72) cont. $J_n(\theta,\eta) = \begin{pmatrix} n/\theta^2 & 0 \\ 0 & n/\eta^2 \end{pmatrix}$

$$\sqrt{n}\left(\begin{pmatrix}\hat\theta\\ \hat\eta\end{pmatrix} - \begin{pmatrix}\theta\\ \eta\end{pmatrix}\right) \xrightarrow{\mathcal{D}} N_2\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}\theta^2 & 0\\ 0 & \eta^2\end{pmatrix}\right)$$

b) $\sqrt{n}(\hat\theta - \theta) \xrightarrow{\mathcal{D}} N(0, \theta^2)$

c) $\theta \in [\hat\theta \mp u_{1-\alpha/2}\, \hat\theta/\sqrt{n}]$

d) ak $\theta_0 \in CI$ pre θ nezamietam H_0, inak zamietam.

mod d) exponenciálny systém, úplná postač. pre $\frac{1}{\theta}$ je $\sum \frac{y_i}{x_i}$, $E\,\frac{y_i}{x_i} = \theta$ nestr. odhad, fce úplnej postač., najlepší nestr.

8. asymptotické testy bez rušivých parametrov.

73) $X \sim alt(p)$ z Pr. 49+1 vieme

80 (82) $U_n(p) = \frac{\sum X_i}{p} - \frac{n - \sum X_i}{1-p}$ $\qquad \hat p = \bar X \qquad J_n(p) = \frac{n}{p(1-p)}$

83

- $W_n = n(\bar X - p_0)^2 \cdot \frac{1}{\hat p(1-\hat p)} = \left[\sqrt{n}\,\frac{\bar X - p_0}{(\bar X(1-\bar X))^{1/2}}\right]^2$ ⟺ klas. asympt. test

- $R_n = \dfrac{\left(\frac{\sum X_i}{p_0} - \frac{n - \sum X_i}{1-p_0}\right)^2}{n/(p_0(1-p_0))} = \left[\sqrt{n}\,\sqrt{p_0(1-p_0)}\left(\frac{\bar X}{p_0} - \frac{1-\bar X}{1-p_0}\right)\right]^2 = \left[\frac{\sqrt{n}}{\sqrt{p_0(1-p_0)}}(\bar X - p_0)\right]^2$

⟺ Wilsonova metóda

- $LR_n = 2\left[\sum X_i \log \bar X + (n - \sum X_i)\log(1-\bar X) - \sum X_i \log p_0 - (n - \sum X_i)\log(1-p_0)\right]$
$= 2n\left[\bar X \log\frac{\bar X}{p_0} + (1-\bar X)\log\frac{1-\bar X}{1-p_0}\right]$

každý test zamietam ak $T_n > \chi^2_1(1-\alpha)$

74) $X \sim Po(\lambda)$ z Pr 50+1 vieme

81 $U_n(\lambda) = -n + \sum X_i/\lambda \qquad \hat\lambda = \bar X \qquad J(\lambda) = 1/\lambda$

84

- $W_n = n(\bar X - \lambda_0)^2\,\frac{1}{\bar X} = \left(\sqrt{n}\,\frac{\bar X - \lambda_0}{\sqrt{\bar X}}\right)^2$
- $R_n = \left(-n + \sum X_i/\lambda_0\right)^2/(n/\lambda_0) = \left(\sqrt{n}\,\frac{\bar X - \lambda_0}{\sqrt{\lambda_0}}\right)^2$
- $LR_n = 2\left(-n\bar X + \sum X_i \log \bar X + n\lambda_0 - \sum X_i \log \lambda_0\right) = 2n\left[\log\left(\frac{\bar X}{\lambda_0}\right)\cdot \bar X - (\bar X - \lambda)\right]$

každý test zamietame ak $T_n > \chi^2_1(1-\alpha)$

75) $\tilde X \sim Po(\lambda)$ ale $X = \tilde X \mid \tilde X > 0$

82 $P(X = k) = P(\tilde X = k \mid \tilde X > 0) = P(\tilde X = k, \tilde X > 0)/P(\tilde X > 0) = \frac{e^{-\lambda}\lambda^k/k!}{1 - e^{-\lambda}} \qquad k \geq 1$

