NMSA332

Metoda Maximální věrohodnosti

49) $X \sim \mathrm{alt}(p)$

i) $L(p) = \prod p^{x_i}(1-p)^{1-x_i} = p^{\sum x_i}(1-p)^{n-\sum x_i}$

$\ell(p) = \sum x_i \log p + (n - \sum x_i)\log(1-p)$

$\ell'(p) = \sum x_i / p - (n-\sum x_i)/(1-p) \overset{!}{=} 0$ $\qquad \ell''(p) = -\frac{\sum x_i}{p^2} - \frac{(n-\sum x_i)}{(1-p)^2}$

$(1-p)\sum x_i - (n - \sum x_i)p = 0$

$p(-\sum x_i - n + \sum x_i) = -\sum x_i$ $\qquad J_n(p) = \frac{n}{p} + \frac{n}{1-p} = \frac{n}{p(1-p)}$

$\hat p = \bar X$

$\sqrt{n}(\bar X - p) \xrightarrow{D} N(0, p(1-p))$

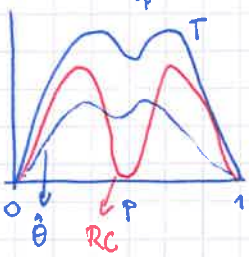
ii) invariancia: $\theta = p(1-p)$ $\quad \hat\theta = \hat p(1-\hat p) = \bar X(1-\bar X)$ $\quad g(t) = t(1-t)$ $\quad g'(t) = (1-t)-t = 1-2t$

$\sqrt{n}(\bar X(1-\bar X) - \theta) \xrightarrow{D} N(0, (1-2p)^2 p(1-p))$ $\quad p \neq 1/2$

iii) $\bar X$ je NNO, NNO θ je $T = \frac{n}{n-1}\bar X(1-\bar X)$, $\hat\theta$ nie je nestranný

je eficientný $\qquad$ nie je eficientný

mse_p



50) $X \sim Po(\lambda)$ $\quad L(\lambda) = e^{-n\lambda}\lambda^{\sum x_i}/\prod x_i!$ $\quad \ell(\lambda) = -n\lambda + \sum x_i \log\lambda + c$

i) $\ell'(\lambda) = -n + \sum x_i/\lambda = 0$ $\qquad \hat\lambda = \bar X$

$\ell''(\lambda) = -\sum x_i/\lambda^2$ $\qquad J_n(\lambda) = n/\lambda$ $\qquad \sqrt{n}(\bar X - \lambda) \xrightarrow{D} N(0,\lambda)$

ii) $\theta = e^{-\lambda}$ $\quad g(t) = e^{-t}$ $\quad g'(t) = -e^{-t}$ $\quad \sqrt{n}(e^{-\bar X} - e^{-\lambda}) \xrightarrow{D} N(0, e^{-2\lambda}\cdot\lambda)$

iii) $\bar X$ je NNO, NNO θ je $T = \left(1-\frac{1}{n}\right)^{\sum x_i}$

je eficientný $\qquad$ nie je eficientný

51) $X \sim Exp(\lambda)$ $\quad L(\lambda) = \lambda^n e^{-\lambda\sum x_i}$ $\quad \ell(\lambda) = n\log\lambda - \lambda\sum x_i$ $\quad \ell'(\lambda) = n/\lambda - \sum x_i = 0$

i) $\hat\lambda = 1/\bar X$ $\qquad \ell''(\lambda) = -n/\lambda^2$ $\qquad J_n(\lambda) = n/\lambda^2$

ii) $\sqrt{n}(1/\bar X - \lambda) \xrightarrow{D} N(0, \lambda^2)$

iii) $\frac{n-1}{\sum x_i}$ je NNO λ ale nie je eficientný

52) $X \sim Ge(p)$ $\quad L(p) = p^n(1-p)^{\sum x_i}$ $\quad \ell(p) = n\log p + \sum x_i \log(1-p)$

i) $\ell'(p) = n/p - \sum x_i/(1-p) = 0$ $\qquad \ell''(p) = -n/p^2 - \sum x_i/(1-p)^2$

