Ortogonální polynomy (spec. Legendreovy)

H ... Hilb. prostor (spec. $L^2(-1,1)$)

$A: \mathcal{D} \xrightarrow{lin} H$ samoadjungovaný: $\langle Ay, z\rangle = \langle y, Az\rangle \ \forall y, z \in \mathcal{D}$, $\mathcal{D} \subset H$

spec: $A_L(y) = -((1-x^2)y')'$, na $\mathcal{D} = C^2([-1,1])$

$$\int_{-1}^{1} -\frac{d}{dx}\left((1-x^2)\frac{dy}{dx}\right) z\,dx \overset{p.p.}{=} \int_{-1}^{1}(1-x^2)\frac{dy}{dx}\frac{dz}{dx} = -\int_{-1}^{1}\frac{d}{dx}\left((1-x^2)\frac{dz}{dx}\right)\frac{dy}{dx}\,dx \Rightarrow A_L \text{ je samoadjung.}$$

$\Rightarrow$ vlastní vektory pro různá vlastní čísla jsou OG; $Ay_1 = \lambda_1 y_1$, $Ay_2 = \lambda_2 y_2$, $\lambda_1 \neq \lambda_2 \Rightarrow \langle y_1, y_2\rangle = 0$

Spec: $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left((x^2-1)^n\right)$, $\lambda_n = n(n+1)$, $A_L P_n = \lambda_n P_n$. $\{P_n\}$ navíc úplné

PZN: $\frac{1}{\sqrt{1-2rh+r^2}} = \sum_{n=0}^{\infty} r^n P_n(h)$, $|r|<1$ $\rightarrow$ užitečné pro rozvoj potenciálu

· úplný systém ON funkcí: y_n, $\langle y_n, y_n\rangle = 1$, $\langle y_n, y_m\rangle = 0 \ \forall n \neq m$, $y = \sum_{n=0}^{\infty}\langle y, y_n\rangle y_n \ \forall y \in H$

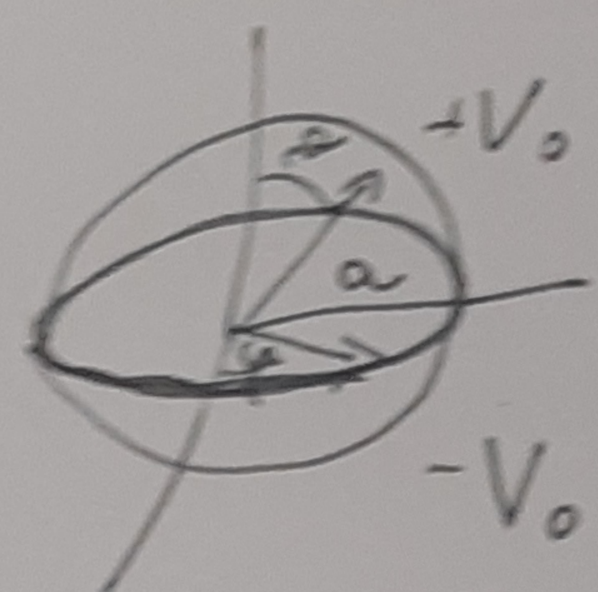
Spec. Rozvoj $\operatorname{sgn}(x)$ na $[-1,1]$ do P_n

1) $\langle P_n, P_n\rangle = \int_{-1}^{1} P_n^2\,dx \overset{...}{=} \frac{2}{2n+1}$ $\Rightarrow \operatorname{sgn} x = \sum_{n=0}^{\infty}\frac{2n+1}{2}\langle \operatorname{sgn}, P_n\rangle P_n(x)$

2) $\langle \operatorname{sgn}, P_n\rangle = \int_{-1}^{1}\operatorname{sgn}(x)P_n(x)\,dx \overset{...}{=} \begin{cases} 0 & n \text{ sudé} \\ -\frac{2}{2^n n!}\binom{n}{\frac{n-1}{2}}(-1)^{n-\frac{n-1}{2}}(n-1)! & \text{pro } n \text{ liché, tj. } n = 2\ell+1 \end{cases}$

$= -2\binom{2\ell+1}{\ell}\frac{(2\ell)!}{2^{2\ell+1}(2\ell+1)!}(-1)^{\ell+1}$

$\Rightarrow \operatorname{sgn} x = \sum_{\ell=0}^{\infty}\frac{(4\ell+3)(-1)^{\ell}}{2^{2\ell+1}}\frac{(2\ell)!}{\ell!(\ell+1)!}P_{2\ell+1}(x)$



kulová plocha rozdělená rovinou na dvě polokoule. Určete potenciál vně a uvnitř.

