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SOLUTION OF AUGMENTED SYSTEMS FROM A MIXED-HYBRID FINITE ELEMENT DISCRETIZATION OF THE POTENTIAL FLUID FLOW PROBLEM: ASYMPTOTIC RATES OF CONVERGENCE

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Abstract. In the paper we consider several approaches for solving linear systems from a mixed-hybrid finite element discretization of the Darcy's law and the continuity equation describing the potential fluid flow in porous media. Spectral properties of resulting symmetric but indefinite systems in terms of a mesh size parameter are discussed and the asymptotic rate of convergence of iterative solvers applied either to whole indefinite system, to successive Schur complement systems or to systems projected onto certain null-spaces is estimated.

Key words. potential fluid flow problem, mixed-hybrid finite element approximation, symmetric indefinite linear systems, iterative solution, Schur complement system, null-space method, conjugate gradient-type methods, asymptotic rate of convergence

AMS subject classifications. 35J25, 65F10, 65F50

1. Introduction.

The potential fluid flow problem is one of the most important and most frequently solved problems in such applications as underground water flow modelling and hydraulics, oil reservoir engineering or modelling the environmental impacts of pollution [13], [16]. The fluid flow in porous medium is usually described by the Darcy's law which gives a relation between the potential head (pressure) and the fluid velocity which must also satisfy the continuity equation representing the mass conservation law in the domain. Mixed-hybrid finite element discretization seems to be very efficient and popular approximation technique for this type of problems, especially when an accurate approximation of the fluid velocity is required [2]. The lowest order Raviart-Thomas approximation on general prismatic elements [5] [6] with five faces leads to a system of linear equations for components of the velocity vector u, for components of the pressure vector p and for Lagrange multipliers λ in the form

$$\begin{pmatrix} A & B & (C_1\ C_2) \\ B^T & & \\ (C_1\ C_2)^T & & \end{pmatrix} \begin{pmatrix} u \\ p \\ \lambda \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}. \tag{1}$$

Here the square matrix block A is a discrete form of Darcy's law tensor; the matrix block B^T enforces the continuity equation on every element; the block C_1^T ensures the continuity of the velocity vector across the interior inter-element faces and C_2^T stands for the fullfilment of Neumann boundary conditions. For details we refer to [6] or [5]. The matrix block A is element-wise block-diagonal and symmetric positive definite.

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It was shown in [6], [7] that its eigenvalues are in the interval

$$\sigma(A) \subset [\frac{c_1}{h}, \frac{c_2}{h}], \tag{2}$$

where h is a discretization parameter (mesh size) and where c_1 and c_2 are positive constants independent of the discretization and of the parameters of resulting linear systems (like its dimension). However, they do depend on the properties and numerical values of Darcy's tensor and on the geometry of a considered domain. The condition number $\kappa(A) = c_2/c_1$ is therefore not dependent on h. The block B is a face-element incidence matrix (with weights equal -1), the block C_1 is an interior face incidence matrix (with weights 1 and -1 in each column) and C_2 is a boundary condition incidence matrix. The matrix blocks B and $C = (C_1\ C_2)$ are thus, up to some normalization coefficients, orthogonal matrices with $B^TB = 5 \star I$, $C_1^TC_1 = 2 \star I$ and $C_2^TC_2 = I$. Nevertheless, the whole off-diagonal block $(B\ C)$ in the system (1) is no longer orthogonal and its condition number depends on the discretization parameter h. It was shown in [7] that assuming at least one Dirichlet condition imposed on a boundary the singular values of $(B\ C)$ satisfy the conditions

$$sv(B\ C) \subset [c_3 h, c_4]; \tag{3}$$

where again c_3 and $c_4 = \sqrt{10}$ are positive constants independent of the system parameters and dependent on physical and geometrical properties of a given domain. Using the result of Rusten and Winther [11] the eigenvalues of the whole indefinite matrix in the system (1) can be related to the eigenvalues of the block A and to the singular values of the block $(B\ C)$.