85 $P(\tilde X > 0) = 1 - e^{-\lambda}\lambda^0/0! = 1 - e^{-\lambda}$

$$L(\lambda) = \prod \frac{e^{-\lambda}\lambda^{x_i}/x_i!}{(1-e^{-\lambda})} = \frac{e^{-n\lambda}\lambda^{\Sigma x_i}/\prod x_i!}{(1-e^{-\lambda})^n}$$

$$\ell(\lambda) = -n\lambda + \Sigma x_i \log\lambda - \Sigma \log x_i! - n\log(1-e^{-\lambda})$$

$$U(\lambda) = -n + \Sigma x_i/\lambda + - \frac{n e^{-\lambda}}{1-e^{-\lambda}} \overset{!}{=} 0$$

$$\hookrightarrow \lambda \frac{e^{\lambda}}{e^{\lambda}-1} = \bar{X}$$

$$H_\ell(\lambda) = -\Sigma x_i/\lambda^2 - \frac{n e^{\lambda}}{(e^{\lambda}-1)^2} \qquad EX = \frac{1}{1-e^{-\lambda}} \sum_{x=1}^{\infty} x\, e^{-\lambda}\frac{\lambda^x}{x!} =$$

$$J_n(\lambda) = \frac{n}{\lambda(1-e^{-\lambda})} + \frac{n e^{\lambda}}{(1-e^{\lambda})^2} \qquad = \frac{\lambda e^{-\lambda}}{1-e^{-\lambda}} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = \frac{\lambda}{1-e^{-\lambda}}$$

a) $\sqrt{n}(\hat{\lambda}-\lambda) \xrightarrow{\mathcal{D}} N\left(0, \left[\frac{1}{\lambda(1-e^{-\lambda})} + \frac{e^{\lambda}}{(1-e^{\lambda})^2}\right]^{-1}\right)$

b) numericky $\hat{\lambda} \approx 0{,}31$ $\quad \sqrt{n}(\hat{\lambda}-\lambda) \overset{as}{\approx} N(0, 0{,}033)$ Mathematica

c) $\theta = P(\tilde{X}=0) = e^{-\lambda}$ $\quad \hat{\theta} = e^{-\hat{\lambda}} \doteq 0{,}73$

interval Δ-metodou $g(t) = e^{-t}$ $\quad g'(\lambda) = -e^{-\lambda}$

$$\sqrt{n}(\hat{\theta}-\theta) \xrightarrow{d} N\left(0, e^{-2\lambda}\left[\frac{1}{\lambda(1-e^{-\lambda})} + \frac{e^{\lambda}}{(1-e^{-\lambda})^2}\right]^{-1}\right)$$

CI: $\theta \in [0{,}725; 0{,}735]$ $\quad$ naměřeno $\theta = 0{,}75$.

d) $\hat{\theta} = 0{,}73$, nap θ v intervalu hore v c)

86 83 76) $X \sim Exp(\lambda)$ z příkladu On 51.

$U_n(\lambda) = n/\lambda - \Sigma x_i$ $\quad \hat{\lambda} = 1/\bar{X}$ $\quad J(\lambda) = 1/\lambda^2$

- $W_n = n\left(\frac{1}{\bar{X}} - \lambda_0\right)^2 \bar{X}^2$
- $R_n = (n/\lambda_0 - \Sigma x_i)^2/(n/\lambda_0^2)$
- $LR_n = 2n(-\log\bar{X} - (\bar{X})^{-1}\bar{X} - \log\lambda_0 + \bar{X}\lambda_0)$

87 84 77) $X \sim Ge(p)$ On 52

- $W_n = n\left(\frac{1}{1+\bar{X}} - p_0\right)^2\left(\frac{1}{\hat{p}^2} - \frac{1}{\hat{p}(1-\hat{p})}\right)$
- $R_n = \left(\frac{n}{p_0} - \frac{\Sigma x_i}{1-p_0}\right)^2 \Big/ \left(n\left(\frac{1}{p_0^2} - \frac{1}{p_0(1-p_0)}\right)\right)$
- $LR_n = 2n(\log\hat{p} + \bar{X}\log(1-\hat{p}) - \log p_0 - \bar{X}\log(1-p_0))$

88 85 78) $\binom{X}{Y} \overset{iid}{\sim} f(x,y) = \beta x e^{-\beta x y}$ $\quad x \in (0,1), y > 0$

a) $L(\beta) = \beta^n \prod x_i \cdot e^{-\beta\Sigma x_i y_i}$

$\ell(\beta) = n\log\beta + c - \beta\Sigma x_i y_i$

$U(\beta) = \frac{n}{\beta} - \Sigma x_i y_i = 0$

$$\hat{\beta} = \underline{\underline{\frac{n}{\Sigma x_i y_i}}}$$

88 78) cont. b) $\frac{\partial U}{\partial \beta}(\beta) = -m/\beta^2$ $\qquad J(\beta) = 1/\beta^2$

