Homogeneous graphs: construction, examples, and applications

Alexey Barsukov\textsuperscript{1}

17 October 2023

\textsuperscript{1}\textsuperscript{1}Funded by the European Union (ERC, POCOCOP, 101071674). Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Research Council Executive Agency. Neither the European Union nor the granting authority can be held responsible for them.
Table of contents

1 Introduction

2 Construction of Infinite Graphs

3 Examples

4 Constraint Satisfaction Problems
Introduction
Graph homomorphism

Definition

For graphs $G$ and $H$, a mapping $f : V(G) \rightarrow V(H)$ is a **homomorphism** if, for all $x, y \in V(G)$, $xy$ is an edge in $G \Rightarrow f(x)f(y)$ is an edge in $H$. 
Graph isomorphism

Definition

For graphs $G$ and $H$, a bijective mapping $f : V(G) \rightarrow V(H)$ is an isomorphism if both $f$ and $f^{-1}$ are homomorphisms.
Graph automorphism

**Definition**

An isomorphism \( \alpha : G \rightarrow G \) from a graph \( G \) to itself is an **automorphism**. The automorphisms of \( G \) form a group denoted \( \text{Aut}(G) \).
Induced subgraph

Definition

- A graph $G$ is an *induced subgraph* of a graph $H$ if $V(G) \subseteq V(H)$ and $E(G) = E(H) \cap (V(G))^2$.
- Let $\text{Age}(H)$ denote the set of all induced subgraphs of $H$ up to isomorphism.
- If $G$ is isomorphic to an induced subgraph of $H$, then $G$ *embeds* into $H$. 
Homogeneous graph

**Definition**

A graph $\mathbb{H}$ is *homogeneous* if it has countably many vertices and, for any finite induced subgraphs $A, B$ of $\mathbb{H}$ and any isomorphism $f : A \to B$, there exists $\alpha \in \text{Aut}(\mathbb{H})$ such that $\alpha|_{V(A)} = f$. 
Example: Finite homogeneous graphs

Theorem ([Gardiner, Golfand & Klin])

Let $G$ be a finite homogeneous graph. Then either $G$ or $\overline{G}$ is isomorphic to a disjoint union of complete graphs all of the same size, or to the pentagon, or to the $3 \times 3$ rook’s graph.
Construction of Infinite Graphs
The infinite homogeneous graph is constructed from a countable class $\mathcal{C}$ of finite graphs which is assumed to be closed under taking isomorphisms, taking induced subgraphs, and to have the two following properties.
A class $\mathcal{C}$ has the *joint embedding* property if, for any $A, B \in \mathcal{C}$, there exists $C \in \mathcal{C}$ and embeddings $f : A \to C$ and $g : B \to C$. 

**Definition**

A class $\mathcal{C}$ has the *joint embedding* property if, for any $A, B \in \mathcal{C}$, there exists $C \in \mathcal{C}$ and embeddings $f : A \to C$ and $g : B \to C$. 

**Joint Embedding Property (JEP)**
**Amalgamation Property (AP)**

**Definition**

\( \mathcal{C} \) has the *amalgamation* property if, for any \( \mathbb{A}, \mathbb{B}_1, \mathbb{B}_2 \in \mathcal{C} \) and any embeddings \( f_1 : \mathbb{A} \to \mathbb{B}_1, f_2 : \mathbb{A} \to \mathbb{B}_2 \), there exists \( \mathbb{C} \in \mathcal{C} \) and embeddings \( g_1 : \mathbb{B}_1 \to \mathbb{C}, g_2 : \mathbb{B}_2 \to \mathbb{C} \) such that \( g_1 \circ f_1 = g_2 \circ f_2 \).
Theorem ([Fraïssé])

1. Let $\mathcal{M}$ be a homogeneous graph. Then $\text{Age}(\mathcal{M})$ has the amalgamation property.
Fraïssé’s Theorem

