Asymptotic results

Assume that

(A.1) $X_1, \ldots, X_n$ are i.i.d. random variables with $EX_i = 0$, $0 < var\, X_i = \sigma^2$ and $E|X_i|^\nu < \infty$ with some $\nu > 2$ (mostly $\nu = 4$).

Under this assumption the processes

$$V_{n,1}(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor (n+1)t \rfloor} X_i, \quad t \in (0,1),$$

converge in distribution to a Wiener process $\{W(t), t \in (0,1)\}$

and the processes

$$V_{n,2}(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor (n+1)t \rfloor} (X_i - \overline{X}_n), \quad t \in (0,1),$$

converge in distribution to the Brownian bridge $\{B(t), t \in (0,1)\}$.

Theorem 1 (Darling, Erdős , 1953) Under the above assumptions for all t

$$\lim_{n\to\infty} P\Big(a(\log n) \max_{1\le k<n} \sqrt{\frac{n}{k(n-k)}}\Big|\sum_{i=1}^{k}(X_i - \overline{X}_n)\Big| \tag{.1}$$

$$\le t + b_1(\log n)\Big) = \exp\{-2e^{-t}\}$$

$$a(t) = \sqrt{2\log t}, \quad \log t > 0, \tag{.2}$$

$$b_1(t) = 2\log t + \frac{1}{2}\log\log t - \frac{1}{2}\log(\pi), \quad \log t > 0. \tag{.3}$$

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Theorem 2 Under the above assumptions for all x

$$\lim_{n\to\infty} P\Big(\max_{1\le k<n}\Big(\frac{k}{n}\Big(1-\frac{k}{n}\Big)\Big)^{-\beta}\frac{1}{\sqrt{n}}\Big|\sum_{i=1}^{k}(X_i-\overline{X}_n)\Big|\le x\Big) \tag{.4}$$

$$= P\Big(\sup_{0<t<1}\frac{|B(t)|}{(t(1-t))^{\beta}}\le x\Big),$$

where $\beta \in \langle 0, 1/2)$, $\{B(t); t \in (0,1)\}$ is a Brownian bridge and

$$\lim_{n\to\infty} P\Big(\int_0^1 \frac{\left|\sum_{i=1}^{\lfloor (n+1)t\rfloor}(X_i-\overline{X}_n)/\sqrt{n}\right|}{(t(1-t))^\gamma}dt \le x\Big) \tag{.5}$$

$$= P\Big(\int_{0<t<1} \frac{|B(t)|}{(t(1-t))^\gamma}dt \le x\Big),$$

where $\beta \in \langle 0, 1/2)$, $\gamma < 3/2$, $\{B(t); t \in (0,1)\}$ is a Brownian bridge.

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Auxiliary results

Law of iterated logarithm

Hájek-Rényi-Chow inequality

Invariance principles:

- Komlós-Major-Tusnády (1975,1976)
- Einmahl (1987, 1989)
- Skorochod representation
- Strassen (1967)

Extreme value theorems

Hájek’s inequality for rank statistics (1961)

Law of iterated logarithm and its consequences

Let $X_1, \ldots, X_n, \ldots$ be i.i.d. random variables with zero mean, unit variance, then, as $n \to \infty$,

$$\lim \sup_{n\to\infty} \frac{1}{\sqrt{n \log \log n}} |\sum_{i=1}^{n} X_i| \leq C$$

with some $C > 0$.

Law of iterated logarithm and its consequences

Let $X_1, \ldots, X_n, \ldots$ be i.i.d. random variables with zero mean, unit variance, then, as $n \to \infty$,

$$\lim_{n\to\infty} \sup \frac{1}{\sqrt{n \log \log n}} \left| \sum_{i=1}^{n} X_i \right| \le C$$

with some $C > 0$.