- $W_m = (\hat\beta - \beta)^2 m / \hat\beta^2$
- $R_m = \left(\frac{m}{\beta_0} - \sum x_i y_i\right)^2 / \left(m/\beta_0^2\right)$
- $LR_m = m2\left(\log\hat\beta - \hat\beta \frac{1}{m}\sum x_i y_i - \log\beta_0 + \beta_0 \frac{1}{m}\sum x_i y_i\right)$

79) $Y|X \sim Po(e^{\beta X})$ $\quad$ R $\quad$ Pr 60+1 $\quad \theta = e^{\beta}$

86

89 $L(\theta) = \prod \frac{\theta^{y_i x_i} e^{-\theta^{x_i}}}{\cancel{x_i}\, y_i!} \cdot c = \theta^{\sum y_i x_i} e^{-\sum \theta^{x_i}} \cdot c'$

$\ell(\theta) = \sum y_i x_i \log\theta - \sum \theta^{x_i}$ ANO $\quad$ nedokážeme riešiť

a) $U(\theta) = \sum x_i y_i/\theta - \sum x_i \theta^{x_i - 1} = 0$

$U'(\theta) = -\sum x_i y_i/\theta^2 - \sum x_i(x_i-1)\theta^{x_i-2}$

$J_m(\theta) = E\left[\frac{m XY}{\theta^2} + m X(X-1)\theta^{X-2}\right]$ $\qquad EXY = E[E(XY|X)] = E X E(Y|X) = E X e^{\beta X}$

b) explicitne vyjadrím iba

- $R_m = \left[\sum X_i Y_i/\theta_0 - \sum X_i \theta_0^{X_i-1}\right]^2 / J_m(\theta_0)$

c) $\{\theta \in \mathbb{R} : R_m(\theta_0) \le \chi^2_1(1-\alpha)\}$ je interval pre $\theta = e^{\beta}$

d) pre β urobíme iba numericky, prípadne $\log\theta = \beta$ teda log-transformáciou množiny z c).

80) $X \sim Logist(\theta)$ $\quad$ R $\quad$ Pr 58+1

87

90 explicitne iba

- $R_m = \left(m - \sum_{i=1}^{m} \frac{2e^{-x_i}}{e^{-\theta} - e^{-x_i}}\right)^2 / \left(\frac{m}{3}\right)$

81) $X \sim N(\mu, \sigma^2)$ $\quad$ R $\quad$ Os 63+1

88

91

- $W_m = \left(\left(\bar X, \frac{1}{m}\sum (X_i - \bar X)^2\right) - (\mu, \sigma^2)\right) \begin{pmatrix} m/\hat\sigma^2 & 0 \\ 0 & m/2\hat\sigma^4 \end{pmatrix} \begin{pmatrix} \bar X - \mu \\ \frac{1}{m}\sum(X_i - \bar X)^2 - \sigma^2 \end{pmatrix}$

$= \frac{(\bar X - \mu)^2 m}{\hat\sigma^2} + \frac{(\hat\sigma^2 - \sigma^2) m}{2\hat\sigma^4}$

- $R_m = \left(\frac{1}{\sigma_0^2}\sum(X_i - \mu_0),\ -\frac{m}{2\sigma_0^2} + \frac{1}{2\sigma_0^4}\sum(X_i-\mu_0)^2\right) \begin{pmatrix} \sigma_0^2/m & 0 \\ 0 & 2\sigma_0^4/m \end{pmatrix} \begin{pmatrix} \frac{1}{\sigma_0^2}\sum(X_i - \mu_0) \\ -\frac{m}{2\sigma_0^2} + \frac{1}{2\sigma_0^4}\sum(X_i - \mu_0)^2 \end{pmatrix}$

$= \frac{\sigma_0^2}{m}\left(\sum(X_i - \mu_0)\right)^2 + \frac{2}{m}\left(-\frac{m}{2} + \frac{1}{2\sigma_0^2}\sum(X_i - \mu_0)^2\right)^2$

