

TIME SERIES AND BOOTSTRAP

I. AR(1) : $X_{n+1} = \mu + \rho(X_n - \mu) + \sigma \cdot \varepsilon_{n+1}$, $n \in \mathbb{Z}$, $\{\varepsilon_n\}_{n \in \mathbb{Z}}$ i.i.d.

GOAL: ESTIMATE μ

$$\mathbb{E} \varepsilon_n = 0$$

Simulation : $\rho = 0.5$, $\mu = 10$, $\sigma^2 = 4$

NONP BOOT : $\text{var S.E.}(\bar{X}_T) \approx \frac{S_T^2}{\sqrt{T}}$ $S_T^2 = \frac{1}{T-1} \sum_{n=1}^T (X_n - \bar{X}_T)^2$

MODEL BASED BOOTSTRAP

- RESIDUALS : $\hat{\varepsilon}_{n+1} = X_{n+1} - \hat{\mu} - \hat{\rho}(X_n - \hat{\mu})$, $n = 1, \dots, T-1$

- CENTER RESIDUALS : $\tilde{\varepsilon}_{n+1} = \hat{\varepsilon}_{n+1} - \frac{1}{T-1} \sum_{n=2}^T \hat{\varepsilon}_n$

FOR $b = 1, \dots, B$

~~$X_{1,b}^*$~~ DO NONP. BOOT ON $\tilde{\varepsilon}_{1,1}, \dots, \tilde{\varepsilon}_{T-1,1} \rightarrow \tilde{\varepsilon}_{1,b}^*, \dots, \tilde{\varepsilon}_{T,b}^*$

~~$X_{1,b}^*$~~ $X_{1,b}^* = \hat{\varepsilon}_{1,b}^*$, $X_2 = \hat{\rho} X_{1,b}^* + \tilde{\varepsilon}_{2,b}^*$,

$\dots$, $X_{T,b}^* = X_{T-1,b}^* + \hat{\rho} \tilde{\varepsilon}_{T,b}^*$

$\leadsto \bar{X}_{T,b}^* , \dots , \bar{X}_{T,B}^*$

ANALOGOUSLY FOR HIGHER-ORDER AR(q) FOR $q > 1$.

JACK - KNIFE

↓ COF LOEF

$X_1, \dots, X_n \stackrel{i.i.d.}{\sim}$? $\rightarrow$ The Ges $T_n = T(X_1, \dots, X_n)$ be the estimate of the quantity of interest $\theta = \frac{\text{cov}(X_{n1}, X_{n2})}{\sqrt{\text{var}(X_{n1})\text{var}(X_{n2})}}$

- jack-knife

- bias of T_n : $\text{bias}(T_n) = \mathbb{E}T_n - \theta$ $\widehat{\text{bias}}(T_n) = (n-1)(\bar{T}_n - T_n)$,
where $\bar{T}_n = \frac{1}{n} \sum_{i=1}^n T_{(-i)}$

the second

- estimate of S.E.:

sk-pseudovalue $\widetilde{T}_{ni} = n T_n - (n-1) T_{(-i)}$, $i=1, \dots, n$

$$S.E.(\widehat{T}_n) = \sqrt{\frac{1}{n} \frac{1}{n-1} \sum_{i=1}^n (\widehat{T}_{ni} - \widetilde{T}_n)^2}, \quad \widetilde{T}_n = \frac{1}{n} \sum_{i=1}^n \widetilde{T}_{ni}$$

HISTOGRAM

$$x_1, \dots, x_n$$

- breaks: $a_0 < a_1 < \dots < a_L$

- bar heights $\sum_{i=1}^n \mathbb{1}\{x_i \in (a_{l-1}, a_l]\}$

freq = TRUE

freq = FALSE

$$\frac{1}{n} \text{ --- } || \text{ ---}$$

if $a_0 = 0$ and $a_L = 1$ then $a_1 = 0.5$

NAIVE KERNEL ESTIMATOR

$$\hat{F}_n(x) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}\{x_i \leq x\} \quad \leftarrow \text{cdf}$$

Remember that $f(x) = \lim_{h \rightarrow 0} \frac{F(x+h) - F(x-h)}{2h}$

$$\hat{f}_n(x) = \lim_{h \rightarrow 0} \frac{\hat{F}_n(x+h) - \hat{F}_n(x-h)}{2h} = \frac{1}{n \cdot 2h} \sum_{i=1}^n \mathbb{1}\{x_i \in (x-h, x+h)\}$$

$$= \frac{1}{n \cdot h} \sum_{i=1}^n \mathbb{1}\left\{\frac{x_i - x}{h} \in (-1, 1)\right\} \cdot \frac{1}{2}$$

$$= \frac{1}{n \cdot h} \sum_{i=1}^n K\left(\frac{x_i - x}{h}\right) \quad \text{where } K(y) = \frac{1}{2} \mathbb{1}\{-1 \leq y \leq 1\}$$

