NMSA 434 Ex. session 5 - 2018/2019

30) Neyman - Scott $Y_{ij} \sim N(\mu_i, \sigma^2)$ $i = 1, \dots, N$, $j = 1, 2$ independent.

$$L(\mu, \sigma^2) = \prod_{i=1}^{N} \prod_{j=1}^{2} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(y_{ij} - \mu_i)^2\right\}$$

$$\ell(\mu, \sigma^2) = c - \frac{2N}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{N}\sum_{j=1}^{2}(y_{ij} - \mu_i)^2$$

$$\frac{\partial \ell}{\partial \mu_i} = +\frac{1}{2\sigma^2}\sum_{j=1}^{2} 2(y_{ij} - \mu_i) \stackrel{!}{=} 0 \Rightarrow \hat{\mu}_i = \frac{y_{i1} + y_{i2}}{2}$$

$$\frac{\partial \ell}{\partial \sigma^2} = -\frac{N}{\sigma^2} + \frac{1}{2\sigma^4}\sum_i\sum_j (y_{ij} - \mu_i)^2 \stackrel{!}{=} 0 \Rightarrow \hat{\sigma}^2 = \frac{1}{2N}\sum_i\sum_j (y_{ij} - \hat{\mu}_i)^2$$

[R1] is not satisfied, system of densities is not regular.

$\hat{\mu}_i \sim N(\mu_i, \frac{\sigma^2}{2})$ not a consistent estimator of μ_i

Further $\sum_{j=1}^{2}(y_{ij} - \hat{\mu}_i)^2 = \left(y_{i1} - \frac{y_{i1} + y_{i2}}{2}\right)^2 + \left(y_{i2} - \frac{y_{i1} + y_{i2}}{2}\right)^2 =$

$= \frac{2}{4}(y_{i1} - y_{i2})^2$, $\hat{\sigma}^2 = \frac{1}{2N}\sum_{i=1}^{N}\frac{1}{2}(y_{i1} - y_{i2})^2 \stackrel{d}{=} \frac{1}{4N}\sum_{i=1}^{N}\left(\sqrt{2\sigma^2}\,N(0,1)\right)^2$

$= \frac{\sigma^2}{2}\cdot\frac{\chi^2_N}{N} \xrightarrow[N \to \infty]{as} \frac{\sigma^2}{2}$ (LLN for X_i^2, $X_i \sim N(0,1)$)

$\hat{\sigma}^2$ is not consistent estimator of σ^2.

31) a) marginal likelihood. $Z_i := \frac{Y_{i1} - Y_{i2}}{\sqrt{2}} \sim N(0, \sigma^2)$ iid

MLE: $\hat{\sigma}^2_{(n)} = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{Y_{i1} - Y_{i2}}{\sqrt{2}}\right)^2 = \frac{1}{2N}\sum_{i=1}^{N}(Y_{i1} - Y_{i2})^2 = \sigma^2\frac{\chi^2_N}{N} \xrightarrow{as} \sigma^2$

Furthermore, the regularity conditions are satisfied and $\hat{\sigma}^2_{(n)}$ is as. normal

b) conditional likelihood. In L μ_i relates to y only through

$(y_{i1} - \mu_i)^2 + (y_{i2} - \mu_i)^2 = y_{i1}^2 + y_{i2}^2 - 2\mu(y_{i1} + y_{i2}) + \mu_i^2 . 2$

$\Rightarrow Y_{i1} + Y_{i2}$ is sufficient for μ_i. $\begin{pmatrix} Y_{i1} \\ Y_{i1} + Y_{i2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} Y_{i1} \\ Y_{i2} \end{pmatrix}$

$(Y_{i1} \mid Y_{i1} + Y_{i2} = s_i)$ $\sim N_2\left(\begin{pmatrix} \mu_i \\ 2\mu_i \end{pmatrix}, \sigma^2\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\right)$

$$(Y_{i1} \mid Y_{i1}+Y_{i2}=s_i) \underset{\rho = 1/\sqrt{2}}{\sim} N\left(\frac{s_i}{2}, \frac{\sigma^2}{2}\right)$$