$$\Delta V = 0 \overset{\text{sférické souřadnice}}{\Longrightarrow} 0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\vartheta}\frac{\partial}{\partial \vartheta}\left(\sin\vartheta \frac{\partial V}{\partial \vartheta}\right) + \underbrace{\frac{1}{r^2 \sin^2\vartheta}\frac{\partial^2 V}{\partial \varphi^2}}_{=0 \text{ ze symetrie}}$$

Ansatz: $V(r,\vartheta) = R(r)\,\Theta(\vartheta) \Rightarrow 0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right)\Theta + \frac{1}{r^2 \sin\vartheta}\frac{\partial}{\partial \vartheta}\left(\sin\vartheta \frac{\partial \Theta}{\partial \vartheta}\right) R \Big/ \cdot \frac{r^2}{\Theta R}$

$$\underbrace{\frac{1}{R}\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right)}_{\text{nezávisí na } \vartheta} + \underbrace{\frac{1}{\Theta}\frac{1}{\sin\vartheta}\frac{\partial}{\partial \vartheta}\left(\sin\vartheta \frac{\partial \Theta}{\partial \vartheta}\right)}_{\text{nezávisí na } r} = 0$$

$\Rightarrow = \text{const.}$

$$\frac{1}{\Theta}\frac{1}{\sin\vartheta}\frac{\partial}{\partial \vartheta}\left(\sin\vartheta \frac{\partial \Theta}{\partial \vartheta}\right) = \lambda \overset{x=\cos\vartheta}{\Longrightarrow} \left[\frac{\partial}{\partial\vartheta} = \underbrace{\frac{\partial x}{\partial \vartheta}}_{=-\sin\vartheta}\frac{\partial}{\partial x}\right] \quad \overbrace{\frac{d}{dx}\left((1-x^2)\frac{d\Theta}{dx}\right)}^{-A_L\Theta} = \lambda\,\Theta \Rightarrow \Theta \text{ jsou Leg. pol.}$$

$\lambda_l = -l(l+1)$, $\Theta_l = P_l(\cos\vartheta)$

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) = -\lambda \Rightarrow 2rR' + r^2R'' + \lambda R = 0, \quad R = r^\alpha, \quad \alpha(\alpha-1) + 2\alpha + \lambda = 0$$

$$\alpha(\alpha+1) = l(l+1) \Rightarrow \alpha = \begin{cases} l \\ -(l+1) \end{cases}$$

$\Rightarrow V(r,\vartheta) = \sum_{l=0}^{\infty}\left(A_l r^l + B_l r^{-(l+1)}\right) P_l(\cos\vartheta)$ je obecné axisymetrické řešení Laplaceovy rce

a) vnitřní potenciál: V konečný $\Rightarrow B_l = 0\ \forall l$, navíc $V(a,\vartheta) = V_0\, \mathrm{sgn}(\cos\vartheta)$

tedy $V_0 \sum_{l=0}^{\infty} \frac{4l+3}{2^{2l+1}}(-1)^l \frac{(2l)!}{l!(l+1)!} P_{2l+1}(\cos\vartheta) = \sum_{l=0}^{\infty} A_l a^l P_l(\cos\vartheta) \Rightarrow A_{2l} = 0$

$V = \sum_{l=0}^{\infty} (-1)^l \frac{4l+3}{2^{2l+1}}\left(\frac{r}{a}\right)^{2l+1} \frac{(2l)!}{l!(l+1)!} P_{2l+1}(\cos\vartheta) \Leftarrow \Big| = \sum_{l=0}^{\infty} A_{2l+1} a^{2l+1} P_{2l+1}(\cos\vartheta)$

b) vnější potenciál konečnost $\Rightarrow A_l = 0$, $V(r,\vartheta) = \sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos\vartheta)$

$$V(a,\vartheta) = V_0 \operatorname{sgn}(\cos\vartheta) \Rightarrow V_0 \sum_{l=0}^{\infty} \frac{4l+3}{2^{2l+1}} (-1)^l \frac{(2l)!}{l!(l+1)!} P_{2l+1}(\cos\vartheta) = \sum_{l=0}^{\infty} B_{2l+1} a^{-(2l+2)} P_l(\cos\vartheta)$$

$$\Rightarrow V(r,\vartheta) = \sum_{l=0}^{\infty} (-1)^l \left(\frac{a}{r}\right)^{2l+2} \frac{4l+3}{2^{2l+1}} \frac{(2l)!}{l!(l+1)!} P_{2l+1}(\cos\vartheta)$$