$\|R_\lambda\|_{\mathcal{L}(X)} \leq \frac{1}{\lambda}, \qquad \lambda > 0.$$

■

Remark 9.1.21. For the problem

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}t} &= Au + f(t) \\ u(0) &= u_0 \in X.\end{aligned} \tag{9.3}$$

we can under suitable assumptions on f write

$$u(t) = S(t)u_0 + \int_0^t S(t - s) f(s) \, \mathrm{d}\lambda_1(s) \qquad \forall t \in [0, T],$$

where $\{S(t)\}_{t\geq 0}$ is the semigroup generated by A with its domain $D(A)$ and f is Bochner integrable. The obtained solution depends significantly on the regularity of f.

9.2 Linear parabolic equation with a nonlinear compact perturbation

We now consider two examples of nonlinear equations of parabolic type which can be studied using combination of methods presented above. First, we look at a nonlinear parabolic problem, where the nonlinearity can be overcome by the application of the Aubin–Lions Theorem 8.5.2.

We consider a system of parabolic equations

$$\begin{aligned}\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \frac{1}{2}\mathbf{u}\operatorname{div}\mathbf{u} - \Delta\mathbf{u} &= \mathbf{f} && \text{in } (0, T)\times\Omega, \\ \mathbf{u}(0, \cdot) &= \mathbf{u}_0 && \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} && \text{on } (0, T)\times\partial\Omega,\end{aligned} \tag{9.4}$$

where $\Omega \subset \mathbb{R}^3$ is bounded, open, $\mathbf{u} = (u_1, u_2, u_3)$ is the unknown (vector-valued) function and $\mathbf{f} = (f_1, f_2, f_3)$ is the given right-hand side. The system above can be viewed as a simplified model for the three-dimensional incompressible Navier–Stokes equations.

Definition 9.2.1 The function $\mathbf{u} \in L^2(0, T; W^{1,2}_0(\Omega; \mathbb{R}^3)) \cap L^\infty(0, T; L^2(\Omega; \mathbb{R}^3))$ such that its time derivative $\partial_t\mathbf{u} \in L^q(0, T; W^{-1,2}(\Omega; \mathbb{R}^3))$ for some $q > 1$ is called a weak solution to (9.4), if

$$\langle\partial_t\mathbf{u}, \mathbf{v}\rangle_{W^{1,2}_0(\Omega;\mathbb{R}^3)} + \int_\Omega \Big((\mathbf{u}\cdot\nabla\mathbf{u})\cdot\mathbf{v} + \frac{1}{2}\mathbf{u}\cdot\mathbf{v}\operatorname{div}\mathbf{u} + \nabla\mathbf{u} : \nabla\mathbf{v}\Big) \, \mathrm{d}x = \langle\mathbf{f}, \mathbf{v}\rangle_{W^{1,2}_0(\Omega;\mathbb{R}^3)} \tag{9.5}$$

holds for all $\mathbf{v} \in W^{1,2}_0(\Omega; \mathbb{R}^3)$ and a.e. $t \in (0, T)$, as well as

$$\mathbf{u}(0, \cdot) = \mathbf{u}_0.$$

Remark 9.2.2. 1. Note that for $q < 2$ (which will be our case), we cannot expect that $\mathbf{u} \in C([0, T]; L^2(\Omega; \mathbb{R}^3))$ (recall that $W^{1,2}_0(\Omega; \mathbb{R}^3)$, $L^2(\Omega; \mathbb{R}^3)$ and $W^{-1,2}(\Omega; \mathbb{R}^3)$ form the Gelfand triple), since q and 2 are not Hölder conjugate exponents. Therefore we only have $\mathbf{u} \in C([0, T]; L^2_w(\Omega; \mathbb{R}^3))$ (see Lemma 8.4.9) and thus

$$\lim_{t\to 0^+} \int_\Omega \mathbf{u}(t, \cdot)\cdot\mathbf{v} \, \mathrm{d}x = \int_\Omega \mathbf{u}_0\cdot\mathbf{v} \, \mathrm{d}x$$

for all $\mathbf{v} \in L^2(\Omega;\mathbb{R}^3)$. We will, however, construct a solution with slightly better properties.

2. Using formally as test function in (9.5) $\mathbf{v} := \mathbf{u}$ (the solution) and assuming that we may perform all necessary operations below, we end up with

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\nabla\mathbf{u}\|^2_{L^2(\Omega;\mathbb{R}^9)} + \int_\Omega \Big((\mathbf{u}\cdot\nabla\mathbf{u})\cdot\mathbf{u} + \frac{1}{2}|\mathbf{u}|^2 \mathrm{div}\,\mathbf{u}\Big)\,\mathrm{d}x = \langle \mathbf{f},\mathbf{u}\rangle_{W_0^{1,2}(\Omega;\mathbb{R}^3)}.$$

For $\mathbf{u}$ sufficiently smooth we can further compute

$$\int_\Omega \Big((\mathbf{u}\cdot\nabla\mathbf{u})\cdot\mathbf{u} + \frac{1}{2}|\mathbf{u}|^2 \mathrm{div}\,\mathbf{u}\Big)\,\mathrm{d}x = \frac{1}{2}\int_\Omega \big(\mathbf{u}\cdot|\mathbf{u}|^2 + \mathrm{div}\,\mathbf{u}|\mathbf{u}|^2\big)\,\mathrm{d}x = \frac{1}{2}\int_\Omega \big(-\mathrm{div}\,\mathbf{u}|\mathbf{u}|^2 + \mathrm{div}\,\mathbf{u}|\mathbf{u}|^2\big)\,\mathrm{d}x = 0.$$

Thus

$$\frac{1}{2}\|\mathbf{u}(t,\cdot)\|^2_{L^2(\Omega;\mathbb{R}^3)} + \int_0^t \|\nabla\mathbf{u}\|^2_{L^2(\Omega;\mathbb{R}^9)}\,\mathrm{d}s = \int_0^t \langle \mathbf{f},\mathbf{u}\rangle_{W_0^{1,2}(\Omega;\mathbb{R}^3)} + \frac{1}{2}\|\mathbf{u}_0\|^2_{L^2(\Omega;\mathbb{R}^3)}. \tag{9.6}$$

We call this identity energy equality. In fact, we will be able to prove a weaker relation, so called energy inequality

$$\frac{1}{2}\|\mathbf{u}(t,\cdot)\|^2_{L^2(\Omega;\mathbb{R}^3)} + \int_0^t \|\nabla\mathbf{u}\|^2_{L^2(\Omega;\mathbb{R}^9)}\,\mathrm{d}s \le \int_0^t \langle \mathbf{f},\mathbf{u}\rangle_{W_0^{1,2}(\Omega;\mathbb{R}^3)}\,\mathrm{d}s + \frac{1}{2}\|\mathbf{u}_0\|^2_{L^2(\Omega;\mathbb{R}^3)}. \tag{9.7}$$

We show the following result

Theorem 9.2.3 Let $\mathbf{u}_0 \in L^2(\Omega;\mathbb{R}^3)$, $\mathbf{f} \in L^2(0,T;W^{-1,2}(\Omega;\mathbb{R}^3))$, $T \in (0,\infty)$. Then there exists a weak solution to (9.4) in the sense of Definition 9.2.1. Moreover, the weak solution fulfills the energy inequality (9.7) for every $t \in (0,T]$ and the initial condition is satisfied in the sense

$$\lim_{t\to 0_+} \|\mathbf{u}(t) - \mathbf{u}_0\|_{L^2(\Omega;\mathbb{R}^3)} = 0.$$

Remark 9.2.4. Note, however, that $\mathbf{u}$ is generally not a continuous function with values in $L^2(\Omega;\mathbb{R}^3)$, the continuity holds only from the right at $t = 0$, otherwise the function is only continuous in the weak topology in $L^2(\Omega;\mathbb{R}^3)$. Moreover, no uniqueness is claimed.

Proof of Theorem 9.2.3. **Step 1:** Galerkin approximation
We follow the strategy developed in the proof of existence of a solution for the parabolic problems. We consider a basis $\{\mathbf{w}^k\}_{k=1}^\infty$ such that $-\Delta w_i^k = \lambda_k w_i^k$, $i = 1,2,3$, $k \in \mathbb{N}$ with homogeneous Dirichlet boundary conditions. Then we look for a solution in the form

$$\mathbf{u}^n(t,x) := \sum_{k=1}^n c_k^n(t)\mathbf{w}^k(x), \qquad n \in \mathbb{N}$$

such that

$$\begin{aligned} &\int_\Omega \partial_t \mathbf{u}^n \cdot \mathbf{w}^l\,\mathrm{d}x + \int_\Omega \Big(\mathbf{u}^n\cdot\nabla\mathbf{u}^n + \frac{1}{2}\mathrm{div}\,\mathbf{u}^n\mathbf{u}^n\Big)\cdot\mathbf{w}^l\,\mathrm{d}x + \int_\Omega \nabla\mathbf{u}^n : \nabla\mathbf{w}^l\,\mathrm{d}x = \langle \mathbf{f},\mathbf{w}^l\rangle_{W_0^{1,2}(\Omega;\mathbb{R}^3)}, \qquad l = 1,2,\dots,n \\ &\mathbf{u}^n(0) = P_n\mathbf{u}_0 = \sum_{k=1}^n \Big(\int_\Omega \mathbf{u}_0\cdot\mathbf{w}^k\,\mathrm{d}x\Big)\mathbf{w}^k, \qquad n \in \mathbb{N}. \end{aligned} \tag{9.8}$$

We may rewrite the problem above as a system of first order ordinary differential equations

$$\begin{aligned} c_l^{n\prime}(t) &+ \int_\Omega \Big[\Big(\sum_{k=1}^n c_k^n(t)\mathbf{w}^k\Big)\cdot\nabla\Big(\sum_{m=1}^n c_m^n(t)\mathbf{w}^m\Big)\cdot\mathbf{w}^l + \frac{1}{2}\Big(\sum_{k=1}^n c_k^n(t)\mathbf{w}^k\Big)\mathrm{div}\,\Big(\sum_{m=1}^n c_m^n(t)\mathbf{w}^m\Big)\cdot\mathbf{w}^l\Big]\,\mathrm{d}x \\ &+ \int_\Omega \nabla\Big(\sum_{k=1}^n c_k^n(t)\mathbf{w}^k\Big) : \nabla\mathbf{w}^l\,\mathrm{d}x = \langle \mathbf{f},\mathbf{w}^l\rangle_{W_0^{1,2}(\Omega;\mathbb{R}^3)}, \qquad l = 1,2,\dots,n \\ c_l^n(0) &= \int_\Omega \mathbf{u}_0\cdot\mathbf{w}^l\,\mathrm{d}x. \end{aligned} \tag{9.9}$$

Step 2: Existence of a solution for Galerkin approximation
We may use again the Carathéodory theory (even the classical one if we regularize the right-hand side in time) and get, now only *locally in time*, existence of a unique solution to the Galerkin approximation such that $c_l^n \in AC_{\mathrm{loc}}([0,T^*))$, $l = 1,2,\dots,n$ for some $0 < T^* \le T$. The question is whether $T^* = T$. If $T^* < T$, then due to the theory of ordinary differential equations it holds $\limsup_{t\to T^*-} \sum_{l=1}^n |c_l^n(t)| = \infty$. We exclude this possibility by proving suitable a priori estimates below.
Step 3: A priori estimates

We multiply each equation in (9.9) by $c_l^n(t)$ and sum from 1 to n (i.e., we use as test function the solution $\vec{u}^n$). We get

$$(\partial_t \mathbf{u}^n, \mathbf{u}^n)_{L^2(\Omega;\mathbb{R}^3)} + \int_\Omega \Big(\mathbf{u}^n \cdot \nabla \mathbf{u}^n + \frac{1}{2}\mathrm{div}\, \mathbf{u}^n \mathbf{u}^n\Big) \cdot \mathbf{u}^n \,\mathrm{d}x + \int_\Omega |\nabla \mathbf{u}^n|^2 \,\mathrm{d}x = \langle \mathbf{f}, \mathbf{u}^n\rangle_{W_0^{1,2}(\Omega;\mathbb{R}^3)}.$$

We now repeat the computations from the formal proof of the energy inequality

$$\int_\Omega (\mathbf{u}^n \cdot \nabla \mathbf{u}^n)\cdot \mathbf{u}^n \,\mathrm{d}x = \int_\Omega \sum_{i,j=1}^3 u_i^n \frac{\partial u_j^n}{\partial x_i} u_j^n \,\mathrm{d}x = \frac{1}{2}\int_\Omega \sum_{i,j=1}^3 u_i^n \frac{\partial (u_j^n)^2}{\partial x_i}\,\mathrm{d}x$$
$$= -\frac{1}{2}\int_\Omega \sum_{i,j=1}^3 \frac{\partial u_i^n}{\partial x_i}(u_j^n)^2 \,\mathrm{d}x = -\frac{1}{2}\int_\Omega |\mathbf{u}^n|^2 \mathrm{div}\, \mathbf{u}^n \,\mathrm{d}x.$$

Hence

$$\int_\Omega \Big(\mathbf{u}^n \cdot \nabla \mathbf{u}^n + \frac{1}{2}\mathrm{div}\, \mathbf{u}^n \mathbf{u}^n\Big)\cdot \mathbf{u}^n \,\mathrm{d}x = 0.$$

Therefore we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}^n\|^2_{L^2(\Omega;\mathbb{R}^3)} + \int_\Omega |\nabla \mathbf{u}^n|^2 \,\mathrm{d}x = \langle \mathbf{f}, \mathbf{u}^n\rangle_{W_0^{1,2}(\Omega;\mathbb{R}^3)}.$$

This yields

$$\frac{1}{2}\|\mathbf{u}^n(t,\cdot)\|^2_{L^2(\Omega;\mathbb{R}^3)} + \int_0^t\int_\Omega |\nabla \mathbf{u}^n|^2 \,\mathrm{d}x\,\mathrm{d}s = \int_0^t \langle \mathbf{f}, \mathbf{u}^n\rangle_{W_0^{1,2}(\Omega;\mathbb{R}^3)}\,\mathrm{d}s + \frac{1}{2}\|P_n \mathbf{u}_0\|^2_{L^2(\Omega;\mathbb{R}^3)} \tag{9.10}$$

and consequently

$$\max_{t\in[0,\tau]} \frac{1}{2}\|\mathbf{u}^n(t,\cdot)\|^2_{L^2(\Omega;\mathbb{R}^3)} + \int_0^\tau\int_\Omega |\nabla \mathbf{u}^n|^2 \,\mathrm{d}x\,\mathrm{d}t \le \|\mathbf{f}\|_{L^2(0,\tau;W^{-1,2}(\Omega;\mathbb{R}^3))}\|\mathbf{u}^n\|_{L^2(0,\tau;W_0^{1,2}(\Omega;\mathbb{R}^3))} + \frac{1}{2}\|P_n\mathbf{u}_0\|^2_{L^2(\Omega;\mathbb{R}^3)}$$
$$\le \frac{1}{2}\int_0^\tau\int_\Omega |\nabla \mathbf{u}^n|^2 \,\mathrm{d}x\,\mathrm{d}t + C\int_0^\tau \|\mathbf{f}\|^2_{W^{-1,2}(\Omega;\mathbb{R}^3)}\,\mathrm{d}t + \frac{1}{2}\|P_n\mathbf{u}_0\|^2_{L^2(\Omega;\mathbb{R}^3)}.$$

However

$$\|P_n\mathbf{u}_0\|_{L^2(\Omega;\mathbb{R}^3)} \le \|\mathbf{u}_0\|_{L^2(\Omega:\mathbb{R}^3)}$$
$$\int_0^\tau \|\mathbf{f}\|^2_{W^{-1,2}(\Omega;\mathbb{R}^3)}\,\mathrm{d}t \le \int_0^T \|\mathbf{f}\|^2_{W^{-1,2}(\Omega;\mathbb{R}^3)}\,\mathrm{d}t,$$

and we have

$$\max_{t\in[0,\tau]} \|\mathbf{u}^n(t,\cdot)\|^2_{L^2(\Omega;\mathbb{R}^3)} + \int_0^\tau\int_\Omega |\nabla \mathbf{u}^n|^2 \,\mathrm{d}x\,\mathrm{d}t \le C,$$

where C is independent of $\tau < T^*$ (and also of n), depends linearly on $\|\mathbf{u}_0\|^2_{L^2(\Omega:\mathbb{R}^3)}$ and on $\|\mathbf{f}\|^2_{L^2(0,T;W^{-1,2}(\Omega;\mathbb{R}^3))}$. Thus the solution remains bounded when $\tau \to T^*_-$ and hence $T = T^*$. We proved

$$\|\mathbf{u}^n\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))} + \|\mathbf{u}^n\|_{L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3))} \le C,$$

where C depends on the data of the problem, but is independent of n. Above, we use $\|\mathbf{v}\|_{W_0^{1,2}(\Omega;\mathbb{R}^3)} := \|\nabla \mathbf{v}\|_{L^2(\Omega;\mathbb{R}^9)}$ which is on $W_0^{1,2}(\Omega;\mathbb{R}^3)$ an equivalent norm with the standard $W^{1,2}(\Omega;\mathbb{R}^3)$-norm.

Next we need to estimate the time derivative of the approximate solutions. We have, exactly as in the linear problem, that arbitrary $\mathbf{v} \in L^2(\Omega;\mathbb{R}^3)$ can be written as $\mathbf{v}_1^n + \mathbf{v}_2^n$, where $\mathbf{v}_1^n$ belongs to the linear hull of $\{\mathbf{w}^k\}_{k=1}^n$ and $\mathbf{v}_2^n$ is perpendicular to it in $L^2(\Omega;\mathbb{R}^3)$. Moreover,

$$\|\mathbf{v}_1^n\|_{L^2(\Omega;\mathbb{R}^3)} \le \|\mathbf{v}\|_{L^2(\Omega;\mathbb{R}^3)} \quad \text{and} \quad \|\mathbf{v}_1^n\|_{W_0^{1,2}(\Omega;\mathbb{R}^3)} \le \|\mathbf{v}\|_{W_0^{1,2}(\Omega;\mathbb{R}^3)}.$$

Furthermore,

$$\langle \partial_t \mathbf{u}^n, \mathbf{v}\rangle_{W_0^{1,2}(\Omega;\mathbb{R}^3)} = (\partial_t \mathbf{u}^n, \mathbf{v})_{L^2(\Omega;\mathbb{R}^3)} = (\partial_t \mathbf{u}^n, \mathbf{v}_1^n)_{L^2(\Omega;\mathbb{R}^3)}$$

and we may for $\mathbf{v} \in W_0^{1,2}(\Omega)$ use the Galerkin approximation to deduce

$$\sup_{\substack{\mathbf{v}\in W_0^{1,2}(\Omega;\mathbb{R}^3)\\ \|\mathbf{v}\|_{W_0^{1,2}(\Omega;\mathbb{R}^3)}\le 1}} \langle \partial_t \mathbf{u}^n, \mathbf{v}\rangle_{W_0^{1,2}(\Omega;\mathbb{R}^3)} = \sup_{\substack{\mathbf{v}\in W_0^{1,2}(\Omega;\mathbb{R}^3)\\ \|\mathbf{v}\|_{W_0^{1,2}(\Omega;\mathbb{R}^3)}\le 1}} (\partial_t \mathbf{u}^n, \mathbf{v}_1^n)_{L^2(\Omega;\mathbb{R}^3)}$$
$$= \sup_{\substack{\mathbf{v}\in W_0^{1,2}(\Omega;\mathbb{R}^3)\\ \|\mathbf{v}\|_{W_0^{1,2}(\Omega;\mathbb{R}^3)}\le 1}} \Big(-\int_\Omega \Big[\big(\mathbf{u}^n\cdot\nabla \mathbf{u}^n + \frac{1}{2}\mathrm{div}\,\mathbf{u}^n\mathbf{u}^n\big)\cdot \mathbf{v}_1^n + \nabla \mathbf{u}^n : \nabla \mathbf{v}_1^n\Big]\,\mathrm{d}x + \langle \mathbf{f}, \mathbf{v}_1^n\rangle_{W_0^{1,2}(\Omega;\mathbb{R}^3)}\Big)$$
$$\le C\big(\|\mathbf{u}^n\|_{L^3(\Omega;\mathbb{R}^3)}\|\nabla \mathbf{u}^n\|_{L^2(\Omega;\mathbb{R}^9)} + \|\nabla \mathbf{u}^n\|_{L^2(\Omega;\mathbb{R}^9)} + \|\mathbf{f}\|_{W^{-1,2}(\Omega;\mathbb{R}^3)}\big).$$

Hence

$$\begin{aligned}
&\|\partial_t \mathbf{u}^n\|_{L^{\frac{4}{3}}(0,T;W^{-1,2}(\Omega;\mathbb{R}^3))}\\
&\le \Big\| \|\mathbf{u}^n\|_{L^3(\Omega;\mathbb{R}^3)}\|\nabla \mathbf{u}^n\|_{L^2(\Omega;\mathbb{R}^9)} + \|\nabla \mathbf{u}^n\|_{L^2(\Omega;\mathbb{R}^9)} + \|\mathbf{f}\|_{W^{-1,2}(\Omega;\mathbb{R}^3)}\Big\|_{L^{\frac{4}{3}}(0,T)}\\
&\le C\Big(\|\mathbf{u}^n\|^{\frac{3}{2}}_{L^2(0,T;W^{1,2}_0(\Omega;\mathbb{R}^3))}\|\mathbf{u}^n\|^{\frac{1}{2}}_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))} + \|\mathbf{u}^n\|_{L^2(0,T;W^{1,2}_0(\Omega;\mathbb{R}^3))} + \|\mathbf{f}\|_{L^2(0,T;W^{-1,2}(\Omega;\mathbb{R}^3))}\Big).
\end{aligned}$$

Above, we used that

$$\|\mathbf{u}^n\|_{L^3(\Omega;\mathbb{R}^3)} \le C\|\mathbf{u}^n\|^{\frac{1}{2}}_{L^2(\Omega;\mathbb{R}^3)}\|\mathbf{u}^n\|^{\frac{1}{2}}_{W^{1,2}_0(\Omega;\mathbb{R}^3)}.$$

Therefore we proved

$$\|\partial_t \mathbf{u}^n\|_{L^{\frac{4}{3}}(0,T;W^{-1,2}(\Omega;\mathbb{R}^3))} \le C(\|\mathbf{u}_0\|_{L^2(\Omega;\mathbb{R}^3)}, \|\mathbf{f}\|_{L^2(0,T;W^{-1,2}(\Omega;\mathbb{R}^3))})$$

(C depends on the data nonlinearly).

