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The Potential Fluid Flow Problem and the Convergence Rate of the Minimal Residual Method

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In the paper the potential fluid flow problem in porous media using the Darcy's law and the continuity equation is solved. Mixed-hybrid finite element formulation based on general trilateral prismatic elements is considered. Spectral properties of resulting symmetric indefinite system of linear equations are examined. Minimal residual method for the solution of systems with a symmetric indefinite matrix is applied. The rate of convergence and the asymptotic convergence factor which depend on the eigenvalue distribution of the system matrix are estimated.

KEY WORDS   Potential flow problem in porous media Mixed-hybrid formulation trilateral prismatic elements Symmetric indefinite linear systems Minimal residual method Rate of convergence Asymptotic convergence factor

1. Introduction

The potential flow of polluted groundwater in porous media is usually modelled by a mixed or mixed-hybrid formulations which allow more accurate treatment of the velocities of the studied contaminant in comparison to primal formulation. Moreover, in some applications, the model must reflect the complex geographical and geological structure of the considered area. Layers of stratified rocks with substantially different physical properties must be modelled using the appropriate discretization of geological region. The flow domain is usually fairly large (several squared kilometres) in comparison to the vertical thickness of sedimented layers (several metres). These characteristics can be correspondingly described by trilateral prismatic elements with vertical faces and general nonparallel bases. Detailed description of this element type and a corresponding mixed-hybrid formulation of the problem has been presented in [1], where existence and uniqueness of the approximate solution from the resulting system of linear equations is proved. The mixed-hybrid formulation gives rise to the solution of a linear system with a matrix which is symmetric, but due to the

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zero diagonal block necessarily indefinite. Since the classical conjugate gradient method for such symmetric indefinite systems is inapplicable, the related minimal residual method (MINRES) can be applied. Because the system matrix is normal, the rate of convergence of the MINRES method depends on its eigenvalue distribution. In particular, the estimate for the relative residual norm is given by the best minimal polynomial approximations on a set of eigenvalues of the system matrix. These are usually estimated by the solution of a continuous problem on an optimal inclusion set for the spectrum of the matrix. The solution of such approximation problem for the case of inclusion set with two disjoint intervals on the positive and negative part of real axis, respectively, is well defined, but unfortunately there is no explicit expression for the optimal polynomial in general. Instead, the asymptotic convergence factor is considered. It is given by the Green's function for the complement of the corresponding inclusion set. For the particular case of two disjoint intervals, this function can be expressed in terms of elliptic functions. This approach enables the determination of the asymptotic convergence rate of the MINRES method and its dependence on the mesh size from the discretization of the formulation can be derived.

The outline of the paper is as follows. In Section 2, we briefly recall the definition of the problem and describe the finite element discretization resulting in the system with symmetric indefinite matrix. In Subsection 2.1, the spectral properties of the positive definite diagonal block are analyzed. Using graph theory tools the singular values of the off-diagonal block of the system matrix are estimated. In particular, the dependence on the mesh size parameter is studied. In Subsection 2.3 these quantities are related to the eigenvalue distribution of the whole indefinite matrix. Section 3 introduces the minimal residual method. In Subsection 3.1 the theoretical rate of the method applied to the system is estimated using the derived eigenvalue distribution. Subsection 3.2 includes the estimation of the asymptotic convergence factor similar to one used by Wathen, Fischer and Silvester in [2] for our particular case of an inclusion set dependent on the mesh size parameter. In Section 4 numerical experiments illustrating the previous theoretical results are presented. Finally, Section 5 contains some concluding remarks, open questions and directions for the future work.

2. Mixed-hybrid finite element approximation of the problem

Let Ω be a bounded domain in R^3 with a Lipschitz continuous boundary $\partial\Omega$. Potential fluid flow in saturated porous media can be modelled by the velocity $\mathbf{u}$ given by Darcy's law

$$\mathbf{A}\mathbf{u} = -\nabla p, \tag{2.1}$$

where p is the piezometric potential and $\mathbf{A}$ is the symmetric and uniformly positive definite second rank tensor of hydraulic permeability of the porous medium. We assume $[\mathbf{A}(\mathbf{x})]_{ij} \in L^\infty(\Omega)$ for all $i, j \in \{1, 2, 3\}$. Consider also the continuity equation for the incompressible flow

$$\nabla \cdot \mathbf{u} = q, \tag{2.2}$$

where q represents the density of potential sources in the medium. The boundary conditions are given by

$$p = p_D \quad \text{on} \quad \partial\Omega_D, \tag{2.3}$$

$$\mathbf{u} \cdot \mathbf{n} = u_N \quad \text{on} \quad \partial\Omega_N, \tag{2.4}$$

where $\partial\Omega = \overline{\partial\Omega_D} \cup \overline{\partial\Omega_N}$ are such that $\partial\Omega_D \cap \partial\Omega_N = \emptyset$ and $\mathbf{n}$ is the outward normal vector defined (almost everywhere) on the boundary $\partial\Omega$.

Assume that the domain Ω is a polyhedron and is subdivided into a collection of subdomains such that every subdomain is a trilateral prism with the suitably chosen vertices $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5, \mathbf{x}_6$ (see Figure 1),

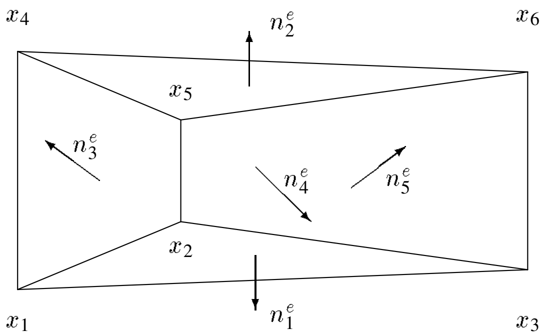
$$\begin{array}{lll} \mathbf{x}_1 = (x_1, y_1, z_1)\,, & \mathbf{x}_2 = (x_2, y_2, z_2)\,, & \mathbf{x}_3 = (x_3, y_3, z_3)\,, \\ \mathbf{x}_4 = (x_1, y_1, z_4)\,, & \mathbf{x}_5 = (x_2, y_2, z_5)\,, & \mathbf{x}_6 = (x_3, y_3, z_6)\,. \end{array}$$

We will denote the discretization of the domain Ω by $\mathcal{E}_h$ and for every element $e \in \mathcal{E}_h$ we define $h_1^e = \max_{\{\mathbf{x}_i, \mathbf{x}_j \in e\}} [(x_{i1}-x_{j1})^2+(x_{i2}-x_{j2})^2]^{\frac{1}{2}}$ and $h_2^e = \max_{\{\mathbf{x}_i, \mathbf{x}_j \in e\}} |x_{i3}-x_{j3}|$. Define the parameters of horizontal and vertical discretization by

$$h_1 = \max_{e\in\mathcal{E}_h} h_1^e, \;\; h_2 = \max_{e\in\mathcal{E}_h} h_2^e,$$

and denote by σ the maximal angle of the prismatic discretization $\sigma = \max_{e\in\mathcal{E}_h} \sigma^e$, where σ^e is the maximal angle between the lower and upper base of the element e. Introduce also the collection of faces of elements $e \in \mathcal{E}_h$ which are not adjacent to the boundary $\partial\Omega_D$ by $\Gamma_h = \cup_{e\in\mathcal{E}_h} \partial e - \partial\Omega_D$.