- $LR_m = 2m\left(-\frac{\log\hat\sigma^2}{2} - \frac{1}{2\hat\sigma^2}\frac{1}{m}\sum(X_i - \bar X)^2 + \frac{\log\sigma_0^2}{2} + \frac{1}{2\sigma_0^2}\frac{1}{m}\sum(X_i-\mu_0)^2\right)$

porovnávam s $\chi^2_2(1-\alpha)$

82) $X \sim \log N(\mu, \sigma^2)$ Pn 64

12 89) a) $\hat{\mu} = \frac{1}{m} \sum \log X_i \quad \hat{\sigma}^2 = \frac{1}{m} \sum (\log X_i - \hat{\mu})^2$

b) $\sqrt{m}\left(\begin{pmatrix}\hat{\mu} \\ \hat{\sigma}^2\end{pmatrix} - \begin{pmatrix}\mu \\ \sigma^2\end{pmatrix}\right) \xrightarrow{d} N_2\left(\begin{pmatrix}0 \\ 0\end{pmatrix}, \begin{pmatrix}\sigma^2 & 0 \\ 0 & 2\sigma^4\end{pmatrix}\right)$

c) $\left\{ \mu, \sigma^2 : m(\hat{\mu} - \mu, \hat{\sigma}^2 - \sigma^2) \begin{pmatrix} 1/\hat{\sigma}^2 & 0 \\ 0 & 1/2\hat{\sigma}^4 \end{pmatrix} \begin{pmatrix} \hat{\mu} - \mu \\ \hat{\sigma}^2 - \sigma^2 \end{pmatrix} \leq \chi^2_2(1-\alpha) \right\}$

d) $(\mu, \sigma) = (0,1) : H_0$

- $W_m = m(\hat{\mu} - \mu_0, \hat{\sigma}^2 - \sigma^2) \begin{pmatrix} 1/\hat{\sigma}^2 & 0 \\ 0 & 1/2\hat{\sigma}^4 \end{pmatrix} \begin{pmatrix} \hat{\mu} - \mu_0 \\ \hat{\sigma}^2 - \sigma^2 \end{pmatrix}$

- $R_m = \left(\sum \frac{\log X_i - \mu_0}{\sigma_0^2}, -\frac{m}{2\sigma_0^2} + \frac{1}{2\sigma_0^4} \sum (\log X_i - \mu_0)^2 \right) \begin{pmatrix} \sigma_0^2/m & 0 \\ 0 & 2\sigma_0^4/m \end{pmatrix} \begin{pmatrix} \sum \log X_i \\ -\frac{m}{2} + \frac{1}{2} \sum (\log X_i)^2 \end{pmatrix}$

$$= \frac{(\sum \log X_i)^2}{m} + \frac{2}{m}\left(-\frac{m}{2} + \frac{1}{2} \sum (\log X_i)^2\right)^2$$

- $LR_m = 2m\left(-\frac{\log \hat{\sigma}^2}{2} - \frac{1}{2\hat{\sigma}^2} \frac{1}{m} \sum (\log X_i - \hat{\mu})^2 + 0 + \frac{1}{2m} \sum (\log X_i)^2\right)$

porovnávam s $\chi^2_2(1-\alpha)$

e) $\sqrt{m}(\hat{\mu} - \mu) \xrightarrow{d} N(0, 2\sigma^4)$

zamietam $H_0 : \mu = \mu_0$ ak $\mu_0 \notin \left[\hat{\mu} \mp u_{1-\alpha/2} \frac{\sqrt{2\hat{\sigma}^4}}{\sqrt{m}}\right]$

90) 83) $\equiv$ Pn 78

91) 84) $Y | X \sim N(\beta_1 X + \beta_2 X^2, 1)$

93

$L(\beta) = \prod c \cdot \exp\{-\frac{1}{2}(y_i - \beta_1 x_i - \beta_2 x_i^2)^2\} = c \cdot \exp\{-\frac{1}{2} \sum (y_i - \beta_1 x_i - \beta_2 x_i^2)^2\}$

$\ell(\beta) = -\frac{1}{2} \sum (y_i - \beta_1 x_i - \beta_2 x_i^2)^2$

$U(\beta) = \left(\sum \frac{2}{2}(y_i - \beta_1 x_i - \beta_2 x_i^2) x_i, \sum x_i^2 (y_i - \beta_1 x_i - \beta_2 x_i^2) \right)$

$U'(\beta) = \begin{pmatrix} -\sum x_i^2 & -\sum x_i^3 \\ -\sum x_i^3 & -\sum x_i^4 \end{pmatrix} \qquad J_m(\beta) = m \begin{pmatrix} EX^2 & EX^3 \\ EX^3 & EX^4 \end{pmatrix}$