Step 4: Limit passage in Galerkin approximation

We know that there exists a subsequence $\{\mathbf{u}^{n_k}\}_{k\in\mathbb{N}}$ such that it holds

$$\begin{aligned}
\mathbf{u}^{n_k} &\rightharpoonup \mathbf{u} && \text{in } L^2(0,T;W^{1,2}_0(\Omega;\mathbb{R}^3))\\
\mathbf{u}^{n_k} &\rightharpoonup^* \mathbf{u} && \text{in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))\\
\partial_t\mathbf{u}^{n_k} &\rightharpoonup \partial_t\mathbf{u} && \text{in } L^{\frac{4}{3}}(0,T;W^{-1,2}(\Omega;\mathbb{R}^3))
\end{aligned} \tag{9.11}$$

as $k\to\infty$. This is not enough to pass to the limit in the nonlinear terms. We therefore apply the Aubin–Lions Theorem 8.5.2, where we take $X_0 = W^{1,2}_0(\Omega;\mathbb{R}^3)$, $X_1 = W^{-1,2}(\Omega;\mathbb{R}^3)$ and $X = L^2(\Omega;\mathbb{R}^3)$, $\alpha_0 = 2$, $\alpha_1 = \frac{4}{3}$. Then we have, in addition to (9.11) (taking, if necessary, another subsequence which we relabel)

$$\mathbf{u}^{n_k} \to \mathbf{u} \qquad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^3)). \tag{9.12}$$

It further yields, when combined with the a priori estimates

$$\begin{aligned}
\mathbf{u}^{n_k} &\to \mathbf{u} && \text{in } L^q(0,T;L^2(\Omega;\mathbb{R}^3)), && 1\le q<\infty\\
\mathbf{u}^{n_k} &\to \mathbf{u} && \text{in } L^2(0,T;L^r(\Omega;\mathbb{R}^3)), && 1\le r<6\\
\mathbf{u}^{n_k} &\to \mathbf{u} && \text{in } L^s((0,T)\times\Omega;\mathbb{R}^3), && 1\le s<\frac{10}{3}.
\end{aligned} \tag{9.13}$$

The last convergence follows from the interpolation inequality

$$\|\mathbf{u}^{n_k}\|_{L^{\frac{10}{3}}((0,T)\times\Omega;\mathbb{R}^3)} \le C\|\mathbf{u}^{n_k}\|^{\frac{2}{5}}_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))}\|\mathbf{u}^{n_k}\|^{\frac{3}{5}}_{L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))}.$$

We may now start to deal with the limit passage in the Galerkin approximation. As in the linear problem, we multiply the Galerkin approximation by $\psi\in C^\infty_0((0,T))$ and integrate the resulted identities over $(0,T)$. We have

$$\begin{aligned}
&\int_0^T \langle \partial_t\mathbf{u}^{n_k}, \mathbf{w}^l\rangle_{W^{1,2}_0(\Omega;\mathbb{R}^3)}\psi\,\mathrm{d}t + \int_0^T\int_\Omega\Big(\mathbf{u}^{n_k}\cdot\nabla\mathbf{u}^{n_k} + \frac{1}{2}\mathrm{div}\,\mathbf{u}^{n_k}\mathbf{u}^{n_k}\Big)\cdot\mathbf{w}^l\,\mathrm{d}x\psi\,\mathrm{d}t\\
&\quad+\int_0^T\int_\Omega \nabla\mathbf{u}^{n_k}:\nabla\mathbf{w}^l\,\mathrm{d}x\psi\,\mathrm{d}t = \int_0^T\langle \mathbf{f},\mathbf{w}^l\rangle_{W^{1,2}_0(\Omega;\mathbb{R}^3)}\psi\,\mathrm{d}t \qquad \forall l=1,2,\dots,n_k.
\end{aligned} \tag{9.14}$$

We consider each term separately. First

$$\int_0^T \langle \partial_t\mathbf{u}^{n_k}, \mathbf{w}^l\rangle_{W^{1,2}_0(\Omega)}\psi\,\mathrm{d}t \to \int_0^T \langle \partial_t\mathbf{u}, \mathbf{w}^l\rangle_{W^{1,2}_0(\Omega)}\psi\,\mathrm{d}t.$$

Next

$$\begin{aligned}
\int_0^T\int_\Omega(\mathbf{u}^{n_k}\cdot\nabla\mathbf{u}^{n_k})\cdot\mathbf{w}^l\,\mathrm{d}x\psi\,\mathrm{d}t &= \int_0^T\int_\Omega(\mathbf{u}^{n_k}-\mathbf{u})\cdot\nabla\mathbf{u}^{n_k})\cdot\mathbf{w}^l\,\mathrm{d}x\psi\,\mathrm{d}t\\
&+\int_0^T\int_\Omega(\mathbf{u}\cdot\nabla(\mathbf{u}^{n_k}-\mathbf{u}))\cdot\mathbf{w}^l\,\mathrm{d}x\psi\,\mathrm{d}t + \int_0^T\int_\Omega(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\mathbf{w}^l\,\mathrm{d}x\psi\,\mathrm{d}t.
\end{aligned}$$

The first term on the right-hand side can be estimated by

$$\begin{aligned}
\Big|\int_0^T\int_\Omega(\mathbf{u}^{n_k}-\mathbf{u})\cdot\nabla\mathbf{u}^{n_k})\cdot\mathbf{w}^l\,\mathrm{d}x\psi\,\mathrm{d}t\Big| &\le \int_0^T \|\mathbf{u}^{n_k}-\mathbf{u}\|_{L^3(\Omega;\mathbb{R}^3)}\|\nabla\mathbf{u}^{n_k}\|_{L^2(\Omega;\mathbb{R}^9)}\|\mathbf{w}^l\|_{L^6(\Omega;\mathbb{R}^3)}|\psi|\,\mathrm{d}t\\
&\le C\|\mathbf{u}^{n_k}-\mathbf{u}\|_{L^2(0,T;L^3(\Omega;\mathbb{R}^3))}\|\nabla\mathbf{u}^{n_k}\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^9))}\to 0
\end{aligned}$$

for $k \to \infty$, while the second term goes to zero as $\nabla \mathbf{u}^{n_k} \rightharpoonup \nabla \mathbf{u}$ in $L^2((0,T)\times\Omega;\mathbb{R}^9)$ and

$$\|\mathbf{u}\mathbf{w}^l\|_{L^2((0,T)\times\Omega;\mathbb{R}^9)} \leq \|\mathbf{u}\|_{L^2(0,T;L^6(\Omega;\mathbb{R}^3))}\|\mathbf{w}^l\|_{L^3(\Omega;\mathbb{R}^3)} < +\infty.$$

Exactly in the same way we may treat the third term in (9.14) and get that for $k \to \infty$

$$\frac{1}{2}\int_0^T\int_\Omega \operatorname{div}\mathbf{u}^{n_k}\mathbf{u}^{n_k}\cdot\mathbf{w}^l\,\mathrm{d}x\psi\,\mathrm{d}t \to \frac{1}{2}\int_0^T\int_\Omega \operatorname{div}\mathbf{u}\mathbf{u}\cdot\mathbf{w}^l\,\mathrm{d}x\psi\,\mathrm{d}t.$$

Finally

$$\int_0^T\int_\Omega \nabla\mathbf{u}^{n_k}\cdot\nabla\mathbf{w}^l\,\mathrm{d}x\psi\,\mathrm{d}t \to \int_0^T\int_\Omega \nabla\mathbf{u}\cdot\nabla\mathbf{w}^l\,\mathrm{d}x\psi\,\mathrm{d}t$$

due to the weak convergence of $\nabla\mathbf{u}^{n_k}$ in $L^2((0,T)\times\Omega;\mathbb{R}^9)$ and we end up with

$$\int_0^T \langle\partial_t\mathbf{u},\mathbf{w}^l\rangle_{W_0^{1,2}(\Omega;\mathbb{R}^3)}\psi\,\mathrm{d}t + \int_0^T\int_\Omega\Big[\Big(\mathbf{u}\cdot\nabla\mathbf{u}+\frac{1}{2}\operatorname{div}\mathbf{u}\mathbf{u}\Big)\cdot\mathbf{w}^l+\nabla\mathbf{u}:\nabla\mathbf{w}^l\Big]\,\mathrm{d}x\psi\,\mathrm{d}t = \int_0^T\langle\mathbf{f},\mathbf{w}^l\rangle_{W_0^{1,2}(\Omega;\mathbb{R}^3)}\psi\,\mathrm{d}t$$

for all $l \in \mathbb{N}$ and all $\psi \in C_0^\infty((0,T))$. Now, as for the linear parabolic problem, it is an easy matter to verify that

$$\langle\partial_t\mathbf{u},\mathbf{v}\rangle_{W_0^{1,2}(\Omega;\mathbb{R}^3)} + \int_\Omega\Big[\Big(\mathbf{u}\cdot\nabla\mathbf{u}+\frac{1}{2}\operatorname{div}\mathbf{u}\mathbf{u}\Big)\cdot\mathbf{v}+\nabla\mathbf{u}:\nabla\mathbf{v}\Big]\,\mathrm{d}x = \langle\mathbf{f},\mathbf{v}\rangle_{W_0^{1,2}(\Omega;\mathbb{R}^3)}$$

for all $\mathbf{v} \in W_0^{1,2}(\Omega)$ and almost every $t \in (0,T)$.
Step 5: Initial condition
As in the linear parabolic problem we may show that $\mathbf{u}(0,\cdot) = \mathbf{u}_0$ in the sense that

$$\lim_{t\to 0_+}(\mathbf{u}(t,\cdot),\mathbf{v})_{L^2(\Omega;\mathbb{R}^3)} = (\mathbf{u}_0,\mathbf{v})_{L^2(\Omega;\mathbb{R}^3)} \qquad \forall\mathbf{v}\in W_0^{1,2}(\Omega;\mathbb{R}^3).$$

Moreover, since $\mathbf{u} \in C([0,T];L^2_w(\Omega;\mathbb{R}^3))$ (recall that $\mathbf{u} \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^3)) \cap C([0,T];W^{-1,2}(\Omega;\mathbb{R}^3)))$ we easily see that the limit above holds even for all $\mathbf{v} \in L^2(\Omega;\mathbb{R}^3)$ as well as

$$\lim_{t\to t_{0+}}(\mathbf{u}(t,\cdot),\mathbf{v})_{L^2(\Omega;\mathbb{R}^3)} = (\mathbf{u}(t_0,\cdot),\mathbf{v})_{L^2(\Omega;\mathbb{R}^3)} \qquad \forall\mathbf{v}\in L^2(\Omega;\mathbb{R}^3) \quad \text{and } t_0\in(0,T].$$

Step 6: Energy inequality
We return back to (9.10), multiply it by a non-negative $\psi \in C_0^\infty((0,T))$ and integrate over $(0,T)$. We get

$$\begin{aligned}&\int_0^T\frac{1}{2}\|\mathbf{u}^{n_k}(t,\cdot)\|^2_{L^2(\Omega;\mathbb{R}^3)}\psi\,\mathrm{d}t + \int_0^T\Big(\int_0^t\int_\Omega|\nabla\mathbf{u}^{n_k}|^2\,\mathrm{d}x\,\mathrm{d}s\Big)\psi\,\mathrm{d}t\\ &= \int_0^T\Big(\int_0^t\langle\mathbf{f},\mathbf{u}^n\rangle_{W_0^{1,2}(\Omega;\mathbb{R}^3)}\,\mathrm{d}s\Big)\psi\,\mathrm{d}t + \int_0^T\psi\,\mathrm{d}t\frac{1}{2}\|P_n\mathbf{u}_0\|^2_{L^2(\Omega;\mathbb{R}^3)}.\end{aligned}$$

We now pass with $k \to \infty$. Since

$$\liminf_{k\to\infty}\int_0^t\int_\Omega|\nabla\mathbf{u}^{n_k}|^2\,\mathrm{d}x\,\mathrm{d}s \geq \int_0^t\int_\Omega|\nabla\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}s,$$

due to the Fatou lemma we have

$$\liminf_{k\to\infty}\int_0^T\Big(\int_0^t\int_\Omega|\nabla\mathbf{u}^{n_k}|^2\,\mathrm{d}x\,\mathrm{d}s\Big)\psi\,\mathrm{d}t \geq \int_0^T\Big(\liminf_{k\to\infty}\int_0^t\int_\Omega|\nabla\mathbf{u}^{n_k}|^2\,\mathrm{d}x\,\mathrm{d}s\Big)\psi\,\mathrm{d}t \geq \int_0^T\Big(\int_0^t\int_\Omega|\nabla\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}s\Big)\psi\,\mathrm{d}t.$$

The limit passage in the other terms is trivial and we end up with

$$\int_0^T\frac{1}{2}\|\mathbf{u}(t)\|^2_{L^2(\Omega;\mathbb{R}^3)}\psi\,\mathrm{d}t + \int_0^T\Big(\int_0^t\int_\Omega|\nabla\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}s\Big)\psi\,\mathrm{d}t \leq \int_0^T\Big(\int_0^t\langle\mathbf{f},\mathbf{u}\rangle_{W_0^{1,2}(\Omega;\mathbb{R}^3)}\,\mathrm{d}s\Big)\psi\,\mathrm{d}t + \int_0^T\psi\,\mathrm{d}t\frac{1}{2}\|\mathbf{u}_0\|^2_{L^2(\Omega;\mathbb{R}^3)},$$

hence

$$\frac{1}{2}\|\mathbf{u}(t,\cdot)\|^2_{L^2(\Omega;\mathbb{R}^3)} + \int_0^t\int_\Omega|\nabla\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}s \leq \int_0^t\langle\mathbf{f},\mathbf{u}\rangle_{W_0^{1,2}(\Omega;\mathbb{R}^3)}\,\mathrm{d}s + \frac{1}{2}\|\mathbf{u}_0\|^2_{L^2(\Omega;\mathbb{R}^3)} \tag{9.15}$$

almost everywhere in $(0,T)$. Due to the weak lower semicontinuity of the norm, as $\mathbf{u}(t_n,\cdot) \rightharpoonup \mathbf{u}(t,\cdot)$ in $L^2(\Omega;\mathbb{R}^3)$ for $t_n \to t$ (one-sided for $t=0$ or $t=T$), we get

$$\liminf_{t_n\to t}\|\mathbf{u}(t_n,\cdot)\|_{L^2(\Omega;\mathbb{R}^3)} \geq \|\mathbf{u}(t,\cdot)\|_{L^2(\Omega;\mathbb{R}^3)}.$$

Moreover, as $\int_0^{t_n}\int_\Omega g\,\mathrm{d}x\,\mathrm{d}s \to \int_0^t g\,\mathrm{d}x\,\mathrm{d}s$ for any $g \in L^1((0,T)\times\Omega)$, we see that (9.15) holds in fact for any $t \in (0,T]$.

In particular, we have

$$\liminf_{t\to 0_+} \|\mathbf{u}(t,\cdot)\|^2_{L^2(\Omega;\mathbb{R}^3)} \ge \|\mathbf{u}_0\|^2_{L^2(\Omega;\mathbb{R}^3)}.$$

On the other hand, the energy inequality yields

$$\limsup_{t\to 0_+} \|\mathbf{u}(t,\cdot)\|^2_{L^2(\Omega;\mathbb{R}^3)} \le \|\mathbf{u}_0\|^2_{L^2(\Omega;\mathbb{R}^3)}.$$

Hence

$$\lim_{t\to 0_+} \|\mathbf{u}(t,\cdot)\|^2_{L^2(\Omega;\mathbb{R}^3)} = \|\mathbf{u}_0\|^2_{L^2(\Omega;\mathbb{R}^3)}$$

which together with the weak continuity implies

$$\lim_{t\to 0^+} \|\mathbf{u}(t,\cdot) - \mathbf{u}_0\|^2_{L^2(\Omega;\mathbb{R}^3)} = 0.$$

Here we used that if $v_n \rightharpoonup v$ in a Hilbert space H and $\|v_n\|_H = (v_n, v_n)_H \to \|v\|_H$ for $n \to \infty$, then $u_n \to u$ in H. ■

9.3 Rothe's method for parabolic monotone operator problem

Let us consider the following problem

$$\begin{aligned} \partial_t u - \operatorname{div} \mathbf{a}(x, u(x), \nabla u(x)) &= f && \text{in } (0,T)\times\Omega \\ u(0,\cdot) &= u_0 && \text{in } \Omega \\ u &= 0 && \text{on } (0,T)\times\partial\Omega, \end{aligned} \tag{9.16}$$

where $\mathbf{a}\colon \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is a given function.

We assume that for some $p \in (1,\infty)$, almost every $x \in \Omega$ and all $v \in \mathbb{R}$, $\mathbf{V} \in \mathbb{R}^d$ it holds

$$\begin{aligned} &|\vec{a}(x,v,\mathbf{V})| \le C(1 + |v|^{p-1} + |\mathbf{V}|^{p-1}) \\ u_0 \in L^2(\Omega), \quad &f \in L^{p'}(0,T; W^{-1,p'}(\Omega)), \end{aligned} \tag{9.17}$$

where $W^{-1,p'}(\Omega) = (W_0^{1,p}(\Omega))^*$. Then

Definition 9.3.1 We say that $u \in L^\infty(0,T;L^2(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega))$ with $\partial_t u \in L^{p'}(0,T;W^{-1,p'}(\Omega))$ is a weak solution to (9.16), provided

$$\langle \partial_t u, v\rangle_{W_0^{1,p}(\Omega)} + \int_\Omega \vec{a}(\cdot, u, \nabla u)\cdot\nabla v \,\mathrm{d}x = \langle f, v\rangle_{W_0^{1,p}(\Omega)}$$

for all $v \in W_0^{1,p}(\Omega)$ and almost every $t \in (0,T)$. Futhermore, $u(0,\cdot) = u_0$.

Remark 9.3.2. Note that $u \in C([0,T];L^2(\Omega))$ (as $W_0^{1,p}(\Omega)\cap L^2(\Omega)$, $L^2(\Omega)$, $(W_0^{1,p}(\Omega)\cap L^2(\Omega))^*$ form a Gelfand triple). Note also that the intersection with $L^2(\Omega)$ is important only for small p's, more precisely, for $p < \frac{2d}{d+2}$; for larger $p's$, $W^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$.

In order to prove existence of a solution, we will need additional assumptions

$$\begin{aligned} &\{a_i(\cdot,\cdot,\cdot)\}_{i=1}^d \text{ are Carathéodory functions} \\ (\vec{a}(x,v,\mathbf{V}) - \vec{a}(x,v,\mathbf{W}))\cdot(\mathbf{V}-\mathbf{W}) \ge 0 \quad &\text{almost everywhere in } \Omega,\ \forall v\in\mathbb{R} \text{ and } \mathbf{V},\mathbf{W}\in\mathbb{R}^d \\ \vec{a}(x,v,\mathbf{V})\cdot\mathbf{V} \ge \alpha|\mathbf{V}|^p - C, \quad &\text{for some } \alpha>0,\ \text{almost every } x\in\Omega,\ \text{all } v\in\mathbb{R}, \mathbf{V}\in\mathbb{R}^d. \end{aligned} \tag{9.18}$$

Theorem 9.3.3 Under assumptions (9.17) and (9.18), there exists a weak solution to (9.16) in the sense of Definition 9.3.1. Moreover, if $\mathbf{a}$ is independent of the second variable (i.e., $\mathbf{a} = \mathbf{a}(x,\nabla u)$), then the solution is unique.

Proof. **Step I:** Uniqueness
Let u, w be two (possibly) different solutions corresponding to the same data. Then we have

$$\begin{aligned} \langle \partial_t u, v\rangle_{W_0^{1,p}(\Omega)} + \int_\Omega \mathbf{a}(\cdot,\nabla u)\cdot\nabla v\,\mathrm{d}x &= \langle f, v\rangle_{W_0^{1,p}(\Omega)} \\ \langle \partial_t w, v\rangle_{W_0^{1,p}(\Omega)} + \int_\Omega \mathbf{a}(\cdot,\nabla w)\cdot\nabla v\,\mathrm{d}x &= \langle f, v\rangle_{W_0^{1,p}(\Omega)} \end{aligned}$$

for all $v \in W_0^{1,p}(\Omega)$. Denote $U = u - w$, subtract the equalities above and use as test function $v := u - w = U$. Then

$$\langle \partial_t U, U \rangle_{W_0^{1,p}(\Omega)} + \int_\Omega \big(\mathbf{a}(\cdot, \nabla u) - \mathbf{a}(\cdot, \nabla w)\big) \cdot \nabla(u - w)\,\mathrm{d}x = 0.$$

Due to property $(9.18)_2$ we have

$$\langle \partial_t U, U \rangle_{W_0^{1,p}(\Omega)} \le 0,$$

i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\|U\|^2_{L^2(\Omega)} \le 0.$$

As $U(0, \cdot) = 0$, we have $U(t, \cdot) = 0$ for all $t \ge 0$, i.e. $u = w$ almost everywhere in $(0, T) \times \Omega$.

Step II: Existence of a solution

We present here another method how to construct solutions to evolutionary problems. The method uses as approximate solution a variant of the corresponding steady problem and therefore we may take advantage that we already know how to construct such solutions. On the other hand, the limit passage is slightly more difficult.

We divide the time interval $(0, T)$ into a finite number of subintervals of the length $\frac{T}{m}$, $m \in \mathbb{N}$ and on each subinterval we consider a steady problem based on the fact that we replace the time derivative by the corresponding differences. We may use Theorem 7.2.5 to construct approximate solutions on all subintervals. Then we pass with $m \to \infty$ and show that the corresponding limit of the approximate solutions is our weak solution to the evolutionary problem.