Figure 1. General trilateral prismatic element



The velocity function $\mathbf{u}$ is approximated by the vector functions linear on every element $e \in \mathcal{E}_h$. We define the Raviart-Thomas space

$$\mathbf{RT}^0_{-1}(\mathcal{E}_h) = \{\mathbf{v}_h \in \mathbf{L}^2(\Omega);\; \mathbf{v}_h^e \in \mathbf{RT}^0(e),\, \forall e \in \mathcal{E}_h\}, \tag{2.5}$$

where $\mathbf{RT}^0(e)$ is the subspace spanned by the linearly independent basis functions $\mathbf{v}_j^e,\ j = 1, ..., 5$, of the form

$$\mathbf{v}_i^e = k_i^e \begin{bmatrix} 0 \\ 0 \\ x_3 - \alpha_{i3}^e \end{bmatrix},\ i = 1, 2,\;\; \mathbf{v}_i^e = k_i^e \begin{bmatrix} x_1 - \alpha_{i1}^e \\ x_2 - \alpha_{i2}^e \\ \beta_i^e x_3 - \alpha_{i3}^e \end{bmatrix},\ i = 3, 4, 5 \tag{2.6}$$

such that

$$\mathcal{F}_j(\mathbf{v}_i^e) = \int_{f_j^e} \mathbf{n}_j^e \cdot \mathbf{v}_i^e dS = \delta_{ij}, \; i, j = 1, ..., 5. \tag{2.7}$$

Here f_j^e denotes the j-th face of the element e and $\mathbf{n}_j^e = (n_{j1}^e, n_{j2}^e, n_{j3}^e)$ its outward normal vector (with respect to the element e).

From the conditions (2.7) for $i = 1, 2$ we have

$$\alpha_{i3}^e = x_{T3}(f_j^e), \; k_i^e = \{n_{i3}^e[x_{T3}(f_i^e) - x_{T3}(f_j^e)]|f_i^e|\}^{-1}, \; j = 1, 2, \; j \neq i \tag{2.8}$$

where $|f_i^e|$ denotes area of the i-th face of the element e and $x_{Ti}(f_j^e)$ denotes the i-th component of the centre of gravity of the j-th face of the element e.

For $i = 3, 4, 5$ we obtain the system of equations for the unknowns α_{i1}^e, α_{i2}^e, α_{i3}^e and β_i^e:

$$\begin{gathered} n_{j1}^e\alpha_{i1}^e + n_{j2}^e\alpha_{i2}^e + n_{j3}^e\alpha_{i3}^e - n_{j3}^e x_{T3}(f_j^e)\beta_i^e = n_{j1}^e x_{T1}(f_j^e) + n_{j2}^e x_{T2}(f_j^e), \; j = 1, 2 \\ n_{j1}^e\alpha_{i1}^e + n_{j2}^e\alpha_{i2}^e = 0, \; j = 3, 4, 5; \; j \neq i. \end{gathered} \tag{2.9}$$

The piezometric potential p will be approximated by the space $M_{-1}^0(\mathcal{E}_h)$ which consists the element-wise constant functions and by the space $M_{-1}^0(\Gamma_h)$ of the functions constant on every face from Γ_h. Let $e_j \in \mathcal{E}_h$, $j = 1, \ldots, J$, be the numbered system of prismatic elements and f_k, $k = 1, \ldots, K$, be the numbered system of interelement faces from Γ_h. The finite-dimensional space $\mathbf{RT}_{-1}^0(\mathcal{E}_h)$ is then spanned by $5 \times J$ linearly independent basis functions $\mathbf{v}_i$, $i = 1, \ldots, I = 5 \times J$. Consequently, from the Raviart-Thomas approximation of the problem (for the details we refer to [1]), we obtain the system of linear algebraic equations of the form

$$\begin{pmatrix} A & B & C \\ B^T & & \\ C^T & & \end{pmatrix} \begin{pmatrix} u \\ p \\ \lambda \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}, \tag{2.10}$$

where $u = (u_1, ..., u_I)^T$, $p = (p_1, ..., p_J)^T$, $\lambda = (\lambda_1, ..., \lambda_K)^T$ are unknowns and the matrix blocks are given by

$$A_{ij} = (\mathbf{A}\mathbf{v}_i, \mathbf{v}_j)_{0,\Omega}; \; i = 1, \ldots, I, \; j = 1, \ldots, I, \tag{2.11}$$

$$B_{ij} = -(\nabla \cdot \mathbf{v}_i, 1)_{0,e_j}; \; i = 1, \ldots, I, \; j = 1, \ldots, J, \tag{2.12}$$

$$C_{ik} = < \mathbf{n}_k \cdot \mathbf{v}_i, 1 >_{f_k}; \; i = 1, \ldots, I, \; k = 1, \ldots, K. \tag{2.13}$$

Here $\mathbf{n}_k$ is the outward normal vector to the face f_k with respect to the element corresponding to the support of the function $\mathbf{v}_i$. The right-hand side vector satisfies

$$(q_1)_i = - < p_{D,h}, \mathbf{n}_i \cdot \mathbf{v}_i >_{\partial\Omega_D}; i = 1, \ldots, I, \tag{2.14}$$

$$(q_2)_j = -(q, 1)_{0,e_j}; \; j = 1, \ldots, J, \tag{2.15}$$

$$(q_3)_k = < u_{N,h}, 1 >_{f_k}; \; k = 1, \ldots, K. \tag{2.16}$$

2.1. The spectrum of the symmetric positive definite block A

The matrix block A is symmetric positive definite and is composed of diagonal blocks $A^e \in R^{5,5}$, where $[A^e]_{ij} = (\mathbf{Av}_i^e, \mathbf{v}_j^e)_{0,e},\ i,j = 1,\dots,5$. Since the matrix A^e is independent of the translation and rotation, we choose the coordinate system so that for some i the vertices $\mathbf{x}_i,\ \mathbf{x}_{i+3}$ belong to axis x_3 and the centre of gravity of lower base belongs to the plane spanned by x_1 and x_2. From (2.7) for some positive constants c_α^e, $c_{k,1}^e$ and $c_{k,2}^e$ we have

$$\alpha_{13}^e = 0,\ \alpha_{23}^e = c_\alpha^e h_2,\ k_i^e = c_{k,i}^e (h_1^2 h_2)^{-1},\ i = 1,2. \tag{2.17}$$

From (2.9) it follows $\alpha_{i1}^e = \alpha_{i2}^e = 0$ for $i = 3,4,5$. Using (2.9) and that $x_{T3}(f_1^e) = 0$ we have

$$n_{13}^e \alpha_{i3}^e = n_{11}^e x_{T1}(f_1^e) + n_{12}^e x_{T2}(f_1^e) = |\mathbf{x}_T|.\sin\sigma^e,$$
$$(n_{13}^e + n_{23}^e)\alpha_{i3}^e - n_{23}^e x_{T3}(f_2^e)\beta_i^e = (n_{11}^e + n_{21}^e)x_{T1}(f_1^e) + (n_{12}^e + n_{22}^e)x_{T2}(f_1^e).$$