Theorem ([Fraïssé])

1. Let $\mathcal{M}$ be a homogeneous graph. Then $\text{Age}(\mathcal{M})$ has the amalgamation property.

2. Let $\mathcal{C}$ be a non-empty class of finite graphs such that it is closed under taking isomorphisms and subgraphs, and that it has JEP and AP. Then there is a homogeneous graph $\mathcal{M}$ with $\text{Age}(\mathcal{M}) = \mathcal{C}$. $\mathcal{M}$ is called the Fraïssé limit of $\mathcal{C}$. 
Fraïssé’s Theorem

Theorem ([Fraïssé])

1. Let $\mathcal{M}$ be a homogeneous graph. Then $\text{Age}(\mathcal{M})$ has the amalgamation property.

2. Let $\mathcal{C}$ be a non-empty class of finite graphs such that it is closed under taking isomorphisms and subgraphs, and that it has JEP and AP. Then there is a homogeneous graph $\mathcal{M}$ with $\text{Age}(\mathcal{M}) = \mathcal{C}$. $\mathcal{M}$ is called the Fraïssé limit of $\mathcal{C}$.

3. Let $\mathcal{M}$ and $\mathcal{M}'$ be two homogeneous graphs such that $\text{Age}(\mathcal{M}) = \text{Age}(\mathcal{M}')$. Then $\mathcal{M}'$ is isomorphic to $\mathcal{M}'$. 

Construction of Infinite Graphs
Proof of (1)

Without loss of generality, let $A$ be an induced subgraph of $B_1$. Let $f$ be an embedding of $A$ into $B_2$, i.e. $A \cong f(A)$. Then there is $\alpha \in \text{Aut}(M)$ such that $\alpha|_A = f$. The desired graph $C$ is induced on the union of $\alpha(B_1)$ and $B_2$. 
Proof of (2) – Construction

Let $\theta: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ be a bijection such that $\theta(i, j) \geq i$ for all $i, j \in \mathbb{N}$. 

■ Construction of Infinite Graphs
Proof of (2) – Construction

- Let \( \theta : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) be a bijection such that \( \theta(i, j) \geq i \) for all \( i, j \in \mathbb{N} \).

- Construct the chain \( M_0 \subseteq M_1 \subseteq \ldots \) by induction and put \( M := \bigcup_{k=0}^{\infty} M_k \).
Proof of (2) – Construction

- Let $\theta : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ be a bijection such that $\theta(i, j) \geq i$ for all $i, j \in \mathbb{N}$.
- Construct the chain $M_0 \subseteq M_1 \subseteq \ldots$ by induction and put $M := \bigcup_{k=0}^{\infty} M_k$.
- Step 0. Take arbitrary $M_0 \in \mathcal{C}$.
Proof of (2) – Construction

- Let $\theta : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a bijection such that $\theta(i, j) \geq i$ for all $i, j \in \mathbb{N}$.

- Construct the chain $\mathcal{M}_0 \subseteq \mathcal{M}_1 \subseteq \ldots$ by induction and put $\mathcal{M} := \bigcup_{k=0}^{\infty} \mathcal{M}_k$.

- Step 0. Take arbitrary $\mathcal{M}_0 \in \mathcal{C}$.

- Step $k + 1$. $\mathcal{M}_k$ is associated with a countably infinite sequence $(A_{kj}, B_{kj}, f_{kj})_j$ of all triples $A_{kj}, B_{kj}, f_{kj}$ such that
Proof of (2) – Construction

- Let $\theta : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a bijection such that $\theta(i, j) \geq i$ for all $i, j \in \mathbb{N}$.
- Construct the chain $\mathbb{M}_0 \subseteq \mathbb{M}_1 \subseteq \ldots$ by induction and put $\mathbb{M} := \bigcup_{k=0}^{\infty} \mathbb{M}_k$.
- Step 0. Take arbitrary $\mathbb{M}_0 \in \mathcal{C}$.
- Step $k + 1$. $\mathbb{M}_k$ is associated with a countably infinite sequence $(A_{k,j}, B_{k,j}, f_{k,j})_j$ of all triples $A_{k,j}, B_{k,j}, f_{k,j}$ such that
  (a) $A_{k,j} \subseteq \mathbb{M}_k$, 

Proof of (2) – Construction

Let $\theta : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ be a bijection such that $\theta(i, j) \geq i$ for all $i, j \in \mathbb{N}$.