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N_2\left(\begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}, \begin{pmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{pmatrix}\right) \Rightarrow X \mid Y \sim N\left(\mu_x + \frac{\sigma_x}{\sigma_y}\rho(y-\mu_y), (1-\rho^2)\sigma_x^2\right)$$

Likelihood of $(Y_{i1}, Y_{i2} \mid Y_{i1}+Y_{i2}=s) = \delta_{\{s\}} \rightsquigarrow Y_{i1} \mid Y_{i1}+Y_{i2} \overset{as}{=} s - Y_{i2} \mid Y_{i1} + \ldots$ is the same as the likelihood of $Y_{i1} \mid Y_{i1}+Y_{i2}$ (the other component is a deterministic function of the first)

$$L^{(c)}(\sigma^2) = \left(\prod_{i=1}^{N} \frac{1\cdot\sqrt{2}}{\sqrt{2\pi\sigma^2}}\right) \exp\left\{-\frac{1}{\sigma^2}\sum_{i=1}^{N}\left(y_{i1} - s_i/2\right)^2\right\}$$

$$\ell^{(c)}(\sigma^2) = c - \frac{m}{2}\log\sigma^2 - \frac{1}{\sigma^2}\sum_{i=1}^{N}\left(y_{i1} - s_i/2\right)^2$$

$$\frac{\partial \ell^{(c)}(\sigma^2)}{\partial\sigma^2} = -\frac{m}{2\sigma^2} + \frac{1}{\sigma^4}\sum\left(y_{i1} - s_i/2\right)^2 \overset{!}{=} 0 \Rightarrow \hat{\sigma}^2_{(c)} = \frac{2}{m}\sum_{i=1}^{N}\left(y_{i1} - \frac{s_i}{2}\right)^2$$

$$\hat{\sigma}^2_{(c)} = \frac{2}{m}\sum\left(y_{i1} - \frac{y_{i1}+y_{i2}}{2}\right)^2 = \frac{1}{2m}\sum\left(y_{i1}-y_{i2}\right)^2 = \hat{\sigma}^2_{(M)}$$

32) $Y_{ij} \sim \mathcal{E}xp\left(u_i\, I[j=2]\, \tau^{\wedge}(\cdot)\right) \quad \left(f(y) = \frac{1}{u_i \tau^{I[j=2]}} \exp\left(-u_i^{-1}\tau^{-I[j=2]}\, y\right)\right)$

$Y_{ij} \sim \Gamma\left(1, u_i \tau^{I[j=2]}\right)$ independent. $\quad i = 1, \ldots, N,\ j = 1, 2$

$$\begin{matrix} Y_1 \sim \Gamma(\alpha_1, \theta_1) \\ Y_2 \sim \Gamma(\alpha_2, \theta_2) \end{matrix} \Longrightarrow \frac{Y_1}{Y_2} \sim F(2\alpha_1, 2\alpha_2)\left[\frac{\alpha_1\theta_1}{\alpha_2\theta_2}\right]^{-1}$$

indep. $\quad \chi^2_k = \Gamma\left(\frac{k}{2}, 2\right) \quad c\,\Gamma(\alpha,\theta) = \Gamma(\alpha, c\theta) \quad$ (*)

In our case $\left[\frac{Y_{i1}}{Y_{i2}}\right] \sim F(2,2)\cdot\tau \qquad V_i := \left[\frac{Y_{i2}}{Y_{i1}}\right]^{-1}$

density of $F_{2,2}$ is $f(t) = (1+t)^{-2}\, I(t>0]$

density of $\tau \cdot F_{2,2}$ is $f(t;\tau) = (1+t/\tau)^{-2}\frac{1}{\tau}\, I[t>0]$

$$L^{(M)}_m(\tau) = \prod f(v_i, \tau) = \frac{1}{\tau^m}\left[\prod\left(\frac{\tau}{v_i+\tau}\right)\right]^2 I[\forall v_i > 0]$$

$$\ell^{(M)}_m(\tau) = -m\log\tau + 2m\log\tau - 2\sum\log(v_i+\tau) + c$$

$$\partial \ell^{(M)}_m(\tau) = \frac{m}{\tau} - 2\sum\frac{1}{v_i+\tau} \overset{!}{=} 0 \qquad \text{To be solved numerically.}$$