Consequently, for some constant c_σ^e

$$\alpha_{i3}^e = |\mathbf{x}_T|\tan\sigma^e = c_\sigma^e h_1,\ \ -n_{23}^e x_{T3}(f_2^e)\beta_i^e = \tag{2.18}$$
$$= (n_{11}^e + n_{21}^e)x_{T1}(f_1^e) + (n_{12}^e + n_{22}^e)x_{T2}(f_1^e) - (n_{13}^e + n_{23}^e)|\mathbf{x}_T|\tan\sigma^e \tag{2.19}$$

hold. For the coefficient β_i^e we obtain

$$\beta_i^e = c_\beta^e \frac{h_1}{h_2}\sin\sigma^e, \tag{2.20}$$

where c_β^e is a positive constant independent of the discretization. Note that $\beta_i^e \neq 0$ if and only if the bases are nonparallel. Finally we evaluate the coefficients k_i^e, which are equal to

$$k_i^e = [\mathbf{n}_i^e \cdot \mathbf{x}_T(f_i^e)]^{-1}|f_i^e|^{-1} = c_{k,i}^e \frac{1}{h_1^2 h_2} \tag{2.21}$$

for some positive constants $c_{k,i}^e$, $i = 3,4,5$. Assuming the diagonal tensor of permeability we finally get the estimates for components of matrix A^e in the form

$$(\mathbf{Av}_i^e, \mathbf{v}_j^e)_{0,e} = \hat{c}_{i,j}^e \frac{h_2}{h_1^2},\ i,j = 1,2, \tag{2.22}$$

$$(\mathbf{Av}_i^e, \mathbf{v}_j^e)_{0,e} = \hat{c}_{i,j}^e \frac{1}{h_2},\ i,j = 3,4,5, \tag{2.23}$$

$$(\mathbf{Av}_i^e, \mathbf{v}_j^e)_{0,e} = \hat{c}_{i,j}^e \frac{1}{h_1}\sin\sigma,\ i = 1,2,\ j = 3,4,5. \tag{2.24}$$

where $\hat{c}_{i,j}^e$ are positive constants independent of the discretization parameters h_1, h_2 and σ. For any vector function $\mathbf{u} = (u^1, u^2, u^3, u^4, u^5)$ such that $\sum_{i=1}^5 (u^i)^2 = 1$ it follows

$$(A^e\mathbf{u}, \mathbf{u}) = \sum_{i,j=1}^{5} u^i u^j (\mathbf{Av}_i^e, \mathbf{v}_j^e)_{0,e} \le \frac{\hat{c}^e}{h_2}\sum_{i,j=1}^{5} u^i u^j \le \frac{5\hat{c}^e}{h_2}, \tag{2.25}$$

where $\hat{c}^e$ is given as $\hat{c}^e = max_{i,j}\hat{c}^e_{i,j},\ i,j = 1,\ldots,5$. The minimal eigenvalue of the matrix A^e will be estimated using the estimate for the maximal eigenvalue of the matrix inverse $(A^e)^{-1}$. The maximal eigenvalue of the latter is bounded by the maximum norm. Let us estimate the maximal element of the matrix $(A^e)^{-1}$ using the direct approximation of algebraic complements. It is easy to see that for some positive constants $\tilde{c}^e$

$$|(A^e)^{-1}_{ij}| \le \tilde{c}^e \frac{h_1^2}{h_2},\ i,j = 1,\ldots,5. \tag{2.26}$$

Consequently, for any vector $\mathbf{u} \in R^5$ with $||\mathbf{u}|| = 1$ we have

$$((A^e)^{-1}\mathbf{u},\mathbf{u}) \le 5\tilde{c}^e \frac{h_1^2}{h_2}. \tag{2.27}$$

Using the estimates (2.25) and (2.27) for the extremal eigenvalues of the matrix A^e we obtain

$$\sigma(A^e) \subset [\frac{h_2}{5\tilde{c}^e h_1^2}, 5\hat{c}^e\frac{1}{h_2}]. \tag{2.28}$$

Introducing the parameter $\varrho = h_2/h_1 < 1$, which relates the parameters of vertical and horizontal discretization and denoting $h = h_1$ we get the inclusion set for the eigenvalues of the matrix block A in the form

$$\sigma(A) \subset [c_1\frac{1}{h}, c_2\frac{1}{h}], \tag{2.29}$$

where $c_1 = min_{e\in\mathcal{E}_h}\varrho/(5\tilde{c}^e)$ and $c_2 = 5max_{e\in\mathcal{E}_h}\hat{c}^e$.

2.2. *Singular values of the matrix block* $(B|C)$

It is easy to see that the system matrix of the system (2.10) is nonsingular if and only if the block $(B|C)$ has a full column rank. In order to get the eigenvalues of the symmetric indefinite matrix (2.10) it is important to estimate the extremal singular values of the matrix block $(B|C)$.

From the definition of the blocks B and C one can easily derive that their euclidean norms satisfy the relations

$$||B|| = \sqrt{\lambda_{max}(B^TB)} = \sqrt{5}\ and\ ||C|| = \sqrt{\lambda_{max}(C^TC)} = \sqrt{2},$$

where $\lambda_{max}(X)$ denotes the maximal eigenvalue of the matrix X. Consequently, for the euclidean norm of the block $(B|C)$ we get the estimate of the maximal singular value σ_1 of the matrix $(B|C)$ in the form

$$||(B|C)|| = \sigma_1((B|C)) \le \sqrt{5} + \sqrt{2}.$$

The standard approach in the mixed finite element method for estimating the minimal singular value of the corresponding nondiagonal block is to apply the discrete version of the Babuška-Brezzi condition (see e.g. [3], [4]). The situation with the hybrid mixed formulation is much more complicated, the Babuška-Brezzi condition cannot be used and direct analysis of the structure of the matrix block $(B|C)$ must be considered. We estimate the minimal singular value of the rectangular matrix $(B|C)$ by the expression

$$\inf_{\|(p^T,\lambda^T)\|=1} \sup_{\|v\|\leq 1} ((B|C)(p^T,\lambda^T)^T, v). \tag{2.30}$$

Consider i-th row of the matrix $(B|C)$ which defines the situation on i-th face of the element e. The nonzeros in the i-th row are in the column of B which corresponds to the element e (with the value -1) and in the columns of C that correspond to faces of the element e (with the value 1). Note that the outer faces with the Dirichlet boundary conditions do not induce the columns of C (see [1]). Let us extend the matrix C to cover also the faces that correspond to Dirichlet part of boundary $\partial\Omega_D$ and denote the extension of the matrix block C with zero columns by $\tilde{C}$. Further, define the functional with the extended variable vector $\tilde{\lambda}$

$$\tilde{\mathcal{F}}(p,\tilde{\lambda},v) = ((B|\tilde{C})(p^T,\tilde{\lambda}^T)^T, v). \tag{2.31}$$