Construct the chain $\mathcal{M}_0 \subseteq \mathcal{M}_1 \subseteq \ldots$ by induction and put $\mathcal{M} := \bigcup_{k=0}^{\infty} \mathcal{M}_k$.

Step 0. Take arbitrary $\mathcal{M}_0 \in \mathcal{C}$.

Step $k + 1$. $\mathcal{M}_k$ is associated with a countably infinite sequence $(A_{kj}, B_{kj}, f_{kj})_j$ of all triples $A_{kj}, B_{kj}, f_{kj}$ such that

(a) $A_{kj} \subseteq \mathcal{M}_k$, (b) $B_{kj} \in \mathcal{C}$,
Proof of (2) – Construction

- Let $\theta : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ be a bijection such that $\theta(i, j) \geq i$ for all $i, j \in \mathbb{N}$.

- Construct the chain $\mathbb{M}_0 \subseteq \mathbb{M}_1 \subseteq \ldots$ by induction and put $\mathbb{M} := \bigcup_{k=0}^{\infty} \mathbb{M}_k$.

- Step 0. Take arbitrary $\mathbb{M}_0 \in \mathcal{C}$.

- Step $k + 1$. $\mathbb{M}_k$ is associated with a countably infinite sequence $(A_{kj}, B_{kj}, f_{kj})_j$ of all triples $A_{kj}, B_{kj}, f_{kj}$ such that (a) $A_{kj} \subseteq \mathbb{M}_k$, (b) $B_{kj} \in \mathcal{C}$, and (c) $f : A_{kj} \rightarrow B_{kj}$ is an embedding.
Proof of (2) – Construction

- Let $\theta : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a bijection such that $\theta(i, j) \geq i$ for all $i, j \in \mathbb{N}$.

- Construct the chain $\mathcal{M}_0 \subseteq \mathcal{M}_1 \subseteq \ldots$ by induction and put $\mathcal{M} := \bigcup_{k=0}^{\infty} \mathcal{M}_k$.

- Step 0. Take arbitrary $\mathcal{M}_0 \in \mathcal{C}$.

- Step $k + 1$. $\mathcal{M}_k$ is associated with a countably infinite sequence $(A_{kj}, B_{kj}, f_{kj})_j$ of all triples $A_{kj}, B_{kj}, f_{kj}$ such that
  (a) $A_{kj} \subseteq \mathcal{M}_k$,
  (b) $B_{kj} \in \mathcal{C}$, and
  (c) $f : A_{kj} \to B_{kj}$ is an embedding.

Assuming $k = \theta(i, j)$, choose $\mathcal{M}_{k+1}$ to be the amalgamation of $\mathcal{M}_k$ and $B_{ij}$ over $A_{ij}$.
Proof of (2) – Checking the properties

\[ \text{Age}(\mathbb{M}) = \mathcal{C} \]

\( \supseteq \) By JEP, for every \( A \) in \( \mathcal{C} \) there is \( B \) in \( \mathcal{C} \) such that both \( \mathbb{M}_0 \) and \( A \) embed into \( B \). The graph \( B \) is amalgamated to some \( \mathbb{M}_k \) over \( \mathbb{M}_0 \), therefore \( A \) embeds into \( \mathbb{M}_{k+1} \).
Proof of (2) – Checking the properties

\( \mathcal{M}_0 \supseteq \mathcal{M}_k \supseteq A \supseteq \mathcal{M}_k+1 \)

\[ \text{Age}(\mathcal{M}) = C \]

(\( \supseteq \)) By JEP, for every \( A \) in \( C \) there is \( B \) in \( C \) such that both \( \mathcal{M}_0 \) and \( A \) embed into \( B \). The graph \( B \) is amalgamated to some \( \mathcal{M}_k \) over \( \mathcal{M}_0 \), therefore \( A \) embeds into \( \mathcal{M}_k+1 \).