Consequences:

$$\max_{1\le k<n} \frac{1}{\sqrt{k}} \left| \sum_{i=1}^{k} X_i \right| = O_P(\sqrt{\log \log n})$$

$$\max_{1\le k<n} \frac{1}{\sqrt{n-k}} \left| \sum_{i=k+1}^{n} X_i \right| = O_P(\sqrt{\log \log n})$$

and as $n_1 \to \infty$, as $n_2 - n_1 \to \infty$,

$$\max_{n_1 \le k < n_2} \frac{1}{\sqrt{k}} \left| \sum_{i=1}^{k} X_i \right| = O_P\Big(1 + \sqrt{\log\log(n_2/n_1)}\Big)$$

eventually, it is valid for triangular arrays

Then

$$P\Big(\max_{1\le k<n}\frac{1}{\sqrt{k}}|\sum_{i=1}^{k}X_i| = \max_{\log n\le k<n/\log n}\frac{1}{\sqrt{k}}|\sum_{i=1}^{k}X_i|\Big)\Bigg|$$

$$\to 1.$$

$$P\Big(\max_{1\le k<n}\frac{1}{\sqrt{(n-k)}}|\sum_{i=k+1}^{n}X_i|$$

$$=\max_{n-n/\log n\le k<n-\log n}\frac{1}{\sqrt{n-k}}|\sum_{i=k+1}^{n}X_i|\Big)\to 1.$$

Hájek-Rényi-Chow inequality

Chow, Teicher Probability Theory, 1988,p. 247 – 248.

(i) Let $\{S_n = \sum_{i=1}^n X_i, \mathcal{F}_{\setminus}, \setminus \geq \infty\}$ be a nonnenaga-tive submartingale, ES_n be finite, $v_n \geq v_{n+1} \geq 0$, then for any $A > 0$

$$P\Big(\max_{1\leq k\leq n} v_k S_k \geq A\Big) \leq A^{-1} \sum_{i=1}^n E v_i X_i,$$

(ii) Let $\{S_n = \sum_{i=1}^n X_i, \mathcal{F}_{\setminus}, \setminus \geq\}$ be martingale, let $v_n \geq v_{n+1} \geq 0$, then for any $A > 0$

$$P\Big(\max_{1\leq k\leq n} |v_k S_k| \geq A\Big) \leq A^{-2} \sum_{i=1}^n E v_i^2 X_i^2$$

(iii) Let $\{S_n = \sum_{i=1}^n X_i, \mathcal{F}_\backslash, \backslash \geq \infty\}$ be a submartingale, let h be a nonnegative, increasing convex function, then for any $A > 0$ and $t > 0$

$$P\Big(\max_{1\leq k\leq n} S_k \geq A\Big) \leq \frac{Eh(tS_n)}{h(tA)},$$

e.g. $h(t) = t^{1+p}$, $t > 0$, $p > 0$ or $h(t) = \exp\{t\}$, $t > 0$

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Remark

1. Notice that for $1 \leq m < n < \infty$:

$$\sum_{j=m}^{n} v_j a_j = \sum_{j=m}^{n} (v_j - v_{j+1}) a_j + v_m a_{m-1} + v_n a_n$$

2. If $X_1, \ldots, X_n$ satisfy (A.1), then $|\sum_{i=1}^{k} X_i|^\nu, k = 1, \ldots, n$ forms submartingale

$$\left(E(|\sum_{i=1}^{k+1} X_i|^\nu | X_1, \ldots, X_k) \geq |\sum_{i=1}^{k} X_i|^\nu, a.s. \right)$$

and, as $n \to \infty$,

$$E|\sum_{i=1}^{n} X_i|^\nu = O_P(n^{\nu/2}).$$

3. Under (A.1) for any $A > 0$

$$P\Big(\max_{m \le k < n} \frac{1}{\sqrt{k}} |\sum_{i=1}^{k} X_i| \ge A\Big)$$
$$\le CA^{-\nu}(1 + \log(n/m))$$

with some $C > 0$

Invariance principles

Komlós- Major-Tusnády type results(1975, 1976):

1. Let $X_1, \ldots, X_n, \ldots$ be i.i.d random variables with zero mean, unit variance and

$$E \exp\{tX_1\} < \infty$$

in a neighborhood of 0. The there exists i.i.d. $N(0,1)$ random variables $N_1, \ldots$ such that for all $n \geq 1$ and all $x \geq 0$

$$P(\max_{1 \leq k \leq n} |\sum_{i=1}^{k} (X_i - N_i)| > C_1 \log n + x) \leq C_2 \exp\{-C_3 x\}$$

with some positive C_1, C_2, C_3 depending only of d.f. of X_1.