$H_0 : \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$R_m = \left(\sum (Y_i - \beta_1 X_i - \beta_2 X_i^2) \cdot X_i, \sum X_i^2 Y_i \right) \frac{1}{m} \begin{pmatrix} EX^2 & EX^3 \\ EX^3 & EX^4 \end{pmatrix}^{-1} \begin{pmatrix} \sum Y_i X_i \\ \sum Y_i X_i^2 \end{pmatrix}$

porovnávam s $\chi^2_2(1-\alpha)$

85) $X \sim Mult(4, p_1, p_2, p_3, p_4)$ z Pr 67

$H_0: (p_1, p_2, p_3, p_4) = (1/4, 1/4, 1/4, 1/4)$

$\hat{p} = \sum_{i=1}^{m} \frac{X_{ij}}{m}$ vieme že $\sqrt{m}(\hat{p} - p) \xrightarrow{D} N_4\left(0, \underbrace{\begin{pmatrix} p_1(1-p_1) & -p_1p_2 & -p_1p_3 & -p_1p_4 \\ & \cdots & & \\ & & \ddots & p_4(1-p_4) \end{pmatrix}}_{\Sigma(p)}\right)$

$T = m(\hat{p} - p_0)' \underbrace{\Sigma(\hat{p})^{-1}}_{\text{neexistuje}} (\hat{p} - p_0) \xrightarrow{H_0} \chi^2_3$ lebo rank $\Sigma = 3$ (veta 12.3, anděl)

zamietam ak $T > \chi^2_3(1-\alpha)$

daj $V^- = diag\left(\frac{1}{mp}\right)$. Potom podľa V12.3 z Anděla je $\Sigma V^- \Sigma = \Sigma$, tj. V^- je pseudoinverzia ku Σ a podľa V4.15 z Anděla je

(viz V12.5)

$T_m = m(\hat{p} - p_0)' V^-(\hat{p})(\hat{p} - p_0) = \sum_{i=1}^{4} \frac{(X_i - mp_i)^2}{mp_i} \xrightarrow{H_0} \chi^2_3$ $\Rightarrow$ χ^2-test Multinom. rozdelenia

86) $Y|N \sim Bi(N, p)$ $N \sim Po(\lambda)$ z Pr 69

92) a) $\hat{p} = \frac{\sum m_i}{\sum y_i}$ $\hat{\lambda} = \frac{1}{m}\sum m_i$

94 b) $\sqrt{m}\left(\begin{pmatrix}\hat{p}\\ \hat{\lambda}\end{pmatrix} - \begin{pmatrix}p\\ \lambda\end{pmatrix}\right) \xrightarrow{D} N_2\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix} p(1-p)/\lambda & 0 \\ 0 & \lambda \end{pmatrix}\right)$

c) $W_m = (\hat{p} - p, \hat{\lambda} - \lambda)\begin{pmatrix} \frac{m\hat{\lambda}}{\hat{p}(1-\hat{p})} & 0 \\ 0 & m/\hat{\lambda} \end{pmatrix}\begin{pmatrix} \hat{p} - p \\ \hat{\lambda} - \lambda \end{pmatrix}$

z Pr 50 a 49

$R_m = \left(\sum Y_i/p_0 - \sum \frac{N_i - Y_i}{1-p_0}, -m + \frac{\sum N_i}{\lambda_0}\right)\begin{pmatrix} \frac{p_0(1-p_0)}{m\lambda_0} & 0 \\ 0 & (m/\lambda_0)^{-1} \end{pmatrix}\begin{pmatrix} \sum Y_i/p_0 - \sum \frac{N_i - Y_i}{1-p_0} \\ -m + \sum N_i/\lambda_0 \end{pmatrix}$

$LR_m = 2m\Big(\frac{1}{m}\sum Y_i \log \hat{p}_0 - \frac{1}{m}\sum (N_i - Y_i)\log(1-\hat{p}_0) - 1 + \sum \frac{N_i}{\hat{\lambda}}\frac{1}{m} - \frac{1}{m}\sum Y_i \log p_0 +$
$\frac{1}{m}\sum (N_i - Y_i)\log(1-p_0) + 1 - \sum \frac{N_i}{\lambda_0}\frac{1}{m}\Big)$

porovnávam s $\chi^2_2(1-\alpha)$