Step 1: Approximate solution u_m

Let us fix $m \in \mathbb{N}$, denote $\tau = \frac{T}{m}$, $t_k = k\tau$, $k = 0, 1, \ldots, m-1$ and define

$$f_k = \frac{1}{\tau}\int_{t_k}^{t_{k+1}} f\,\mathrm{d}t \quad \in W^{-1,p'}(\Omega) \quad \text{(Bochner integral mean).}$$

We define $u_m(t_0) := u_0$ and look for functions constant on each interval $(t_k, t_{k+1}]$. We define $\eta_k := u_m(t_k) \in W_0^{1,p}(\Omega) \cap L^2(\Omega)$ and look for η_{k+1} as weak solution to

$$\int_\Omega \frac{\eta_{k+1} - \eta_k}{\tau}\varphi\,\mathrm{d}x + \int_\Omega \mathbf{a}(\cdot, \eta_{k+1}, \nabla\eta_{k+1}) \cdot \nabla\varphi\,\mathrm{d}x = \langle f_k, \varphi\rangle_{W_0^{1,p}(\Omega)}$$

for all $\varphi \in W_0^{1,p}(\Omega) \cap L^2(\Omega)$. Using the method from Theorem 7.2.5 together with Theorem 7.3.1 (for the lower order part) we can prove existence of a solution $\eta_{k+1}(x)$. Note that, using $\varphi := \eta_{k+1}$, we get the following a priori estimate

$$\frac{1}{\tau}\int_\Omega |\eta_{k+1}|^2\,\mathrm{d}x + \alpha\int_\Omega |\eta_{k+1}|^p\,\mathrm{d}x \le C|\Omega| + \langle f_k, \eta_{k+1}\rangle_{W_0^{1,p}(\Omega)} + \frac{1}{\tau}\int_\Omega \eta_k\eta_{k+1}\,\mathrm{d}x. \tag{9.19}$$

Since the right-hand side can be easily bounded by the left-hand side and the data f_k, η_k, we see that $\eta_{k+1} \in W_0^{1,p}(\Omega) \cap L^2(\Omega)$. Repeating the construction for $k = 0, 1, \ldots, m-1$ we obtain u_m. Note that generally, u_m may be non-unique.

Step 2: A priori estimates

We take (9.19) for $k = 0, 1, \ldots, M < m$ with $\varphi := \eta_{k+1}$ and sum it up. We get

$$\int_\Omega \Big[\sum_{k=0}^M (\eta_{k+1} - \eta_k)\eta_{k+1} + \tau\sum_{k=0}^M \mathbf{a}(\cdot, \eta_{k+1}, \nabla\eta_{k+1}) \cdot \nabla\eta_{k+1}\Big]\,\mathrm{d}x = \tau\sum_{k=0}^M \langle f_k, \eta_{k+1}\rangle_{W_0^{1,p}(\Omega)}.$$

We use that

$$(\eta_{k+1} - \eta_k)\eta_{k+1} = \frac{(\eta_{k+1})^2}{2} - \frac{(\eta_k)^2}{2} + \frac{(\eta_{k+1} - \eta_k)^2}{2},$$

which yields

$$\int_\Omega \Big(\frac{(\eta_{M+1})^2}{2} + \sum_{k=0}^M \frac{(\eta_{k+1} - \eta_k)^2}{2} + \tau\sum_{k=0}^M \mathbf{a}(\cdot, \eta_{k+1}, \nabla\eta_{k+1}) \cdot \nabla\eta_{k+1}\Big)\,\mathrm{d}x = \int_\Omega \frac{(\eta_0)^2}{2}\,\mathrm{d}x + \tau\sum_{k=0}^M \langle f_k, \eta_{k+1}\rangle_{W_0^{1,p}(\Omega)}.$$

Note that η_k are constant on $(t_k, t_{k+1}]$, u_m is defined as η_k on $(t_k, t_{k+1}]$ and $\tau = t_{k+1} - t_k$. We have

$$\begin{aligned}\int_\Omega \tau\sum_{k=0}^M \mathbf{a}(\cdot, \eta_{k+1}, \nabla\eta_{k+1}) \cdot \nabla\eta_{k+1}\,\mathrm{d}x &= \sum_{k=0}^M \int_{t_k}^{t_{k+1}}\int_\Omega \mathbf{a}(\cdot, \eta_{k+1}, \nabla\eta_{k+1}) \cdot \nabla\eta_{k+1}\,\mathrm{d}x\,\mathrm{d}t \\ &= \sum_{k=0}^M \int_{t_k}^{t_{k+1}}\int_\Omega \mathbf{a}(\cdot, u_m, \nabla u_m) \cdot \nabla u_m\,\mathrm{d}x\,\mathrm{d}t = \int_0^{(M+1)\tau}\int_\Omega \mathbf{a}(\cdot, u_m, \nabla u_m) \cdot \nabla u_m\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

and

$$\tau\sum_{k=0}^{M}\langle f_k,\eta_{k+1}\rangle_{W_0^{1,p}(\Omega)} = \sum_{k=0}^{M}\Big\langle\int_{t_k}^{t_{k+1}} f\,\mathrm{d}t,\eta_{k+1}\Big\rangle_{W_0^{1,p}(\Omega)}$$
$$= \sum_{k=0}^{M}\int_{t_k}^{t_{k+1}}\langle f,u_m\rangle_{W_0^{1,p}(\Omega)}\,\mathrm{d}t = \int_0^{(M+1)\tau}\langle f,u_m\rangle_{W_0^{1,p}(\Omega)}\,\mathrm{d}t.$$

Therefore we have

$$\int_\Omega\Big(\frac{\big(u_m(t_{M+1})\big)^2}{2}+\sum_{k=0}^{M}\frac{\big(u_m(t_{k+1})-u_m(t_k)\big)^2}{2}\Big)\,\mathrm{d}x+\alpha\int_0^{(M+1)\tau}\|\nabla u_m\|^p_{L^p(\Omega;\mathbb{R}^d)}\,\mathrm{d}t$$
$$\le C+\frac{1}{2}\|u_0\|^2_{L^2(\Omega)}+\int_0^{(M+1)\tau}\|f\|_{W^{-1,p'}(\Omega)}\|u_m\|_{W_0^{1,p}(\Omega)}\,\mathrm{d}t.$$

The Young inequality yields

$$\int_\Omega\Big(\big(u_m(t_{M+1})\big)^2+\sum_{k=0}^{M}\big(u_m(t_{k+1})-u_m(t_k)\big)^2\Big)\,\mathrm{d}x+\alpha\int_0^{(M+1)\tau}\|\nabla u_m\|^p_{L^p(\Omega;\mathbb{R}^d)}\,\mathrm{d}t$$
$$\le C\Big(1+\|u_0\|^2_{L^2(\Omega)}+\int_0^T\|f\|^{p'}_{W^{-1,p'}(\Omega)}\,\mathrm{d}t\Big)\le C(DATA).$$

Since M was arbitrary, we have

$$\sum_{k=0}^{m-1}\|u_m(t_{k+1})-u_m(t_k)\|^2_{L^2(\Omega)}\,\mathrm{d}t\le C$$
$$\|u_m\|_{L^\infty(0,T;L^2(\Omega))}\le C$$
$$\|u_m\|_{L^p(0,T;W_0^{1,p}(\Omega))}\le C.$$

Moreover, we also have

$$\|\mathbf{a}(\cdot,u_m,\nabla u_m)\|_{L^{p'}(0,T;L^{p'}(\Omega;\mathbb{R}^d))}\le C.$$

Thus we can extract subsequences such that

$$\begin{aligned} u_{m_k} &\rightharpoonup u && \text{in } L^p(0,T;W_0^{1,p}(\Omega))\\ u_{m_k} &\rightharpoonup^* u && \text{in } L^\infty(0,T;L^2(\Omega))\\ \mathbf{a}(\cdot,u_{m_k},\nabla u_{m_k}) &\rightharpoonup \mathbf{A} && \text{in } L^{p'}(0,T;L^{p'}(\Omega;\mathbb{R}^d)).\end{aligned}$$

In what follows, we write without loss of generality u_m instead of u_{m_k} (we relabel the sequence).
Step 3: Another sequence and its limit
We now modify u_m by redefining it on (t_k,t_{k+1}) to be linear. Hence

$$\widetilde{u}_m(t)=u_m(t_k)+\frac{t-t_k}{\tau}\big(u_m(t_{k+1})-u_m(t_k)\big)=\eta_k+\frac{t-t_k}{\tau}(\eta_{k+1}-\eta_k)$$

for $t\in(t_k,t_{k+1})$. We easily see that

$$\|\widetilde{u}_m\|_{L^\infty(0,T;L^2(\Omega))}\le C,\qquad \|\widetilde{u}_m\|_{L^p(0,T;W_0^{1,p}(\Omega))}\le C$$

and

$$\partial_t\widetilde{u}_m=\frac{1}{\tau}\big(u_m(t_{k+1})-u_m(t_k)\big),\qquad t\in(t_k,t_{k+1}).$$

We now compute an estimate of the time derivative. For $t\in(t_k,t_{k+1})$ it holds

$$\int_\Omega\partial_t\widetilde{u}_m\varphi\,\mathrm{d}x=-\int_\Omega\mathbf{a}(\cdot,\eta_{k+1},\nabla\eta_{k+1})\cdot\nabla\varphi\,\mathrm{d}x+\langle f_k,\varphi\rangle_{W_0^{1,p}(\Omega)}$$
$$\le C\big((1+\|\eta_{k+1}\|^{p-1}_{W^{1,p}(\Omega)})\|\nabla\varphi\|_{L^p(\Omega;\mathbb{R}^d)}+\|f_k\|_{W^{-1,p'}(\Omega)}\|\varphi\|_{W_0^{1,p}(\Omega)}\big).$$

Thus on the same time interval

$$\|\partial_t\widetilde{u}_m\|_{W^{-1,p'}(\Omega)}=\sup_{\|\varphi\|_{W_0^{1,p}(\Omega)}\le 1}\langle\partial_t\widetilde{u}_m,\varphi\rangle_{W_0^{1,p}(\Omega)}=:\sup_{\|\varphi\|_{W_0^{1,p}(\Omega)}\le 1}\int_\Omega\partial_t\widetilde{u}_m\varphi\,\mathrm{d}x$$
$$\le C\big(1+\|\eta_{k+1}\|^{p-1}_{W^{1,p}(\Omega)}+\|f_k\|_{W^{-1,p'}(\Omega)}\big)$$

which yields

$$\int_0^T \|\partial_t\widetilde{u}_m\|_{W^{-1,p'}(\Omega)}^{p'}\,\mathrm{d}t \le C\Big(1+\int_0^T \|\nabla u_m\|_{L^p(\Omega;\mathbb{R}^d)}^p\,\mathrm{d}t + \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} \|f_k\|_{W^{-1,p'}(\Omega)}^{p'}\,\mathrm{d}t\Big) \le C,$$

since

$$\begin{aligned}\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} \|f_k\|_{W^{-1,p'}(\Omega)}^{p'}\,\mathrm{d}t &= \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} \Big\|\frac{1}{\tau}\int_{t_k}^{t_{k+1}} f\,\mathrm{d}s\Big\|_{W^{-1,p'}(\Omega)}^{p'}\,\mathrm{d}t\\ &\le \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} \Big(\frac{1}{\tau}\int_{t_k}^{t_{k+1}} \|f\|_{W^{-1,p'}(\Omega)}\,\mathrm{d}s\Big)^{p'}\,\mathrm{d}t\\ &\le \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} \frac{1}{\tau}\int_{t_k}^{t_{k+1}} \|f\|_{W^{-1,p'}(\Omega)}^{p'}\,\mathrm{d}s\,\mathrm{d}t = \int_0^T \|f\|_{W^{-1,p'}(\Omega)}^{p'}\,\mathrm{d}t \le C,\end{aligned}$$

where we used the Hölder inequality. Thus

$$\begin{aligned}\widetilde{u}_m &\rightharpoonup \widetilde{u} &&\text{in } L^p(0,T;W_0^{1,p}(\Omega))\\ \partial_t\widetilde{u}_m &\rightharpoonup \partial_t\widetilde{u} &&\text{in } L^{p'}(0,T;W^{-1,p'}(\Omega))\end{aligned}$$

and due to the Aubin–Lions Theorem 8.5.2 applied on spaces $W_0^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega) \hookrightarrow W^{-1,p'}(\Omega)$ also

$$\widetilde{u}_m \to \widetilde{u} \qquad \text{in } L^p(0,T;L^p(\Omega)).$$

Let us show that $\widetilde{u} = u$. We have

$$\begin{aligned}\int_0^T \|\widetilde{u}_m - u_m\|_{L^2(\Omega)}^2\,\mathrm{d}t &= \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} \|\widetilde{u}_m - u_m\|_{L^2(\Omega)}^2\,\mathrm{d}t = \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} \Big\|\eta_{k+1} - \Big(\eta_k + \frac{t-t_k}{\tau}(\eta_{k+1}-\eta_k)\Big)\Big\|_{L^2(\Omega)}^2\,\mathrm{d}t\\ &= \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} \|\eta_{k+1}-\eta_k\|_{L^2(\Omega)}^2\Big(1-\frac{t-t_k}{\tau}\Big)^2\,\mathrm{d}t \le \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} \|\eta_{k+1}-\eta_k\|_{L^2(\Omega)}^2\,\mathrm{d}t\\ &= \tau\sum_{k=0}^{m-1} \|\eta_{k+1}-\eta_k\|_{L^2(\Omega)}^2 \le \frac{CT}{m} \to 0\end{aligned}$$

for $m \to \infty$. Therefore $\widetilde{u}_m - u_m \to 0$ in $L^2(0,T;L^2(\Omega))$. Since $u_m \rightharpoonup u$ in $L^p(0,T;W_0^{1,p}(\Omega))$ and $\widetilde{u}_m \rightharpoonup \widetilde{u}$ in $L^p(0,T;W_0^{1,p}(\Omega))$, we easily see that $u = \widetilde{u}$. Moreover, we also have $u_m \to u$ and $\widetilde{u}_m \to u$ in $L^q((0,T)\times\Omega)$ for any $q < \max\{2,p\}$.

Step 4: Initial condition

First, since $u \in L^p(0,T;W_0^{1,p}(\Omega)\cap L^2(\Omega))$ and $\partial_t u \in L^{p'}(0,T;(W_0^{1,p}(\Omega)\cap L^2(\Omega))^*)$ (the former follows from the a priori estimates, the latter from the fact that $(W_0^{1,p}(\Omega))^* \hookrightarrow (W_0^{1,p}(\Omega)\cap L^2(\Omega))^*$), we may use properties of the Gelfand triple to verify that $u \in C([0,T];L^2(\Omega))$. Let us show that $u(0,\cdot) = u_0$ in this sense. We fix δ and $h > 0$ and define

$$\Phi(t) = \begin{cases} 1 & t \in [0,\delta]\\ 1-\frac{1}{h}(t-\delta) & t \in (\delta,\delta+h]\\ 0 & t > \delta+h\end{cases}$$

and multiply the weak formulation for u_m by $\Phi(t)$ and integrate over $(0,T)$. It yields

$$\begin{aligned}\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} \langle f_k,\varphi\rangle_{W_0^{1,p}(\Omega)}\Phi\,\mathrm{d}t &= \int_0^T\int_\Omega \partial_t\widetilde{u}_m\varphi\Phi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \mathbf{a}(\cdot,u_m,\nabla u_m)\cdot\nabla\varphi\Phi\,\mathrm{d}x\,\mathrm{d}t\\ &= -\int_0^T\int_\Omega \widetilde{u}_m\varphi\partial_t\Phi\,\mathrm{d}x\,\mathrm{d}t - \Phi(0)\int_\Omega u_m(0,\cdot)\varphi\,\mathrm{d}x + \int_0^T\int_\Omega \mathbf{a}(\cdot,u_m,\nabla u_m)\cdot\nabla\varphi\Phi\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

We now pass with $m \to \infty$

$$-\frac{1}{h}\int_0^{\delta+h}\int_\Omega u\varphi\partial_t\Phi\,\mathrm{d}x\,\mathrm{d}t - \Phi(0)\int_\Omega u_0\varphi\,\mathrm{d}x + \int_0^{\delta+h}\int_\Omega \mathbf{A}\cdot\nabla\varphi\Phi\,\mathrm{d}x\,\mathrm{d}t = \int_0^{\delta+h}\langle f,\varphi\rangle_{W_0^{1,p}(\Omega)}\Phi\,\mathrm{d}t \tag{9.20}$$

(recall that $\widetilde{u}_m \rightharpoonup u$ in $L^2((0,T)\times\Omega)$, $u_m(0,\cdot) = u_0$, $\mathbf{a}(\cdot,u_m,\nabla u_m) \rightharpoonup \mathbf{A}$ in $L^{p'}(0,T;L^{p'}(\Omega;\mathbb{R}^d))$, the function $\nabla\varphi\Phi \in L^p(0,T;L^p(\Omega;\mathbb{R}^d))$) and finally

$$\begin{aligned}\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} \Big\langle\frac{1}{\tau}\int_{t_k}^{t_{k+1}} f(s)\,\mathrm{d}s,\varphi\Big\rangle_{W_0^{1,p}(\Omega)}\Phi(t)\,\mathrm{d}t &= \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} \Big(\int_{t_k}^{t_{k+1}} \langle f(s),\varphi\rangle_{W_0^{1,p}(\Omega)}\,\mathrm{d}s\Big)\frac{1}{\tau}\Phi(t)\,\mathrm{d}t\\ &= \sum_{k=0}^{m-1}\Big(\frac{1}{\tau}\int_{t_k}^{t_{k+1}} \langle f(s),\varphi\rangle_{W_0^{1,p}(\Omega)}\,\mathrm{d}s\frac{1}{\tau}\int_{t_k}^{t_{k+1}} \Phi(t)\,\mathrm{d}t(t_{k+1}-t_k)\Big) \to \int_0^T \langle f(t),\varphi\rangle_{W_0^{1,p}(\Omega)}\Phi(t)\,\mathrm{d}t \qquad \text{as } m\to\infty.\end{aligned}$$

The last limit passage above uses the fact that Φ is uniformly continuous on $[0,T]$. Passing with $h \to 0_+$

$$\int_\Omega u(\delta,\cdot)\varphi \,\mathrm{d}x - \int_\Omega u_0\varphi\,\mathrm{d}x + \int_0^\delta \int_\Omega \mathbf{A}\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t = \int_0^\delta \langle f,\varphi\rangle_{W_0^{1,p}(\Omega)}\,\mathrm{d}t.$$

Finally, passing with $\delta \to 0_+$ (recall, $u \in C([0,T];L^2(\Omega))$)

$$\int_\Omega u(0,\cdot)\varphi\,\mathrm{d}x = \int_\Omega u_0\varphi\,\mathrm{d}x$$

for all $\varphi \in W_0^{1,p}(\Omega)\cap L^2(\Omega)$; in particular for all $\varphi \in C_0^\infty(\Omega)$ which due to the fact that $u \in C([0,T];L^2(\Omega))$ implies the equality also for all $\varphi \in L^2(\Omega)$.

Step 5: Weak formulation

Let us multiply the weak formulation for u_m ($m \in \mathbb{N}$) by $\psi \in C_0^\infty(0,T)$. We get

$$\int_0^T\int_\Omega \partial_t\widetilde{u}_m\varphi\psi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \mathbf{a}(\cdot,u_m,\nabla u_m)\cdot\nabla\varphi\psi\,\mathrm{d}x\,\mathrm{d}t = \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} \langle f_k,\varphi\rangle_{W_0^{1,p}(\Omega)}\psi\,\mathrm{d}t.$$

As above (the first term, which is different, can be treated easily) we get

$$\int_0^T\int_\Omega \langle\partial_t u,\varphi\rangle_{W_0^{1,p}(\Omega)}\psi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \mathbf{A}\cdot\nabla\varphi\psi\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \langle f,\varphi\rangle_{W_0^{1,p}(\Omega)}\psi\,\mathrm{d}t.$$

It remains to show that $\mathbf{A} = \mathbf{a}(\cdot,u,\nabla u)$. Let us first show that

$$u_m(t,\cdot) \rightharpoonup u(t,\cdot) \qquad \text{in } L^2(\Omega) \tag{9.21}$$

for almost every $t \in (0,T)$. Recall that $\widetilde{u}_m \to u$ in $L^p((0,T)\times\Omega)$ and $\widetilde{u}_m - u_m \to 0$ in $L^2((0,T)\times\Omega)$. Thus

$$u_m \to u \qquad \text{in } L^q((0,T)\times\Omega)$$

with $q = \max\{2,p\}$. Therefore also

$$u_m(t,\cdot) \to u(t,\cdot) \qquad \text{in } L^q(\Omega) \tag{9.22}$$

for almost every $t \in (0,T)$ (possibly for a subsequence which we relabel). However, for almost every $t \in (0,T)$ the sequence $\{u_m(t)\}_{t\in(0,T)}$ is bounded in $L^2(\Omega)$ and thus it contains a weakly convergent subsequence in this space. If t is such that (9.22) holds, then $u_m(t) \rightharpoonup u(t)$ in $L^2(\Omega)$. Since all this can be repeated for arbitrary subsequence, (9.21) holds for all t's such that (9.22) is valid, for the whole sequence.