GENERALLY:

$$\hat{f}_n(x) = \frac{1}{n \cdot h} \sum_{i=1}^n K\left(\frac{x_i - x}{h}\right)$$

NORMAL MIXTURES

- densities of the form

$$f(x) = \frac{w}{\sigma_1} \phi\left(\frac{x - \mu_1}{\sigma_1}\right) + (1-w) \frac{1}{\sigma_2} \phi\left(\frac{x - \mu_2}{\sigma_2}\right)$$

$D = 1/\sigma_1^2 \quad D = 1/\sigma_2^2$

THE CHOICE OF h_n

I. CROSS-VALIDATION

- usual measuring $L(h_n) = \int (\hat{f}_n(x) - f(x))^2 dx$

ESTIM. TRUE

$$\leadsto L(h_n) = \int [\hat{f}_n(x)]^2 dx - 2 \int \hat{f}_n(x) f(x) dx + \int f^2(x) dx$$

$$\leadsto L(h_n) = \int [\hat{f}_n(x)]^2 dx - \frac{2}{n} \sum_{i=1}^n \hat{f}_n(x_i) f(x_i)$$

$$\hat{f}_n(x_i) = \frac{1}{n h_n} \sum_{j=1}^n K\left(\frac{x-x_j}{h_n}\right)$$

II. (normal) reference rule

- usual measuring $MISE(\hat{f}_n(x)) = \int \mathbb{E}(\hat{f}_n(x) - f(x))^2 dx$

$$= \int \mathbb{E}(\hat{f}_n(x) - f(x))^2 dx$$

$$MSE(\hat{f}_n(x)) = \mathbb{E}(\hat{f}_n(x) - f(x))^2 = \text{var}(\hat{f}_n(x)) + (\mathbb{E}\hat{f}_n(x) - f(x))^2$$

$$\text{var}(\hat{f}_n(x)) = \text{var}\left(\frac{1}{n h_n} \sum_{i=1}^n K\left(\frac{x-x_i}{h_n}\right)\right) \stackrel{i.i.d.}{=} \frac{1}{n h_n^2} \text{var}\left(K\left(\frac{x-x_1}{h_n}\right)\right)$$

$$= \frac{1}{n h_n^2} \left[\mathbb{E} K^2\left(\frac{x-x_1}{h_n}\right) - (\mathbb{E} K\left(\frac{x-x_1}{h_n}\right))^2 \right] = \frac{1}{n h_n^2} \int K^2(u) du - \left(\frac{1}{h_n}\right)^2$$

$\neq$

$$\mathbb{E} K^2\left(\frac{x-x_1}{h_n}\right) = \int K^2\left(\frac{x-y}{h_n}\right) f(y) dy \stackrel{\text{sub } u = \frac{x-y}{h_n}}{=} \int K^2(u) f(x - u h_n) du$$

$$\stackrel{\text{sub } u = \frac{x-y}{h_n}}{=} h_n \int K^2(u) f(x - u h_n) du$$

$$\mathbb{E}\hat{f}_n(x) = \frac{1}{h_n} \mathbb{E} K\left(\frac{x-x_1}{h_n}\right) = \frac{1}{h_n} \int K(u) f(x - u h_n) du$$

$$\stackrel{\text{sub } u = \frac{x-y}{h_n}}{=} f(x) - h_n f'(x) + \frac{h_n^2}{2} f''(x)$$

$$= f(x) + \frac{h_n^2}{2} f''(x) \int \frac{u^2 K(u)}{h_n^2} du$$

ok

$$MISE(\hat{f}_n) \doteq \int \frac{b(x)}{m_{2n}} R(x) + \frac{h_n^4}{4} [f''(x)]^2 \otimes m_{2n}^2 dx =$$

$$= \frac{R(x)}{m_{2n}} + \frac{h_n^4}{4} [f''(x) R(x)] m_{2n}$$

$$h_n^{(opt)} = \argmin_{h_n} \{MISE(\hat{f}_n)\} = m^{-1/5} \left[\frac{f(x) R(x)}{f''(x) m_{2n}^2} \right]^{1/5}$$

- normal reference rule:

- for a moment assume that $\varphi(x) = \gamma \left(\frac{x-\mu}{\sigma} \right) \cdot \frac{1}{\sigma}$ & $K(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

$$\leadsto h_n^{(opt)} = \frac{1.06}{m^{1/5}} m^{-1/5} \left(\frac{1}{3} \right)^{1/5} = 1.06$$

$$h_n = 1.06 m^{-1/5} \min \left\{ S_n, \frac{F_n^{-1}(\frac{3}{2}) - F_n^{-1}(\frac{1}{2})}{\bar{F}(\frac{3}{2}) - \bar{F}(\frac{1}{2})} \right\}$$

↑
low and 0

modified normal ref. rule

$$h_n = 0.9 m^{-1/5} \min \{ S_n, \dots \} \quad \text{low and 0}$$