V1MST434 exercise session 5: conditional likelihood

Ex 30) 33) Y_{ij} $i = 1..I$, $j = 0,1$ independent $Y_{ij} \sim Bi(m_{ij}, p_{ij})$, $\log(p_{ij}/1-p_{ij}) = \psi_i + \tau I[j=1]$

$H_0: \tau = 0$, $H_1: \tau \neq 0$; i.e. $\tau = \log(p_{i1}/1-p_{i1}) - \log(p_{i0}/1-p_{i0})$ $\forall i = 1..I$
$= \log[p_{i1}(1-p_{i0})/p_{i0}(1-p_{i1})]$ log odds-ratio

H_0: $j=1$ has no effect, treatment has no effect in groups
MLE works only for m_{ij} large for all i,j, otherwise as $I \to \infty$ or for I large
MLE is seriously biased $\rightarrow$ for $m_{ij} = 1$ $\forall ij$ it can be shown that as $I \to \infty$
$\hat{\tau} \rightarrow 2\tau$.
Reason: too many nuisance parameters. ψ_i $i = 1..I$.

conditional likelihood: condition out the nuisance parameters

complete likelihood:

$$L(\psi, \tau) = \prod_{i=1}^{I} \binom{m_{i0}}{y_{i0}} p_{i0}^{y_{i0}} (1-p_{i0})^{m_{i0}-y_{i0}} \binom{m_{i1}}{y_{i1}} p_{i1}^{y_{i1}} (1-p_{i1})^{m_{i1}-y_{i1}}$$

$$= h(y) \prod_{i=1}^{I} \left(\frac{p_{i0}}{1-p_{i0}}\right)^{y_{i0}} \left(\frac{p_{i1}}{1-p_{i1}}\right)^{y_{i1}} (1-p_{i0})^{m_{i0}} (1-p_{i1})^{m_{i1}}$$

$$= h(y) \left[\prod_{i=1}^{I} e^{\psi_i y_{i0}} e^{(\psi_i+\tau) y_{i1}}\right] \cdot a(\tau, \psi)$$

$$= h(y)\, a(\tau, \psi) \exp\left\{ \sum_{i=1}^{I} \psi_i (y_{i0} + y_{i1}) + \tau \sum_{i=1}^{I} y_{i1} \right\}$$

exponential family, sufficient for ψ_i is $\underset{\overset{\shortparallel}{Y_{i+}}}{Y_{i0} + Y_{i1}}$ $\Rightarrow$ distribution of $Y_{ij} \mid Y_{i+}$ does not depend on $\underline{\underline{\psi}}$

$Y_{i+} := Y_{i0} + Y_{i1}$ i fixed, notation supressed

$$P(Y_1 = z \mid Y_+ = y_+) = \frac{P(Y_1 = z, Y_0 = y_+ - z)}{P(Y_+ = y_+)} \overset{Y_1 \perp Y_2}{=} \frac{\binom{m_1}{z} p_1^{z}(1-p_1)^{m_1-z} \binom{m_0}{y_+-z} p_0^{y_+-z}(1-p_0)^{m_0-(y_+-z)}}{\sum_{s=0}^{y_+} \binom{m_1}{s} p_1^{s}(1-p_1)^{m_1-s} \binom{m_0}{y_+-s} p_0^{y_+-s} (1-p_0)^{m_0-(y_+-s)}}$$

$Y_1 \le m_1$ $(z \le m_1)$

$Y_0 \le m_0$ $(Y_+ + Y_1 - m_0 \le Y_1 = z)$

$z = 0, 1, \dots, m_{i1}$
$y_+ = 0, 1, \dots, m_{i1} + m_{i0}$

$$= \frac{\binom{m_1}{z}\binom{m_0}{y_+-z} (e^{\psi+\tau})^{z} \cancel{(e^{\psi})^{y_+-z}}\, \cancel{(1-p_1)^{m_1}}\, \cancel{(1-p_0)^{m_0}}}{\cancel{(1-p_0)^{m_0}}\, \cancel{(1-p_1)^{m_1}} \sum_{s=0}^{y_+} \binom{m_1}{s}\binom{m_0}{y_+-s} (e^{\psi+\tau})^{s} \cancel{(e^{\psi})^{y_+-s}}}$$