Let us show that

$$\limsup_{m\to\infty}\int_0^t\int_\Omega \mathbf{a}(\cdot,u_m,\nabla u_m)\cdot\nabla u_m\,\mathrm{d}x\,\mathrm{d}s \le \int_0^t\int_\Omega \mathbf{A}\cdot\nabla u\,\mathrm{d}x\,\mathrm{d}s$$

for almost every $t \in (0,T)$. Let us take t such that (9.21) holds. For fixed $m \in \mathbb{N}$ take $k = k(m)$ such that $t_k^m < t \le t_{k+1}^m$. Since $t_{k+1}^m - t_k^m \to 0$ for $m \to \infty$, we have

$$\begin{aligned}
\int_0^t\int_\Omega \mathbf{a}(\cdot,u_m,\nabla u_m)\cdot\nabla u_m\,\mathrm{d}x\,\mathrm{d}s &= \int_0^{t_{k+1}^m}\int_\Omega \big(\mathbf{a}(\cdot,u_m,\nabla u_m)-\mathbf{a}(\cdot,u_m,\mathbf{0})\big)\cdot(\nabla u_m-\mathbf{0})\,\mathrm{d}x\,\mathrm{d}s \\
&- \int_t^{t_{k+1}^m}\int_\Omega \big(\mathbf{a}(\cdot,u_m,\nabla u_m)-\mathbf{a}(\cdot,u_m,\mathbf{0})\big)\cdot(\nabla u_m-\mathbf{0})\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_\Omega \mathbf{a}(\cdot,u_m,\mathbf{0})\cdot\nabla u_m\,\mathrm{d}x\,\mathrm{d}s \\
&\le \int_0^{t_{k+1}^m}\int_\Omega \mathbf{a}(\cdot,u_m,\nabla u_m)\cdot\nabla u_m\,\mathrm{d}x\,\mathrm{d}s - \int_t^{t_{k+1}^m}\int_\Omega \mathbf{a}(\cdot,u_m,\mathbf{0})\cdot\nabla u_m\,\mathrm{d}x\,\mathrm{d}s \\
&\le \int_0^{t_{k+1}^m}\int_\Omega \mathbf{a}(\cdot,u_m,\nabla u_m)\cdot\nabla u_m\,\mathrm{d}x\,\mathrm{d}s + o(1) \qquad \text{as } m\to\infty,
\end{aligned}$$

where we used the monotonicity of $\mathbf{a}$ in the last variable and

$$\Big|\int_t^{t_{k+1}^m}\int_\Omega \mathbf{a}(\cdot,u_m,\mathbf{0})\cdot\nabla u_m\,\mathrm{d}x\,\mathrm{d}s\Big| \le C\int_t^{t_{k+1}^m}\int_\Omega (1+|u_m|^{p-1})|\nabla u_m|\,\mathrm{d}x\,\mathrm{d}s.$$

Note that ∇u_m is bounded in $L^p((0,T)\times\Omega;\mathbb{R}^d)$ while for u_m we can combine the Sobolev embedding of $W^{1,p}(\Omega)$ to the corresponding Lebesgue space with the $L^\infty(0,T;L^2(\Omega))$ estimate which yields the convergence to zero due to the fact that the time interval degenerates to zero. We now have (recall that $u_m(t)$ is constant in $(t_k^m, t_{k+1}^m]$)

$$\begin{aligned}\int_0^{t_{k+1}^m}\int_\Omega \mathbf{a}(\cdot,u_m,\nabla u_m)\cdot\nabla u_m \,\mathrm{d}x\,\mathrm{d}s &= \frac12\int_\Omega u_0^2\,\mathrm{d}x + \tau\sum_{j=0}^k \langle f_j, u_m\rangle_{W_0^{1,p}(\Omega)}\\ &\quad -\frac12\int_\Omega u_m^2(t_{k+1}^m,\cdot)\,\mathrm{d}x - \frac12\int_\Omega\sum_{j=0}^k \big(u_m(t_{j+1}^m,\cdot)-u_m(t_j^m,\cdot)\big)^2\,\mathrm{d}x\\ &\le \frac12\big(\|u_0\|^2_{L^2(\Omega)} - \|u_m(t_{k+1}^m,\cdot)\|^2_{L^2(\Omega)}\big) + \int_0^{t_{k+1}^m}\langle f, u_m\rangle_{W_0^{1,p}(\Omega)}\,\mathrm{d}s\\ &\le \frac12\big(\|u_0\|^2_{L^2(\Omega)} - \|u_m(t,\cdot)\|^2_{L^2(\Omega)}\big) + \int_0^{t}\langle f, u_m\rangle_{W_0^{1,p}(\Omega)}\,\mathrm{d}s + o(1)\end{aligned}$$

(as $m\to\infty$). Hence, using the weak lower semicontinuity of the norm in $L^2(\Omega)$ and our choice of t, we have

$$\begin{aligned}\limsup_{m\to\infty}\int_0^t\int_\Omega \mathbf{a}(\cdot,u_m,\nabla u_m)\cdot\nabla u_m\,\mathrm{d}x\,\mathrm{d}s &\le \frac12\big(\|u_0\|^2_{L^2(\Omega)} - \|u(t,\cdot)\|^2_{L^2(\Omega)}\big) + \int_0^t\langle f,u\rangle_{W_0^{1,p}(\Omega)}\,\mathrm{d}s\\ = \frac12\big(\|u_0\|^2_{L^2(\Omega)} - \|u(t,\cdot)\|^2_{L^2(\Omega)}\big) + \int_0^t\langle \partial_t u,u\rangle_{W_0^{1,p}(\Omega)}\,\mathrm{d}s &+ \int_0^t\int_\Omega \mathbf{A}\cdot\nabla u\,\mathrm{d}x\,\mathrm{d}s = \int_0^t\int_\Omega \mathbf{A}\cdot\nabla u\,\mathrm{d}x\,\mathrm{d}s,\end{aligned} \tag{9.23}$$

where we used the properties of the Gelfand triple, in particular, that the function $u\in C([0,T];L^2(\Omega))$.

We now finish the proof by employing Minty's trick. We fix $\mathbf{B}\in L^p(0,T;L^p(\Omega;\mathbb{R}^d))$ and get

$$\begin{aligned}0 &\le \limsup_{m\to\infty}\int_0^t\int_\Omega \big(\mathbf{a}(\cdot,u_m,\nabla u_m)-\mathbf{a}(\cdot,u_m,\mathbf{B})\big)\cdot(\nabla u_m-\mathbf{B})\,\mathrm{d}x\,\mathrm{d}s\\ &\le \limsup_{m\to\infty}\int_0^t\int_\Omega \mathbf{a}(\cdot,u_m,\nabla u_m)\cdot\nabla u_m\,\mathrm{d}x\,\mathrm{d}s\\ &\quad -\liminf_{m\to\infty}\int_0^t\int_\Omega\big(\mathbf{a}(\cdot,u_m,\nabla u_m)\cdot\mathbf{B}+\mathbf{a}(\cdot,u_m,\mathbf{B})\cdot(\nabla u_m-\mathbf{B})\big)\,\mathrm{d}x\,\mathrm{d}s\\ &\le \int_0^t\int_\Omega \mathbf{A}\cdot\nabla u\,\mathrm{d}x\,\mathrm{d}s - \int_0^t\int_\Omega\big(\mathbf{A}\cdot\mathbf{B}+\mathbf{a}(\cdot,u,\mathbf{B})\cdot(\nabla u-\mathbf{B})\big)\,\mathrm{d}x\,\mathrm{d}s\\ &= \int_0^t\int_\Omega\big(\mathbf{A}-\mathbf{a}(\cdot,u,\mathbf{B})\big)(\nabla u-\mathbf{B})\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

Note that we used (9.23) as well as $\mathbf{a}(\cdot,u_m,\mathbf{B})\to\mathbf{a}(\cdot,u,\mathbf{B})$ in $L^{p'}((0,T)\times\Omega;\mathbb{R}^d)$ due to the continuity of the Nemytskii operator, as $u_m\to u$ in $L^p((0,T)\times\Omega)$ and $|\mathbf{a}(\cdot,u,\mathbf{B})|\le C(1+|v|^{p-1}+\mathbf{B}|^{p-1})$.

We now set $\mathbf{B}:=\nabla u-\lambda\mathbf{H}$, $\lambda>0$ and $\mathbf{H}\in L^p(0,T;L^p(\Omega;\mathbb{R}^d))$, arbitrary. Then we have

$$0\le\int_0^t\int_\Omega\big(\mathbf{A}-\mathbf{a}(\cdot,u,\nabla u-\lambda\mathbf{H})\big)\cdot\mathbf{H}\,\mathrm{d}x\,\mathrm{d}s.$$

We let $\lambda\to 0_+$ and use the properties of Carathéodory functions (continuity of the Nemytskii operator) to conclude

$$0\le\int_0^t\int_\Omega\big(\mathbf{A}-\mathbf{a}(\cdot,u,\nabla u)\big)\cdot\mathbf{H}\,\mathrm{d}x\,\mathrm{d}s \tag{9.24}$$

for all $\mathbf{H}\in L^p(0,T;L^p(\Omega;\mathbb{R}^d))$. Therefore, equality in (9.24) holds and, moreover, $\mathbf{A}=\mathbf{a}(\cdot,u,\nabla u)$. The proof is finished. ■

Appendix A

Function spaces

A.1 Introduction and notation

The basic function spaces for solving the linear and nonlinear partial differential equations and their systems of elliptic type are the Sobolev spaces $W^{k,p}(\Omega)$, and for the evolutionary equations the Sobolev–Bochner spaces $W^{k,p}(0,T;X)$ or the Lebesgue–Bochner spaces $L^p(0,T;X)$, where X stands usually for the Lebesgue space $L^q(\Omega)$ or the Sobolev space $W^{l,q}(\Omega)$. The Sobolev, Lebesgue–Bochner and Sobolev–Bochner spaces are introduced in detail in Chapters 6 and 8, respectively. The starting point for their study are the continuous and continuously differentiable functions studied in Section A.2 and the Lebesgue spaces studied in Section A.3. The reader should have met these spaces in the basic courses on mathematical analysis and measure theory, however, for the sake of completeness and for the reader's comfort, we recall in this appendix the basic properties of these function spaces. They will be mostly presented without proofs which can be found, e.g., in Kufner et al. (1977), Pick et al. (2013), Lukeš and Malý (1995) or also in Czech in Černý and Pokorný (2023). More special results needed in these Lecture Notes are presented including their proofs.

Before we start to deal with the corresponding function spaces, we recall the standard notation of multiindeces and a shorten description of partial derivatives.

Notation A.1.1 (Multiindex)**.** The ordered d-tuple $\alpha = (\alpha_1, \dots, \alpha_d)$, $\alpha_i \in \mathbb{N}_0$, is called a multiindex. The length of the multiindex is denoted by $|\alpha|$ and is defined as $|\alpha| = \alpha_1 + \dots + \alpha_d$.

Notation A.1.2 (Partial derivative written by a multiindex)**.** The symbol $D^\alpha \phi$ denotes the partial derivative of a function ϕ

$$D^\alpha \phi(x) := \frac{\partial^{|\alpha|}\phi(x)}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}.$$

A.2 Spaces of continuous, Hölder continuous and continuously differentiable functions

We recall in this sections basic properties of spaces of continuous, Hölder continuous and continuously differentiable functions. The theorems are mostly presented without proofs, the interested reader can find them, e.g., in monographs Kufner et al. (1977) or Pick et al. (2013).

Definition A.2.1 — Continuous and continuously differentiable functions. Let $\Omega \subset \mathbb{R}^d$ be open.

1. The set of all continuous functions in Ω is denoted as $\mathcal{C}(\Omega)$ or $\mathcal{C}^0(\Omega)$, respectively.

2. Let $k \in \mathbb{N}$, then $\mathcal{C}^k(\Omega)$ denotes the set of all functions u which have all (classical) partial derivatives up to the order k in the set Ω and for any multiindex α such that $|\alpha| \le k$ it holds that $D^\alpha u \in \mathcal{C}(\Omega)$.

3. The space $\mathcal{C}^\infty(\Omega)$ denotes the set of all infinitely times differentiable functions in Ω, i.e., $\mathcal{C}^\infty(\Omega) = \bigcap_{k\in\mathbb{N}} \mathcal{C}^k(\Omega)$.

4. Denote $\operatorname{supp} u := \overline{\{x \in \Omega \,|\, u(x) \neq 0\}}$ the support of u. Then for any $k \in \mathbb{N}_0 \cup \{\infty\}$ we define the space $\mathcal{C}_0^k(\Omega) := \big\{u \in \mathcal{C}^k(\Omega) \,|\, \operatorname{supp} u \subset \Omega\,, \operatorname{supp} u \text{ is compact}\big\}$.

5. The space $\mathcal{C}(\overline{\Omega})$ or $\mathcal{C}^0(\overline{\Omega})$, respectively, contains all functions from $\mathcal{C}(\Omega)$ which are bounded and uniformly continuous in Ω.

6. Let $k \in \mathbb{N}$, then we define the space $\mathcal{C}^k(\overline{\Omega}) := \big\{u \in \mathcal{C}^k(\Omega) \,|\, \forall\alpha,\ |\alpha| \le k : D^\alpha u \in \mathcal{C}(\overline{\Omega})\big\}$.

7. The space $\mathcal{C}^\infty(\overline{\Omega}) = \bigcap_{k\in\mathbb{N}} \mathcal{C}^k(\overline{\Omega})$ contains all infinitely times differentiable functions in Ω such that all the derivatives including the function itself belong to $\mathcal{C}(\overline{\Omega})$.

Remark A.2.2. If Ω is a bounded set, then Point 5. in the definition above is equivalent with the fact that there exists a (uniquely defined) continuous extension of the corresponding function to the set $\overline{\Omega}$. In what follows we therefore assume that the function has already been extended.

Exercise A.2.3. Prove the claim of Remark A.2.2.

The basic properties of these function spaces are summarized in the following theorem.

Theorem A.2.4 — Properties of the space $\mathcal{C}^k(\overline{\Omega})$. Let $k \in \mathbb{N}_0$. Denote for $u \in \mathcal{C}^k(\overline{\Omega})$

$$\|u\|_{\mathcal{C}^k(\overline{\Omega})} := \sum_{\{\alpha\,|\,|\alpha|\le k\}} \sup_{x\in\overline{\Omega}} |D^\alpha u(x)|. \tag{A.1}$$

Then the following holds.

1. The functional $\|\cdot\|_{\mathcal{C}^k(\overline{\Omega})}$ is a norm on $\mathcal{C}^k(\overline{\Omega})$.
2. The space $\mathcal{C}^k(\overline{\Omega})$ is with respect to the above defined norm a Banach space.
3. If additionally Ω is bounded, then the space $\mathcal{C}^k(\overline{\Omega})$ is separable.
4. The space $\mathcal{C}^k(\overline{\Omega})$ is not reflexive.

Proof. The proof can be found in (Kufner et al., 1977, Sections 1.3–1.7) or (Pick et al., 2013, Sections 2.3–2.6). ∎

Exercise A.2.5. Prove that the space $\mathcal{C}^0(\overline{\Omega})$ is with respect to the norm introduced in (A.1) for $k = 0$ a Banach space which is separable if Ω is bounded.

Exercise A.2.6. Prove that the space $\mathcal{C}_B(\mathbb{R}) := \{u \in C(\mathbb{R})\,|\,u \text{ is bounded on } \mathbb{R}\}$ with the norm (A.1) for $k = 0$ is not separable.

Let us now present the characterization of the dual space to $\mathcal{C}(\overline{\Omega})$. This characterization will be useful to describe properties of certain special Sobolev spaces.

Theorem A.2.7 — Representation of the dual space to $\mathcal{C}(\overline{\Omega})$. Let $\Omega \subset \mathbb{R}^d$ be an open set. Let ϕ be a continuous linear functional on $\mathcal{C}(\overline{\Omega})$. Then there exists exactly one finite regular Radon measure μ on $\overline{\Omega}$ such that it holds

$$\forall u \in \mathcal{C}(\overline{\Omega})\colon\ \phi(u) := \langle \phi, u\rangle = \int_{\overline{\Omega}} u \,\mathrm{d}\mu. \tag{A.2}$$

Moreover, $\|\phi\|_{(\mathcal{C}(\overline{\Omega}))^*} = |\mu|(\overline{\Omega})$, where $|\mu|(\overline{\Omega})$ is the total variation of the measure μ on $\overline{\Omega}$.

Proof. The proof can be found in (Dunford and Schwartz, 1988, Section 4.6.3, Theorem 2) or (Lukeš and Malý, 1995, Theorem 16.5). ∎

The following theorem gives an equivalent characterization of precompact sets in $\mathcal{C}(\overline{\Omega})$.

Theorem A.2.8 — Arzelà–Ascoli. Let $\Omega \subset \mathbb{R}^d$ be an open bounded set. Let $A \subset \mathcal{C}(\overline{\Omega})$. The set A is totally bounded, if and only if it is

1. uniformly bounded, i.e.,
$$\exists C \in \mathbb{R}^+\colon\ \sup_{f\in A} \|f\|_{\mathcal{C}(\overline{\Omega})} = \sup_{f\in A}\max_{x\in\overline{\Omega}} |f(x)| \le C$$
2. equally uniformly continuous, i.e.,
$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall f \in A\ \forall x_1, x_2 \in \overline{\Omega}\colon\ |x_1 - x_2| < \delta \implies |f(x_1) - f(x_2)| < \varepsilon.$$

Proof. See (Kufner et al., 1977, Theorem 1.5.3) or (Pick et al., 2013, Theorem 2.5.3). ∎

We now introduce the spaces of Hölder continuous functions which can be understood as intermediate spaces between the continuous and continuously differentiable functions.

Definition A.2.9 — Hölder continuous functions. Let $\Omega \subset \mathbb{R}^d$ be an open set, $k \in \mathbb{N}_0$ and $u \in \mathcal{C}^k(\overline{\Omega})$. Let us denote for arbitrary $\lambda \in (0,1]$ and arbitrary multiindex α such that $|\alpha| \le k$

$$H_{\alpha,\lambda}(u) := \sup_{x,y\in\overline{\Omega},\ x\neq y} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\lambda}. \tag{A.3}$$

We then define $\mathcal{C}^{k,\lambda}(\overline{\Omega}) := \big\{u \in \mathcal{C}^k(\overline{\Omega}) \mid \forall\alpha,\ |\alpha| = k,\ H_{\alpha,\lambda}(u) < \infty\big\}$.

The basic properties of the Hölder continuous functions are summarized in the following theorem.

Theorem A.2.10 — Properties of the space $\mathcal{C}^{k,\lambda}(\overline{\Omega})$. Let $\Omega \subset \mathbb{R}^d$ be an open set, $k \in \mathbb{N}_0$ and $\lambda \in (0,1]$. We denote for $u \in \mathcal{C}^{k,\lambda}(\overline{\Omega})$

$$\|u\|_{\mathcal{C}^{k,\lambda}(\overline{\Omega})} := \|u\|_{\mathcal{C}^k(\overline{\Omega})} + \sum_{\{\alpha \mid |\alpha|=k\}} H_{\alpha,\lambda}(u). \tag{A.4}$$

Then the following holds.

1. The functional $\|\cdot\|_{\mathcal{C}^{k,\lambda}(\overline{\Omega})}$ is a norm on the space $\mathcal{C}^{k,\lambda}(\overline{\Omega})$.
2. The vector space $\mathcal{C}^{k,\lambda}(\overline{\Omega})$ is with respect to the above defined norm a Banach space.
3. The space $\mathcal{C}^{k,\lambda}(\overline{\Omega})$ is not separable.

Proof. The proof can be found in (Kufner et al., 1977, Sections 1.3–1.5) or (Pick et al., 2013, Sections 2.3–2.5). ■

Exercise A.2.11. Show that the space of Hölder continuous functions $C^{0,\lambda}(\overline{\Omega})$, $0 < \lambda \le 1$ introduced above is not separable.

The space $\mathcal{C}^{0,1}(\overline{\Omega})$ has special properties. We therefore introduce

Notation A.2.12 (Lipschitz continuous functions)**.** If $\alpha = (0,\dots,0)$, we shall use instead of $H_{(0,\dots,0),\lambda}(u)$ the notation $H_{0,\lambda}(u)$. Furthermore, if $\lambda = 1$, we call $\mathcal{C}^{0,1}(\overline{\Omega})$ the space of Lipschitz continuous functions. If $\lambda = 0$, we identify $\mathcal{C}^{0,0}(\overline{\Omega}) := \mathcal{C}(\overline{\Omega})$.

Similarly as in the case of continuous functions we may study subsets of Hölder continuous functions which are precompact in the space of continuous functions. The following theorem claims that any bounded subset has already this property.

Theorem A.2.13 — Compact embedding of $\mathcal{C}^{0,\lambda}(\overline{\Omega})$ into $\mathcal{C}(\overline{\Omega})$. Let Ω be an open bounded set. Then it holds for any $\lambda \in (0,1]$

$$\mathcal{C}^{0,\lambda}(\overline{\Omega}) \hookrightarrow\hookrightarrow \mathcal{C}(\overline{\Omega}).$$

Proof. The claim is a direct consequence of Arzelà–Ascoli Theorem A.2.8. ■

We have even a stronger result.

Theorem A.2.14 — Compact embedding of $\mathcal{C}^{0,\beta}(\overline{\Omega})$ into $\mathcal{C}^{0,\alpha}(\overline{\Omega})$. Let Ω be an open bounded set. Then it holds for any $\alpha, \beta \in [0,1]$ such that $0 \le \alpha < \beta \le 1$

$$\mathcal{C}^{0,\beta}(\overline{\Omega}) \hookrightarrow\hookrightarrow \mathcal{C}^{0,\alpha}(\overline{\Omega}).$$

Proof. The claim is an easy corollary of the interpolation inequality from the following exercise applied on the difference $u_n - u_m$, where $\{u_n\}_{n=1}^\infty$ is a Cauchy sequence in $\mathcal{C}(\overline{\Omega})$. ■

Exercise A.2.15. Show that for any $0 \le \lambda_1 \le \lambda_2 \le 1$ it holds

$$H_{0,\lambda_1}(u) \le \left(2\|u\|_{\mathcal{C}(\overline{\Omega})}\right)^{\frac{\lambda_2-\lambda_1}{\lambda_2}} \left(H_{0,\lambda_2}(u)\right)^{\frac{\lambda_1}{\lambda_2}}.$$

Theorem A.2.16 — Rademacher. Any function from $\mathcal{C}^{0,1}(\overline{\Omega})$ is differentiable (in the classical sense) almost everywhere in Ω and $|D^\alpha u(x)| \le H_{0,1}(u)$ holds for $|\alpha| = 1$ almost everywhere in Ω.

Proof. The proof can be found in (Lukeš and Malý, 1995, Theorem 30.3). ■

Remark A.2.17. The notation "almost everywhere" corresponds to the d-dimensional Lebesgue measure.

A.3 Lebesgue spaces

We expect that the reader is well acquainted with the basics of the Lebesgue measure and Lebesgue integral theory. More detailed information can be found in the Lecture Notes Lukeš and Malý (1995) or also in Czech in Černý and Pokorný (2023). Therein, most of the results given in this sections are proved. Another possible source of information are the monographs Kufner et al. (1977) or Pick et al. (2013).

If not stated otherwise, Ω denotes in this section an arbitrary non-empty Lebesgue measurable set in $\mathbb{R}^d$. Under an integral we understand the Lebesgue integral and under a measure the Lebesgue measure.

A.3.1 Basic properties of measurable functions and Lebesgue integral

Let us first recall characterization of measurable functions.

Theorem A.3.1 — Luzin. Let $f\colon \Omega \to \mathbb{R}$, f be almost everywhere finite. Then the following assertions are equivalent.

1. The function f is measurable.
2. For any $\varepsilon > 0$ there exists an open set $G \subset \Omega$ such that $|G| < \varepsilon$ and $f|_{\Omega\setminus G}$ is continuous.

Proof. The proof can be found in (Lukeš and Malý, 1995, Theorem 18.1) or (Černý and Pokorný, 2023, Theorem 15.5.9). ■

Next we formulate the basic results connected with properties of integral of a sequence (series) of measurable functions and the limit passage through the integral.