(\( \subseteq \)) Every \( \mathcal{M}_k \) is a result of finitely many amalgamations, so it is in \( C \). Every finite \( A \subseteq \mathcal{M} \) is an induced subgraph of some \( \mathcal{M}_k \). As \( C \) is closed under taking induced subgraphs, \( A \) is also in \( C \).
Proof of (2) – Checking the properties

The homogeneity of $\mathbb{M}$ follows from this claim.

**Claim**

For all $A, B \in \mathcal{C}$ and for all embeddings $e : A \to B$ and $f : A \to M$, there is an embedding $g : B \to M$ such that $g \circ e = f$. 
Proof of (2) – Checking the properties

The homogeneity of $\mathbb{M}$ follows from this claim.

**Claim**

For all $A, B \in C$ and for all embeddings $e: A \to B$ and $f: A \to M$, there is an embedding $g: B \to M$ such that $g \circ e = f$.

**Proof.**

For some $k \in \mathbb{N}$, $A$ embeds into $\mathbb{M}_k$. Then, the sequence $(A_{kj}, B_{kj}, f_{kj})_j$ contains the triple $(A, B, e)$. Then, for some $\ell \geq k$, $\mathbb{M}_{\ell+1}$ is the amalgamation of $\mathbb{M}_\ell$ and $B$ over $A$. □
Proof of (3) – Back-and-forth argument

\[ \text{\( M \) and \( M' \) are homogeneous and \( \text{Age}(M) = \text{Age}(M') \). Let} \]
\[ V(M) = \{0, 1, \ldots\} \text{ and } V(M') = \{0', 1', \ldots\}. \]
Proof of (3) – Back-and-forth argument

The isomorphism $f: \mathcal{M} \rightarrow \mathcal{M}'$ is constructed by induction. Suppose that $f$ is a partial isomorphism between $\mathcal{M}_i$ and $\mathcal{M}'_i$. 
Proof of (3) – Back-and-forth argument

Take the least \( j \) in \( M \) which is not assigned to any element from \( M' \) and consider the subgraph induced on \( M_i \cup \{j\} \).
Proof of (3) – Back-and-forth argument

As $\text{Age}(\mathcal{M}) = \text{Age}(\mathcal{M}')$, the graph $\mathcal{M}'$ contains an induced subgraph isomorphic to $\mathcal{M}_i \cup \{j\}$. 
Proof of (3) – Back-and-forth argument

As \( M' \) is homogeneous, there exists \( \alpha \in \text{Aut}(M') \) that maps \( M'_i \) to this induced subgraph.
After assigning $f(j)$ to $j$, take the smallest unassigned $j'$ in $\mathbb{M}'$ and similarly find the suitable $f^{-1}(j')$ in $\mathbb{M}$.
Independent sets and cliques

The class of finite independent sets and the class of all cliques both have AP. The Fraïssé limits are the countable independent set and the countable clique.
The Fraïssé limit $\mathbb{M}$ of the class of all finite graphs is called Rado graph or Erdős-Rényi graph, or random graph. For $x, y \in V(\mathbb{M})$, $xy$ is an edge with some fixed probability $p \in (0, 1)$.
Universal clique-free graph

\[ K_\ell = \triangle \]

The class of graphs omitting \( K_\ell \) for some \( \ell \) has AP, so there exists the universal homogeneous \( K_\ell \)-free graph that is unique up to isomorphism.
Ordered sets

