NMST332 Neyman-Pearson a LR test.

73) $X_1 \dots X_n \sim Po(\lambda)$

i) $T_n = \frac{L_n(\lambda_1)}{L_n(\lambda_0)} = \frac{e^{-n\lambda_1}\lambda_1^{\Sigma x_i}}{e^{-n\lambda_0}\lambda_0^{\Sigma x_i}} = e^{-n(\lambda_1-\lambda_0)}\left(\lambda_1/\lambda_0\right)^{\Sigma x_i} \geq c$

$-n(\lambda_1-\lambda_0) + \Sigma x_i \log(\lambda_1/\lambda_0) \geq \log c$ $\qquad \lambda_1 > \lambda_0$, $\log \lambda_1/\lambda_0 > 0$

$\Sigma x_i \geq [\log c + n(\lambda_1 - \lambda_0)] / \log(\lambda_1/\lambda_0) =: k$

k: aby $\alpha \overset{\geq}{=} P_{H_0}(\Sigma X_i \geq k) \Rightarrow k =$ kvantil $(1-\alpha)$ $Po(n\lambda_0)$

$\Sigma X_i \overset{H_0}{\sim} Po(n\lambda_0)$

ii) $\Sigma x_i \leq k \Rightarrow k = \alpha$-kvantil $Po(n\lambda_0)$ $\qquad$ přesný test

iii) $\hat\lambda = \bar X$ (Gn.50) $\qquad \sqrt{n}(\bar X - \lambda) \xrightarrow{d} N(0,\lambda)$

$\sqrt{n}\,\frac{\bar X - \lambda_0}{\sqrt{\lambda_0}} \underset{as}{\overset{H_0}{\sim}} N(0,1)$ $\qquad$ test: $\sqrt{n}\,\frac{|\bar X - \lambda_0|}{\sqrt{\lambda_0}} \overset{\text{zamít.}}{\leq} u_{1-\alpha/2}$

resp. Slutsky: $\sqrt{n}\,\frac{|\bar X - \lambda_0|}{\sqrt{\bar X}} \geq u_{1-\alpha/2}$ $\quad$ resp $\quad \frac{n(\bar X - \lambda_0)^2}{\bar X} \geq \chi^2_{1,1-\alpha}$ $\qquad$ asympt. test.

iv) $LR_n \Rightarrow \frac{L_n(\bar X)}{L_n(\lambda_0)} \Rightarrow$ $\qquad LR_n = 2\left[-n(\bar X - \lambda_0) + \Sigma X_i \log\left(\bar X/\lambda_0\right)\right] \geq \chi^2_{1,1-\alpha}$

76) $X_1 \dots X_n \sim N(\mu_x, \sigma_x^2)$

i) MV: $\hat\mu = \bar X$, $\hat\sigma^2 = \frac{1}{n}\Sigma(X_i - \bar X)^2$

$\ell_n(\hat\mu, \hat\sigma^2) = c - \frac{n}{2}\log\hat\sigma^2 - \frac{1}{2\hat\sigma^2}\Sigma(X_i-\bar X)^2 = c - \frac{n}{2}\log\hat\sigma^2 - \frac{n}{2}$

na $H_0: \mu_x = \mu_0$

$\tilde\ell_n(\sigma^2) = c - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\Sigma(x_i-\mu_0)^2$

$\tilde U_n(\sigma^2) = -\frac{n}{2\sigma^2} + \frac{\Sigma(x_i-\mu_0)^2}{2\sigma^4} = 0 \Rightarrow \tilde\sigma^2 = \frac{1}{n}\Sigma(X_i-\mu_0)^2$

$\tilde\ell(\tilde\sigma^2) = c - \frac{n}{2}\log\tilde\sigma^2 - \frac{1}{2\tilde\sigma^2}\Sigma(x_i-\mu_0)^2 = c - \frac{n}{2}\log\tilde\sigma^2 - \frac{n}{2}$