LEMMA 1.1. *Let $0 < \mu_{min} \leq \ldots \leq \mu_{max}$ be the eigenvalues of the symmetric positive definite matrix block A, $\sigma_{max} \geq \ldots \geq \sigma_{min} > 0$ be the singular values of the matrix block $(B\ C)$. Then for the spectrum of the whole indefinite matrix*

$$\mathsf{A} = \begin{pmatrix} A & (B\ C) \\ (B\ C)^T & \end{pmatrix} \tag{4}$$

it follows

$$\begin{aligned} \sigma(\mathsf{A}) \subset & [\frac{1}{2}(\mu_{min} - \sqrt{\mu_{min}^2 + 4\sigma_{max}^2}), \frac{1}{2}(\mu_{max} - \sqrt{\mu_{max}^2 + 4\sigma_{min}^2})] \\ \cup \quad & [\mu_{min}, \frac{1}{2}(\mu_{max} + \sqrt{\mu_{max}^2 + 4\sigma_{max}^2})]. \end{aligned}$$

Thus the inclusion set for the eigenvalues of the system matrix A in (1) has the form

$$[-\frac{c_4^2}{c_1}h, -\frac{c_3^2}{c_2}h^3] \cup [\frac{c_1}{h}, \frac{c_2}{h}]. \tag{5}$$

We note here that since $h \to 0$ in (5) we have omitted higher order terms. We will use the same approach throughout the paper.

For practically feasible discretization parameters h in our application [7] more realistic conditions are represented by $c_1/h \ll c_4$, $c_2/h \ll c_4$ and $c_2/h \gg c_3 h$. Then for the eigenvalues of the symmetric indefinite matrix in (1) (provided that above stated conditions are valid for fairly small values of discretization parameter h) we have

$$[-c_4, -\frac{c_3^2}{c_2}h^3] \cup [\frac{c_1}{h}, c_4]. \tag{6}$$

In our paper we will consider two diagonal scalings of the matrix in the system (1). The first scaling in the form

$$\begin{pmatrix} h^{1/2}I & \\ & I \end{pmatrix} \begin{pmatrix} A & (B\ C) \\ (B\ C)^T & \end{pmatrix} \begin{pmatrix} h^{1/2}I & \\ & I \end{pmatrix}$$

$$= \begin{pmatrix} hA & h^{1/2}(B\ C) \\ h^{1/2}(B\ C)^T & \end{pmatrix} = \begin{pmatrix} \mathbf{A} & (\mathbf{B}\ \mathbf{C}) \\ (\mathbf{B}\ \mathbf{C})^T & \end{pmatrix} \tag{7}$$

leads to the symmetric positive block $\mathbf{A}$ independent of h with $\sigma(\mathbf{A}) \in [c_1, c_2]$ and for the off-diagonal block $(\mathbf{B}\ \mathbf{C})$ it follows $sv(\mathbf{B}\ \mathbf{C}) \subset [c_3 h^{3/2}, c_4 h^{1/2}]$. Using Lemma 1.1 the eigenvalues of the scaled symmetric indefinite matrix (7) are in the inclusion set

$$[-\frac{c_4^2}{c_1}h, -\frac{c_3^2}{c_2}h^3] \cup [c_1, c_2]. \tag{8}$$

The second diagonal scaling

$$\begin{pmatrix} h^{1/2}I & \\ & h^{-1/2}I \end{pmatrix} \begin{pmatrix} A & (B\ C) \\ (B\ C)^T & \end{pmatrix} \begin{pmatrix} h^{1/2}I & \\ & h^{-1/2}I \end{pmatrix}$$

$$= \begin{pmatrix} hA & (B\ C) \\ (B\ C)^T & \end{pmatrix} = \begin{pmatrix} \mathbf{A} & (B\ C) \\ (B\ C)^T & \end{pmatrix}, \tag{9}$$

leads to the diagonal block $\mathbf{A}$, while the off-diagonal block $(B\ C)$ remains untouched. The eigenvalues of the scaled matrix (9) are then in the set

$$[\frac{1}{2}(c_1 - \sqrt{c_1^2 + 4c_4^2}), -\frac{c_3^2}{c_2}h^2] \cup [c_1, \frac{1}{2}(c_2 + \sqrt{c_2^2 + 4c_4^2}).]. \tag{10}$$