2. Let $X_1, \ldots, X_n, \ldots$ be i.i.d random variables with zero mean, unit variance and

$$E\,|X_1|^\nu < \infty$$

with some $\nu > 2$. Then there exists i.i.d. $N(0,1)$ random variables $N_1, \ldots$ such that for all $n \geq 1$ and all $x \geq 0$

$$P(\max_{1\leq k\leq n} |\sum_{i=1}^{k}(X_i - N_i)| > x) \leq C_4 n x^{-\nu}$$

with some positive C_4 depending only of d.f. of X_1.

3. Define the proces $S(t) = S_{[t]}, t \geq 0, S(0) = 0$. Then under the assumptions in 1. there exists a Wiener process $\{W)(t), t \geq 0\}$ such that for all $T \geq 1$ and $x \geq 0$ we have

$$P(\max_{0 \leq t \leq T} |S(t) - W(t))| > C_5 \log T + x) \leq C_2 \exp\{-C_7 x\}$$

and analogously when only moment assumptions are considered.

Reference e.g. Weighted Approximations in probability and Statistics, Csörgő and Horváth (1993).

4. There is multivariate version, expected assertions holds true. Einmahl 1989, 1987, Strassen 1967

Consequences

(a) 2. implies that there exists Wiener process $\{W(t); t \geq 0\}$ such that, as $n \to \infty$,

$$\sum_{i=1}^{n} X_i - W(n) =^{a.s.} o(n^{1/\nu})$$

and

$$\max_{1 \leq k \leq n} k^{-1/\nu} \left| \sum_{i=1}^{k} X_i - W(k)) \right| = O_P(1).$$

(b) We can consider the triangular array $X_{1n}, \ldots, X_{nn}$ i.i.d. for each fixed n, $EX_{1n} = 0$, $var\, X_{1n} = 1$, $E|X_{1n}|^{\nu} \leq C$ with some $D > 0$. Then there exists a sequence of Wiener processes $\{W(t); t \geq 0\}_n$ such that, as $n \to \infty$,

$$\max_{1 \leq k \leq n} k^{-1/\eta} | \sum_{i=1}^{k} X_{in} - W_n(k))| = O_P(1).$$

for any $\eta < \nu$.

5.

Lemma. Let $X_{1,n}, \ldots, X_{n,n}$ be i.i.d. random variables for each $n = 1, 2, \ldots,$. Let $EX_{i,n} = 0$, $var X_{i,n} = 1$ and $E|X_{i,n}|^4 \leq D < \infty$ with some $D > 0$. Then we can define a sequence of Wiener processes $\{W_{n,1}(t), 0 \leq t < \infty\}_n$ and $\{W_{n,2}(t),\ 0 \leq t < \infty\}n$ such that $\{W_{n,1}(t),\ 0 \leq t < \infty\}$ and $\{W_{n,2}(t),\ 0 \leq t < \infty\}$ are independent for each n and

$$n^{r-\frac{1}{2}} \max_{1\leq k\leq n/2} \left| \sum_{1\leq i\leq k} X_{i,n} - W_{n,1}(k) \right| \qquad (.6)$$

$$\times \left(\frac{k}{n}\left(1 - \frac{k}{n}\right)\right)^{-\frac{1}{2}+r} = O_P(1),$$