$LR_n = 2\left[-\frac{n}{2}\log\hat\sigma^2 + \frac{n}{2}\log\tilde\sigma^2\right] = n\log\left[\frac{\tilde\sigma^2}{\hat\sigma^2}\right] \geq \chi^2_1(1-\alpha)$

t-test: $\sqrt{n}(\bar X - \mu) \xrightarrow{d} N(0,\sigma^2)$ $\qquad \sqrt{n}\,\frac{\bar X - \mu_0}{\sqrt{S^2}} \overset{H_0}{\sim} t_{n-1}$

ii) na $H_0: \sigma_x^2 = \sigma_0^2$

$\tilde\ell_n(\mu) = c - \frac{n}{2}\log\sigma_0^2 - \frac{1}{2\sigma_0^2}\Sigma(X_i-\mu)^2$

$\tilde U_n(\mu) = \frac{1}{\sigma_0^2}\Sigma(X_i-\mu) = 0 \Rightarrow \tilde\mu = \frac{1}{n}\Sigma X_i$

$\tilde\ell_n(\tilde\mu) = c - \frac{n}{2}\log\sigma_0^2 - \frac{1}{2\sigma_0^2}\Sigma(X_i-\bar X)^2$

$LR_n = 2\left[-\frac{n}{2}\log\hat\sigma^2 + \frac{n}{2}\log\sigma_0^2 - \frac{n}{2} + \frac{1}{2\sigma_0^2}\Sigma(X_i-\bar X)^2\right] = n\log\left[\frac{\sigma_0^2}{\hat\sigma^2}\right] + n\left[\frac{\hat\sigma^2}{\sigma_0^2} - 1\right] \geq \chi^2_1(1-\alpha)$

opačně tomu $\frac{(n-1)S_n^2}{\sigma_0^2} \overset{H_0}{\sim} \chi^2_{n-1}$ $\qquad \frac{(n-1)S_n^2}{\sigma_0^2} = \frac{\sum(X_i-\bar{X})^2}{\sigma_0^2} = \frac{n\hat{\sigma}^2}{\sigma_0^2}$

iii) na H_0: $\ell_n(\tilde{\mu}, \tilde{\sigma}^2) = c - \frac{n}{2}\log\sigma_0^2 - \frac{1}{2\sigma_0^2}\sum(X_i-\mu_0)^2$

$LR_n = 2\left[-\frac{n}{2}(\log\hat{\sigma}^2 - \log\sigma_0^2) - \frac{n}{2} + \frac{1}{2\sigma_0^2}\sum(X_i-\mu_0)^2\right] = n\log\left[\frac{\sigma_0^2}{\hat{\sigma}^2}\right] - n + \frac{\sum(X_i-\mu_0)^2}{\sigma_0^2}$

$\geq \chi^2_2(1-\alpha)$

iv) z Př 63. $\sqrt{n}\left(\begin{pmatrix}\bar{X}\\ \hat{\sigma}^2\end{pmatrix} - \begin{pmatrix}\mu\\ \sigma^2\end{pmatrix}\right) \xrightarrow{d} N_2\left(\begin{pmatrix}0\\0\end{pmatrix}, \overbrace{\begin{pmatrix}\sigma^2 & 0\\ 0 & 2\sigma^4\end{pmatrix}}^{J^{-1}(\mu,\sigma^2)}\right)$

Anděl Věta 4.15: $X \sim N_2(\mu, V)$, $h(V) = 2 \Rightarrow (X-\mu)'V^{-1}(X-\mu) \sim \chi^2_2$

obecně $X \sim N_n(\mu, V)$, $h(V) = r \Rightarrow (X-\mu)'V^{-}(X-\mu) \sim \chi^2_r$