Theorem A.3.2 — Lebesgue monotone convergence. Let $\{f_n\}_{n=1}^\infty$ be a sequence of measurable functions such that for any n and almost every $x \in \Omega$ it holds $f_n(x) \le f_{n+1}(x)$. Moreover, let

$$\int_\Omega f_1 \,\mathrm{d}x > -\infty.$$

Then there exists a measurable function f such that

$$\lim_{n\to\infty} f_n(x) = f(x) \qquad \text{for almost every } x \in \Omega \text{ and}$$
$$\lim_{n\to\infty} \int_\Omega f_n \,\mathrm{d}x = \int_\Omega f \,\mathrm{d}x.$$

Proof. The proof can be found in (Lukeš and Malý, 1995, Theorem 8.11) or (Černý and Pokorný, 2023, Theorem 15.8.19). ■

Remark A.3.3. The theorem is also known under the name Levi Theorem.

Theorem A.3.4 — Lebesgue dominated convergence. Let $\{f_n\}_{n=1}^\infty$ be a sequence of measurable functions and the function f be such that for almost every $x \in \Omega$ it holds

$$\lim_{n\to\infty} f_n(x) = f(x).$$

Furthermore, assume that there exists a measurable function g such that for all $n \in \mathbb{N}$ and almost every $x \in \Omega$ it holds

$$|f_n(x)| \le g(x) \qquad \text{and} \qquad \int_\Omega g \,\mathrm{d}x < \infty.$$

Then

$$\lim_{n\to\infty} \int_\Omega f_n \,\mathrm{d}x = \int_\Omega f \,\mathrm{d}x.$$

Proof. The proof can be found in (Lukeš and Malý, 1995, Theorem 8.13) or (Černý and Pokorný, 2023, Theorem 15.8.21). ■

Theorem A.3.5 — Vitali. Let Ω be a measurable set with finite measure, $\{f_n\}_{n=1}^\infty$ be a sequence of measurable functions and the function f be such that for almost every $x \in \Omega$ it holds

$$\lim_{n\to\infty} f_n(x) = f(x).$$

Furthermore, let the sequence be equally uniformly integrable, i.e.,

$$\forall \varepsilon > 0\, \exists \delta > 0\, \forall n \in \mathbb{N}\, \forall H \subset \Omega \colon\; |H| \le \delta \implies \int_H |f_n| \,\mathrm{d}x \le \varepsilon.$$

Then

$$\lim_{n\to\infty} \int_\Omega f_n \,\mathrm{d}x = \int_\Omega f \,\mathrm{d}x.$$

Proof. The proof can be found in (Černý and Pokorný, 2023, Theorem 15.14.4). ■

Theorem A.3.6 — Fatou Lemma. Let $\{f_n\}_{n=1}^\infty$ be a sequence of measurable non-negative functions. Then

$$\int_\Omega \liminf_{n\to\infty} f_n \,\mathrm{d}x \le \liminf_{n\to\infty} \int_\Omega f_n \,\mathrm{d}x.$$

Proof. The proof can be found in (Lukeš and Malý, 1995, Theorem 8.15) or (Černý and Pokorný, 2023, Lemma 15.7.10). ■

Theorem A.3.7 — Egorov. Let Ω be a measurable set with finite measure. Let f and $\{f_n\}_{n=1}^\infty$ be measurable functions which are finite almost everywhere in Ω. Then the following assertions are equivalent.

1. For almost every $x \in \Omega$
$$\lim_{n\to\infty} f_n(x) = f(x).$$
2. For any $\varepsilon > 0$ there exists an open set $G \subset \Omega$ such that $|G| < \varepsilon$ and
$$f_n \rightrightarrows f \text{ uniformly in } \Omega \setminus G.$$

Proof. The proof can be found in (Lukeš and Malý, 1995, Theorem 12.5) or (Černý and Pokorný, 2023, Theorem 15.5.8). ■

A.3.2 Basic properties of Lebesgue spaces. Hölder's inequality and its consequences

We now define the basic functionals in the theory of Lebesgue spaces.

Definition A.3.8 — $\mathcal{L}^p$ classes. Let $p \in [1, \infty)$. We denote

$$\mathcal{L}^p(\Omega) := \left\{ f \text{ measurable in } \Omega \,\middle|\, \int_\Omega |f|^p \,\mathrm{d}x < \infty \right\}$$
$$\|f\|_{\mathcal{L}^p} = \|f\|_p := \left(\int_\Omega |f|^p \,\mathrm{d}x \right)^{\frac{1}{p}}. \tag{A.5}$$

For $p = \infty$ we denote

$$\mathcal{L}^\infty(\Omega) := \left\{ f \text{ measurable in } \Omega \mid \exists C \in \mathbb{R}^+,\, |f(x)| \le C \text{ a.e. in } \Omega \right\}$$
$$\|f\|_{\mathcal{L}^\infty} = \|f\|_\infty := \operatorname*{ess\,sup}_{x\in\Omega} |f(x)| = \inf_{\{E\subset\Omega \,|\, |E|=0\}} \sup_{x\in\Omega\setminus E} |f(x)| = \inf_{\alpha\in\mathbb{R}} \{\alpha \mid |f(x)| \le \alpha \text{ a.e. in } \Omega\}. \tag{A.6}$$

The change of a function f on a set of zero measure does not change either whether $f \in \mathcal{L}^p(\Omega)$, $p \in [1, \infty]$ or the value of the functional $\|\cdot\|_{\mathcal{L}^p}$. For functions $f \in \mathcal{L}^p(\Omega)$ the functional $\|\cdot\|_{\mathcal{L}^p}$ is therefore not a norm. Hence we consider instead of individual functions f the classes of equivalence $[f]$ defined as

$$f_1 \in [f] \iff f_1 = f \text{ a.e. in } \Omega.$$

Definition A.3.9 — Lebesgue spaces. Let $p \in [1, \infty]$. Denote

$$L^p(\Omega) := \{[f] \mid f \in \mathcal{L}^p(\Omega)\}.$$

Then we call $L^p(\Omega)$ the Lebesgue space. For Ω open we further introduce

$$L^p_{\text{loc}}(\Omega) := \{[f] \mid f \text{ is measurable in } \Omega \mid \forall K \subset \Omega, K \text{ compact},\; f \in L^p(K)\}.$$

In what follows, we will speak about functions f instead of the classes of equivalence $[f]$. Even though it is not precise, in most cases it is sufficient. When it will be important to specify the representative f from the class of equivalence, we will do so.

The functionals defined in (A.5) and (A.6) have the following important property.

Theorem A.3.10 — Minkowski inequality. Let $p \in [1, \infty]$ and let $f, g \in L^p(\Omega)$. Then it holds:

1. $f + g \in L^p(\Omega)$
2. $\|f + g\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}$.

Proof. The proof can be found in (Lukeš and Malý, 1995, Theorem 10.4) or in (Černý and Pokorný, 2023, Theorem 16.2.6). ■

This inequality (in fact, the triangle inequality for the Lebesgue spaces) is one of the main tools to prove the following basic result.

Theorem A.3.11 — Completeness of Lebesgue spaces. Let $p \in [1, \infty]$. Then the functional $\| \cdot \|_{\mathcal{L}^p}$ defined in (A.5) and (A.6) is a norm in the space $L^p(\Omega)$ and the space $L^p(\Omega)$ is with respect to this norm a Banach space. Moreover, for $p = 2$, the space $L^2(\Omega)$ is a Hilbert space with the scalar product defined as[a]

$$(u, v)_{L^2(\Omega)} := \int_\Omega uv \, dx.$$

[a]We consider only real-valued function in these Lecture Notes. For complex-valued function we have to replace the function v in the integral by its complex conjugate.

Proof. The proof can be found in (Lukeš and Malý, 1995, Theorem 10.6) or (Černý and Pokorný, 2023, Theorem 16.3.3). ■

The inequality below plays a fundamental role in the theory of partial differential equations. It in fact also implies the Minkowski inequality.

Theorem A.3.12 — Hölder's inequality. Let $p \in [1, \infty]$ and let $f \in L^p(\Omega)$, $g \in L^{p'}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$ (with the convention that $p = 1 \Rightarrow p' = \infty$ and vice versa). Then it holds:

1. $fg \in L^1(\Omega)$
2. $\|fg\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^{p'}(\Omega)}$.

Proof. The proof can be found in (Lukeš and Malý, 1995, Theorem 10.3) or in (Černý and Pokorný, 2023, Theorem 16.2.3). ■

Hölder's inequality has several direct consequences.

Lemma A.3.13 — Hölder's inequality for several functions. Let $i = 1, \ldots, k$ and let $p_i \in [1, \infty]$ be such that $\sum_{i=1}^k \frac{1}{p_i} = 1$ (with the convention that $p_i = \infty \Rightarrow \frac{1}{p_i} = 0$). Further, let $f_i \in L^{p_i}(\Omega)$. Then it holds:

1. $\prod_{i=1}^k f_i \in L^1(\Omega)$
2. $\left\| \prod_{i=1}^k f_i \right\|_{L^1(\Omega)} \leq \prod_{i=1}^k \|f_i\|_{L^{p_i}(\Omega)}$.

Proof. Compare with (Černý and Pokorný, 2023, Exercise 16.2.18). ■

Lemma A.3.14 — Trivial embeddings of $L^p(\Omega)$ spaces. Let[a] $|\Omega| < \infty$, then it holds for all $p_1, p_2 \in [1, \infty]$, $p_2 \geq p_1$:

1. $L^{p_2}(\Omega) \hookrightarrow L^{p_1}(\Omega)$
2. $\|f\|_{L^{p_1}(\Omega)} \leq |\Omega|^{\frac{p_2 - p_1}{p_1 p_2}} \|f\|_{L^{p_2}(\Omega)}$.

[a]The symbol $|\Omega|$ denotes the d-dimensional Lebesgue measure of the set $\Omega \subset \mathbb{R}^d$.

Proof. It can be found in (Černý and Pokorný, 2023, Proposition 16.2.9). ■

Lemma A.3.15 — Interpolation inequality in Lebesgue spaces. Let $p_1, p_2 \in [1,\infty]$, $p_2 > p_1$ and $f \in L^{p_1}(\Omega) \cap L^{p_2}(\Omega)$. Then it holds for all $r \in [p_1, p_2]$:

1. $f \in L^r(\Omega)$
2. $\|f\|_{L^r(\Omega)} \le \|f\|^{\alpha}_{L^{p_1}(\Omega)} \|g\|^{1-\alpha}_{L^{p_2}(\Omega)}$,

where $\alpha \in [0,1]$ fulfils $\frac{1}{r} = \frac{\alpha}{p_1} + \frac{1-\alpha}{p_2}$.

Proof. The proof can be found in (Černý and Pokorný, 2023, Proposition 16.2.12). ∎

Exercise A.3.16. Applying Theorem A.3.12 prove Lemmata A.3.13–A.3.15.

Last we present a characterisation of the space $L^\infty(\Omega)$.

Theorem A.3.17 — Connection of norms in $L^\infty(\Omega)$ and $L^p(\Omega)$. Let $|\Omega| < \infty$. If $f \in L^\infty(\Omega)$, then it holds:

1. $\forall p \in [1,\infty)$: $f \in L^p(\Omega)$
2. $\lim_{p\to\infty} \|f\|_{L^p(\Omega)} = \|f\|_{L^\infty(\Omega)}$.

If there exists a sequence $\{p_i\}_{i=1}^\infty$ such that $\lim_{i\to\infty} p_i = \infty$ and a constant $C \in \mathbb{R}^+$ such that $\sup_{i\in\mathbb{N}} \|f\|_{L^{p_i}(\Omega)} \le C$, then it holds:

1. $f \in L^\infty(\Omega)$
2. $\|f\|_{L^\infty(\Omega)} \le C$.

Proof. The proof is a simple exercise; it can also be found in (Kufner et al., 1977, Theorem 2.11.4–2.11.5) or in (Černý and Pokorný, 2023, Proposition 16.2.11). ∎

A.3.3 Density of continuous functions in the Lebesgue spaces

The construction of the Lebesgue integral provides us the density of simple functions in $L^p(\Omega)$, $p \in [1,\infty)$ (for more details, see, e.g., (Černý and Pokorný, 2023, Theorem 16.4.4)). In the section devoted to mollifying of Lebesgue functions we show the density of smooth functions in these spaces. The main tool is Theorem on Lebesgue points A.3.20 proved below. Its proof, however, needs the density of continuous functions in $L^1(\Omega)$. We first show this result.

Theorem A.3.18 — Density of continuous functions in $L^1(\Omega)$. Let $\Omega \subset \mathbb{R}^d$ be open. Then functions from $\mathcal{C}(\overline{\Omega})$ are dense in $L^1(\Omega)$.

Proof. Choose $u \in L^1(\Omega)$ and fix $\varepsilon > 0$. Using the Lebesgue dominated convergence Theorem A.3.4 it is easy to see that there exists $R > 0$ sufficiently large such that $\|u - u\chi_{B_R(0)}\|_1 < \varepsilon$. It is therefore enough to approximate the function $u_1 = u\chi_{B_R(0)}$ which is defined as 0 outside the domain of u. Due to the properties of the Lebesgue integral there exists a simple function $u_2 \in L^1(\mathbb{R}^d)$ such that $\|u_2 - u_1\|_1 < \varepsilon$. We may clearly assume that $u_2 = 0$ in $\mathbb{R}^d \setminus B_R(0)$. This function has only finite number of nonzero values and their counterimages are measurable subsets of $B_R(0)$. Due to the interior regularity of the Lebesgue measure we may approximate any such set arbitrarily precisely by a compact set.

We can therefore find a simple function u_3 such that $\|u_3 - u\|_1 < 3\varepsilon$ and

$$u_3 = \sum_{i=1}^n a_i \chi_{K_i},$$

where $n \in \mathbb{N}$, a_i are real numbers and $K_i \subset B_R(0)$ are disjoint compact sets. Then

$$d := \min_{1\le i<j\le n} \operatorname{dist}(K_i, K_j) > 0.$$

It is now enough to choose $\delta \in (0, \frac{d}{2})$ and set

$$u_4(x) = \sum_{i=1}^n a_i \max\Big\{1 - \frac{1}{\delta}\operatorname{dist}(x, K_i), 0\Big\}.$$

We constructed a continuous function which for δ sufficiently small fulfills $\|u_3 - u_4\|_1 < \varepsilon$. Indeed, the value of the function u_3 was changed only on sets, where $0 < \operatorname{dist}(x, K_i) < \delta$ for some i. The measure of these sets goes to zero if $\delta \to 0_+$ (continuity of the measure) and the value of the function was changed at most by $\max_{i=1,\dots,n} |a_i|$. Whence $\|u_4 - u_1\|_1 < 4\varepsilon$ and the proof is finished. ∎

A.3.4 Lebesgue points

Let us recall the Luzin Theorem A.3.1 which implies that for Ω bounded we may choose a set G such that $\Omega \setminus G$ is compact and u is continuous on $\Omega \setminus G$ (and thus also uniformly continuous). The measure of $\Omega \setminus G$ can be taken arbitrarily small. For functions continuous in the whole Ω it is not difficult to verify that

$$\forall x \in \Omega\colon \quad \lim_{r\to 0_+} \frac{1}{|B_r(x)|}\int_{B_r(x)} u(y)\,\mathrm{d}y = u(x).$$

Since the above mentioned property is fundamental to build the theory of weak solutions to partial differential equations, we want to know how big is the set for which the equality above holds true if u is only locally integrable. First, we define the following notion.

Definition A.3.19 — Lebesgue point. Let $u\colon \mathbb{R}^d \to \mathbb{R}$ be a locally integrable function. We say that the point $x \in \mathbb{R}^d$ is a Lebesgue point of the function u, if

$$\lim_{r\to 0_+} \frac{1}{|B_r(x)|}\int_{B_r(x)} |u(y) - u(x)|\,\mathrm{d}y = 0. \tag{A.7}$$

The following theorem answers the question how big the set of Lebesgue points is.

Theorem A.3.20 — On Lebesgue points. Let $u\colon \mathbb{R}^d \to \mathbb{R}$ be a locally integrable function. Then almost every points $x \in \mathbb{R}^d$ are Lebesgue points of the function u.

The proof of this claim is usually a part of the basic course on the Lebesgue integral. Even though it can be found in (Lukeš and Malý, 1995, Theorem 23.9) or in (Černý and Pokorný, 2023, Theorem 16.5.14), for completeness, we also include the proof here.

We first show two auxiliary results.

Theorem A.3.21 — Vitali covering Lemma. Let $\{B_{r_i}(x_i)\}_{i=1}^n \subset \mathbb{R}^d$ be a finite system of balls. Then there exists its pairwise disjoint subsystem $\{B_{r_j}(x_j)\}_{j\in J}$, $J \subset \{1,\dots,n\}$ such that

$$\bigcup_{i=1,\dots,n} B_{r_i}(x_i) \subset \bigcup_{j\in J} B_{3r_j}(x_j). \tag{A.8}$$

Proof. We may assume, without loss of generality, that the balls are ordered descending according to its size. The subsystem $\{B_{r_j}(x_j)\}_{j\in J}$, $J \subset \{1,\dots,n\}$ is constructed as follows. We first set $j_1 = 1$ and remove all the balls which have nonempty intersection with the largest ball $B_{r_1}(x_1)$. Let $j_2 > j_1$ be the order of the second largest remaining ball. We now remove all the balls which have a nonempty intersection with the ball $B_{r_{j_2}}(x_{j_2})$ and in the next step we use the third largest ball of the remaining ones. We continue by induction until we obtain a disjoint system. Property (A.8) follows from the fact that if we remove a ball $B_{r_i}(x_i)$, there exists $j \in \{1,\dots,i-1\}$ such that the ball $B_{r_j}(x_j)$ is among the chosen balls and $B_{r_i}(x_i) \cap B_{r_j}(x_j) \neq \emptyset$. However, we also have $r_j \geq r_i$, therefore $B_{r_i}(x_i) \subset B_{3r_j}(x_j)$. ■

Next we introduce an operator which plays an important role in the harmonic analysis.

Definition A.3.22 — Hardy–Littlewood maximal operator. Hardy–Littlewood maximal operator assigns to a locally integrable function $u\colon \mathbb{R}^d \to \mathbb{R}$ the function Mu defined by

$$Mu(x) := \sup_{\{B_r(z)\,|\, x\in B_r(z)\}} \frac{1}{|B_r(z)|}\int_{B_r(z)} |u(y)|\,\mathrm{d}y.$$

Theorem A.3.23 — Hardy–Littlewood. Let $u \in L^1(\mathbb{R}^d)$. Then for any $t > 0$ it holds[a]

$$|\{x \in \mathbb{R}^d \,|\, (Mu)(x) > t\}| \leq \frac{3^d}{t}\|u\|_1.$$

[a] This estimate is called "weak $(1,1)$-estimate".

Proof. The set $G_t := \{x \in \mathbb{R}^d \,|\, (Mu)(x) > t\}$ is open, as follows from the continuous dependence of the integral on the domain of integration (verify carefully!). We choose a compact set $K \subset G_t$. By virtue of the definition of the set K, for any $z \in K$ there exist $x_z \in \mathbb{R}^d$ and $r_z > 0$ such that

$$\int_{B_{r_z}(x_z)} |u(y)|\,\mathrm{d}y > t|B_{r_z}(x_z)|.$$

The system $\{B_{r_z}(x_z)\}_{z\in K}$ covers the set $K \subset \mathbb{R}^d$ and by virtue of the Lindelöf covering Theorem (see (Černý and Pokorný, 2021, Theorem 11.7.2)) there exists its countable subcovering. Since K is additionally compact, this subcovering contains due to the Borel covering Theorem (see (Černý and Pokorný, 2021, Theorem 11.7.3)) a finite subcovering. By virtue of the Vitalli covering Lemma A.3.21 there exists a disjoint system $\{B_{r_i}(x_i)\}_{i=1,\dots,n}$ such that $\{B_{3r_i}(x_i)\}_{i=1}^n$ covers K. Thus

$$t|K| \le t\sum_{i=1}^n |B_{3r_i}(x_i)| = 3^d t\sum_{i=1}^n |B_{r_i}(x_i)| \le 3^d \sum_{i=1}^n \int_{B_{r_i}(x_i)} |u(y)|\,\mathrm{d}y \le 3^d\|u\|_1.$$

Due to the regularity from inside of the Lebesgue measure we may switch to the supremum over $K \subset G_t$ on the left-hand side of the previous inequality. The theorem is proved. ■

We are now ready to proof the Theorem on Lebesgue points.

Proof of Theorem A.3.20. For arbitrary, but fixed $\varepsilon > 0$ we define

$$N_\varepsilon = \Big\{x \in \mathbb{R}^d \mid \limsup_{r\to 0_+} \frac{1}{|B_r(x)|}\int_{B_r(x)} |u(y)-u(x)|\,\mathrm{d}y > 3\varepsilon\Big\}.$$

Our goal is to show that $|N_\varepsilon| = 0$ whenever $\varepsilon > 0$. We fix $\varepsilon \in (0,1)$ and further also $\delta \in (0,1)$. Since the continuous functions are dense in $L^1(\mathbb{R}^d)$, we may find such a continuous $v \in L^1(\mathbb{R}^d)$ that $\|u-v\|_1 < \delta$. We further define

$$N_{\varepsilon,\delta} = \{x \in \mathbb{R}^d \mid M(u-v)(x) \ge \varepsilon\} \cup \{x \in \mathbb{R}^d \mid |u(x)-v(x)| \ge \varepsilon\}.$$

We now take $x \notin N_{\varepsilon,\delta}$. Due to the continuity of v we may find $\varrho > 0$ such that

$$|v(y)-v(x)| < \varepsilon \qquad \text{pro } |y-x| < \varrho.$$

For any $r \in (0,\varrho)$ we have (recall that $x \notin N_{\varepsilon,\delta}$)

$$\begin{aligned}\frac{1}{|B_r(x)|}\int_{B_r(x)} |u(y)-u(x)|\,\mathrm{d}y &\le \frac{1}{|B_r(x)|}\int_{B_r(x)} (|v(y)-v(x)| + |u(y)-v(y)| + |v(x)-u(x)|)\,\mathrm{d}y\\ &\le \varepsilon + M(u-v)(x) + \varepsilon \le 3\varepsilon,\end{aligned}$$

and therefore $N_\varepsilon \subset N_{\varepsilon,\delta}$.