$n - np - p\sum x_i = 0$ $\qquad J_n(p) = n/p^2 - \frac{\cancel{(1-p)}}{p}\frac{n}{(1-p)^2} = n\left(\frac{1}{p^2} - \frac{1}{p(1-p)}\right)$

$p(n + \sum x_i) = n$

$\hat p = \frac{n}{n+\sum x_i} = \frac{1}{1+\bar X}$ $\qquad \sqrt{n}\left(\frac{1}{1+\bar X} - p\right) \xrightarrow{D} N\left(0, \frac{1-2p}{p^2(1-p)}\right)$

ii) $\theta = p(1-p)$ $\quad \hat\theta = \frac{1}{1+\bar X}\left(1 - \frac{1}{1+\bar X}\right) = \frac{1\cdot\bar X}{(1+\bar X)^2}$

$g(t) = t(1-t)$ $\quad g'(t) = 1-2t$ $\quad \sqrt{n}(\hat\theta - \theta) \xrightarrow{D} N\left(0, \frac{(1-2p)^2\cdot(1-2p)}{p(1-p)}\right)$

53) $X \sim R(\theta - 1/2, \theta + 1/2)$ $\quad L(\theta) = \prod I[\theta - 1/2 < x_i < \theta + 1/2] = I[\theta - 1/2 < \min x_i]\, I[\max x_i < \theta + 1/2]$

$= \begin{cases} 1 & \text{ak } \theta < \min X_i + 1/2 \text{ a } \theta > \max X_i - 1/2 \\ 0 & \text{inak} \end{cases}$ $\Rightarrow$ $\hat\theta$ je ľubovoľná z hodnôt $[\max X_i - \frac12, \min X_i + \frac12]$

i)

ii) z MSI: $\min X_i \xrightarrow{P} \theta - 1/2$ $\Rightarrow$ $\max X_i - 1/2 \xrightarrow{P} \theta$ $\Rightarrow$ každý odhad je

$\max X_i \xrightarrow{P} \theta + 1/2$ $\qquad \min X_i + 1/2 \xrightarrow{P} \theta$ $\qquad$ teda konzistentný

54) $X \sim R\{1..M\}$ $\quad L(M) = \prod \frac{1}{M} I[x_i \in \{1..M\}] = \frac{1}{M^n} I[1 \le \min X_i \le \max X_i \le M] =$

i) $= \begin{cases} 1/M^n & \text{ak } M \geq \max X_i \\ 0 & \text{inak} \end{cases}$ $\Rightarrow \hat M = \max X_i$

ii) $P(\hat M \le z) = P(X_i \le z)^n = \left(\frac{z}{M}\right)^n$ $\qquad f_{\hat M}(z) = n z^{n-1}/M^n$ $\qquad = P(\hat M \le M - \lceil\varepsilon\rceil)$

$P(|\hat M - M| > \varepsilon) = P(\hat M < M - \varepsilon) \le P(\hat M \le M - \varepsilon) \overset{\le}{=} \left(\frac{M-\varepsilon}{M}\right)^n = \left(1 - \frac{\varepsilon}{M}\right)^n \xrightarrow[n\to\infty]{} 0$

ak $\varepsilon \in \mathbb{R}$

55) $Y_i \sim N(\theta x_i, 1)$ $\quad L(\theta) = c\exp\{-\frac12 \sum (y_i - \theta x_i)^2\}$ $\quad \ell(\theta) = c - \frac12 \sum (y_i - \theta x_i)^2$

i) $\ell'(\theta) = \sum (y_i - \theta x_i)\, x_i = 0$

$\sum y_i x_i = \theta \sum x_i^2$ $\qquad$ ii) $E\hat\theta = \frac{1}{\sum x_i^2}\sum x_i\, EY_i = \theta$ nestranný

$\frac{\sum y_i x_i}{\sum x_i^2} = \hat\theta$

iii) $\ell'' = -\sum x_i^2$

$RC_n = 1/\sum x_i^2$

$mse\,\hat\theta = \frac{\sum x_i^2\cdot 1}{(\sum x_i^2)^2} = \frac{1}{\sum x_i^2} = RC$