$s \in \mathcal{Y}$

$\mathcal{Y} = \{0 \vee (y_+ - m_0), \dots, y_+ \wedge m_1\}$

$$= \frac{\binom{m_1}{z}\binom{m_0}{y_+-z} e^{\tau z}}{\sum_{s=0}^{y_+} \binom{m_1}{s}\binom{m_0}{y_+-s} e^{\tau s}}$$

$z = \cancel{0, \dots m_1}$ $z \in \mathcal{Y}$ for $\tau = 0$ Hypergeometric distribution

$\Rightarrow$ non-central HG distribution

Conditional likelihood function:

$$L(\tau) = \prod_{i=1}^{I} \frac{\binom{m_{i1}}{y_{i1}}\binom{m_{i0}}{y_{i+}-y_{i1}} \exp\{\tau y_{i1}\}}{\sum_{\cancel{s=0}}^{m_{i1}} \binom{m_{i1}}{s}\binom{m_{i0}}{y_{i+}-s} \exp\{\tau s\}}$$

$s \in \mathcal{Y}_i$

Estimation in non-central HG distribution.

$$P(X_i = x_i) = \frac{\binom{n_i}{x_i}\binom{m_i}{t_i - x_i} e^{\theta x_i}}{\sum_{s=0}^{t}\binom{n_i}{s_i}\binom{m_i}{t_i-s_i} e^{\theta s_i}}$$

$x = 0, \dots, t$ $\quad n, m, t$ constants

$a(n, m, t, x) = \binom{n}{x}\binom{m}{t-x} = a(x)$

$$= \frac{a(n,m,t,x)\, e^{\theta x}}{\sum_s a(n,m,t,s)\, e^{\theta s}}$$

$$L(\theta) = \prod \frac{a(x_i)\, e^{\theta x_i}}{\sum_s a(s) e^{\theta s}} = \frac{\left(\prod a(x_i)\right) e^{\theta \sum x_i}}{\prod\left(\sum_s a(s) e^{\theta s}\right)}$$

$$l(\theta) = \sum \log a(x_i) + \theta \sum x_i - \sum \log\left[\sum_s a(s) e^{\theta s}\right]$$

$$U(\theta) = \sum_i x_i - \sum_i \frac{\sum_s a(s)\, e^{\theta s}\, s}{\sum_s a(s)\, e^{\theta s}} \Big|_{\theta=0} = \sum x_i - N\, EX \quad \Rightarrow U(\tilde{\theta}) = \sum (x_i - EX)$$

$$-\frac{\partial}{\partial\theta} U(\theta) = \sum_i \left[\frac{\sum_s a(s)\, e^{\theta s} s^2 \cdot \sum_s (a(s)) e^{\theta s} - \sum a(s) e^{\theta s} s \sum a(s) e^{\theta s} s}{\left[\sum_s a(s) e^{\theta s}\right]^2} \right]$$

$$= \Big|_{\theta=0} N \cdot \left(\frac{\sum a(s) s^2}{\sum a(s)} - \left[\left(\frac{\sum a(s) s}{\sum a(s)}\right)\right]^2 \right) = N\left(EX^2 - (EX)^2\right)$$

$$= N \cdot \mathrm{var}\, X \quad \Rightarrow I(\tilde{\theta}) = \mathrm{var}\, X$$

Rao's score test $H_0: \theta = 0$
$H_1: \theta \neq 0$

$$\frac{1}{N} \frac{\left(\sum (x_i - EX)\right)^2}{\mathrm{var}\, X} \underset{as}{\overset{H_0}{\sim}} \chi^2_1$$