$$n^{r-\frac{1}{2}} \max_{n/2<k\le n} \left| \sum_{k<i\le n} X_{i,n} - W_{n,2}(n-k) \right| \tag{.7}$$

$$\times \left(\frac{k}{n} \left(1 - \frac{k}{n} \right) \right)^{-\frac{1}{2}+r} = O_P(1)$$

for any $0 \le r < 1/4$,

$$\max_{1<k\le n/2} \left| \sum_{1\le i\le k} X_{i,n} - W_{n,1}(k) \right| = o_P(n^{-1/4+\epsilon}) \tag{.8}$$

and

$$\max_{n/2<k\le n}\left|\sum_{k\le i\le n} X_{i,n} - W_{n,2}(k)\right| = o_P(n^{1/4+\epsilon}) \quad \textbf{(.9)}$$

for any $\epsilon > 0$.

The proof based on the Skorokhod embedding scheme (cf. Breiman(1968)).

By the Skorokhod embedding scheme, there is $W_n^{(1)}(t); 0 \le t < \infty$ and i.i.d. random variables $\tau_{n,1}, \tau_{n,2}, \ldots$ such that

$$\sum_{1\le i\le k} X_{i,n} = W_n^{(1)}\left(\sum_{1\le i\le k} \tau_{n,i}\right), \; 1 \le k \le n/2, \tag{.10}$$

$$E\tau_{n,1} = 1, \quad E\tau_{n,1}^2 \le CEX_{i,n}^4 \tag{.11}$$

with some $C > 0$.

Then by the Hájek–Rényi-Chow inequality for any $\beta > 1/2$ and $x > 0$

$$P\left\{\max_{1\le k\le n/2}\frac{1}{k^\beta}\left|\sum_{1\le i\le k}(\tau_{n,i}-1)\right|\ge x\right\} \qquad (.12)$$

$$\le \frac{1}{x^2}E(\tau_{n,1}-1)^2\sum_{1\le k\le n/2}k^{-2\beta}\le Cx^{-2},$$

with some $C > 0$. Hence

$$\max_{1\le k\le n/2}\frac{1}{k^\beta}\left|\sum_{1\le i\le k}(\tau_{n,i}-1)\right| = O_P(1).$$

Note that the distribution of $W_n^{(1)}$ does not depend on n. Lemma 1.2.1 of Csörgő and Révész (1981) yields that

$$\max_{1\le k\le n/2}\ \sup_{|s|\le C} |W_n^{(1)}(k) - W_n^{(1)}(k+s)|/(k^{\beta/2}(\log k)^{1/2}) \tag{.13}$$

$$= O_P(1).$$

Now (.10) and (.13) yield that

$$\max_{1\le k\le n/2}\left|\sum_{1\le i\le k} X_{i,n} - W_n^{(1)}(k)\right|/k^{\frac{1}{2}-r}$$

$$= O_P\left(\max_{1\le k\le n/2} k^{\beta-(1/2-r)}(\log k)^{1/2}\right) = O_P(1),$$

since $\beta > 1/2$ can be as close to $1/2$ as we wish.
Q.E.D.

5. The following holds true: the following assertions are equivalent

$$\limsup_{t\to 0} W(t)/q(t) < \infty, a.s.$$

$$\limsup_{t\to 0} |W(t)/q(t)| < \infty, a.s.$$

$$\limsup_{t\to 0} W(t)/q(t) = \beta, a.s.$$

with some $0 \le \beta < \infty$. The last assertion holds true for

$$q(t) = (t(1-t))^{\gamma}$$

with $\gamma < 1/2$

Rank statistics

Consider partial sums of simple linear rank statistics. Let $U_{1n}, \ldots, U_{nn}$ be i.i.d. with uniformd.f. on $(0,1)$. Let $R_{1n}, \ldots, R_{nn}$ be corresponding ranks. Let $U_{(1n)}, \ldots, U_{(nn)}$ be the ordered sample. Let $a_n(1), \ldots, a_n(n)$ be a vector of scores such that