56) $P(X=k) = \frac{1}{1-(1-p)^m}\binom{m}{k} p^k (1-p)^{m-k}$ $k=1,2,\dots m$

i) $L(p) = \frac{1}{(1-(1-p)^m)^n}\prod_{i=1}^{n}\binom{m}{x_i}\cdot p^{\sum x_i}(1-p)^{\sum(m-x_i)}$

$\ell(p) = -n\log(1-(1-p)^m) + c + \sum x_i \log p + \sum(m-x_i)\log(1-p)$

$\ell'(p) = \frac{-n}{1-(1-p)^m}\cdot m(1-p)^{m-1} + \frac{\sum x_i}{p} - \frac{\sum(m-x_i)}{1-p} = 0$

$\ell''(p) =$ mathematica script

57) $X\sim \theta x^{\theta-1} e^{-x^\theta} I[X>0]$ $L(\theta) = \theta^n \left(\prod x_i\right)^{\theta-1} e^{-\sum x_i^\theta} I[\min x_i>0]$

$x^\theta = e^{\theta\log x}$

i) $\ell(\theta) = n\log\theta + (\theta-1)\log\left(\prod x_i\right) - \sum x_i^\theta$

$\ell'(\theta) = \frac{n}{\theta} + \sum\log x_i - \sum x_i^\theta \log x_i = 0$ } spojitá klesajúca fce, $\ell'(0)=\infty$, $\ell'(\infty)=-\infty$ jediné riešenie

$\ell''(\theta) = -\frac{n}{\theta^2} - \sum x_i^\theta(\log x_i)^2$

ii) $EX = \int_0^\infty \theta x^\theta e^{-x^\theta}\,dx = \int_0^\infty t^{1/\theta} e^{-t}\,dt = \Gamma\left(\frac{1}{\theta}+1\right)$

$x^\theta = t$
$x = t^{1/\theta}$

$E X^\theta(\log X)^2 =$ mathematica $= c'$

$\sqrt{n}(\hat\theta-\theta) \to N\left(0, \left(\frac{1}{\theta^2}+c'\right)^{-1}\right)$

58) $f(x) = e^{-(x-\theta)}/(1+e^{-(x-\theta)})^2$ $L(\theta) = e^{-\sum x_i + n\theta}/\prod(1+e^{-(x_i-\theta)})^2$

$\ell(\theta) = -\sum x_i + n\theta - \sum\log(1+e^{-(x_i-\theta)})^2$

i) $\ell'(\theta) = n - 2\sum \frac{1\cdot e^{-(x_i-\theta)}}{1+e^{-(x_i-\theta)}} = n - 2\sum\frac{e^{-x_i}}{e^{-\theta}+e^{-x_i}} = 0$ } spojitá kles. fce $\ell'(-\infty)=n$, $\ell(\infty) = -n$ jediné riešenie

$\ell''(\theta) = +2\sum\frac{e^{-x_i}\,(-e^{-\theta})}{(e^{-\theta}+e^{-x_i})^2}$

ii) $J_n(\theta) = \frac{n}{3}$

$E\,\frac{2e^{-X}e^{-\theta}}{(e^{-\theta}+e^{-X})^2} = \int_{\mathbb{R}} \frac{2e^{-x}\cancel{e^{-\theta}}}{(e^{-\theta}+e^{-x})^2}\,\frac{e^{-x}\cancel{e^{\theta}}}{(1+e^{-(x-\theta)})^2}\,dx = \frac{2e^{-2\theta}}{e^{-2\theta}}\int_1^\infty \frac{(t-1)}{t^4} = 2\left[\frac{-1}{2t^2}+\frac{1}{3t^3}\right]_1^\infty = \frac{1}{3}$

$1+e^{-(x-\theta)} = t$ $t-1 = e^{-x}e^{\theta}$
$(-1)\cdot e^{-(x-\theta)}\,dx = dt$ $(t-1)e^{-\theta} = e^{-x}$
$-e^{\theta}e^{-x}\,dx = dt$