From the form of inclusion sets (5) - (10) it is clear that the conditioning of the resulting symmetric indefinite matrix in (1) may depend heavily on the scaling used before actual solution of the system. Indeed, we can see from (5) that in the unscaled case the condition number of the matrix A is of order $O(h^{-4})$, the same is true for the more realistic but still unscaled case of inclusion set (6), while for the first diagonal scaling we see from (8) that the condition number of the matrix in (7) depends on h as $O(h^{-3})$. Finally, in the case of diagonal preconditioning (9) we obtain dependence of order $O(h^{-2})$.

In the case of large systems it is rather obvious that some iterative method must be applied at certain step of a solution process (either to the whole system, to some Schur complement system or to a system projected onto a null-space corresponding to some off-diagonal block). Since the rate of convergence of iterative solvers depends heavily on the eigenvalue distribution of a system matrix [3], it is very important to have a good scaling which would guarantee a reasonable asymptotic rate of convergence. In practical situations, it would be desirable to obtain the asymptotic rate of convergence which is not dependent on the discretization parameter h. In subsequent sections we will show that for our application the asymptotic rate of convergence of conjugate gradient-type methods depends at least linearly on h (with a reasonable scaling) for all approaches considered in this paper.

2. Minimal residual method applied to the whole system. The system of linear algebraic equations (1) can be solved by several approaches. It is straightforward to apply an efficient iterative solver directly to the system resulting from the discretization without any preprocessing or problem reduction technique. The system matrix in (1) is symmetric but due to the zero diagonal block necessarily indefinite. Since for such systems the classical conjugate gradient (CG) method cannot be used in its original form [4] the related minimal residual (MINRES) method [10] must be applied. It is well known fact that the rate of convergence of MINRES depends on the eigenvalue distribution of a system matrix [3]. In particular, its relative residual norm can be estimated via the best minimal polynomial approximation on the spectrum [3]. This discrete approximation problem is then relaxed to the polynomial approximation on a continuous inclusion set, which consists of two disjoint intervals, one in the positive and one in the negative part of real axis (see [3] or [14]). If we denote by r_n, $n = 0, 1, \ldots$ the residual vectors generated by the MINRES method applied to (1) its asymptotic rate of convergence can be satisfactorily described by the asymptotic convergence factor

$$\lim_{n\to+\infty}\left(\frac{\|r_n\|}{\|r_0\|}\right)^{1/n} \tag{11}$$

which can be bounded further by a solution of the following approximation problem

$$\lim_{n\to+\infty}\left(\frac{\|r_n\|}{\|r_0\|}\right)^{1/n} \leq \lim_{n\to+\infty}[\min_{P\in\Pi_n}\max_{\lambda\in\mathcal{G}}|P(\lambda)|]^{\frac{1}{n}}, \tag{12}$$

where $\mathcal{G}$ is an inclusion set for the spectrum of the original system matrix (1) and Π_n is a set of polynomials of degree at most n with $P(0) = 1$. For details of this analysis we refer to papers [15], [14] or [7]. Using the approach of Wathen, Fischer and Silvester it was shown in [7] that if we assume our original problem (1) with the inclusion set in the form

$$\mathcal{G}(h) = [-\frac{c_4^2}{c_1}h, -\frac{c_3^2}{c_2}h^3] \cup [\frac{c_1}{h}, \frac{c_2}{h}], \tag{13}$$

then estimating the optimal polynomial from (12) we get a bound for the asymptotic convergence factor in the form

$$\lim_{n\to+\infty}\left(\frac{\|r_n\|}{\|r_0\|}\right)^{\frac{1}{n}} \leq 1 - c_5h^2, \tag{14}$$

where c_5 is a constant dependent on the coefficients c_1, c_2, c_3 and c_4 and independent of the mesh size parameter h. Considering the more realistic conditions (6) with the inclusion set

$$\mathcal{G}(h) = [-c_4, -\frac{c_3^2}{c_2}h^3] \cup [\frac{c_1}{h}, c_4], \tag{15}$$

we get the bound for the asymptotic convergence factor in the form

$$\lim_{n\to+\infty}\left(\frac{\|r_n\|}{\|r_0\|}\right)^{\frac{1}{n}} \leq 1 - c_6h. \tag{16}$$