It is easy to see that the supremum of the functional $\tilde{\mathcal{F}}(p,\tilde{\lambda},v)$ for a given vector $(p,\tilde{\lambda})$ and $||v|| \leq 1$ is achieved for the vector in the form

$$v^{e(i)} = \frac{\tilde{\lambda}^{e(i)} - p^e}{\sqrt{\sum_{e\in\mathcal{E}_h}\sum_{i=1}^{5}|\tilde{\lambda}^{e(i)} - p^e|^2}}. \tag{2.32}$$

Its value is equal to

$$\tilde{\mathcal{H}}(p,\tilde{\lambda}) = \sqrt{\sum_{e\in\mathcal{E}_h}\sum_{i=1}^{5}|\tilde{\lambda}^{e(i)} - p^e|^2}. \tag{2.33}$$

Here p^e denotes the component of the vector p corresponding to the element e, $v^{e(i)}$ is the component of the vector v which corresponds to the i-th face of the element e and, similarly, $\tilde{\lambda}^{e(i)}$ denotes the component of the unknown vector $\tilde{\lambda}$ corresponding to the $i-th$ face of the element e. The desired estimate of (2.30) can be obtained from the lower bound of $\tilde{\mathcal{H}}(p,\tilde{\lambda})$ with respect to the condition

$$||(p^T,\tilde{\lambda}^T)|| = 1. \tag{2.34}$$

To compute this bound we need to introduce some graph theory tools; for the basic graph terminology we refer, e.g., to [5]. Consider the following definition.

Definition 2.1. *Let $G = (V,E)$ be a mixed graph (a graph with both directed arcs and undirected edges). Suppose that $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$, $V_1 \neq \emptyset$, $|V| = n$, $E = A_1 \cup E_2$, $A_1 \subset V_1 \times V_2$, $E_2 \subset \binom{V_2}{2}$. Let $d : V \longrightarrow R$ and $w : E \longrightarrow R$ be mappings such that $(\forall\vartheta \in V_1)(d(\vartheta) = 0)$ and*

$$(\forall\vartheta_i,\vartheta_j \text{ such that } e = \{\vartheta_i,\vartheta_j\} \in E \text{ or } e = (\vartheta_i,\vartheta_j) \in E)(w(e) = d(\vartheta_j) - d(\vartheta_i)).$$

The mixed graph with these mappings we will call a mixed potential network.

The following lemma uses depth first search trees of graphs. The basic tree terminology can be found in [6].

Lemma 2.1. *Let $G = (V, E)$ be a mixed potential network. Let D_V be its potential defined by $D_V = \sum_{\vartheta \in V} d^2(\vartheta)$. Let W_E be its weight defined by $W_E = \sum_{e \in E} w^2(e)$. Let $T = (V, \hat{E})$ be a set of the disjoint depth-first search trees (a depth-first search forest) rooted in V_1. Then the weight $W_{\hat{E}}$ of the graph T satisfies*

$$W_{\hat{E}} \leq W_E.$$

Proof Since all the terms which contribute to the values of $W_{\hat{E}}$ and W_E are nonnegative and since T is a subgraph of G we get the statement. We note that the potentials of G and T are equal. ∎

The following lemma estimates the value of the weight of a tree with the fixed value of its potential.

Lemma 2.2. *Let $T = (\tilde{V}, \tilde{E})$ be a mixed potential network such that T is a rooted tree rooted in $\vartheta_r \in \tilde{V}_1$. Let k be the length of the longest path in T and $D_V = K$ be the potential of T. Then there exists a constant $\tilde{c}_0$ such that $W_{\tilde{E}} \geq \tilde{c}_0 K k^{-2}$.*

Proof Construct the rooted level structure for ϑ_r (see [6]). Then each vertex of $\tilde{V}$ is in some level set. The level l_{ϑ_r} of ϑ_r is zero. The level l_ϑ of a vertex ϑ being the neighbour of ϑ_r is equal to 1. The level l_ϑ of a vertex ϑ is equal to the length of a unique path from ϑ_r to ϑ in T. Consider potentials of the vertices in $\tilde{V}$. Let $\vartheta \in \tilde{V}$ be the vertex in the l_ϑ-th level of the rooted level structure. Then using the Schwartz inequality we can write

$$d(\vartheta)^2 \leq l_\vartheta \sum_{e \in P(\vartheta_r, \vartheta)} w^2(e),$$

where $P(\vartheta_r, \vartheta)$ is the unique path from ϑ_r to ϑ in T. Summing up these inequalities for all the vertices from $\tilde{V}$ we obtain

$$D_V \leq \sum_{\vartheta \in \tilde{V}} (l_\vartheta \sum_{e \in P(\vartheta_r, \vartheta)} w^2(e)).$$

On the right-hand side, the coefficient by any $w^2(e)$ for $e \in \tilde{E}$ is at most of the order k^2. A trivial manipulation provides the relation $W_{\tilde{E}} \geq \tilde{c}_0 K k^{-2}$ for a positive constant $\tilde{c}_0$. It has to be proved. ∎

Theorem 2.1. *Let $G = (V, E)$ be a mixed potential network with $|V| = n$. Let the potential of G be $D_V = 1$. Let T be its depth-first search tree rooted in V_1 having the maximal depth among all such trees. Denote its value by k. Then there exists a constant c_0 such that*

$$W_E \geq c_0 k^{-2}.$$

Proof The result is a direct consequence of the depth-first forest relaxation from Lemma 2.1. and the estimate of the weight of a tree in Lemma 2.2.. ∎

The structure $\mathcal{E}_h$ can be represented by the graph $G_{\mathcal{E}_h}$. The set V_1 corresponds to the outer faces with the Dirichlet boundary conditions. Set V_2 corresponds to the elements and remaining faces of the structure. We take $V = V_1 \cup V_2$. Edges and arcs correspond to the topology of the structure. Define the mapping $d : V \longrightarrow R$ as follows:

$$d(\vartheta) = \begin{cases} \tilde{\lambda}^{e(i)} & \text{where } \vartheta \text{ corresponds to the } i\text{-th face of the element } e, \\ \tilde{p}^e & \text{where } \vartheta \text{ corresponds to the element } e. \end{cases}$$

Then $G_{\mathcal{E}_h}$ with the defined vertices and edges, with the mapping d and with the mapping w related to d is a mixed potential network.

Using the Theorem 2.1. and its notation we obtain the lower bound for the functional $\tilde{\mathcal{H}}(p, \tilde{\lambda})$ with respect to the the condition (2.34) in the form

$$\tilde{\mathcal{H}}(p, \tilde{\lambda}) \geq \sqrt{c_0 k^{-2}}.$$

Considering uniformly regular mesh refinement, there exists a positive constant $\hat{c}$ such that $k = \hat{c}h^{-1}$. From the definition of the functional $\tilde{\mathcal{H}}(p, \tilde{\lambda})$ it follows

$$\inf_{\|(p^T, \lambda^T)\|=1} \sup_{\|v\| \leq 1} ((B|C)(p^T, \lambda^T)^T, v) \geq \frac{\sqrt{c_0}}{\bar{c}} h. \tag{2.35}$$

Denoting the constants $c_4 = \sqrt{5} + \sqrt{2}$ and $c_3 = \sqrt{c_0}/\bar{c}$ for the singular values of the matrix block $(B|C)$ we have

$$sv(B|C) \subset [c_3 h, c_4]. \tag{2.36}$$

2.3. Eigenvalues of the resulting symmetric indefinite matrix

In this subsection we consider the result presented by Winther and Rusten in [7], which relates the eigenvalues of the resulting symmetric indefinite system (2.10) to the spectrum of the positive definite block A and singular values of the nondiagonal matrix block $(B|C)$. We consider the following lemma.