By virtue of the Tschebyshev inequality (see (Černý and Pokorný, 2023, Remark 15.7.4)) and the Hardy–Littlewood Theorem A.3.23 we have

$$|N_{\varepsilon,\delta}| \le |\{x \in \mathbb{R}^d \mid M(u-v)(x) \ge \varepsilon\}| + |\{x \in \mathbb{R}^d \mid |u-v|(x) \ge \varepsilon\}| \le \frac{3^d}{\varepsilon}\|u-v\|_1 + \frac{1}{\varepsilon}\|u-v\|_1 = \frac{C\delta}{\varepsilon}.$$

Altogether, $|N_\varepsilon| \le |N_{\varepsilon,\delta}| \le \frac{C\delta}{\varepsilon}$ and since $\delta > 0$ was arbitrary, it must hold $|N_\varepsilon| = 0$. ■

Exercise A.3.24. Show the following nontrivial extension of the Theorem on Lebesgue points. Let $u \in L^p(\Omega)$ and $p \in [1,\infty)$. Then it holds for almost every $x \in \Omega$

$$\lim_{r\to 0_+} \frac{1}{|B_r(x)|}\int_{B_r(x)} |u(y)-u(x)|^p\,\mathrm{d}y = 0.$$

A.3.5 Mollifier, p-mean continuity, separability of the $L^p(\Omega)$ spaces

Definition A.3.25 — p-mean continuity. Let $f \in L^p(\Omega)$ and $p \in [1,\infty)$. We define $f(x) = 0$ for $x \notin \Omega$. We say that the function f is p-mean continuous if

$$\forall\varepsilon > 0\ \exists\delta > 0\ \forall\mathbf{h} \in \mathbb{R}^d\colon |\mathbf{h}| < \delta \implies \int_\Omega |f(x+\mathbf{h}) - f(x)|^p\,\mathrm{d}x < \varepsilon^p. \tag{A.9}$$

Theorem A.3.26 — On p-mean continuity. Let $p \in [1,\infty)$. Then any function from $L^p(\Omega)$ is p-mean continuous.

Proof. For $x \in \mathbb{R}^d \setminus \Omega$ we have $f(x) = 0$. We fix arbitrary $\varepsilon > 0$. It is possible to choose $R > 0$ such that for the given $\varepsilon > 0$ it is

$$\int_{\mathbb{R}^d\setminus B_R(0)} |f(x)|^p\,\mathrm{d}x \le \left(\frac{\varepsilon}{6}\right)^p$$

$$\int_{\mathbb{R}^d\setminus B_{R+1}(0)} |f(x+\mathbf{h})|^p\,\mathrm{d}x \le \left(\frac{\varepsilon}{6}\right)^p,$$

where we assumed without loss of generality that for δ from (A.9) we have $\delta \le 1$. Due to the above presented inequalities we may work in the ball $B_{R+1}(0)$.

Since any integrable function is uniformly integrable, we can choose $\eta > 0$ such that $\forall E \subset B_{R+2}(0)$, $|E| < 2\eta$, it holds

$$\int_E |f(x)|^p \,\mathrm{d}x < \left(\frac{\varepsilon}{6}\right)^p. \tag{A.10}$$

The function f is measurable in $B_{R+2}(0)$ and by virtue of the Luzin Theorem A.3.1 there exists a compact set $F_\eta^R \subset B_{R+2}(0)$ such that $f \in \mathcal{C}(F_\eta^R)$ and $|B_{R+2}(0) \setminus F_\eta^R| < \eta$. Furthermore, there exists $\delta \in (0,1)$ such that

$$\forall x, y \in F_\eta^R \colon |x-y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{3|B_{R+1}(0)|^{\frac{1}{p}}}. \tag{A.11}$$

Denote $F_\eta^{R,\mathbf{h}} := \left\{x \in B_{R+2}(0) \,|\, x + \mathbf{h} \in F_\eta^R\right\}$ and $F_\eta := F_\eta^R \cap F_\eta^{R,\mathbf{h}} \cap \overline{B_{R+1}(0)}$. The set F_η is compact and it holds

$$|B_{R+1}(0) \setminus F_\eta| = |\left(B_{R+1}(0) \setminus F_\eta^R\right) \cup \left(B_{R+1}(0) \setminus F_\eta^{R,\mathbf{h}}\right)| < 2\eta;$$

this follows due to the choice of η, since

$$\begin{aligned} |B_{R+1}(0) \setminus F_\eta^R| &\le |B_{R+2}(0) \setminus F_\eta^R| < \eta \\ |B_{R+1}(0) \setminus F_\eta^{R,\mathbf{h}}| &= |B_{R+1}(-\mathbf{h}) \setminus F_\eta^R| \le |B_{R+2}(0) \setminus F_\eta^R| < \eta, \end{aligned}$$

where we used in the second estimate that $|\mathbf{h}| \le 1$. Altogether, from (A.10) it follows

$$2^{p-1} \int_{B_{R+1}(0)\setminus F_\eta} \left(|f(x+\mathbf{h})|^p + |f(x)|^p\right) \mathrm{d}x < \left(\frac{\varepsilon}{3}\right)^p.$$

Next, from (A.11) we have for any $\mathbf{h} \in B_\delta(0)$

$$\int_{F_\eta} |f(x+\mathbf{h}) - f(x)|^p \,\mathrm{d}x < \left(\frac{\varepsilon}{3}\right)^p.$$

Finally, for $|\mathbf{h}| < \delta$ it holds

$$\begin{aligned} \int_{\mathbb{R}^d} |f(x+\mathbf{h}) - f(x)|^p \,\mathrm{d}x \le{} & 2^{p-1} \int_{\mathbb{R}^d \setminus B_{R+1}(0)} \left(|f(x+\mathbf{h})|^p + |f(x)|^p\right) \mathrm{d}x \\ & + 2^{p-1} \int_{B_{R+1}(0)\setminus F_\eta} \left(|f(x+\mathbf{h})|^p + |f(x)|^p\right) \mathrm{d}x \\ & + \int_{F_\eta} |f(x+\mathbf{h}) - f(x)|^p \,\mathrm{d}x < \varepsilon^p \end{aligned}$$

which we wanted to prove. ■

Exercise A.3.27. Modify the proof of the previous theorem and show the following. Let $p \in [1,\infty)$ and $u \in L^p(\Omega)$ be arbitrary. Define for $\tau \in (0,1]$ the function $u_\tau(x) = u(\tau x)$. Then for any $\varepsilon > 0$ there exists $0 < \delta < \tau \le 1$ such that $\|u - u_\tau\|_{L^p(\Omega)} < \varepsilon$.

The following mollifying operator defined as convolution with a suitable mollifying kernel plays an important role in the theory of partial differential equations.

Definition A.3.28 — Mollifier. We say that the function $\eta\colon \mathbb{R}^d \to \mathbb{R}$ is a mollifier (a mollifying kernel), if the following holds:

1. $\eta \in \mathcal{C}_0^\infty\left(\mathbb{R}^d\right)$
2. $\operatorname{supp} \eta \subset \overline{B_1(0)}$
3. $\forall x \in \mathbb{R}^d\colon \eta(x) \ge 0$
4. $\forall x, y \in \mathbb{R}^d\colon |x| = |y| \Rightarrow \eta(x) = \eta(y)$
5. $\int_{\mathbb{R}^d} \eta(x)\,\mathrm{d}x = 1$.

Example A.3.29. A possible choice of the mollifying kernel is the function

$$\eta(x) = \begin{cases} C\mathrm{e}^{\frac{1}{|x|^2-1}} & \text{for } |x| < 1 \\ 0 & \text{for } |x| \ge 1, \end{cases}$$

where the constant C is chosen so that the condition $\int_{\mathbb{R}^d} \eta(x)\,\mathrm{d}x = 1$ is satisfied.

In what follows, for a fixed $\varepsilon > 0$ we denote η_ε the function defined as

$$\eta_\varepsilon(x) := \frac{1}{\varepsilon^d}\eta\left(\frac{x}{\varepsilon}\right).$$

Definition A.3.30 — Mollification of a function. Let $\Omega \subset \mathbb{R}^d$ be open and $u \in L^1_{\text{loc}}(\Omega)$. The function u_ε: $\Omega_\varepsilon \to \mathbb{R}$ defined in $\Omega_\varepsilon := \{x \in \Omega \,|\, \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$ as

$$u_\varepsilon := \eta_\varepsilon \star u \,\left(\text{i.e., } u_\varepsilon(x) = \int_{\mathbb{R}^d} \eta_\varepsilon(x-y)u(y)\,\mathrm{d}y = \int_{B_\varepsilon(x)} \eta_\varepsilon(x-y)u(y)\,\mathrm{d}y\right)$$

is called mollification of the function u.

Theorem A.3.31 — On convolution. Let $p, q \in [1, \infty]$ be such that $\frac{1}{p} + \frac{1}{q} \geq 1$. Then for any $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$ it holds:

1. $f \star g \in L^r(\mathbb{R}^d)$, where $\frac{1}{r} := \frac{1}{p} + \frac{1}{q} - 1$
2. $\|f \star g\|_{L^r(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^q(\mathbb{R}^d)}$.

Proof. The proof can be found in (Lukeš and Malý, 1995, Theorem 26.20). ■

Exercise A.3.32. By virtue of Hölder's inequality prove Estimate 2. from the theorem above.

Theorem A.3.33 — On mollification. Let $\Omega \subset \mathbb{R}^d$ be open and $u \in L^1_{\text{loc}}(\Omega)$. Let u_ε be mollification of u. Then it holds.

1. The function $u_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon)$.
2. For almost every $x \in \Omega$ we have $\lim\limits_{\varepsilon \to 0_+} u_\varepsilon(x) = u(x)$.
3. If $u \in \mathcal{C}(\Omega)$, then $u_\varepsilon \rightrightarrows u$ on every compact subset $K \subset \Omega$.
4. If $u \in L^p_{\text{loc}}(\Omega)$ for $p \in [1, \infty)$, then $u_\varepsilon \to u$ in $L^p_{\text{loc}}(\Omega)$.
5. If $u \in L^p(\mathbb{R}^d)$, then $\|u_\varepsilon\|_{L^p(\mathbb{R}^d)} \leq \|u\|_{L^p(\mathbb{R}^d)}$ for $p \in [1, \infty]$ and $u_\varepsilon \to u$ in $L^p(\mathbb{R}^d)$ for $p \in [1, \infty)$.

Proof. **Step 1:** Proof of Claim 1.
We have from the definition of the convolution

$$u_\varepsilon(x) = \int_{\mathbb{R}^d} \eta_\varepsilon(x-y)u(y)\,\mathrm{d}y,$$

where $\eta_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^d)$. It therefore suffices to apply the theorems on continuity of integral with respect to a parameter and on derivative of integral with respect to a parameter and we immediately obtain the required $u_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon)$.

Step 2: Proof of Claim 2.
We apply the Definition of mollification of a function A.3.30, use properties of the mollifier from the Definition of mollifier A.3.28 and compute for arbitrary $x \in \Omega_\varepsilon$

$$\begin{aligned}
|u(x) - u_\varepsilon(x)| &= \left| u(x) \int_{\mathbb{R}^d} \eta_\varepsilon(x-y)\,\mathrm{d}y - \int_{\mathbb{R}^d} \eta_\varepsilon(x-y)u(y)\,\mathrm{d}y \right| \\
&= \left| \int_{B_\varepsilon(x)} \eta_\varepsilon(x-y)\,(u(x) - u(y))\,\mathrm{d}y \right| \\
&= \frac{1}{\varepsilon^d} \left| \int_{B_\varepsilon(x)} \eta\left(\frac{x-y}{\varepsilon}\right)(u(x) - u(y))\,\mathrm{d}y \right| \\
&\leq \frac{C}{\varepsilon^d} \int_{B_\varepsilon(x)} |u(x) - u(y)|\,\mathrm{d}y.
\end{aligned}$$

Using the Theorem on Lebesgue points A.3.20 it holds for a.e. x that $\frac{1}{\varepsilon^d} \int_{B_\varepsilon(x)} |u(x) - u(y)|\,\mathrm{d}y \to 0$ for $\varepsilon \to 0_+$, and the claim is proved.

Step 3: Proof of Claim 3.
We proceed as in the proof of Claim 2., only when estimating

$$\frac{C}{\varepsilon^d} \int_{B_\varepsilon(x)} |u(x) - u(y)|\,\mathrm{d}y$$

we apply the uniform continuity of u on a compact set.

Step 4: Proof of Claim 4.
The claim is a consequence of the Theorem on p-mean continuity A.3.26 and properties of the mollifier ($\operatorname{dist}(K, \partial\Omega) > \varepsilon$). Indeed, using a suitable change of variables, Hölder's inequality and Fubini's Theorem

$$\begin{aligned}
\|u_\varepsilon - u\|^p_{L^p(K)} &= \int_K |u_\varepsilon(x) - u(x)|^p \,\mathrm{d}x \\
&= \int_K \left| \int_{B_\varepsilon(x)} \eta_\varepsilon(x-y)|u(y) - u(x)| \,\mathrm{d}y \right|^p \mathrm{d}x \\
&= \int_K \left| \int_{B_1(0)} \eta(z)|u(x-\varepsilon z) - u(x)| \,\mathrm{d}z \right|^p \mathrm{d}x \\
&\le C \int_K \left(\int_{B_1(0)} |u(x-\varepsilon z) - u(x)|^p \,\mathrm{d}z \right) \mathrm{d}x \\
&= C \int_{B_1(0)} \left(\int_K |u(x-\varepsilon z) - u(x)|^p \,\mathrm{d}x \right) \mathrm{d}z.
\end{aligned}$$

The inner integral over K converges for $\varepsilon \to 0_+$ to zero due to Theorem on p-mean continuity A.3.26 (uniformly with respect to $z \in B_1(0)$) and the claim is proved.
Step 5: Proof of Claim 5.
Due to the Theorem on convolution A.3.31 we have for arbitrary $p \in [1, \infty]$

$$\|u_\varepsilon\|_p \le \|u\|_p \|\eta_\varepsilon\|_1 = \|u\|_p.$$

Now let $p \in [1, \infty)$ and $u \in L^p(\mathbb{R}^d)$. Choose an arbitrary number $\varrho > 0$. Then there exists $R > 0$ such that

$$\int_{\mathbb{R}^d \setminus B_R(0)} |u|^p \,\mathrm{d}x < \left(\frac{\varrho}{4}\right)^p.$$

By virtue of the Theorem on convolution A.3.31 applied on u_ε extended by zero to $B_{R+1}(0)$ we get for $\varepsilon < 1$

$$\int_{\mathbb{R}^d \setminus B_{R+1}(0)} |u_\varepsilon|^p \,\mathrm{d}x < \left(\frac{\varrho}{4}\right)^p.$$

Finally, as in Claim 4., it is possible to show that there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$ it holds

$$\int_{B_{R+1}(0)} |u - u_\varepsilon|^p \,\mathrm{d}x < \left(\frac{\varrho}{2}\right)^p.$$

Combining the inequalities above we achieve

$$\begin{aligned}
\|u - u_\varepsilon\|_p &\le \|u - u_\varepsilon\|_{L^p(\mathbb{R}^d \setminus B_{R+1}(0))} + \|u - u_\varepsilon\|_{L^p(B_{R+1}(0))} \\
&\le \|u\|_{L^p(\mathbb{R}^d \setminus B_R(0))} + \|u_\varepsilon\|_{L^p(\mathbb{R}^d \setminus B_{R+1}(0))} + \|u - u_\varepsilon\|_{L^p(B_{R+1}(0))} < \varrho.
\end{aligned}$$

■

The previous Theorem on mollification A.3.33 implies the following. If $u \in L^p(\Omega)$, $p \in [1, \infty)$, then $u_\varepsilon \to u$ in $L^p(\Omega)$. It is enough to extend u by zero outside of Ω and apply Claim 5. from the theorem above.

Note that Claim 5. from Theorem A.3.33 cannot hold for $p = \infty$, since the convergence in the $L^\infty(\Omega)$-norm is for smooth functions the uniform convergence. A weaker claim holds true, namely that

$$u_\varepsilon \overset{*}{\rightharpoonup} u \text{ in } L^\infty(\Omega)$$

(i.e., $\forall \varphi \in L^1(\Omega)$: $\int_\Omega u_\varepsilon \varphi \,\mathrm{d}x \to \int_\Omega u\varphi \,\mathrm{d}x$, see Subsections A.3.6 and A.3.7 below devoted to representation of continuous functionals and to different kinds of convergences).

By virtue of the Bernstein Theorem on approximations of continuous functions by polynomials in the $\mathcal{C}^0(\Omega)$-norm (and therefore by polynomials with rational coefficients in the $L^p(\Omega)$-norm, $p \in [1, \infty)$) we get as a consequence the first part of the following theorem.

Theorem A.3.34 — Separability of $L^p(\Omega)$. The space $L^p(\Omega)$ is for $p \in [1, \infty)$ separable. The space $L^\infty(\Omega)$ is not separable.

Proof. See also (Kufner et al., 1977, Theorem 2.6.1) or (Pick et al., 2013, Theorem 3.6.1 and 3.10.4). ■

Exercise A.3.35. Show, using the characteristic functions of intervals, that the space $L^\infty(\Omega)$ cannot be separable.

A.3.6 Continuous linear functionals on $L^p(\Omega)$

Theorem A.3.36 — Representation of continuous bounded functionals on $L^p(\Omega)$. Let Ω be a bounded open set. Let ϕ be a continuous bounded functional on $L^p(\Omega)$, $p \in [1,\infty)$. Then there exists a unique function $g \in L^{p'}(\Omega)$, $\frac{1}{p} + \frac{1}{p'} = 1$ ($p' = \infty$ for $p = 1$) such that it holds

$$\forall f \in L^p(\Omega)\colon\ \phi(f) = \langle \phi, f \rangle_{L^p(\Omega)} = \int_\Omega f g \, \mathrm{d}x. \tag{A.12}$$

Furthermore $\|\phi\|_{(L^p(\Omega))^*} = \|g\|_{L^{p'}(\Omega)}$.

Proof. The proof can be found in (Kufner et al., 1977, Theorems 2.9.5 and 2.11.8) or (Pick et al., 2013, Theorems 3.8.3 and 3.10.11). ■

The following result is based on the previous theorem.

Theorem A.3.37 — Reflexivity of $L^p(\Omega)$. Let Ω be a bounded open set. Let $p \in (1,\infty)$. Then the space $L^p(\Omega)$ is reflexive. The spaces $L^\infty(\Omega)$ and $L^1(\Omega)$ are not reflexive.

Proof. The proofs can be found in (Kufner et al., 1977, Theorems 2.10.1, 2.11.10 and 2.11.11) or (Pick et al., 2013, Theorems 3.9.1, 3.10.13 and 3.10.14). ■

A.3.7 Different types of convergences, relatively compact sets in $L^p(\Omega)$

First, let us have a look at different possibilities in which sense a sequence of functions $\{f_n\}_{n=1}^\infty$ can converge to a limit function f and what are the relations among the different types of convergences.

Definition A.3.38 — Types of convergences. Let $\{f_n\}_{n=1}^\infty$ be a sequence of measurable functions and f be measurable in Ω. Let Ω be open.

1. We say that $\{f_n\}_{n=1}^\infty$ converges to f pointwisely in Ω, if for any $x \in \Omega$ we have $\lim_{n\to\infty} f_n(x) = f(x)$, i.e.,
$$\forall x \in \Omega\, \forall \varepsilon > 0\, \exists n_0 \in \mathbb{N}\, \forall n \geq n_0\colon |f_n(x) - f(x)| \leq \varepsilon.$$
2. We say that $\{f_n\}_{n=1}^\infty$ converges to f uniformly in Ω, if
$$\forall \varepsilon > 0\, \exists n_0 \in \mathbb{N}\, \forall x \in \Omega\, \forall n \geq n_0\colon |f_n(x) - f(x)| \leq \varepsilon.$$
3. We say that $\{f_n\}_{n=1}^\infty$ converges to f locally uniformly in Ω, if $\forall K \subset \Omega$, where K is compact, the sequence $\{f_n\}_{n=1}^\infty$ converges uniformly to f in K.
4. We say that $\{f_n\}_{n=1}^\infty$ converges to f uniformly up to small sets in Ω, if
$$\forall \varepsilon > 0\, \exists M \subset \Omega\, |M| < \varepsilon\colon \{f_n\}_{n=1}^\infty \text{ converges to } f \text{ uniformly in } \Omega \setminus M.$$
5. We say that $\{f_n\}_{n=1}^\infty$ converges to f almost everywhere in Ω, if
$$\exists M \subset \Omega,\ |M| = 0\colon\ \{f_n\}_{n=1}^\infty \text{ converges to } f \text{ pointwisely in } \Omega \setminus M.$$
6. We say that $\{f_n\}_{n=1}^\infty$ converges to f in measure in Ω, if
$$\forall \varepsilon > 0\colon\ \lim_{n\to\infty} |\{x \in \Omega \,|\, |f_n(x) - f(x)| \geq \varepsilon\}| = 0.$$
7. We say that $\{f_n\}_{n=1}^\infty$ converges to f in $L^p(\Omega)$, $p \in [1,\infty]$, if
$$\lim_{n\to\infty} \|f_n - f\|_{L^p(\Omega)} = 0.$$
8. Let $p \in [1,\infty)$ and $p' = \frac{p}{p-1}$, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$ (with the standard convention $p' = \infty$ for $p = 1$). We say that $\{f_n\}_{n=1}^\infty$ converges to f weakly in $L^p(\Omega)$, we write $f_n \rightharpoonup f$, if
$$\forall g \in L^{p'}(\Omega)\colon \lim_{n\to\infty} \int_\Omega f_n g \, \mathrm{d}x = \int_\Omega f g \, \mathrm{d}x.$$

9. We say that $\{f_n\}_{n=1}^{\infty}$ converges to f weakly star (weakly*) in $L^\infty(\Omega)$, we write $f_n \overset{*}{\rightharpoonup} f$, if

$$\forall g \in L^1(\Omega)\colon \lim_{n\to\infty} \int_\Omega f_n g \,\mathrm{d}x = \int_\Omega f g \,\mathrm{d}x.$$

Some of the relations among the convergences are depicted in Figure A.1. These relations can be easily shown from the definitions, using also the theorems on limit passages presented at the beginning of this appendix.