The first diagonal scaling of indefinite system (7) with the inclusion set

$$\mathcal{G}(h) = [-\frac{c_4^2}{c_1}h, -\frac{c_3^2}{c_2}h^3] \cup [c_1, c_2]. \tag{17}$$

leads to the bound for the asymptotic convergence factor in the form

$$\lim_{n\to+\infty}\left(\frac{\|r_n\|}{\|r_0\|}\right)^{\frac{1}{n}} \le 1 - c_7 h^{3/2} \tag{18}$$

while the second diagonal scaling (9) with the inclusion set

$$\mathcal{G}(h) = [\frac{1}{2}(c_1 - \sqrt{c_1^2 + 4c_4^2}), -\frac{c_3^2}{c_2}h^2] \cup [c_1, \frac{1}{2}(c_2 + \sqrt{c_2^2 + 4c_4^2})] \tag{19}$$

gives the bound for the asymptotic convergence factor in the form

$$\lim_{n\to+\infty}\left(\frac{\|r_n\|}{\|r_0\|}\right)^{\frac{1}{n}} \le 1 - c_8 h. \tag{20}$$

The constants c_6, c_7 and c_8 are positive and depend only on the coefficients c_1, c_2, c_3 and c_4. Again, in all our bounds we have omitted higher order terms with the parameter h. The bound (14) indicates that the asymptotic rate of convergence of MINRES can be very slow for the original problem without scaling and it may depend quadratically on the discretization parameter h. It is also clear from (18) and (20) that the convergence of the method can be improved significantly by a diagonal preconditioning of the system (1). Then the asymptotic convergence factor depends at most linearly on h. This is the case also for realistic problems in our application with the inclusion set (15). The detailed analysis of the asymptotic convergence factor in MINRES applied to several types of inclusion sets in various application fields can be found in [14] and [15].

3. Iterative solution of Schur complement systems. Since the matrix blocks in the system (1) are rather sparse the approach based on a partial elimination of certain unknowns may be a very efficient alternative. The elimination of some matrix blocks can be followed by iterative solution of resulting Schur complement systems. In particular, for our system (1) a successive reduction to three symmetric and positive definite Schur complement systems is considered and iterative conjugate residual (CR) method is applied [12]. The first and second Schur complement matrices are formed by elimination of the velocity and pressure unknowns, respectively. It was shown in [8] that they remain reasonably sparse and they can be easily assembled. This approach is known as a process of static condensation (see e.g. [2]). In addition, the third Schur complement system can be obtained without an additional fill-in by elimination of a part of Lagrange multipliers. The rate of convergence of CR applied to the symmetric positive definite Schur complement systems depends on the eigenvalue distribution of corresponding Schur complement matrices [3]. From the spectral analysis in [8] it follows that due to particular block structure of the system (1) the spectral properties of successive Schur complement matrices do not deteriorate and their condition numbers can be bounded in terms of extremal eigenvalues of A and of extremal singular values of $(B\ C)$. In the following we denote by A/A the (first) Schur complement matrix obtained after elimination of the velocity unknowns u, the second Schur complement after elimination of the unknowns p will be denoted by $(-\mathsf{A}/A)/A_{11}$ and $((-\mathsf{A}/A)/A_{11})/B_{22}$ will stand for the third Schur complement system which corresponds also to elimination of unknowns related to the block C_2. We can give the following proposition.