Lemma 2.3. *Let $0 < \mu_1 \leq \mu_2 \leq \ldots \leq \mu_I$ be the eigenvalues of the positive definite matrix $A \in R^{I,I}$, $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{K+J} > 0$ the singular values of the matrix $(B|C) \in R^{I,J+K}$. Then for the eigenvalues of the matrix*

$$\mathsf{A} = \begin{pmatrix} A & B & C \\ B^T & & \\ C^T & & \end{pmatrix} \tag{2.37}$$

we have

$$\begin{aligned} \sigma(\mathsf{A}) \quad \subset \quad & [\frac{1}{2}(\mu_1 - \sqrt{\mu_1^2 + 4\sigma_1^2}), \frac{1}{2}(\mu_I - \sqrt{\mu_I^2 + 4\sigma_{K+J}^2})] \cup \\ & \cup [\mu_1, \frac{1}{2}(\mu_I + \sqrt{\mu_I^2 + 4\sigma_1^2})]. \end{aligned}$$

Proof see [7], Lemma 2.1, pp. 3-4. ■

From results of previous two subsections, the statement of Lemma 2.3. and omitting the terms with higher order of the parameter h one can easily obtain the inclusion set for the eigenvalues of the system matrix (2.10) depending on the mesh size parameter in the form

$$[-\frac{c_4^2}{c_1} h, -\frac{c_3^2}{c_2} h^3] \cup [\frac{c_1}{h}, \frac{c_2}{h}]. \tag{2.38}$$

3. The minimal residual method

One of the most known and widely used iterative schemes for the solution of symmetric indefinite systems is the minimal residual method (MINRES) presented by Paige and Saunders in 1975. Consider the system of linear algebraic equations of the form

$$\mathsf{A}x = b, \tag{3.39}$$

where A is a symmetric indefinite nonsingular matrix of order N and b is an N-dimensional right-hand side vector. The minimal residual method starts with an initial guess of the solution x_0 and computes the n-th approximate solution x_n in the form

$$x_n \in x_0 + K_n(\mathsf{A}, r_0), \tag{3.40}$$

where $r_0 = b - \mathsf{A}x_0$ is the initial residual and $K_n(\mathsf{A}, r_0)$ is the n-th Krylov subspace generated by the matrix A and the residual r_0,

$$K_n(\mathsf{A}, r_0) = span\ \{r_0, \mathsf{A}r_0, \ldots, \mathsf{A}^{n-1}r_0\} \tag{3.41}$$

and satisfying the minimal residual norm property

$$||b - \mathsf{A}x_n|| = \min_{u \in x_0 + K_n(\mathsf{A}, r_0)} ||b - \mathsf{A}u||. \tag{3.42}$$

For implementation details as well as for different possible variants of the method we refer to the original paper [8] and other papers [9], [10] and [7].

3.1. Theoretical rate of convergence

From (3.40) it is clear, that the error of the n-th approximation $x - x_n$ and the n-th residual $r_n = b - \mathsf{A}x_n$ can be written as

$$\begin{aligned} x - x_n &= P_n(\mathsf{A})(x - x_0) \\ r_n &= P_n(\mathsf{A})r_0, \end{aligned} \tag{3.43}$$

for some $P_n \in \Pi_n$, where Π_n denotes the set of polynomials $P(\lambda)$ of degree at most n satisfying $P(0) = 1$. Then (3.42) can be equivalently reformulated to

$$||P_n(\mathsf{A})r_0|| = \min_{P \in \Pi_n} ||P(\mathsf{A})r_0||. \tag{3.44}$$

Consider the set of eigenvalues $\{\lambda_1, \lambda_2, ..., \ \lambda_N\}$ of the symmetric indefinite matrix A in the form

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_p < 0 < \lambda_{p+1} \leq \ldots \leq \lambda_N \tag{3.45}$$

and denote by $\{u_1, u_2, \ldots, u_N\}$ the set of corresponding normalized basis eigenvectors. Considering the expansion of the initial residual

$$r_0 = \sum_{j=1}^{N} \zeta_j u_j \tag{3.46}$$

and substituting to (3.44) for the n-th residual norm we obtain

$$\|r_n\|^2 = \sum_{j=1}^{N} \zeta_j^2 P_n^2(\lambda_j) = \min_{P \in \Pi_n} \sum_{j=1}^{N} \zeta_j^2 P^2(\lambda_j). \tag{3.47}$$

Consequently,

$$\|r_n\| \leq \min_{P \in \Pi_n} \max_{\lambda_i \in \sigma(\mathsf{A})} |P(\lambda_i)| \, \|r_0\| \tag{3.48}$$

holds. The estimate (3.48) is sharp, i.e., that for each n, there is an initial residual vector r^0 (depending on n) for which the equality holds (see e.g. [11]). In view of this sharp estimate the convergence of the minimal residual method depends strongly on the distribution of eigenvalues of the matrix A. To estimate the convergence rate of the MINRES method, we consider the approximation problem

$$\max_{\lambda \in \mathcal{G}} |R_n(\lambda)| = \min_{P \in \Pi_n} \max_{\lambda \in \mathcal{G}} |P(\lambda)|, \tag{3.49}$$

for some polynomial $R_n(\lambda)$ of degree at most n and where $\mathcal{G}$ is the minimal inclusion set for the spectrum $\sigma(\mathsf{A})$ which does not contain the origin ($0 \notin \mathcal{G}$).

Since the matrix A is symmetric indefinite (and nonsingular) the set $\mathcal{G}$ must have the form of two disjoint intervals on the positive and negative real axis, respectively. Assume therefore the inclusion set of the form

$$\mathcal{G} = [-c, -d] \cup [a, b],$$

where a, b, c and d are positive numbers. The approximation problem (3.49) always has a unique solution (cf. e.g. [12]), but unfortunately, it is known only for the special cases of two disjoint intervals. One of them is the case of the inclusion set of the form $\mathcal{G} = [-b, -a] \cup [a, b]$. Then for the relative residual norm generated by the MINRES method at n-th iteration, we have the bound

$$\|r^n\| \leq 2 \, (\frac{\frac{b}{a} - 1}{\frac{b}{a} + 1})^{\lfloor n/2 \rfloor} \, \|r^0\|. \tag{3.50}$$

Considering the symmetric inclusion set for (2.38) we obtain for the relative residual norm the bound

$$\frac{\|r^n\|}{\|r^0\|} \leq 2 \, (\frac{1 - (c_3/c_2)^2 h^4}{1 + (c_3/c_2)^2 h^4})^{\lfloor n/2 \rfloor}. \tag{3.51}$$

3.2. *The asymptotic convergence factor*

The bound for the convergence rate (3.51) is clearly a rough overestimate. The convergence rate can be much more satisfactorily described by the asymptotic convergence factor

$$\gamma(\mathcal{G}) = \lim_{n\to+\infty} [\min_{P\in\Pi_n} \max_{\lambda\in\mathcal{G}} |P(\lambda)|]^{\frac{1}{n}}. \tag{3.52}$$

It is known that the asymptotic convergence factor is given as by the Green's function for the complement of $\mathcal{G}$ with a logarithmic singularity at the point of infinity (for details see [13], [14] and references therein). In the following text assuming the inclusion set of the form (2.38) and using the results of Wathen, Fischer and Silvester in [2] (see also [14]) we give a bound for the asymptotic convergence factor in terms of the discretization parameter h.