~~Exa~~

In our situation: product of non-central HG distrs with different n_i, m_i, t_i leads to test statistic

$$R_m^{(c)} = \frac{\left(\sum_{i=1}^{I} Y_{i1} - E_{H_0}[Y_{i1} \mid Y_{i+}]\right)^2}{\sum_{i=1}^{I} \mathrm{var}_{H_0}[Y_{i1} \mid Y_{i+}]} = \frac{\left(\sum_{i=1}^{I} Y_{i1} - Y_{i+}\frac{m_{i1}}{m_{i+}}\right)^2}{\sum_{i=1}^{I} Y_{i+} \frac{m_{i1} m_{i0}}{m_{i+}^2} \frac{m_{i+} - Y_{i+}}{m_{i+} - 1}} \underset{H_0}{\overset{as}{\sim}} \chi^2_1$$

Cochran - Mantel - Haenszel test.

Ex31) 34) $I = 1$, only two samples from binomial distributions $Y_0 \sim Bi(m_0, p_0)$, $Y_1 \sim Bi(m_1, p_1)$

$H_0: p_0 = p_1$
$H_1: p_0 \neq p_1$

Two sample problem for binary data

usually derive as distribution of $\hat{p}_0 - \hat{p}_1$ (Δ-theorem + CLT)

conditional approach: $\log\frac{p_0}{1-p_0} = \eta \qquad \log\frac{p_1}{1-p_1} = \eta + \tau$

$H_0: \tau = 0$ as above:

$$P_\tau(Y_1 = z \mid Y_+ = y_+) = \frac{\binom{m_1}{z}\binom{m_0}{y_+ - z} e^{\tau z}}{\sum_s \binom{m_1}{s}\binom{m_0}{y_+ - s} e^{\tau s}}$$

exact test can be based on this distribution with $\tau = 0$, reject H_0 if

$$p = 2\min\left\{ P_0(Y_1 \le y_1 \mid Y_+ = y_+),\ P_0(Y_1 \ge y_1 \mid Y_+ = y_+) \right\}$$

for y_1 observed value and $y_+ = y_1 + y_2$. Analogously for $\tau_0 = \tau$: H_0 one-sided alternative

Ex 31 contd.) confidence intervals by inversion of the test

34) CI for τ : τ_0 such that $H_0 : \tau = \tau_0$ is not rejected at α level in R- fisher.test but be careful about the computation of the p-value

[redacted]

Ex 33) $X_1 \dots X_{m_1} \sim Po(\lambda_x) \qquad Y_1 \dots Y_{m_2} \sim Po(\lambda_y) \qquad \theta = \frac{\lambda_x}{\lambda_y}$ is parameter of interest.

36) $$p(x_1 \dots x_{m_1}, y_1 \dots y_{m_2}) = \frac{e^{-m_1\lambda_x}\lambda_x^{\sum x_i}}{\prod x_i!} \frac{e^{-m_2\lambda_y}\lambda_y^{\sum y_i}}{\prod y_i!} = \frac{e^{-m_1\lambda_x - m_2\lambda_y}}{\prod x_i! \prod y_i!} \lambda_x^{\sum x_i} \lambda_y^{\sum y_i}$$

$\Rightarrow (\sum X_i, \sum Y_i) = S$ sufficient for $(\lambda_x, \lambda_y) \qquad S_x = \sum X_i \quad S_y = \sum Y_i$

$$P(S_x = x \mid S_x + S_y = n) = P(S_x + S_y = n \mid S_x = x) \frac{P(S_x = x)}{P(S_x + S_y = n)}$$

$$= \frac{e^{-\lambda'_x} \lambda'^{x}_x \, e^{-\lambda'_y} \lambda'^{n-x}_y}{x! \, (n-x)!} \frac{n!}{e^{-m_1\lambda_x - m_2\lambda_y}\left((\lambda_1 m_1) + (\lambda_2 m_2)\right)^n} = \binom{n}{x} \lambda^x (1-\lambda)^{n-x}$$