LEMMA 3.1. *Let $0 < \mu_{min} \le \ldots \le \mu_{max}$ be the eigenvalues of the symmetric positive definite matrix block A, let $\sigma_{max} \ge \ldots \ge \sigma_{min} > 0$ be the singular values of* *the matrix* $(B\ C)$. *Then for the eigenvalues of the Schur complement matrix* $-\mathsf{A}/A = (B\ C)^T A^{-1}(B\ C)$ *it follows*

$$\sigma(-\mathsf{A}/A) \subset [\frac{\sigma_{min}^2}{\mu_{max}}, \frac{\sigma_{max}^2}{\mu_{min}}]. \tag{21}$$

Moreover, for the condition numbers of Schur complement matrices $(-\mathsf{A}/A)/A_{11}$ *and* $((-\mathsf{A}/A)/A_{11})/B_{22}$ *we have*

$$\kappa(((-\mathsf{A}/A)/A_{11})/B_{22}) \leq \kappa((-\mathsf{A}/A)/A_{11}) \leq \kappa(-\mathsf{A}/A). \tag{22}$$

Considering the previous lemma (for its proof and other details see [8]) the inclusion set for the spectrum of the matrix $-\mathsf{A}/A$ in the case of the original system (1) is

$$\sigma(-\mathsf{A}/A) \subset [\frac{c_3^2}{c_2}h^3, \frac{c_4^2}{c_1}h]. \tag{23}$$

The first diagonal scaling (7) leads to the same inclusion set, while in the case of the diagonal scaling (9) we have for its spectrum

$$\sigma(-\mathsf{A}/A) \subset [\frac{c_3^2}{c_2}h^2, \frac{c_4^2}{c_1}]. \tag{24}$$

The bound for its condition number is, however, for all three cases (the unscaled system and two scaled systems) of order $O(h^{-2})$ and it has the form

$$\kappa(-\mathsf{A}/A) \leq \frac{c_4^2 c_2}{c_3^2 c_1 h^2}. \tag{25}$$

All three resulting Schur complement matrices $-\mathsf{A}/A$, $(-\mathsf{A}/A)/A_{11}$ and $((-\mathsf{A}/A)/A_{11})/B_{22}$ are symmetric positive definite and the conjugate gradient method [4] or the conjugate residual method [12] can be applied. For the CR method applied to the first Schur complement system it follows from [3] that relative residual norm satisfies the bound

$$\frac{\|r_n\|}{\|r_0\|} \leq 2\left(\frac{1 - 1/\sqrt{\kappa(-\mathsf{A}/A)}}{1 + 1/\sqrt{\kappa(-\mathsf{A}/A)}}\right)^n. \tag{26}$$

Using the second statement of Lemma 3.1 the same bound can be given also for CR applied to the second and third Schur complement systems. Consequently, taking into account the bound (25) it follows for the relative residual norm of the CR method

$$\frac{\|r_n\|}{\|r_0\|} \leq 2\left(\frac{1 - \frac{c_3}{c_4}\sqrt{\frac{c_1}{c_2}}h}{1 + \frac{c_3}{c_4}\sqrt{\frac{c_1}{c_2}}h}\right)^n. \tag{27}$$

The asymptotic convergence factor of CR can be thus bounded as follows

$$\lim_{n\to+\infty}\left(\frac{\|r_n\|}{\|r_0\|}\right)^{\frac{1}{n}} \leq 1 - c_9 h, \tag{28}$$

where c_9 is a positive constant depending only on the constants c_1, c_2, c_3 and c_4. For the Schur complement approach we have obtained again the bounds which depend linearly on the discretization parameter h. We note here that while for the MINRES method applied to the whole system the asymptotic rate of convergence depends significantly on the scaling of the system (1), for the Schur complement approach we obtain the same asymptotic bounds for the unscaled case and both diagonal scalings.

4. Iterative solution of systems projected onto a null space. Another possible approach is to construct a null-space basis of a certain off-diagonal block in the system matrix (1) first and then to solve a system projected onto this null-space using an iterative solver. First, assuming we have a null-space basis Z of the whole off-diagonal block $(B\ C)^T$ satisfying $(B\ C)^T Z = 0$ and a solution of the underdetermined system $(B\ C)^T u_1 = (q_2^T q_3^T)^T$, then the unknown velocity vector u can be written as $u = u_1 + Zu_2$, where u_2 is a solution of the projected system $Z^T AZu_2 = Z^T(q_1 - Au_1)$. The projected system with matrix $Z^T AZ$ is symmetric positive definite and so the CR or CG method can be applied. Their rate of convergence depends on the eigenvalue distribution of the projected matrix $Z^T AZ$ [3].