It is well known (see, e.g., [14]) that optimal polynomial $R_n(\lambda)$ defined by (3.49) has at least $n+1$ and at most $n+2$ extremal points on the inclusion set $\mathcal{G}$, where $\mathcal{G}$ is the union of two disjoint intervals. The sets $\mathcal{G}$ for which the optimal polynomial has $n+2$ extremal points are of particular interest for the problem (3.49). These sets are explicitly known and the solutions are given in terms of elliptic functions. For details we refer to [14] and references therein. Consider now, for simplicity, the inclusion set

$$\mathcal{G}(h) = [-ah, -bh^3] \cup [\frac{c}{h}, \frac{d}{h}], \tag{3.53}$$

where $a = c_4^2/c_1$, $b = c_3^2/c_2$, $c = c_1$ and $d = c_2$. Following [2], we normalize the set (3.53) using the map

$$\ell(x) = \frac{2x - (d/h - ah)}{d/h + ah}. \tag{3.54}$$

It is easy to see that if $R_n(\lambda)$ has $n+2$ extremal points on the normalized set

$$\mathcal{I}(h) = [-1, \ell(-bh^3)] \cup [\ell(c/h), 1]$$

then $R_n(\lambda)$ has also $n+2$ extremal points on $\mathcal{G}$. Introduce the modulus

$$m = \frac{2(\ell(c/h) - \ell(-bh^3))}{(1-\ell(-bh^3))(1+\ell(c/h))}$$

and compute the argument of the elliptic sine with respect to m

$$\varrho(m) = -sn^{-1}(\sqrt{(1-\ell(-bh^3))/2}; m). \tag{3.55}$$

The argument $\varrho(m)$ satisfies $\varrho(m) \in (-K(m), 0)$, where $K(m)$ is the complete elliptic integral of the first kind. Considering the Proposition 3.1 in [2] (see also [15]) we have

Theorem 3.1. *The polynomial $R_n(\lambda)$ has $n+2$ extremal points on the set $\mathcal{I}(h)$ if and only if there exists a number $k \in \{1, 2, \ldots, n-1\}$ such that*

$$\varrho(m) = -\frac{k}{n}K(m).$$

Since $\varrho(m) \in (-K(m), 0)$ for every n there exists $j_n \in \{0, 1, \ldots, n-1\}$ such that

$$\varrho(m) \in [-\frac{j_n + 1}{n} K(m), -\frac{j_n}{n} K(m)]. \tag{3.56}$$

Following Wathen et al. in [2] we define

$$\tilde{\varrho}(m) = -\frac{j_n + 1}{n} K(m)$$

and nonnegative quantities ε_n^b and ε_n^c by

$$\ell(-bh^3 - \varepsilon_n^b) = 1 - 2sn^2(\tilde{\varrho}(m)) \tag{3.57}$$

$$\ell(c/h - \varepsilon_n^c) = 2sn^2(K(m) + \tilde{\varrho}(m)) - 1. \tag{3.58}$$

Since sn is a continuous function, we have obtained the following proposition.

Theorem 3.2. *For any h and $\varepsilon > 0$ there exists a number n_* such that the polynomial $R_n(\lambda)$ has $n+2$ extremal points on the set $[-ah, -bh^3 - \varepsilon_n^b] \cup [c/h - \varepsilon_n^c, d/h]$ for all $n \geq n_*$ and some $0 < \varepsilon_n^b, \varepsilon_n^c \leq \varepsilon$. Consequently*

$$[-ah, -bh^3 - \varepsilon_n^b] \cup [c/h - \varepsilon_n^c, d/h] \to \mathcal{G}(h) \ as \ n \to +\infty.$$

Applying this procedure to the slightly enlarged set $[-ah, -bh^3 + h^4] \cup [c/h, d/h]$ and redefining $\varepsilon_n^b = h^4 - \varepsilon_n^b$ we obtain that the optimal polynomial $\tilde{R}_n(\lambda)$ defined on the set

$$\tilde{\mathcal{G}}(h) = [-ah, -bh^3 + \varepsilon_n^b] \cup [\frac{c}{h} - \varepsilon_n^c, \frac{d}{h}] \tag{3.59}$$

has $n+2$ extremal points for all $n \geq n_*$ with $0 \leq \varepsilon_n^b, \varepsilon_n^c \leq h^4$. Moreover, $\mathcal{G}(h) \subset \tilde{\mathcal{G}}(h)$ and therefore

$$\min_{P \in \Pi_n} \max_{\lambda \in \mathcal{G}(h)} |P(\lambda)| \leq \min_{P \in \Pi_n} \max_{\lambda \in \tilde{\mathcal{G}}(h)} |P(\lambda)| = \max_{\lambda \in \tilde{\mathcal{G}}(h)} |\tilde{R}_n(\lambda)| = \tilde{e}_n. \tag{3.60}$$

It is known [2] that the extremal polynomial with $n+2$ extremal points has all four boundary points of the set $\tilde{\mathcal{G}}(h)$ extremal and satisfies the ordinary differential equation

$$\frac{d\tilde{R}_n(\lambda)}{d\lambda} = \frac{-n(\lambda - \tilde{f}_n)(\tilde{R}_n^2(\lambda) - \tilde{e}_n^2)^{1/2}}{[(\frac{d}{h} - \lambda)(\lambda + ah)(\lambda + bh^3 - \varepsilon_n^b)(\frac{c}{h} - \varepsilon_n^c - \lambda)]^{1/2}}, \tag{3.61}$$

where $\tilde{f}_n$ is the unique point in the interval $(-bh^3 + \varepsilon_n^b, c/h - \varepsilon_n^c)$ with $\tilde{R}_n'(\tilde{f}_n) = 0$. Similarly to [2], separating the variables and integrating from y to z, where $-bh^3 + \varepsilon_n^b < y \leq z < c/h - \varepsilon_n^c$ we obtain

$$\int_y^z \frac{(\lambda - \tilde{f}_n)d\lambda}{[(\frac{d}{h} - \lambda)(\lambda + ah)(\lambda + bh^3 - \varepsilon_n^b)(\frac{c}{h} - \varepsilon_n^c - \lambda)]^{\frac{1}{2}}} = ln \left| \frac{\tilde{R}_n(y) + [\tilde{R}_n^2(y) - \tilde{e}_n^2]^{\frac{1}{2}}}{\tilde{R}_n(z) + [\tilde{R}_n^2(z) - \tilde{e}_n^2]^{\frac{1}{2}}} \right|^{\frac{1}{n}}.$$