$$\frac{1}{n} \sum_{1 \le i \le n} (a_n(i) - \overline{a}_n)^2 = 1, \tag{.14}$$

$$\limsup_{n \to \infty} \frac{1}{n^2} \sum_{1 \le i \le n} a_n^4(i) < \infty. \tag{.15}$$

$$\overline{a}_n = \frac{1}{n} \sum_{1 \le i \le n} a_n(i).$$

Denote

$$S_{k,n}(\boldsymbol{R}) = \sum_{1\le i\le k} (a_n(i) - \overline{a}_n), \quad k = 1, \ldots, n. \quad (.16)$$

Notice that $S_{n,n}(\boldsymbol{R}) = 0$,

$$\{\frac{1}{n-k} S_{k,n}(\boldsymbol{R}), \sigma(R_{1n}, \ldots, R_{kn}), \ k = 1, 2, \ldots, n\}$$

$$\{\frac{1}{k}(S_{n,n}(\boldsymbol{R}) - S_{k,n}(\boldsymbol{R}))$$

$$, \sigma(R_{nn}, \ldots, R_{k+1,n}), \ k = n-1, n-2, \ldots, 1\}$$

form martingales.

Lemma

$$n^r \max_{2\le k\le n} \frac{k(n-k)}{n^{3/2}} \Big| \Big(\frac{1}{k}\sum_{1\le i\le k} a_n(R_i) - \frac{1}{n-k}\sum_{k<i\le n} a_n(R_i) \Big) \Big|$$

$$- \Big(\frac{1}{k}\sum_{1\le i\le k} Z_{i,n} - \frac{1}{n-k}\sum_{k<i\le n} Z_{i,n} \Big) \Big|$$

$$\times \Big(\frac{k}{n}\Big(1-\frac{k}{n}\Big)\Big)^{-\frac{1}{2}+r} = O_P(1), 0 \le r < 1/2$$

and

$$= O_P(\log n), r = 1/2.$$

where $Z_{i,n} = a_n(1 + [nU_i^{(n)}]) - \overline{a}_n.$

It is a special case of Theorem 3 of Hušková (1997). $Z_{1,n}, , \ldots , Z_{n,n}$ are i.i.d., we also have

$$\max_{1 \le k < n} \sqrt{\frac{k(n-k)}{n}} \Big| \sum_{1 \le i \le k} (a_n(R_i) - \overline{a}_n)$$
$$- \sum_{1 \le i \le k} (Z_{i,n} - \overline{Z}_n) \Big| = O_P(n^{-\eta})$$

with some $\eta > 0$.

Limiting extreme value distributions

Proposition 3.1. Consider a zero mean, unit variance, stationary Gaussian process $\{X(t) : t \geq 0\}$ with auto-correlation $r(\tau) = E\, X(t)\, X(t+\tau)$, $t,\ \tau \geq 0$. Assume that

$$\lim_{\tau\to\infty} r(\tau) \log \tau = 0,$$

$$r(\tau) = 1 - C\,\tau^{\alpha} + o(\tau^{\alpha}), \quad \text{as } \tau \downarrow 0.$$

Then

$$\lim_{T\to\infty} P\left(a_T \sup_{0\leq t\leq T} X(t) \leq x + b_T\right) = \exp(-e^{-x}), \tag{.17}$$

for all x, where

$$a_T = (2 \log T)^{1/2},$$

$$b_T = 2\log T + \frac{2-\alpha}{2\alpha}\log\log T$$
$$+\log\left(C^{1/\alpha}\, H_\alpha (2\pi)^{-1/2}\, 2^{(2-\alpha)/(2\alpha)}\right),$$

H_α as in Remark 12.2.10 of Leadbetter et al (1983),
$H_1 = 1,\ H_2 = 1/\sqrt{\pi}$.

Proof. Confer Leadbetter et al (1983), Theorem 12.3.5.