LEMMA 4.1. *Let $0 < \mu_{min} \leq \ldots \leq \mu_{max}$ be the eigenvalues of the symmetric positive definite matrix block A, $\sigma_{max} \geq \ldots \geq \sigma_{min} > 0$ the singular values of the matrix Z such that $(B\ C)^T Z = 0$. Then for the eigenvalues of the projected matrix $Z^T AZ$ it follows*

$$\sigma(Z^T AZ) \subset [\sigma_{min}^2 \mu_{min}, \sigma_{max}^2 \mu_{max}]. \tag{29}$$

There exist several approaches how to compute a null-space basis Z. One of them is based on construction of cycles in a certain graph associated with our particular off-diagonal block $(B\ C)^T$. In [1] the fundamental null-space basis using a spanning tree of this graph was constructed. It was also shown that singular values of the matrix Z satisfy

$$sv(Z) \subset [1, \frac{c_{10}}{h^2}], \tag{30}$$

where c_{10} is a positive constant independent of h. Using the statement of Lemma 4.1 and (30) the spectrum of the projected matrix $Z^T AZ$ (for the original unscaled problem) is a subset of the interval

$$\sigma(Z^T AZ) \subset [\frac{c_1}{h}, \frac{c_2 c_{10}^2}{h^5}]. \tag{31}$$

Similarly, since we have $\sigma(\mathbf{A}) \subset [c_1, c_2]$ the scalings (7) and (9) lead to the inclusion set in the form

$$\sigma(Z^T \mathbf{A} Z) \subset [c_1, \frac{c_2 c_{10}^2}{h^4}]. \tag{32}$$

It is clear from (31) and (32) that the ultimate bound for the condition number of the projected matrices $Z^T \mathbf{A} Z$ and $Z^T AZ$ is of order $O(h^{-4})$ and it has a form

$$\kappa(Z^T AZ) \leq \frac{c_2 c_{10}^2}{c_1 h^4}. \tag{33}$$

The relative residual norm of the CR method applied to the projected system can be bounded (see [3]) by

$$\frac{\|r_n\|}{\|r_0\|} \leq 2 \left(\frac{1 - 1/\sqrt{\kappa(Z^T AZ)}}{1 + 1/\sqrt{\kappa(Z^T AZ)}} \right)^n. \tag{34}$$

Considering our bound (32) we have

$$\frac{\|r_n\|}{\|r_0\|} \leq 2 \left(\frac{1 - \frac{1}{c_{10}} \sqrt{\frac{c_1}{c_2}} h^2}{1 + \frac{1}{c_{10}} \sqrt{\frac{c_1}{c_2}} h^2} \right)^n, \tag{35}$$

which leads to the asymptotic convergence factor

$$\lim_{n\to+\infty}\left(\frac{\|r_n\|}{\|r_0\|}\right)^{\frac{1}{n}} \le 1 - c_{11}h^2. \tag{36}$$

The positive constant c_{11} depends only on the constants c_1, c_2 and c_{10}. Note that due to the potential ill-conditioning of the fundamental cycle null-space basis Z, the bound (36) depends quadratically on the discretization parameter h. Indeed, it is clear that the conditioning of the matrix Z depends on the conditioning of the matrix block $(B\ C)^T$ which, due to (3), depends linearly on h.