The classes of finite partially ordered sets and of finite linearly ordered sets both have AP. The Fraïssé limit of finite linear orders is isomorphic to \((\mathbb{Q}, <)\).
Constraint Satisfaction Problems
Definition

Let \( \tau = \{ R_1, \ldots, R_t \} \) be a relational signature. A \( \tau \)-structure \( \mathbb{A} \) is a tuple \((A; R^A_1, \ldots, R^A_t)\), where \( A \) is a set, and, for all \( R \in \tau \) of arity \( k \), we have that \( R^A \subseteq A^k \). The set \( A \) is the *domain* of \( \mathbb{A} \) and each \( R^A \) is a *relation* of \( \mathbb{A} \).
Definition

Let $\tau = \{R_1, \ldots, R_t\}$ be a relational signature. A $\tau$-structure $A$ is a tuple $(A; R_1^A, \ldots, R_t^A)$, where $A$ is a set, and, for all $R \in \tau$ of arity $k$, we have that $R^A \subseteq A^k$. The set $A$ is the domain of $A$ and each $R^A$ is a relation of $A$.

Definition

Let $B$ be a $\tau$-structure. The class $\text{CSP}(B)$ contains all finite $\tau$-structures $A$ such that there is a homomorphism $A \rightarrow B$. The corresponding membership problem is also denoted $\text{CSP}(B)$. 
What is CSP?

Theorem

Let $\tau$ be a finite relational signature and let $C$ be a class of finite $\tau$-structures. Then the following are equivalent.

- $C = CSP(\mathbb{B})$ for a countable $\tau$-structure $\mathbb{B}$. 

What is CSP?

**Theorem**

Let $\tau$ be a finite relational signature and let $C$ be a class of finite $\tau$-structures. Then the following are equivalent.

- $C = CSP(\mathcal{B})$ for a countable $\tau$-structure $\mathcal{B}$.
- $C$ is closed under disjoint unions and inverse homomorphisms.

\[
A, B \in C \implies A \sqcup B \in C
\]

\[
B \in C \text{ and } A \to B \implies A \in C
\]
What do I study?

$F$-free edge-coloring problems

How to color the input graph edges so that no edge-colored graph from $F$ maps homomorphically to it?
Finitely boundedness

Definition

A family $C$ of finite $\tau$-structures is \textit{finitely bounded} if there is a finite set of structures $F$ such that, for any finite $\tau$-structure $A$, $A \in C$ iff no $F \in F$ embeds into $A$, denoted $C = \text{Forb}_{emb}(F)$. A homogeneous structure $\mathbb{M}$ is \textit{finitely bounded} if so is $\text{Age}(\mathbb{M})$. 

Question

Which homogeneous structures from “Examples” are finitely bounded?
Finitely boundedness

**Definition**

A family $C$ of finite $\tau$-structures is *finitely bounded* if there is a finite set of structures $F$ such that, for any finite $\tau$-structure $A$, $A \in C$ iff no $F \in F$ embeds into $A$, denoted $C = \text{Forb}_{\text{emb}}(F)$. A homogeneous structure $M$ is *finitely bounded* if so is $\text{Age}(M)$.

**Question**

Which homogeneous structures from “Examples” are finitely bounded?
First-order reducts of relational structures

\[ \exists z, z' \exists xz \land Ez'y \land Ezz' \land Ez'y \]

**Definition**

A structure $\mathbb{B}$ is a *first-order reduct* of a structure $\mathbb{A}$ if they have the same set of vertices and if all relations of $\mathbb{B}$ are first-order definable from the relations of $\mathbb{A}$. 
First-order reducts of finitely bounded homogeneous structures

**Theorem ([CSS’1999, HN’2019, BMM’2021])**

For every finite class $\mathcal{F}$ of connected finite structures there is a first-order reduct $\mathbb{M}$ of some homogeneous structure such that $\text{Age}(\mathbb{M}) = \text{Forb}_{\text{emb}}(\mathcal{F})$.