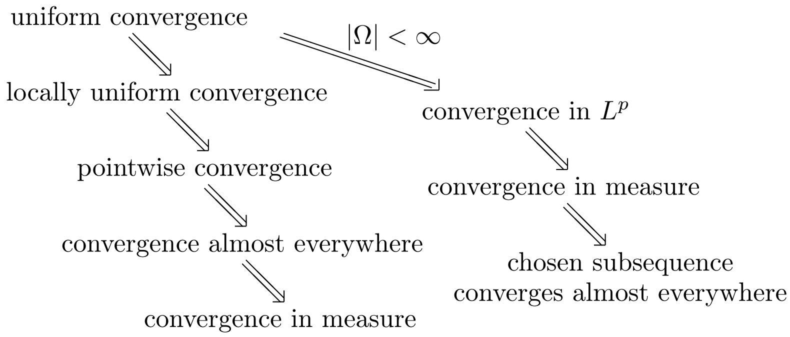

Figure A.1: Some relations among the convergences.

Remark A.3.39. For $p \in [1,\infty)$ the above stated weak convergences coincide with the standard Definition of weak convergence B.2.5 due to Theorem A.3.36. For $p = \infty$ Theorem A.3.36 says that $(L^1(\Omega))^* = L^\infty(\Omega)$ and therefore the weak star convergence defined above corresponds to Definition B.2.5. Even though it is also possible to introduce the weak convergence in $L^\infty(\Omega)$, we do not present it, as Theorem A.3.36 does not say anything about the space[1] $(L^\infty(\Omega))^*$. On the other hand, we do not introduce the weak star convergence on $L^1(\Omega)$, since there is no Banach space X such that it holds $X^* = L^1(\Omega)$.

Furthermore, in $L^p(\Omega)$, $p \in (1,\infty)$, we have at our disposal both weak and weak star convergences, but for reflexive spaces (which is the case for $L^p(\Omega)$ with $p \in (1,\infty)$ according to Theorem A.3.37) both convergences coincide, and there is no reason to introduce the weak star convergence in this situation.

Let us finally have a look at the characterization of totally bounded sets in $L^p(\Omega)$, $p \in [1,\infty)$. It is an analogy of the Arzelà–Ascoli Theorem A.2.8, the role of uniform continuity will be played by the p-mean continuity. Due to Theorem B.2.10 the total boundedness for the $L^p(\Omega)$ spaces is equivalent with the relative compactness; therefore the following theorem gives us a characterization of subsets of the $L^p(\Omega)$ spaces which have the property that any sequence from these subsets contains a subsequence which converges in the $L^p(\Omega)$-norm.

Theorem A.3.40 — Kolmogorov. Let $p \in [1,\infty)$. Denote $r_{\mathbf{h}} f(x) := f(x+\mathbf{h})$ (we extend f by zero outside of the set Ω). The set $A \subset L^p(\Omega)$ is totally bounded if and only if:

1. it is bounded, i.e.,
$$\sup_{f\in A} \|f\|_{L^p(\Omega)} \le C < \infty$$
2. it is equally p-mean continuous, i.e.,
$$\forall \varepsilon > 0\, \exists \delta > 0\, \forall f \in A\colon |\mathbf{h}| < \delta \implies \|r_{\mathbf{h}} f - f\|_{L^p(\Omega)} < \varepsilon,$$
3. it equally uniformly decays at infinity, i.e.,
$$\forall \varepsilon > 0\, \exists R > 0\, \forall f \in A\colon \|f\|_{L^p(\Omega\setminus B_R(0))} < \varepsilon.$$

Proof. **Step 1:** Proof of implication "⇒"
Step 1a: boundedness
This property directly follows from the total boundedness (perform the proof in more details!).

Step 1b: equal p-mean continuity
Let $\{f_i\}_{i=1}^N$ be the $\frac{\varepsilon}{3}$–net in A. Clearly, as the net contains finite number of functions, the set $\{f_i\}_{i=1}^N$ is equally p-mean continuous (each function is p-mean continuous due to Theorem A.3.26). Then

$$\|r_{\mathbf{h}} f - f\|_{L^p(\Omega)} \le \|r_{\mathbf{h}} f - r_{\mathbf{h}} f_i\|_{L^p(\Omega)} + \|r_{\mathbf{h}} f_i - f_i\|_{L^p(\Omega)} + \|f_i - f\|_{L^p(\Omega)} < \varepsilon$$

for suitably chosen $i \in \{1,\dots,N\}$ from the property of the net (recall that we have $\|f_i - f\|_{L^p(\Omega)} < \frac{\varepsilon}{3}$ as well as $\|r_{\mathbf{h}} f_i - r_{\mathbf{h}} f\|_{L^p(\Omega)} < \frac{\varepsilon}{3}$) and for a suitably chosen δ, $|\mathbf{h}| < \delta$ (the equal p-mean continuity of the net implies $\|r_{\mathbf{h}} f_i - f_i\|_{L^p(\Omega)} < \frac{\varepsilon}{3}$).

[1] The space $(L^\infty(\Omega))^*$ is well defined, however, its characterization goes far beyond the scope of these Lecture Notes.

Step 1c: equal uniform decay at infinity
Again, let $\{f_i\}_{i=1}^N$ denote the $\frac{\varepsilon}{2}$–net in A . Since for each i we have $f_i \subset L^p(\Omega)$ and the net contains finite number of functions, it holds

$$\forall \varepsilon > 0\, \exists R > 0\, \forall i \in \{1, \dots, N\}: \quad \|f_i\|_{L^p(\Omega \setminus B_R(0))} < \frac{\varepsilon}{2};$$

this is a direct consequence of the Lebesgue dominated convergence Theorem A.3.4. Then for arbitrary $\varepsilon > 0$ there exists $i \in \{1, \dots, N\}$ and $R > 0$ such that

$$\|f\|_{L^p(\Omega \setminus B_R(0))} \le \|f - f_i\|_{L^p(\Omega \setminus B_R(0))} + \|f_i\|_{L^p(\Omega \setminus B_R(0))} < \varepsilon.$$

Step 2: Proof of implication "$\Leftarrow$"
The proof is slightly longer, so we summarize the main ideas. We first show that it is enough to construct the net on any open bounded set $M \subset \mathbb{R}^d$ (recall that outside of Ω, the functions are extended by zero). In particular, the set $\overline{M}$ is compact.

We next mollify the functions from the subset A and obtain the set A_h of continuous functions in $\overline{M}$. We verify that this set satisfies the equivalent characterization of total boundedness in $\mathcal{C}(\overline{M})$ from the Arzelà–Ascoli Theorem A.2.8 and therefore there exists a finite ε-net of A_h in $\mathcal{C}(\overline{M})$.

The last step then contains the proof of the fact that the functions from A which after mollification form the finite ε-net of A_h in $\mathcal{C}(\overline{M})$, form in fact the finite $\widetilde{\varepsilon}$-net of A in $L^p(\Omega)$. Since the functions can be extended by zero outside of Ω, we can always assume that Ω is open.

Step 2a: it suffices to consider bounded sets
Indeed, let $\{f_i\}_{i=1}^N$ be an $\frac{\varepsilon}{2}$-net in $L^p(\Omega \cap B_R(0))$ for R chosen so that

$$\forall f \in A: \quad \|f\|_{L^p(\Omega \setminus B_R(0))} < \frac{\varepsilon}{4}.$$

This is indeed possible, since we assume the equal uniform decay at infinity. Now it is easy to verify that $\{f_i\}_{i=1}^N$ form an ε-net in $L^p(\Omega)$, as we have for any $f \in A$

$$\|f - f_i\|_{L^p(\Omega)} \le \|f - f_i\|_{L^p(\Omega \cap B_R(0))} + \|f\|_{L^p(\Omega \setminus B_R(0))} + \|f_i\|_{L^p(\Omega \setminus B_R(0))} < \varepsilon$$

for suitably chosen $i \in \{1, \dots, N\}$.

Therefore, in what follows, we can assume that Ω is a bounded open set.

Step 2b: mollification, application of the Arzelà–Ascoli Theorem A.2.8
Denote $A_h = \{f_h \,|\, f \in A\}$, where $f_h = \eta_h \star f$ is the mollification of f in $\mathbb{R}^d$. Then we have that $A_h \subset \mathcal{C}(\overline{\Omega})$. Moreover, A_h is totally bounded in $\mathcal{C}(\overline{\Omega})$, since the assumptions of Theorem A.2.8 are satisfied; in particular, the uniform boundedness

$$|f_h(x)| \le \left| \int_\Omega \eta_h(x-y) f(y) \,\mathrm{d}y \right| \le C(h)\, \|f\|_{L^p(\Omega)} \le C(h)$$

and the equal uniform continuity

$$\begin{aligned} |f_h(x+\mathbf{z}) - f_h(x)| &\le \int_\Omega \left| \frac{1}{|h|^d} \left(\eta\left(\frac{x+\mathbf{z}-y}{h}\right) - \eta\left(\frac{x-y}{h}\right)\right) f(y) \right| \mathrm{d}y \\ &\le \|f\|_{L^p(\Omega)} \frac{1}{h^d} \left(\int_\Omega \left| \left(\eta\left(\frac{x+\mathbf{z}-y}{h}\right) - \eta\left(\frac{x-y}{h}\right)\right)\right|^{p'} \mathrm{d}y \right)^{\frac{1}{p'}} \\ &\le C(h)|\mathbf{z}|\, \|f\|_{L^p(\Omega)} \le C(h)|\mathbf{z}|. \end{aligned}$$

To any $\varepsilon > 0$ there exists a finite $\frac{\varepsilon}{2|\Omega|^{\frac{1}{p}}}$-net of the set A_h in $\mathcal{C}(\overline{\Omega})$. Denote $(f_i)_h$, $i = 1, \dots, N$, the elements of this net. Hence, for arbitrary $f_h \in A_h$ there exists $j \in \{1, \dots, N\}$ such that

$$\|f_h - (f_j)_h\|_{\mathcal{C}(\overline{\Omega})} < \frac{\varepsilon}{2|\Omega|^{\frac{1}{p}}}. \tag{A.13}$$

Step 2c: ε-net of the set A in $L^p(\Omega)$
We show that $\{f_i\}_{i=1}^N$, where f_i are the original functions corresponding to $(f_i)_h$ from the net A_h in $\mathcal{C}(\overline{\Omega})$, form the ε–net of A in $L^p(\Omega)$.

Choose $\delta > 0$ so that

$$\forall f \in A\, \forall \mathbf{z} \in \mathbb{R}^d\, |\mathbf{z}| < \delta: \quad \left(\int_\Omega |f(x+\mathbf{z}) - f(x)|^p \,\mathrm{d}x \right)^{\frac{1}{p}} < \frac{\varepsilon}{4} \frac{1}{\widetilde{C}}, \tag{A.14}$$

where $\widetilde{C}$ is a constant specified below. The existence of $\delta > 0$ ensuring that the inequality above holds true is evident from the assumption on the equal uniform p-mean continuity of the set A.

Then

$$\|f_j - f\|_{L^p(\Omega)} \le \|f_j - (f_j)_h\|_{L^p(\Omega)} + \|(f_j)_h - f_h\|_{L^p(\Omega)} + \|f_h - f\|_{L^p(\Omega)}.$$

The first term on the right-hand side is smaller than $\frac{\varepsilon}{4}$, as by virtue of Hölder's inequality, Fubini Theorem and properties of η we get

$$\begin{aligned}\|f_j - (f_j)_h\|_{L^p(\Omega)} &= \left(\int_\Omega \left|\int_{B_h(x)} (f_j(x) - f_j(y))\,\eta_h(x-y)\,\mathrm{d}y\right|^p \mathrm{d}x\right)^{\frac{1}{p}} \\ &= \left(\int_\Omega \left|\int_{B_1(0)} (f_j(x) - f_j(x+hz))\,\eta(z)\,\mathrm{d}z\right|^p \mathrm{d}x\right)^{\frac{1}{p}} \\ &\le C(\eta)\left(\int_{B_1(0)}\int_\Omega |f_j(x+hz) - f_j(x)|^p\,\mathrm{d}x\,\mathrm{d}z\right)^{\frac{1}{p}} < \frac{\varepsilon}{4};\end{aligned}$$

this follows from (A.14) with the choice $\widetilde{C} = C(\eta)|B_1(0)|^{\frac{1}{p}}$. Similarly we also estimate the last term on the right-hand side. Finally, the second term on the right-hand side is smaller than $\frac{\varepsilon}{2}$ which follows from (A.13). Altogether,

$$\|f_j - f\|_{L^p(\Omega)} < \varepsilon,$$

and $\{f_i\}_{i=1}^N$ is therefore the ε-net of A in $L^p(\Omega)$. ■

A.3.8 Nemytskii operator and weak lower semicontinuity

This last subsection recalls two fundamental results which form basis to build the theory for nonlinear partial differental equations. We presents these results with their proofs since they are not included in the standard courses on measure theory and on theory of integral. We will be interested in behaviour of the composite function $f(x, u(x))$. If f and u are only measurable functions of its variables, the composite function $f(x, u(x))$ may not be measurable. The next definition contains a sufficient condition to ensure the measurability for such a mapping.

Definition A.3.41 — Carathéodory function. Let $\Omega \subset \mathbb{R}^d$ be measurable and $N \in \mathbb{N}$. We say that the function $f\colon \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function, if:

1. the function $f(x, \cdot)\colon \mathbb{R}^N \to \mathbb{R}$ is continuous for almost every $x \in \Omega$
2. the function $f(\cdot, \vec{v}) : \Omega \to \mathbb{R}$ is measurable for all $\vec{v} \in \mathbb{R}^N$.

We are now ready to introduce the basic tool needed in the theory of nonlinear partial differential equations.

Theorem A.3.42 — Nemytskii operator. Let $\Omega \subset \mathbb{R}^d$ be measurable and $f\colon \Omega \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function. For $\vec{u} = (u_1, \dots, u_N)\colon \Omega \to \mathbb{R}^N$ we define the Nemytskii operator

$$[\mathcal{N}(\vec{u})](x) := f(x, \vec{u}(x)).$$

Then the function $\mathcal{N}(\vec{u})$ is measurable provided the function $\vec{u}$ is measurable.

Moreover, let for any $i = 1, \dots, N$ there exist $p_i \in [1, \infty)$ and let there exist $p \in [1, \infty)$, $g \in L^p(\Omega)$ and a constant $C \ge 0$ such that for almost every $x \in \Omega$ and all $\vec{v} := (v_1, \dots, v_N) \in \mathbb{R}^N$ it holds

$$|f(x, \vec{v})| \le g(x) + C\sum_{i=1}^N |v_i|^{\frac{p_i}{p}}. \tag{A.15}$$

Then $\mathcal{N}\colon \vec{u} \mapsto \mathcal{N}(\vec{u})$ is a continuous operator from $L^{p_1}(\Omega) \times \dots \times L^{p_N}(\Omega)$ to $L^p(\Omega)$.

Proof. **Step 1:** Measurability
Since $\vec{u}$ is measurable, there exists a sequence of simple functions $\{\vec{u}^n\}_{n=1}^\infty$ such that $\vec{u}^n \to \vec{u}$ almost everywhere in Ω. We define functions $f_n(x) := f(x, \vec{u}^n(x))$. Then $f_n \to \mathcal{N}(\vec{u})$ almost everywhere in Ω, due to Property 1. for the Carathéodory function f. Moreover, f_n are measurable which can be easily checked applying Property 2. from Definition A.3.41 on the level set of the functions $\vec{u}^n$. Consequently, the function $\mathcal{N}(\vec{u})$ is measurable.

Step 2: Boundedness
If $\vec{u} \in L^{p_1}(\Omega) \times \dots \times L^{p_N}(\Omega)$, then $\|\mathcal{N}(\vec{u})\|_{L^p(\Omega)}$ is finite due to the Minkowski inequality A.3.10 and the growth assumption (A.15), since

$$\|\mathcal{N}(\vec{u})\|_{L^p(\Omega)} \le \left\| g + C\sum_{i=1}^N |u_i|^{\frac{p_i}{p}} \right\|_{L^p(\Omega)} \le \|g\|_{L^p(\Omega)} + C\sum_{i=1}^N \|u_i\|_{L^{p_i}(\Omega)}^{\frac{p_i}{p}} < \infty. \tag{A.16}$$

We also see that $\mathcal{N}$ maps bounded sets in $L^{p_1}(\Omega) \times \cdots \times L^{p_N}(\Omega)$ to bounded sets in $L^p(\Omega)$.

Step 3: Continuity

It remains to show the continuity of $\mathcal{N}$. Let $\vec{u}^n \to \vec{u}$ in $L^{p_1}(\Omega) \times \cdots \times L^{p_N}(\Omega)$. Switching to subsequences we may ensure that $\vec{u}^n \to \vec{u}$ almost everywhere in Ω and

$$\|u_i^n - u_i\|_{L^{p_i}(\Omega)} \leq 2^{-n} \qquad \text{for all } n \in \mathbb{N} \text{ and } i \in \{1, \dots, N\}.$$

Then the functions

$$v_i := |u_i| + \sum_{n=1}^{\infty} |u_i^n - u_i| \qquad i \in \{1, \dots, N\}$$

control the members of the sequence $\{\vec{u}^n\}$ in $L^{p_i}(\Omega)$. Assumption (A.15) and estimate (A.16) yield that

$$g + C \sum_{i=1}^{N} |v_i|^{\frac{p_i}{p}} \in L^p(\Omega)$$

controls the sequence $\{f(\cdot, \vec{u}^n)\}$ from above. Further due to (A.16)

$$\begin{aligned} |f(x, \vec{u}^n(x)) - f(x, \vec{u}(x))|^p &\leq 2^p |f(x, \vec{u}^n(x))|^p + 2^p |f(x, \vec{u}(x))|^p \\ &\leq 2^p \left(g(x) + C \sum_{i=1}^{N} |v_i(x)|^{\frac{p_i}{p}} \right)^p + 2^p |f(x, \vec{u}(x))|^p \in L^1(\Omega); \end{aligned}$$

in other words we constructed an integrable majorant. Since $f(x, \vec{u}^n(x)) \to f(x, \vec{u}(x))$ for almost every $x \in \Omega$ (it follows due to the almost everywhere convergence $\vec{u}^n \to \vec{u}$ and the continuity of f in the second variable), we can use the Lebesgue dominated convergence Theorem A.3.4 and get

$$\int_\Omega |f(x, \vec{u}^n(x)) - f(x, \vec{u}(x))|^p \, \mathrm{d}x \overset{n \to \infty}{\to} 0.$$

It is easy to see (by contradiction argument) that the result holds for the whole sequence. ■

We finally formulate the key property of the Carathéodory functions which are additionally convex in the last variable.

Theorem A.3.43 — Weak lower semicontinuity. Let $\Omega \subset \mathbb{R}^d$ be measurable, $f\colon \Omega \times (\mathbb{R}^N \times \mathbb{R}^M) \to \mathbb{R}$ a Carathéodory function. Moreover, assume that:

1. f is bounded from below by an integrable minorant, i.e., there exists $g \in L^1(\Omega)$ such that for all $(\vec{u}, \vec{v}) \in \mathbb{R}^N \times \mathbb{R}^M$ and almost every $x \in \Omega$ it holds
$$f(x, \vec{u}, \vec{v}) \geq -g$$
2. f is convex in the last variables, i.e, for each $\vec{u} \in \mathbb{R}^N$, each $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^M$, each $\alpha \in [0, 1]$ and almost every $x \in \Omega$ it holds
$$f(x, \vec{u}, \alpha \vec{v}_1 + (1 - \alpha)\vec{v}_2) \leq \alpha f(x, \vec{u}, \vec{v}_1) + (1 - \alpha) f(x, \vec{u}, \vec{v}_2).$$

Then for every sequence $\{\vec{u}^n\}_{n=1}^\infty = \{u_1^n, \dots, u_N^n\}_{n=1}^\infty$ which fulfils for each $i = 1, \dots, N$

$$u_i^n \to u_i \quad \text{strongly in } L^1(\Omega) \tag{A.17}$$

and every sequence $\{\vec{v}^n\}_{n=1}^\infty = \{v_1^n, \dots, v_M^n\}_{n=1}^\infty$ which fulfils for each $i = 1, \dots, M$

$$v_i^n \rightharpoonup v_i \quad \text{weakly in } L^1(\Omega) \tag{A.18}$$

it holds

$$\int_\Omega f(x, \vec{u}(x), \vec{v}(x)) \, \mathrm{d}x \leq \liminf_{n \to \infty} \int_\Omega f(x, \vec{u}^n(x), \vec{v}^n(x)) \, \mathrm{d}x. \tag{A.19}$$

Proof. Let (A.17)–(A.18) hold. We denote

$$L := \liminf_{n \to \infty} \int_\Omega f(x, \vec{u}^n(x), \vec{v}^n(x)) \, \mathrm{d}x.$$

Switching to a subsequence if necessary, we may achieve that

$$\int_\Omega f(x, \vec{u}^n(x), \vec{v}^n(x)) \, \mathrm{d}x \to L,$$

$$\begin{aligned} u_i^n &\to u_i \text{ in } L^1(\Omega) \quad \forall i = 1, \dots, N, \\ u_i^n &\to u_i \text{ a.e. in } \Omega \quad \forall i = 1, \dots, N. \end{aligned}$$

In what follows we consider only this subsequence. Moreover, since $g \in L^1(\Omega)$, switching to $f(x, \vec{u}, \vec{v}) + g(x)$ we may achieve that f is a non-negative Carathéodory function which is convex in the last variable. In what follows, we additionally assume that f is non-negative.

