1. With given parameter $c \in \mathbb{R}$ find all solutions $x \in \mathbb{R}$ of the inequation

$$\frac{\log|x+c|}{c} \leq 2.$$

2. Which planar curve is determined by the equation

$$x^2 - 4y^2 = 2x?$$

3. Compute the limit

$$\lim_{n\to\infty} n \cdot \left(\sqrt{n^2+1} - \sqrt{n^2-1}\right)$$

or prove it does not exists.

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*Results.*

1. Notice that $c \neq 0$. The inequation rearrangements depend on positivity/negativity of $c$.

(a) $c > 0$.

$$\begin{aligned} \log|x+c| &\leq 2c \\ e^{\log|x+c|} = |x+c| &\leq e^{2c} \\ -e^{2c} \leq x+c &\leq e^{2c}, \text{ i.e.} \end{aligned}$$

$x \in \langle -e^{2c} - c,\, e^{2c} - c \rangle$.

(b) $c < 0$.

$$\log|x+c| \geq 2c,$$

the rest is similar, $|x+c| \geq e^{2c}$ gives $x \in (-\infty, -e^{2c} - c\rangle \cup \langle e^{2c} - c, +\infty)$.

2. Since the sign of coefficient of $x^2$ is positive and that of $y^2$ is negative, let us try to rearrange the equation to

$$\frac{(x-x_0)^2}{a^2} - \frac{(y-y_0)^2}{b^2} = 1$$

– the equation of hyperbola.

$x^2 - 2x = x^2 - 2x + 1 - 1 = (x-1)^2 - 1$, thus we get

$$\begin{aligned} (x-1)^2 - 4y^2 &= 1, \text{ i.e.} \\ (x-1)^2 - \frac{y^2}{\frac{1}{4}} &= 1. \end{aligned}$$

This is an equation of hyperbola with center $[1, 0]$, open horizontally, with semiaxes $a = 1$ and $b = \frac{1}{2}$.

3. The straightforward computation gives '$\infty \cdot (\infty - \infty)$' – an undefined expression, thus we have to rearrange the square roots first.

$$\begin{aligned}
\sqrt{n^2+1} - \sqrt{n^2-1} &= \left(\sqrt{n^2+1} - \sqrt{n^2-1}\right) \cdot \frac{\sqrt{n^2+1} + \sqrt{n^2-1}}{\sqrt{n^2+1} + \sqrt{n^2-1}} \\
&= \frac{2}{\sqrt{n^2+1} + \sqrt{n^2-1}},
\end{aligned}$$

consequently

$$\begin{aligned}
\lim_{n\to\infty} n \cdot \left(\sqrt{n^2+1} - \sqrt{n^2-1}\right) &= \lim_{n\to\infty} \frac{2n}{\sqrt{n^2+1} + \sqrt{n^2-1}} \\
&= \lim_{n\to\infty} \frac{2n}{n \cdot \left(\sqrt{1+\frac{1}{n^2}} + \sqrt{1-\frac{1}{n^2}}\right)} \\
&= \lim_{n\to\infty} \frac{2}{\left(\sqrt{1+\frac{1}{n^2}} + \sqrt{1-\frac{1}{n^2}}\right)}.
\end{aligned}$$

Now, $\frac{1}{n^2} \to 0$, and arithmetics of limits applies:

$$\lim_{n\to\infty} n \cdot \left(\sqrt{n^2+1} - \sqrt{n^2-1}\right) = \frac{2}{\sqrt{1} + \sqrt{1}} = 2.$$