From the properties of the optimal polynomial $\tilde{R}_n(\lambda)$ it follows

$$\lim_{z\nearrow c/h-\varepsilon_n^c \; y\searrow -bh^3+\varepsilon_n^b} \int_y^z \frac{(\lambda-\tilde{f}_n)d\lambda}{[(\frac{d}{h}-\lambda)(\lambda+ah)(\lambda+bh^3-\varepsilon_n^b)(\frac{c}{h}-\varepsilon_n^c-\lambda)]^{1/2}} = 0. \tag{3.62}$$

Dividing (3.62) into two parts and applying the integral mean value theorem we obtain

$$\lim_{z\nearrow c/h-\varepsilon_n^c} g_1(h)\int_{\tilde{f}_n}^z \frac{(\lambda-\tilde{f}_n)d\lambda}{[(\lambda+bh^3-\varepsilon_n^b)(\frac{c}{h}-\varepsilon_n^c-\lambda)]^{1/2}} +$$

$$+\lim_{y\searrow -bh^3+\varepsilon_n^b} g_2(h)\int_y^{\tilde{f}_n} \frac{(\lambda-\tilde{f}_n)d\lambda}{[(\lambda+bh^3-\varepsilon_n^b)(\frac{c}{h}-\varepsilon_n^c-\lambda)]^{1/2}} =$$

$$= \left[g_2(h)-g_1(h)\right]\left\{(\tilde{f}_n+bh^3-\varepsilon_n^b)(\frac{c}{h}-\varepsilon_n^c-\tilde{f}_n)\right\}^{1/2}+$$

$$+\left[g_2(h)+g_1(h)\right]\frac{\pi}{2}\left\{\frac{1}{2}(\frac{c}{h}-\varepsilon_n^c)-\frac{1}{2}(bh^3-\varepsilon_n^b)-\tilde{f}_n\right\}+$$

$$+\left[g_2(h)-g_1(h)\right]\left\{\frac{1}{2}(\frac{c}{h}-\varepsilon_n^c-bh^3+\varepsilon_n^b)-\tilde{f}_n\right\} arcsin\left(\frac{\frac{c}{h}-\varepsilon_n^c-bh^3+\varepsilon_n^b-2\tilde{f}_n}{\frac{c}{h}-\varepsilon_n^c+bh^3-\varepsilon_n^b}\right),$$

where $g_1(h)$ and $g_2(h)$ satisfy the inequalities

$$g_1(h),\ g_2(h)\in\left[\frac{h}{[dc]^{1/2}}+O(h^2),\frac{1}{[(d-c)a]^{1/2}}+O(h)\right]. \tag{3.63}$$

Lemma 3.1. *The unique turning point $\tilde{f}_n$ for large n satisfies $\tilde{f}_n = O(\frac{1}{h})$.*

Proof Assuming that $\tilde{f}_n$ expressed as a power of h in the form $\tilde{f}_n = O(h^\psi)$ and since $\tilde{f}_n \in (-bh^3+\varepsilon_n^b, c/h-\varepsilon_n^c)$ we have $\psi\in[-1,3]$. Supposing $\psi>-1$ it follows that

$$arcsin\left(\frac{\frac{c}{h}-\varepsilon_n^c-bh^3+\varepsilon_n^b-2\tilde{f}_n}{\frac{c}{h}-\varepsilon_n^c+bh^3-\varepsilon_n^b}\right)=\frac{\pi}{2}+O(h^{min(2,\sqrt{\psi+1})}).$$

Consequently, (3.62) cannot be satisfied. ■

Since the turning point satisfies $\tilde{f}_n>0$ taking the limits $y\searrow -bh^3+\varepsilon_n^b$ and $z=0$ we obtain

$$\lim_{y\searrow -bh^3+\varepsilon_n^b}\int_y^0 \frac{(\lambda-\tilde{f}_n)d\lambda}{[(\frac{d}{h}-\lambda)(\lambda+ah)(\lambda+bh^3-\varepsilon_n^b)(\frac{c}{h}-\varepsilon_n^c-\lambda)]^{\frac{1}{2}}} = ln\frac{\tilde{e}_n^{\frac{1}{n}}}{[1+(1-\tilde{e}_n^2)^{\frac{1}{2}}]^{\frac{1}{n}}}.$$

Applying the integral mean value theorem to the previous equality we get

$$ln\frac{\tilde{e}_n^{1/n}}{[1+(1-\tilde{e}_n^2)^{1/2}]^{1/n}} = -g_1(h)\left\{(bh^3-\varepsilon_n^b)(\frac{c}{h}-\varepsilon_n^c)\right\}^{1/2}+$$

$$+g_1(h)\left\{\frac{1}{2}(\frac{c}{h}-\varepsilon_n^c)-\frac{1}{2}(bh^3-\varepsilon_n^b)-\tilde{f}_n\right\}\left\{\frac{\pi}{2}-arcsin\left(\frac{\frac{c}{h}-\varepsilon_n^c-bh^3+\varepsilon_n^b}{\frac{c}{h}-\varepsilon_n^c+bh^3-\varepsilon_n^b}\right)\right\}.$$

Table 1. Number of elements and matrix dimension

mesh size h	elements	dimension N
1/2	16	144
1/3	54	477
1/4	128	1120
1/6	432	3744
1/8	1024	8832
1/12	3456	29664

Considering the standard asymptotics for arcsin function we have

$$arcsin\left(\frac{\frac{c}{h}-\varepsilon_n^c-bh^3+\varepsilon_n^b}{\frac{c}{h}-\varepsilon_n^c+bh^3-\varepsilon_n^b}\right)=arcsin\left(1-\frac{2b}{c}h^4+O(h^5)\right)=\frac{\pi}{2}+O(h^2).$$

Using Lemma 3.1. and providing that $\varepsilon_n^b,\ \varepsilon_n^c\le h^4$ for the asymptotic convergence factor

$$\gamma(\mathcal{G})=\lim_{n\to+\infty}[\min_{P\in\Pi_n}\max_{\lambda\in\mathcal{G}}|P(\lambda)|]^{\frac{1}{n}}\le 1-c_\gamma h^2 \tag{3.64}$$

holds for some constant c_γ dependent on the coefficients a, b, c and d and independent of the mesh size parameter h.

4. Numerical example

In this section we present a numerical example that illustrates the theoretical results from the previous sections. Our test problem is the potential flow problem (2.1) in a rectangular domain. Dirichlet boundary conditions on two opposite faces and homogeneous Neumann conditions on the other faces of the domain are prescribed. Different values of mesh size are considered. The number of elements used in the discretization and the dimension of the resulting system for different values of the parameter h are included in the Table 1. We note that the dimension of the problem depends linearly on the number of elements used, and consequently, dependence on the discretization parameter can be estimated as $N\sim 22/h^3$.