**Corollary**

Every $\mathcal{F}$-free edge-coloring problem is exactly $\text{CSP}(\mathbb{B})$, where $\mathbb{B}$ is a first-order reduct of a finitely bounded homogeneous structure.

Why?

If $\mathbb{B}$ is a first-order reduct of a finitely bounded homogeneous structure, then $\text{CSP}(\mathbb{B})$ is in NP.
First-order reducts of finitely bounded homogeneous structures

**Theorem ([CSS’1999, HN’2019, BMM’2021])**

For every finite class $\mathcal{F}$ of connected finite structures there is a first-order reduct $\mathbb{M}$ of some homogeneous structure such that $\text{Age}(\mathbb{M}) = \text{Forb}_{\text{emb}}(\mathcal{F})$.

**Corollary**

Every $\mathcal{F}$-free edge-coloring problem is exactly $\text{CSP}(\mathbb{B})$, where $\mathbb{B}$ is a first-order reduct of a finitely bounded homogeneous structure.
First-order reducts of finitely bounded homogeneous structures

Theorem ([CSS’1999, HN’2019, BMM’2021])

For every finite class $\mathcal{F}$ of connected finite structures there is a first-order reduct $\mathbb{M}$ of some homogeneous structure such that $\text{Age}(\mathbb{M}) = \text{Forb}_{\text{emb}}(\mathcal{F})$.

Corollary

Every $\mathcal{F}$-free edge-coloring problem is exactly $\text{CSP}(\mathbb{B})$, where $\mathbb{B}$ is a first-order reduct of a finitely bounded homogeneous structure.

Why?

If $\mathbb{B}$ is a first-order reduct of a finitely bounded homogeneous structure, then $\text{CSP}(\mathbb{B})$ is in NP.
How can infinite CSPs help me?

A family of NP-problems has a *dichotomy* if every its problem is either solvable in polynomial time or NP-hard.

**Definition**

A problem $P_1$ is *contained* in a problem $P_2$ if every finite input is accepted by $P_1$ only if it is accepted by $P_2$. For a class of problems $\mathcal{L}$, the containment is *decidable* if there is an algorithm running in finite time that checks for any given $P_1, P_2 \in \mathcal{L}$ whether $P_1 \subseteq P_2$.

<table>
<thead>
<tr>
<th></th>
<th>Dichotomy</th>
<th>Decidability of Containment</th>
</tr>
</thead>
<tbody>
<tr>
<td>CSP(finite)</td>
<td>2017</td>
<td>decidable (obvious)</td>
</tr>
<tr>
<td>CSP(FORFBHS)</td>
<td>open</td>
<td>open</td>
</tr>
<tr>
<td>edge-coloring</td>
<td>open</td>
<td>decidable (2023) (binary)</td>
</tr>
</tbody>
</table>
References

- R. Fraïssé
  Sur certains relations qui généralisent l'ordre des nombres rationnels

- M. Bhattacharjee and R. G. Möller and D. Macpherson and P. M. Neumann
  Notes on infinite permutation groups
  *Springer*, 2006

- A. Gardiner
  Homogeneous graphs
  *Journal of Combinatorial Theory*, 1976

- Y. Golfand and M. Klin
  On k-homogeneous graphs
  *Algorithmic studies in combinatorics (Russian)*, 1978
References

- G. Cherlin and S. Shelah and N. Shi
  Universal Graphs with Forbidden Subgraphs and Algebraic Closure
  *Advances in Applied Mathematics, 1999*

- J. Hubička and J. Nešetřil
  All those Ramsey classes (Ramsey classes with closures and forbidden homomorphisms)
  *Advances in Mathematics, 2019*

- M. Bodirsky and F. R. Madelaine and A. Mottet
  A Proof of the Algebraic Tractability Conjecture for Monotone Monadic SNP
  *SIAM J. Comput., 2021*