Proposition 3.2. Consider $\{X(t) : t \geq 0\}$ as in Proposition 3.1, and let $1 - r(\tau)$ be regularly varying of index 2, as $\tau \downarrow 0$. Define $v = v_T$ to be the largest solution of the equation

$$(2\log T)\left(1 - r\left(\frac{1}{v}\right)\right) = 1, \qquad (.18)$$

and set

$$u = u_T = \left(2\log\frac{Tv\sqrt{2\pi}}{\sqrt{2\log T}}\right)^{1/2}. \qquad (.19)$$

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Then,

$$\lim_{T\to\infty} P\left(u_T \sup_{0\le t\le T} X(t) \le x + u_T^2\right) = \exp\left(-e^{-x}/\sqrt{\pi}\right), \tag{.20}$$

for all x.

Proof. This is a special case of Berman (1992), Theorem 10.6.1. Note that, in (3.3.24) of Berman (1992), $\omega = 1/\sqrt{\pi}$.

Proof of Theorem 3.1. We first note that, as $n \to \infty$,

$$\max_{2 \le k \le \log n} \frac{\left| \sum_{i=1}^{k-1} (k-i)^{\alpha} Z_i \right|}{\left(\sum_{i=1}^{k-1} i^{2\alpha} \right)^{1/2}}$$

$$= \max_{2 \le k \le \log n} \frac{\left| \sum_{i=1}^{k-1} (i^{\alpha} - (i-1)^{\alpha}) \sum_{j=1}^{k-i} Z_j \right|}{\left(\sum_{i=1}^{k-1} i^{2\alpha} \right)^{1/2}}$$

$$= O_P\left(\max_{2 \le k \le \log n} \frac{k^{\alpha} (k \log \log k)^{1/2}}{k^{\alpha+1/2}} \right) = O_P\left((\log \log \log n)^{1/2} \right)$$

$$= o_P\left((\log \log n)^{1/2} \right),$$

which means that it suffices to investigate the max over $k \in [\log n, n]$ only.

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We now put $s = e^t$, and consider the process $\{X(t) : t \geq 0\}$ given by

$$X(t) = \frac{(2\alpha+1)^{1/2}\alpha \int_0^{e^t} x^{\alpha-1} W(e^t - x)\, dx}{e^{t(\alpha+1/2)}} \quad (.21)$$

$$= \frac{(2\alpha+1)^{1/2}\alpha \int_0^{e^t} (e^t - y)^{\alpha}\, dW(y)}{e^{t(\alpha+1/2)}}.$$

Note that $\{X(t) : t \geq 0\}$ is a zero mean Gaussian process with covariance function

$$cov(X(s), X(t)) = (2\alpha+1)\frac{\int_0^{e^s} (e^s - y)^{\alpha} (e^t - y)^{\alpha}\, dy}{e^{s(\alpha+1/2)}\, e^{t(\alpha+1/2)}}$$

$$= (2\alpha+1)\, e^{(s-t)/2} \int_0^1 (1-z)^{\alpha} (1 - z\, e^{s-t})^{\alpha}\, dz,$$

Clearly $\{X(t) : t \geq 0\}$ is stationary with unit vari

(3.7) is satisfied for any $\alpha > 0$.

Proof of Theorem 3.1. We first note that, as $n \to \infty$,

$$\max_{2\le k\le \log n} \frac{\left|\sum_{i=1}^{k-1}(k-i)^{\alpha} Z_i\right|}{\left(\sum_{i=1}^{k-1} i^{2\alpha}\right)^{1/2}}$$

$$= \max_{2\le k\le \log n} \frac{\left|\sum_{i=1}^{k-1}(i^{\alpha}-(i-1)^{\alpha}) \sum_{j=1}^{k-i} Z_j\right|}{\left(\sum_{i=1}^{k-1} i^{2\alpha}\right)^{1/2}}$$

$$= O_P\left(\max_{2\le k\le \log n} \frac{k^{\alpha}(k\log\log k)^{1/2}}{k^{\alpha+1/2}}\right) = O_P\left((\log\log\log n)^{1/2}\right)$$

$$= o_P\left((\log\log n)^{1/2}\right),$$

which means that it suffices to investigate the max over $k \in [\log n, n]$ only.