Extremal eigenvalues of the symmetric positive definite block A and extremal singular values of the off-diagonal block $(B|C)$ were calculated using the double precision implementations of LAPACK and NAG subroutines [16]. These values are tabulated in Table 2. The extremal eigenvalues of the matrix block A presented in 2 correspond to the theoretical results in Subsection 2.1 The minimal singular value decreases linearly with the discretization parameter h for this problem and thus the bound for the singular values of the block $(B|C)$ derived in Subsection 2.2 seems to be realistic.

We calculated also the extremal eigenvalues of the resulting symmetric indefinite matrix in the form (2.10). These values are included in Table 3 and they illustrate that estimates given by Lemma 2.3. are nearly sharp. We note here that $c_1, c_2 \ll c_3, c_4$ holds for this example and so the inclusion set (2.38) is valid only for very small values of discretization parameter h.

The minimal residual method (MINRES) was applied. For the implementation details of the method we refer to the original paper [8]. The initial approximation x_0 was set to zero

Table 2. Extremal eigenvalues and singular values of the matrix blocks

mesh size h	spectrum of A	sing. values of $(B\|C)$
1/2	[0.11e-2, 0.66e-2]	[0.276, 2.577]
1/3	[0.16e-2, 0.10e-1]	[0.197, 2.611]
1/4	[0.22e-2, 0.13e-1]	[0.152, 2.624]
1/6	[0.33e-2, 0.19e-1]	[0.104, 2.635]
1/8	[0.44e-2, 0.26e-1]	[0.079, 2.639]
1/12	[0.66e-2, 0.40e-1]	——

Table 3. Extremal eigenvalues of the whole indefinite matrix

mesh size h	negative part	positive part
1/2	[-2.57, -0.27]	[0.13e-2, 2.57]
1/3	[-2.60, -0.19]	[0.18e-2, 2.61]
1/4	[-2.62, -0.15]	[0.23e-2, 2.62]
1/6	[-2.63, -0.13]	[0.34e-2, 2.63]
1/8	[-2.63, -0.10]	[0.44e-2, 2.64]

and an unpreconditioned algorithm was used. The convergence behavior of the method for different values of the parameter h is presented in Figure 2.

Similarly to [2] in Table 4 we consider the iteration counts after relative residual norm reduction of ε and the average contraction factor (in brackets) computed as $\gamma_n = (\|r_n\|/\|r_0\|)^{1/n}$.

We have extrapolated the dependence of the average contraction factors on the mesh size parameter h for the relative residual norm $\frac{\|r_n\|}{\|r_0\|} = 10^{-12}$ by the function $y = 1 - \alpha x^{\beta}$. We have formulated the task as a general nonlinear least squares problem with constraints for the unknown variables α, β. The 'envelope' achieved in the form

$$\gamma_n \leq 1 - 1.875h^{2.95}$$

overestimates the bound for the asymptotic convergence factor (3.64). The values of discretization parameters h considered are relatively large so the convergence rates observed (especially for $h = 1/2$) may differ substantially from the asymptotic convergence rate.

Table 4. Number of iterations (average contraction factor)

h	$\frac{\|r_n\|}{\|r_0\|} = 10^{-4}$	$\frac{\|r_n\|}{\|r_0\|} = 10^{-8}$	$\frac{\|r_n\|}{\|r_0\|} = 10^{-12}$
1/2	58 (0.853)	78 (0.789)	98 (0.754)
1/3	164 (0.945)	245 (0.927)	365 (0.927)
1/4	229 (0.960)	389 (0.953)	520 (0.948)
1/6	350 (0.974)	669 (0.973)	1033 (0.973)
1/8	395 (0.977)	805 (0.977)	1168 (0.977)
1/12	529 (0.982)	1083 (0.983)	1682 (0.983)

5. Concluding remarks

We have concentrated on the spectral properties of the symmetric indefinite matrices that arise from the mixed-hybrid finite element discretization of the potential fluid flow problem. The minimal residual method (MINRES) was applied and the asymptotic convergence rate, using the approach of Wathen et al. in [2] was estimated.

The numerical experiment in Section 4 indicates that the bounds for the eigenvalue set of the positive definite block A as well as the bound for the singular values of the off-diagonal block $(B|C)$ are realistic. Also the bounds for the eigenvalue inclusion set of the resulting symmetric indefinite matrix (2.10) are nearly sharp. It is not clear, as was also reported in [2], whether the bound for the asymptotic convergence factor is pessimistic or not. We note here again that our values of the discretization parameter in numerical experiments may not correspond to the asymptotic behavior given by the bound (3.64).

The convergence rate of the method can be improved by preconditioning. As was pointed out by Silvester and Wathen in a personal communication, simple diagonal preconditioning of the block A with the matrix hI_I (I_I denotes the identity matrix of order I) gives rise to a matrix block which has eigenvalues in a fixed interval independent of discretization parameter h. For example, two sided preconditioning of (2.10) by the diagonal matrix $diag(h^{1/2}I_I, I_{J+K})$ would lead to the bound for the asymptotic convergence factor in the form $\gamma(\mathcal{G}) \leq 1 - c_\gamma h^{3/2}$. Two different degrees of freedom in the choice of proper diagonal preconditioner for two corresponding matrix blocks may give a more tight analysis. Other preconditioning techniques for indefinite systems and particularly the analysis of the symmetric indefinite factorizations of indefinite matrices remain an open question.

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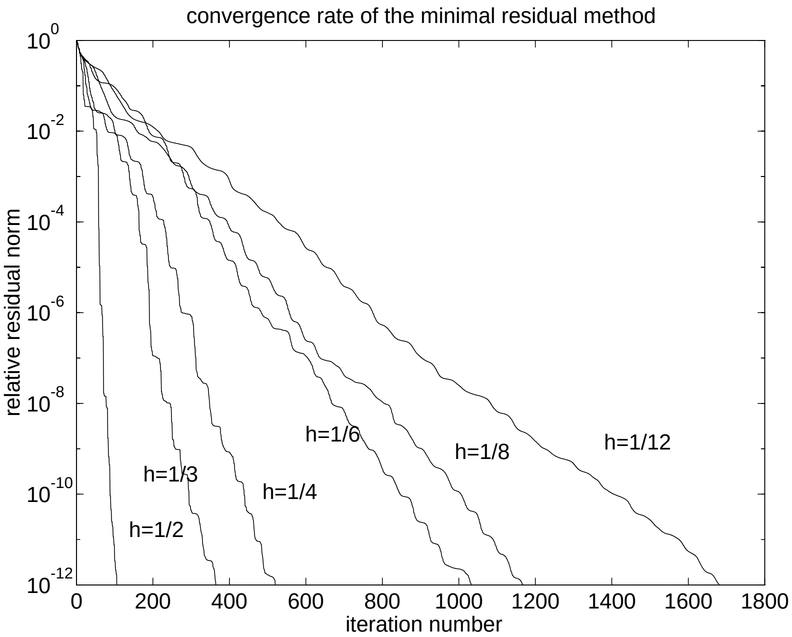

Figure 2. Convergence rate of the MINRES method