$\sqrt{n}(\hat\theta-\theta) \xrightarrow{D} N(0,3)$

59) $X\sim N(\theta,\theta^2)$ $L(\theta) = c\cdot\theta^{-n}\exp\left\{-\frac{1}{2\theta^2}\sum(x_i-\theta)^2\right\}$ $\ell(\theta) = -n\log\theta + c - \frac{1}{2\theta^2}\sum(x_i-\theta)^2$

i) $\ell'(\theta) = -\frac{n}{\theta} + \frac{1\cdot\cancel{2}}{\cancel{2}\theta^2}\sum(x_i-\theta) + \frac{1\cdot\cancel{2}}{\cancel{2}\theta^3}\sum(x_i-\theta)^2 = 0$

$-n + \frac{\sum x_i - n\theta}{\theta} + \frac{\sum x_i^2 - 2\theta\sum x_i + n\theta^2}{\theta^2} = 0$

$-n + \frac{\sum x_i}{\theta} - \cancel{n} + \frac{\sum x_i^2}{\theta^2} - \frac{2\sum x_i}{\theta} + \cancel{n} = 0$

$-n\theta^2 + \theta\sum x_i - 2\theta\sum x_i + \sum x_i^2 = 0$

$\theta^2 n + \theta\sum x_i - \sum x_i^2 = 0$

$\frac{-\sum x_i \pm\sqrt{(\sum x_i)^2 + 4\sum x_i^2\, n}}{2n} = \hat\theta$

ii) $E\hat\theta = E\left[-\frac{1}{2}\bar X + \frac{1}{2}\sqrt{(\bar X)^2 + 4\frac{1}{n}\sum x_i^2}\right] \xrightarrow[n\to\infty]{s.j.} -\frac{\theta}{2} + \frac{1}{2}\sqrt{\theta^2+4\cdot 2\theta^2} = -\frac{\theta}{2}+\frac{1}{2}\sqrt{9\theta^2} = \theta$

$EX^2 = 2\theta^2$

iii) $\ell''(\theta) = \frac{n}{\theta^2} + \left(-\frac{2}{\theta^3}\sum(x_i-\theta) + \frac{1(-n)}{\theta^2}\right) + \left(\frac{-3}{\theta^4}\sum(x_i-\theta)^2 - \frac{2\sum(x_i-\theta)}{\theta^3}\right)$

$J_n(\theta) = -\frac{n}{\theta^2} + \frac{n}{\theta^2} + \frac{3n}{\theta^2} = \frac{3n}{\theta^2}$ $\sqrt{n}(\hat\theta-\theta)\xrightarrow{D} N\left(0,\frac{\theta^2}{3}\right)$

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60) $P(Y=k \mid X) = \dfrac{\lambda(x)^k e^{-\lambda(x)}}{k!}$ $\quad \lambda(x) = e^{Bx}$, X nezávisí na B

i) $L(B) = \prod P(Y_i = y_i \mid X = x_i) \cdot P(X = x_i) \left(= \prod P(Y_i = y_i, X_i = x_i)\right)$

$= \prod_{i=1}^{m} \dfrac{(e^{Bx_i})^{y_i} e^{-e^{Bx_i}}}{y_i!} P(X = x_i)$

$\ell(B) = \sum \left(y_i x_i B - e^{Bx_i} - \log(y_i!) + \log P(X = x_i)\right)$

$= \sum y_i x_i \cdot B - \sum e^{Bx_i} + c$

$\ell'(B) = \sum x_i y_i - \sum e^{Bx_i} \cdot x_i \overset{!}{=} 0$

$\ell''(B) = -\sum e^{Bx_i} x_i^2 \qquad J_m(B) = m\, E\, e^{BX} \cdot X^2 \qquad \sqrt{m}(\hat{B} - B) \xrightarrow{\mathcal{D}} N\left(0, \dfrac{1}{E e^{BX} X^2}\right)$