$\lambda'_x = m_1 \lambda_x$
$\lambda'_y = m_2 \lambda_y$ $\qquad S_x + S_y \sim Po(m_1\lambda_x + m_2\lambda_y) \qquad \lambda := \frac{m_1\lambda_x}{m_1\lambda_x + m_2\lambda_y}$

$$\Rightarrow S_x \mid S_x + S_y \sim Bi\left(S_x + S_y, \frac{m_1 \lambda_x}{m_1 \lambda_x + m_2 \lambda_y}\right) = Bi\left(S_x + S_y, \frac{m_1 \theta}{m_1 \theta + m_2}\right)$$

$$\lambda = \frac{m_1 \lambda_x}{m_1 \lambda_x + m_2 \lambda_y} = \frac{m_1 \theta}{m_1 \theta + m_2} \quad \text{for } \theta = \frac{\lambda_x}{\lambda_y}$$

$H_0: \lambda_x = \lambda_y \Leftrightarrow \theta = 1$

$T := \sum X_i$ reject if $\sum X_i /_{S_x + S_y} \overset{H_0}{\sim} Bi\left(S_x + S_y, \frac{m_1}{m_1 + m_2}\right)$ is too large or too small

$p_0 := 2 \min \left\{ P_0\left(\sum X_i \le t \mid S_x + S_y = s\right), P_0\left(\sum X_i \ge t \mid S_x + S_y = s\right) \right\}$

for t observed $\sum X_i$ and s observed $S_x + S_y$, reject if $p < 0{,}05$

exact CI: θ such that $p(\theta) > \alpha$

- sufficiency: $p(x, y) = \frac{e^{-m_1 \lambda_x - m_2 \lambda_y}}{\prod x_i! \prod y_i!} \lambda_x^{\sum x_i} \lambda_y^{\sum y_i} = g(\lambda_x, \lambda_y)\, r(x, y) \left(\frac{\lambda_x}{\lambda_y}\right)^{\sum x_i} \lambda_y^{\sum y_i + \sum x_i}$

$$\underset{\theta = \frac{\lambda_x}{\lambda_y}}{=} g(\lambda_x, \lambda_y)\, r(x, y)\, \theta^{\sum x_i} \lambda_y^{\sum y_i + \sum x_i} \Rightarrow \sum X_i + \sum Y \text{ is suff for } \lambda_y$$

Ex 32 contd) $-E \frac{\partial^2 \ell_m^{(w)}(\tau)}{\partial \tau^2} = -E\left(-\frac{m}{\tau^2} + 2 \sum \frac{1}{(V_i + \tau)^2}\right) \overset{\text{multiplication}}{=} m\left(\frac{1}{\tau^2} - \frac{2}{3\tau^2}\right) = \frac{m}{3\tau^2} = I_m(\tau)$

$\sqrt{m}\,(\hat{\tau} - \tau) \xrightarrow{d} N(0, 3\tau^2)$

$$E \frac{1}{(V_i + \tau)^2} = \int_0^\infty \frac{1}{(v + \tau)^2} \cdot \frac{1}{\tau} \frac{1}{\left(1 + \frac{v}{\tau}\right)^2} \, dv$$

$$= \tau \int_0^\infty \frac{1}{(v + \tau)^4} \, dv \underset{v + \tau = u}{=} \tau \int_\tau^\infty u^{-4} \, du = \left[\frac{\tau u^{-3}}{-3}\right]_\tau^\infty = \frac{1}{3\tau^2}$$

$c\Gamma(\alpha, \theta) = \Gamma(\alpha, c\theta)$

multiplication

(*) $$\frac{\Gamma(1, 4)}{\Gamma(1, 4\tau)} = \frac{4\tau}{4} \frac{\Gamma(1,1)}{\Gamma(1,1)} = \frac{24\tau}{24} \frac{\Gamma(1,2)}{\Gamma(1,2)} = \tau \cdot \frac{\chi_2^2}{\chi_2^2} = \tau F_{2,2}$$

$\chi_k^2 = \Gamma\left(\frac{k}{2}, 2\right) \qquad F_{d_1, d_2} = \frac{\chi_{d_1}^2 / d_1}{\chi_{d_2}^2 / d_2}$