The off-diagonal matrix block C in (1) is, however, orthogonal up to normalization constants. We can thus easily construct a null-space of the matrix C^T only, and consider a different variant of null-space method. Due to a special structure of nonzeros in the matrix block C the basis matrix Z can be given explicitly and it will remain very sparse. The velocity unknown u can be then written as $u = u_1 + Zu_2$, where u_1 is a solution of the underdetermined system $C^T u_1 = q_3$ and the vectors u_2 and p are obtained from an associated projected system

$$\begin{pmatrix} Z^T\mathbf{A}Z & Z^TB \\ B^TZ & \end{pmatrix}\begin{pmatrix} u_2 \\ p \end{pmatrix} = \begin{pmatrix} Z^T(q_1 - \mathbf{A}u_1) \\ q_2 - B^Tu_1 \end{pmatrix} \tag{37}$$

with a symmetric indefinite matrix which can be written as follows

$$\begin{pmatrix} Z^T & \\ & I \end{pmatrix}\begin{pmatrix} \mathbf{A} & B \\ B^T & \end{pmatrix}\begin{pmatrix} Z & \\ & I \end{pmatrix} = \begin{pmatrix} Z^T\mathbf{A}Z & Z^TB \\ B^TZ & \end{pmatrix}. \tag{38}$$

Note that in order to avoid the direct dependence on h also in the positive definite block A the second diagonal scaling (9) is considered here. For details we refer to paper [1]. Moreover, the matrix Z is due to its construction an orthogonal matrix and the projected matrix (38) is therefore an orthogonal transformation of the first two by two leading block submatrix in the system (1). It was shown in [1] that there exist positive constants c_{12} and c_{13} such that for the singular values of the matrix block Z^TB it follows

$$sv(Z^TB) \subset [c_{12}h, c_{13}]. \tag{39}$$

Using the fact $\sigma(\mathbf{A}) \subset [c_1, c_2]$, the relation (39) and the statement of Lemma 1.1 we can for the spectrum of the projected matrix (38) write

$$[\frac{1}{2}(c_1 - \sqrt{c_1^2 + 4c_{13}^2}, -\frac{c_{12}^2}{c_2}h^2] \cup [c_1, \frac{1}{2}(c_2 + \sqrt{c_2^2 + 4c_{13}^2})]. \tag{40}$$

We have obtained the result which is completely analogous to the case of the inclusion set (19). If we apply now the MINRES method to the projected symmetric indefinite system (37), the bound for its asymptotic convergence factor will be be analogous to (20) and it will be in the form

$$\lim_{n\to+\infty}\left(\frac{\|r_n\|}{\|r_0\|}\right)^{\frac{1}{n}} \le 1 - c_{14}h, \tag{41}$$

where the positive constant c_{14} depends only on the constants c_1, c_2, c_{12} and c_{13}. Indeed, if the diagonal scaling (9) together with this variant of the null-space method is considered, the asymptotic rate of convergence of the MINRES method depends at most linearly on the parameter h. For details we refer to [1]. The case of unscaled problem, the problem with more realistic conditions and the problem with the first scaling (7) can be examined using the same approach as done in Section 2 and results analogous to (14), (16) and (18) can be derived.

5. Conclusions. We have considered several approaches for solving the symmetric indefinite systems (1) that arise from mixed-hybrid finite element discretization of the potential fluid problem. We analyzed the fully iterative approach using the indefinite MINRES solver, the approach based on the successive Schur complement reductions with subsequent iterative solution and the null-space (dual variable) approach based on iterative solution of the system projected onto a certain null-space. We presented an analysis of convergence of the iterative solvers used in the algorithms and gave bounds for their asymptotic convergence factor. We have analyzed this quantity in terms of the discretization parameter h. We have shown that scaling of the system may affect the asymptotic rate of convergence in the case of iterative solution of the whole indefinite system, while for the Schur complement approach we obtained the same bounds for all scalings considered in this paper. The diagonal scaling (9) leads in all approaches to bounds with linear dependence on h.

Although the best asymptotic convergence factor of approaches considered in the paper is the same, one must take into account total algorithmic cost when attempting to compare their computational efficiency. This includes not only the cost of the iterative part but also the initial overheads of the Schur complement reduction, computation of a null space-basis and of the back-substitution processes and other transformations. Thorough computational experiments and comparison of results are out of the scope of this paper and they will be published elsewhere.

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