61) $P(X = x) = \begin{cases} p & x \in \{-1, 1\} \\ 1-2p & x = 0 \end{cases}$ $\qquad$ dg: $y := \#\{X_i = 0\}$

$L(p) = (1-2p)^y p^{m-y} \qquad \ell(p) = y \log(1-2p) + (m-y) \log p$

$\ell'(p) = \dfrac{-2y}{1-2p} + \dfrac{m-y}{p} = 0$

$\cancel{-2py} + m - 2mp - y + \cancel{2py} = 0$

$\hat{p} = (y - m)/(-2m) = (m - y)/2m = \sum I[X_i \neq 0]/2m$

$\ell''(p) = \dfrac{-2y \cdot 2}{(1-2p)^2} - \dfrac{m-y}{p^2} \qquad EY = E \sum I[X = 0] = m P(X = 0) = m(1-2p)$

$J_m(p) = \dfrac{4m\cancel{(1-2p)}}{(1-2p)^{\cancel{2}}} + \dfrac{m - m(1-2p)}{p^2} = \dfrac{4m}{1-2p} + \dfrac{2m}{p} =$

$= \dfrac{4mp + 2m - 4mp}{(1-2p)p} = m \cdot \dfrac{2}{p(1-2p)}$

$\sqrt{m}(\hat{p} - p) \xrightarrow{\mathcal{D}} N(0, p(1-2p)/2)$

62) $L(\theta) = \prod \frac{1}{2} \exp(-|x_i - \theta|) = \frac{1}{2^m} \exp\left\{-\sum |x_i - \theta|\right\}$

$\ell(\theta) = c - \sum |x_i - \theta| \Rightarrow$ maximalizovať $-\sum |x_i - \theta|$ vo θ

riešením je medián $\{x_i\}$. (riešenie online) Lemma 3.4 MSI

MLE: Vektorový parameter

63) $X \sim N(\mu, \sigma^2)$

i) $L(\mu, \sigma^2) = c \cdot (\sigma^m)^{-1} \exp\left\{-\frac{1}{2\sigma^2} \sum (x_i - \mu)^2\right\}$

$\ell(\mu, \sigma^2) = c - \frac{m}{2} \log \sigma^2 - \frac{1}{2\sigma^2} \sum (x_i - \mu)^2$

$\nabla \ell = \left(\frac{1}{\sigma^2} \sum (x_i - \mu) \; ; \; -\frac{m}{2\sigma^2} + \frac{1}{2(\sigma^2)^2} \sum (x_i - \mu)^2\right)$

$\dfrac{\partial \ell}{\partial \mu} = 0 \Rightarrow \sum x_i = m\mu, \; \hat{\mu} = \overline{X} \qquad \dfrac{\partial \ell}{\partial \sigma^2} = 0 \Rightarrow m = \dfrac{\sum (x_i - \mu)^2}{\sigma^2}, \; \hat{\sigma}^2 = \frac{1}{m} \sum (x_i - \mu)^2 \Big|_{\mu = \hat{\mu}}$

ii) $H_\ell(\mu, \sigma^2) = \begin{pmatrix} -\dfrac{m}{\sigma^2} & -\dfrac{\sum (x_i - \mu)}{\sigma^4} \\ -\dfrac{\sum (x_i - \mu)}{\sigma^4} & \dfrac{m}{2\sigma^4} - \dfrac{\sum (x_i - \mu)^2}{(\sigma^2)^3} \end{pmatrix}$

$J_m(\mu, \sigma^2) = \begin{pmatrix} m/\sigma^2 & 0 \\ 0 & \dfrac{m}{2\sigma^4} \end{pmatrix} \qquad \sqrt{m}\left(\begin{pmatrix} \overline{X} \\ \frac{m-1}{m} S^2 \end{pmatrix} - \begin{pmatrix} \mu \\ \sigma^2 \end{pmatrix}\right) \xrightarrow{\mathcal{D}} N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma^2 & 0 \\ 0 & 2\sigma^4 \end{pmatrix}\right)$