$V := J^{-1}(\mu,\sigma^2)/n \qquad V^{-1} = n\,J(\mu,\sigma^2)$

$n\left(\begin{pmatrix}\bar{X}\\ \hat{\sigma}^2\end{pmatrix} - \begin{pmatrix}\mu_0\\ \sigma_0^2\end{pmatrix}\right)' J(\mu_0,\sigma_0^2)\left(\begin{pmatrix}\bar{X}\\ \hat{\sigma}^2\end{pmatrix} - \begin{pmatrix}\mu_0\\ \sigma_0^2\end{pmatrix}\right) \overset{H_0}{\underset{as}{\sim}} \chi^2_2$

odhad $J(\hat{\mu},\hat{\sigma}^2) \Rightarrow W_n = n\left[\frac{(\bar{X}-\mu_0)^2}{\hat{\sigma}^2} + \frac{(\hat{\sigma}^2-\sigma_0^2)^2}{2\hat{\sigma}^4}\right] \geq \chi^2_2(1-\alpha)$

77) $X \sim Mult(n; p_1, p_2, p_3, p_4)$ $\qquad H_0: p_1 = p_2 = p_3 = p_4 = 1/4 \qquad H_1: \neg H_0$

$L_n(p) = c\, p_1^{x_1} p_2^{x_2} p_3^{x_3} p_4^{x_4}$ $\qquad$ MV: P 67 $\quad \hat{p}_j = \underbrace{\sum_{i=1}^{n} X_{ij}}_{x_j}/n \qquad \hat{p} = \underset{x}{\sum X_i}/n$

$L_n(p_0) = c\,(1/4)^n$

$LR_n = 2\left[\sum x_j \log\left(\frac{x_j}{n}\right) - n\log 1/4\right] = 2\sum x_j \log\left[\frac{4x_j}{n}\right] \overset{H_0}{\sim} \chi^2_3(1-\alpha)$

$\doteq 327{,}15 > \chi^2_3(1-\alpha) = 7{,}8$ $\qquad$ zamítáme H_0

78) $L_n(\alpha,\beta) = \prod \frac{\lambda(x_i)^{y_i} e^{-\lambda(x_i)}}{y_i!}$ $\qquad f_x(x_i) = \frac{\prod\lambda(x_i)^{y_i} \cdot e^{-\sum\lambda(x_i)}}{\prod y_i!}$ $\qquad \frac{\partial\lambda}{\partial\alpha} = \lambda \quad \frac{\partial\lambda}{\partial\beta} = \lambda x$

$\ell_n(\alpha,\beta) = \sum y_i \log\lambda(x_i) - \sum\lambda(x_i) + c$ $\qquad \lambda(x) = \cancel{\alpha+\beta' x}\ e^{\alpha+\beta' x}$

$\frac{\partial\ell_n}{\partial\alpha} = \sum \frac{y_i\,\cancel{x_i}}{\cancel{\lambda(x_i)}\,x_i} - \sum\lambda_i = 0$ $\qquad$ na H_0: $\tilde{\ell}_n(\alpha,0) = \log\alpha\sum y_i - n\alpha$

$\cancel{\sum \frac{y_i}{\alpha+\beta x_i}}$ $\qquad$ } numericky $\qquad \frac{\partial\tilde{\ell}}{\partial\alpha} = \frac{\sum y_i}{\alpha} - n = 0$

$\tilde{\alpha} = \bar{Y}$

$\frac{\partial\ell_n}{\partial\beta} = \sum\frac{y_i\,\cancel{x_i}\,x}{\cancel{\lambda(x_i)}} - \sum\lambda x_i = 0$

$LR_n = 2\left[\sum y_i \log(\hat{\alpha} + \hat{\beta}' x_i) - \sum\exp(\hat{\alpha} + \hat{\beta}' x_i) - \sum y_i \log\bar{y} + n\bar{y}\right] \geq \chi^2_q(1-\alpha)$

$\hat{\alpha} = \cancel{-0{,}52}\ {-1{,}85} \qquad \hat{\beta}_1 = \cancel{-0{,}51}\ 0{,}51 \qquad \hat{\beta}_2 = \cancel{1{,}87}\ 0{,}33$

$LR_n = 11{,}\cancel{81}\,21 > \chi^2_2(0{,}95) = 5{,}99$ $\qquad$ zamítáme $H_0: (\beta_1,\beta_2)' = (0,0)'$

$\chi^2_1(1-\alpha)$