Step 1: Main idea of the proof

Assume for a moment that f does not depend on the second variable, i.e., $f(x, \vec{u}, \vec{v}) = f(x, \vec{v})$. The main idea is connected with application of the Mazur Theorem B.2.14. Due to it and due to (A.18) we can find subsequences $\{\vec{w}^n\}_{n=1}^{\infty} \subset L^1(\Omega)$, $\{k^n\}_{n=1}^{\infty}$, and numbers $a_k^n \in [0,1]$ such that

$$k^n \geq n, \quad \vec{w}^n := \sum_{k=n}^{k^n} a_k^n \vec{v}^k, \quad \sum_{k=n}^{k^n} a_k^n = 1,$$

and

$$\forall i = 1, \dots, M\colon\ w_i^n \to v_i \text{ strongly in } L^1(\Omega).$$

Finally, by virtue of the convexity of f in the last variable we obtain

$$\int_\Omega f(x, \vec{w}^n(x))\,\mathrm{d}x \leq \sum_{k=n}^{k^n} a_k^n \int_\Omega f(x, \vec{v}^k(x))\,\mathrm{d}x \leq \sup_{k \geq n} \int_\Omega f(x, \vec{v}^k(x))\,\mathrm{d}x.$$

Passing to the limit $n \to \infty$, where on the left-hand side we use the pointwise convergence (for a suitable subsequence), non-negativity of f and the Fatou Lemma A.3.6 we get

$$\begin{aligned}\int_\Omega f(x, \vec{v}(x))\,\mathrm{d}x &\leq \liminf_{n\to\infty} \int_\Omega f(x, \vec{w}^n(x))\,\mathrm{d}x \leq \liminf_{n\to\infty} \sup_{k\geq n} \int_\Omega f(x, \vec{v}^k(x))\,\mathrm{d}x \\ &\leq \limsup_{n\to\infty} \int_\Omega f(x, \vec{v}^n(x))\,\mathrm{d}x = L\end{aligned}$$

which is precisely inequality (A.19). It remains to explore the case when f also depends on $\vec{u}$.

Step 2: "ε–perturbation due to the dependence on $\vec{u}$"

Let $\widetilde{\Omega} \subset \Omega$ be an arbitrary bounded measurable set and $\varepsilon > 0$ be arbitrary. We define

$$\widetilde{\Omega}_{\varepsilon,n} = \{x \in \widetilde{\Omega} \,|\, |f(x, \vec{u}^n(x), \vec{v}^n(x)) - f(x, \vec{u}(x), \vec{v}^n(x))| \geq \varepsilon\}.$$

We show that

$$|\widetilde{\Omega}_{\varepsilon,n}| \overset{n\to\infty}{\to} 0. \tag{A.20}$$

For contradiction, assume that (A.20) does not hold. Switching to a subsequence and making $\varepsilon > 0$ smaller, if necessary, we get

$$|\widetilde{\Omega}_{\varepsilon,n}| \geq 4\varepsilon \qquad \text{for all } n \in \mathbb{N}. \tag{A.21}$$

Due to (A.18) there exists $C_1 > 0$ such that $\|\vec{v}^n\|_{L^1(\Omega)} \leq C_1$. If we set $C_\varepsilon := \frac{C_1}{\varepsilon}$ and define

$$\widetilde{\Omega}_n^{\vec{v}} := \{x \in \widetilde{\Omega} \,|\, |\vec{v}^n(x)| \geq C_\varepsilon\},$$

we have for all $n \in \mathbb{N}$ the estimate

$$|\widetilde{\Omega}_n^{\vec{v}}| \leq \frac{1}{C_\varepsilon} \int_{\widetilde{\Omega}} |\vec{v}^n(x)|\,\mathrm{d}x \leq \frac{C_1}{C_\varepsilon} = \varepsilon. \tag{A.22}$$

By virtue of the Egorov Theorem A.3.7 we can find $\widetilde{\Omega}^{\vec{u}}$ such that $\vec{u} \in \mathcal{C}(\overline{\Omega \setminus \widetilde{\Omega}^{\vec{u}}})$ and

$$\begin{aligned} &|\widetilde{\Omega}^{\vec{u}}| \leq \varepsilon \\ &\quad \vec{u}^n \to \vec{u} \text{ uniformly in } \widetilde{\Omega} \setminus \widetilde{\Omega}^{\vec{u}}. \end{aligned} \tag{A.23}$$

Therefore there exists $C_2 > 0$ such that for all $n \in \mathbb{N}$, $\|\vec{u}^n\|_{L^\infty(\widetilde{\Omega}\setminus\widetilde{\Omega}^{\vec{u}})} \leq C_2$. Finally, using the Luzin Theorem A.3.1 and the fact that f is a Carathéodory function, we can find $\widetilde{\Omega}^f$ such that

$$|\widetilde{\Omega}^f| \leq \varepsilon, \quad f \text{ is uniformly continuous in } (\widetilde{\Omega} \setminus \widetilde{\Omega}^f) \times B_{C_2}(0) \times B_{C_1}(0). \tag{A.24}$$

Combining (A.21)–(A.24) it is not difficult to see that

$$|\widetilde{\Omega}_{\varepsilon,n} \setminus (\widetilde{\Omega}^f \cup \widetilde{\Omega}^{\vec{u}} \cap \widetilde{\Omega}_n^{\vec{v}})| \geq \varepsilon \quad \text{ for any } n \in \mathbb{N}. \tag{A.25}$$

Finally, applying the uniform convergence (A.23) and the uniform continuity (A.24) we have

$$\|f(\cdot, \vec{u}^n, \vec{v}^n) - f(\cdot, \vec{u}, \vec{v}^n)\|_{L^\infty(\widetilde{\Omega}\setminus(\widetilde{\Omega}^f\cup\widetilde{\Omega}^{\vec{u}}\cap\widetilde{\Omega}_n^{\vec{v}}))} \overset{n\to\infty}{\to} 0.$$

By virtue of the definition of $\widetilde{\Omega}_{\varepsilon,n}$ we get a contradiction with (A.21). Hence, claim (A.20) is shown.

Step 3: Conclusion of the proof

Due to (A.20) we can choose a subsequence which we relabel and thus we may again assume that $|\widetilde{\Omega}_{\varepsilon,n}| < 2^{-n}\varepsilon$. We set $\widetilde{\Omega}_\varepsilon := \bigcup_{n=1}^\infty \widetilde{\Omega}_{\varepsilon,n}$. Then $|\widetilde{\Omega}_\varepsilon| < \varepsilon$ and for any $x \in \widetilde{\Omega} \setminus \Omega_\varepsilon$ and any $n \in \mathbb{N}$ it holds

$$|f(x, \vec{u}^{\,n}(x), \vec{v}^{\,n}(x)) - f(x, \vec{u}(x), \vec{v}^{\,n}(x))| < \varepsilon. \tag{A.26}$$

We now proceed as in Step 1. We first repeat the procedure to find the sequence $\{\vec{w}^{\,n}\}_{n=1}^\infty$ and applying the convexity of f in the last variable we get the estimate

$$\begin{aligned} f(x, \vec{u}(x), \vec{w}^{\,n}(x)) &\le \sum_{k=n}^{k^n} a_k^n f(x, \vec{u}(x), \vec{v}^{\,k}(x)) \\ &\le \sum_{k=n}^{k^n} a_k^n |f(x, \vec{u}(x), \vec{v}^{\,k}(x)) - f(x, \vec{u}^k(x), \vec{v}^{\,k}(x))| \\ &\quad + \sum_{k=n}^{k^n} a_k^n f(x, \vec{u}^{\,k}(x), \vec{v}^{\,k}(x)). \end{aligned}$$

Integrating over $\widetilde{\Omega} \setminus \Omega_\varepsilon$ and using (A.26) together with $\sum_{k=n}^{k^n} a_k^n = 1$ we get

$$\begin{aligned} \int_{\widetilde{\Omega}\setminus\Omega_\varepsilon} f(x, \vec{u}(x), \vec{w}^{\,n}(x))\,\mathrm{d}x &\le \varepsilon|\widetilde{\Omega} \setminus \Omega_\varepsilon| + \int_{\widetilde{\Omega}\setminus\Omega_\varepsilon} \sum_{k=n}^{k^n} a_k^n f(x, \vec{u}^{\,k}(x), \vec{v}^{\,k}(x))\,\mathrm{d}x \\ &\le \varepsilon|\widetilde{\Omega}| + \sup_{k\ge n} \int_\Omega f(x, \vec{u}^{\,k}(x), \vec{v}^{\,k}(x))\,\mathrm{d}x. \end{aligned}$$

We may now, exactly as in Step 1, pass to the limit on the right-hand side and apply the Fatou Lemma A.3.6 on the left-hand side to conclude

$$\int_{\widetilde{\Omega}\setminus\Omega_\varepsilon} f(x, \vec{u}(x), \vec{v}(x))\,\mathrm{d}x \le \varepsilon|\widetilde{\Omega}| + L.$$

We now pass with $\varepsilon \to 0_+$. We use the boundedness of $\widetilde{\Omega}$ on the right-hand side and the Lebesgue monotone convergence Theorem A.3.2 on the right-hand side to show (recall that $|\widetilde{\Omega}_\varepsilon| \le \varepsilon$)

$$\int_{\widetilde{\Omega}} f(x, \vec{u}(x), \vec{v}(x))\,\mathrm{d}x \le L.$$

Finally, let $\widetilde{\Omega}^n \subset \Omega$ be a sequence of bounded sets such that $\widetilde{\Omega}^n \nearrow \Omega$. Then passing to the limit in the inequality above (by virtue of the Lebesgue monotone convergence Theorem A.3.2) we get (A.19). ■

Appendix B

Several basic results from functional analysis

The results presented below summarize only several most important results needed in these Lecture Notes. Further details can be found e.g. in Edwards (1995), Yosida (1980), Dunford and Schwartz (1988) or in Czech in Lukeš (1998).

B.1 Banach and Hilbert spaces

Definition B.1.1 — Norm, normed space. A mapping $x \mapsto \|x\|_V$, where x belongs to a certain real (or complex) vector space V, is called a norm, if:

1. $\|x\|_V \geq 0$ for all $x \in V$, where $\|x\|_V = 0$, if and only if $x = 0$
2. $\|\lambda x\|_V = |\lambda|\|x\|_V$ for all $\lambda \in \mathbb{R}$ ($\lambda \in \mathbb{C}$) and all $x \in V$
3. $\|x + y\|_V \leq \|x\|_V + \|y\|_V$ for all x and $y \in V$.

A vector space V on which we define a norm is called a normed vector space.

Definition B.1.2 — Scalar product, unitary space. A mapping $(x, y) \mapsto (x, y)_H$, where x, y belong to a certain real (or complex) vector space H, is called a scalar product, if:

1. $(x, x)_H \geq 0$ for all $x \in H$, where $(x, x)_H = 0$, if and only if $x = 0$
2. $(x, y)_H = (y, x)_H$ (or $(x, y)_H = \overline{(y, x)_H}$) for all x, $y \in H$
3. $(\lambda x, y)_H = \lambda(x, y)_H$ for all x, $y \in H$, $\lambda \in \mathbb{R}$ ($\lambda \in \mathbb{C}$).

A vector space with a scalar product is called a unitary space.

Remark B.1.3. Recall that the unitary space is also normed and the norm can be taken as $\|u\|_H := \sqrt{(u, u)_H}$. We call this norm the associated norm to the scalar product.

Definition B.1.4 — Strong convergence (or convergence in norm). We say that a sequence $\{v_n\}_{n\in\mathbb{N}}$ of elements of a normed vector space V converges strongly (i.e., converges in norm) to an element $v \in V$, if

$$\lim_{n\to\infty} \|v_n - v\|_V = 0.$$

Definition B.1.5 — Cauchy sequence. We say that a sequence $\{v_n\}_{n\in\mathbb{N}}$ of elements of a normed vector space V is a Cauchy sequence, if for any $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for any m, $n \geq n_0$ it holds

$$\|v_n - v_m\|_V < \varepsilon.$$

While any strongly convergent sequence is a Cauchy sequence (which a kind reader can surely verify himself or herself), the opposite implication is in general not true.

Definition B.1.6 — Complete spaces, Banach and Hilbert spaces. A normed vector space V which fulfils that any Cauchy sequence of its elements has a strong limit in V is called complete. Complete normed spaces are called Banach spaces, complete unitary spaces (with respect to the norm associated to the scalar product) are called Hilbert spaces.

Definition B.1.7 — Separable space. A normed vector space V is called separable, if it contains a countable dense subset, i.e., a set $M = \{v_n\}_{n\in\mathbb{N}} \subset V$ such that for any $\varepsilon > 0$ and any $v \in V$ there exists an element $v_k \in M$ such that $\|v_k - v\|_V < \varepsilon$.

B.2 Dual spaces, weak convergence

Definition B.2.1 — Dual space. The set of all continuous linear functionals on the Banach space X is called a dual space and it is denoted as X^*.

The dual space is again a Banach space with respect to the norm

$$\|L\|_{X^*} = \sup_{v\in X;\|v\|_X\le 1} \langle L, v\rangle_X,$$

see, e.g., (Lukeš, 1998, Theorem 2.4) or (Yosida, 1980, Section III.6). The dual space to $L^p(\Omega)$, $1 \le p < \infty$, can be identified with the space $L^{p'}(\Omega)$, $\frac{1}{p} + \frac{1}{p'} = 1$, the dual space to $C(\overline{\Omega})$ with the space of Radon measures $\overline{\Omega}$. For the case of the complete normed space endowed with a scalar product it holds the following characterization of the dual space.

Theorem B.2.2 — Riesz representation. Let L be a continuous linear functional on the Hilbert space H. Then there exists a unique element $v \in H$ such that

$$\langle L, u\rangle_H = (v, u)_H.$$

Moreover, $\|L\|_{H^*} = \|v\|_H$.

Proof. The proof can be found, e.g., in (Lukeš, 1998, Theorem 2.9) or (Yosida, 1980, Section III.6). ■

For a Banach space X it is possible to define a dual space X^{**} to the space X^* (the second dual space to X). We can define the canonical mapping E: $X \to X^{**}$ as

$$Eu = u^{**}, \quad \text{where } \langle u^{**}, L\rangle_{X^*} = \langle L, u\rangle_X \quad \text{for all } L \in X^*.$$

Definition B.2.3 — Reflexive space. A Banach space X is called reflexive, if the canonical mapping satisfies $E(X) = X^{**}$.

While any Hilbert space is reflexive, for example on the scale of the Lebesgue space it holds only for $p \in (1, \infty)$. The following result is trivial.

Theorem B.2.4 — Properties of closed subspaces. Let X be a Banach space and Y its closed subspace. Then it holds.

1. If X is separable, then Y is separable.
2. If X is reflexive, then Y is reflexive.

Definition B.2.5 — Weak and weak star convergence. We say that a sequence $\{v_n\}_{n\in\mathbb{N}}$ in the Banach space X converges weakly to an element $v \in X$, we denote it $v_n \rightharpoonup v$, if

$$\lim_{n\to\infty} \langle L, v_n\rangle_X = \langle L, v\rangle_X$$

for any $L \in X^*$. We say that a sequence $\{L_n\}_{n\in\mathbb{N}} \subset X^*$ converges weakly* (weakly star) to an element $L \in X^*$, we denote it $L_n \overset{*}{\rightharpoonup} L$, if

$$\lim_{n\to\infty} \langle L_n, v\rangle_X = \langle L, v\rangle_X$$

for any $v \in X$.

Theorem B.2.6 — Alaoglu–Bourbaki–Banach. Let X be a Banach space. If there exists a separable Y such that $Y^* = X$, then from any bounded sequence in X it is possible to extract a weakly star convergent subsequence.

Proof. The proof can be found, e.g., in (Lukeš, 1998, Theorem 16.6). ■

Theorem B.2.7 — Eberlein–Smulyan. Let X be a Banach space. Then the following two claims are equivalent.

1. From any bounded sequence $\{v_n\}_{n\in\mathbb{N}} \subset X$ it is possible to extract a weakly convergent subsequence.

2. The space X is reflexive.

Proof. The proof can found, e.g., in (Lukeš, 1998, Theorem 16.9) or in (Yosida, 1980, Section V.4). ■

Definition B.2.8 — ε-net. Let X be a Banach space. We say that a nonempty set $A \subset X$ is an ε-net in X, if the system of open balls $\{B_\varepsilon(x) \mid x \in A\}$ covers X.

Definition B.2.9 — Totally bounded set. We say that a non-empty set $M \subset X$, X Banach space, is totally bounded, if for any $\varepsilon > 0$ there exists a finite ε-net, i.e.,

$$\forall \varepsilon > 0 \, \exists N \in \mathbb{N} \, \exists \, \{m_i\}_{i=1}^N \subset M \text{ so that } \bigcup_{i=1}^N m_i \text{ forms an } \varepsilon\text{-net.}$$

Theorem B.2.10 — Relation between relatively compact and totally bounded sets. Let X be a Banach space. Then its subset Y is relatively compact (i.e., its closure is a compact set), if and only if the set Y is totally bounded.

Proof. The proof can be found, e.g., in (Lukeš, 1998, Paragraph 24.4). ■

Definition B.2.11 — Continuous embedding. Let X, Y be Banach spaces. We say that X is continuously embedded into Y (we denote it as $X \hookrightarrow Y$), if:

1. $X \subset Y$

2. the identity I, as a mapping $I\colon X \to Y$, is a continuous mapping, i.e.,
   $\exists C > 0 \, \forall x \in X\colon \|x\|_Y \le C \|x\|_X$.

Definition B.2.12 — Compact operator. Let X, Y be Banach spaces. Let $A\colon X \to Y$ be an operator. This operator is called compact, if it maps bounded sets to relatively compact sets.

Therefore a compact operator maps bounded sequences in X to relatively compact ones in Y. It is therefore possible to choose from them in Y a convergent subsequence.

Definition B.2.13 — Compact embedding. Let X, Y be Banach spaces. We say that X is compactly embedded into Y (we denote it as $X \hookrightarrow\hookrightarrow Y$), if:

1. $X \hookrightarrow Y$

2. the identity I, as a mapping $I\colon X \to Y$, is compact (each bounded subset $B \subset X$ is relatively compact in Y).

Theorem B.2.14 — Mazur. Let X be a Banach space and $u_n \rightharpoonup u$ in X. Then there exists a sequence $\{k_n\} \subset \mathbb{N}$, $k_n \ge n$ for all $n \in \mathbb{N}$, and numbers $a_n^k \ge 0$, $n \in \mathbb{N}$, $k \in \{n, \dots, k_n\}$ such that for all $n \in \mathbb{N}$

$$\sum_{k=n}^{k_n} a_n^k = 1 \qquad \text{and} \qquad v_n := \sum_{k=n}^{k_n} a_n^k u_k \to u \text{ strongly in } X.$$

Proof. The proof can be found, e.g., in (Yosida, 1980, Section V.1) or in Brezis (1993). ■

The theorem directly implies the following characterization of weakly closed sets.

Theorem B.2.15 — Characterization of weakly closed sets. A convex set in a Banach space is weakly closed, if and only if it is (strongly) closed.

B.3 Spectrum of operators in Hilbert spaces, Fredholm alternative

Let us consider a linear mapping (operator) $A\colon H \to H$, where H is a real Hilbert space; so

$$A(\lambda u + \mu v) = \lambda A(u) + \mu A(v)$$

for any u, $v \in D(A)$ (the domain of A) and arbitrary λ, $\mu \in \mathbb{R}$. Let us further assume that A is bounded (and thus continuous), it means that there exists $C > 0$ such that

$$\|A(u)\|_H \leq C\|u\|_H$$

for all $u \in D(A)$.

Definition B.3.1 — Resolvent, spectrum. Let A: $H \to H$ be a bounded linear operator. Then the resolvent of A, denoted by $\rho(A)$, is the set of all $\eta \in \mathbb{R}$ such that $A - \eta I$, where I: $H \to H$ is the identity mapping, is injective and surjective. The spectrum of the operator A is the set $\sigma(A) = \mathbb{R} \setminus \rho(A)$.

Definition B.3.2 — Eigenvalue, eigenfunction, point spectrum. Let A: $H \to H$ be a bounded linear operator. The number $\lambda \in \sigma(A)$ is called an eigenvalue of the operator A, if the kernel of the operator $A - \lambda I$ is not trivial (it means that there exists a nontrivial $x_\lambda \in H$ such that $A(x_\lambda) = \lambda x_\lambda$). Such x_λ is called an eigenfunction of the operator A. The union of all eigenvalues of A is called the point spectrum of the operator A and we denote it $\sigma_p(A)$.

Definition B.3.3 — Adjoint operator. Let A: $H \to H$ be a linear operator. The adjoint operator A^*: $H \mapsto H$ is the operator which for all u, $v \in H$ fulfils

$$(Au, v)_H = (u, A^* v)_H.$$

Note that if the compact operator is additionally linear, it is also bounded (and thus continuous).

Theorem B.3.4 — Compactness of adjoint operator. Let the operator A: $H \to H$ be compact. Then also its adjoint operator A^*: $H \to H$ is compact.

Proof. The proof can be found, e.g., in (Evans, 1998, Appendix D Theorem 4). ■

Definition B.3.5 — Selfadjoint operator. The operator A: $H \to H$ is called selfadjoint, if $A = A^*$ (including the domains).

Theorem B.3.6 — Fredholm alternative. Let H be a Hilbert space and let $K: H \to H$ be a compact operator. Then it holds:

1. $N(I - K)$ is finite dimensional
2. $R(I - K)$ is closed
3. $R(I - K) = N(I - K^*)^\perp$
4. $N(I - K) = \{0\} \iff R(I - K) = H$
5. $\dim N(I - K) = \dim N(I - K^*)$
6. The spectrum of K is at most countable and contains 0. If the spectrum is infinite, then 0 is the only accumulation point.

Proof. The proof can be found, e.g., in (Evans, 1998, Appendix D Theorem 5). ■

Theorem B.3.7 — Spectrum of a compact operator. Let $\dim H = \infty$, A: $H \to H$ be a linear compact operator. Then:

1. $0 \in \sigma(A)$
2. $\sigma(A) \setminus \{0\} = \sigma_p(A) \setminus \{0\}$
3. Either $\sigma(A) \setminus \{0\}$ is a finite set or $\sigma(A) \setminus \{0\}$ is a sequence with the limit 0.

Proof. The proof can be found, e.g., in (Evans, 1998, Appendix D Theorem 6). ■

Theorem B.3.8 — Estimate of the spectrum. Let A: $H \to H$ be a linear, bounded, selfadjoint operator.

Denote

$$m = \inf_{\{u \in H \mid \|u\|_H = 1\}} (Au, u)_H, \qquad M = \sup_{\{u \in H \mid \|u\|_H = 1\}} (Au, u)_H.$$

Then $\sigma(A) \subset [m, M]$ and m, $M \in \sigma(A)$.

Proof. The proof can be found, e.g., in (Evans, 1998, Appendix D Lemma in D6). ■

Theorem B.3.9 — Spectrum of a compact selfadjoint operator. Let A: $H \to H$ be a linear, compact, selfadjoint operator and let H be a separable Hilbert space. Then there exists a countable orthonormal basis of H formed by eigenfunctions of the operator A.

Proof. The proof can be found, e.g., in (Evans, 1998, Appendix D Theorem 7). ■

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