In view of Corollary 2.1, the corresponding asymptotics can be derived from those of

$$\sup_{\frac{\log n}{n} \leq t \leq 1} \frac{\left| \alpha \int_0^t y^{\alpha-1} W_n(t-y)\, \underset{\cdot}{\mathrm{y}} \right|}{\left(\int_0^t y^{2\alpha}\, \underset{\cdot}{\mathrm{y}} \right)^{1/2}}$$

$$= \sup_{\frac{\log n}{n} \leq t \leq 1} \frac{\left| \alpha \int_0^t y^{\alpha-1} W_n(t-y)\, \underset{\cdot}{\mathrm{y}} \right|}{\left(t^{2\alpha+1} \right)^{1/2}} (2\alpha+1)^{1/2}.$$

This is equivalent to investigating

$$\sup_{\log n \le s \le n} \frac{\left|\alpha \int_0^s x^{\alpha-1} W(s-x)\, \underset{\cdot}{\mathbf{x}}\right|}{\left(s^{2\alpha+1}\right)^{1/2}} (2\alpha+1)^{1/2}$$

by substitution $s = nt,\ x = ny$.

This is equivalent to investigating

$$\sup_{\log n \le s \le n} \frac{\left|\alpha \int_0^s x^{\alpha-1} W(s-x)\, \underset{\cdot}{\mathrm{x}}\right|}{\left(s^{2\alpha+1}\right)^{1/2}} (2\alpha+1)^{1/2}$$

by substitution $s = nt, \; x = ny$.

We now put $s = e^t$, and consider the process $\{X(t) : t \ge 0\}$ given by

$$X(t) = \frac{(2\alpha+1)^{1/2} \alpha \int_0^{e^t} x^{\alpha-1} W(e^t - x)\, \underset{\cdot}{\mathrm{x}}}{e^{t(\alpha+1/2)}} \quad (.23)$$

$$= \frac{(2\alpha+1)^{1/2} \alpha \int_0^{e^t} (e^t - y)^{\alpha} \underset{\cdot}{\mathrm{W}}(y)}{e^{t(\alpha+1/2)}}.$$

Note that $\{X(t) : t \geq 0\}$ is a zero mean Gaussian process with covariance function

$$cov(X(s),\, X(t)) = (2\alpha+1)\frac{\int_0^{e^s} (e^s - y)^\alpha\, (e^t - y)^\alpha\, dy}{e^{s(\alpha+1/2)}\, e^{t(\alpha+1/2)}}$$

$$= (2\alpha+1)\, e^{(s-t)/2} \int_0^1 (1-z)^\alpha\, (1 - z\, e^{s-t})^\alpha\, dz,$$

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Clearly, $\{X(t) : t \geq 0\}$ is stationary with unit variance, and, in order to get the limiting extreme value distribution, it will be enough to check the behaviour of $r(t)$, where

$$\begin{aligned}
r(t) - 1 &= cov(X(s), X(s+t)) - 1 \\
&= (2\alpha+1)\, e^{-t/2} \int_0^1 (1-z)^\alpha\, (1 - z\, e^{-t})^\alpha\, \d{z} - 1 \\
&= (2\alpha+1) \Big\{ e^{-t/2} \int_0^1 (1-z)^\alpha \Big((1 - z\, e^{-t})^\alpha - (1-z)^\alpha \\
&\qquad + (e^{-t/2} - 1) \int_0^1 (1-z)^{2\alpha}\, \d{z} \Big\} .
\end{aligned}$$

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Obviously, since $r(t) = O(e^{-t/2})$ as $t \longrightarrow \infty$, condition (3.7) is satisfied for any $\alpha > 0$.

Concerning the asymptotics for $t \longrightarrow 0$, we prove the following lemma.