Solution of problem 11, establish existence or non-existence of periodic orbits for the van der Pol equation $x'' + \mu(x^2-1)x' + x = 0$, $\mu > 0$.

Step 1. Rewrite as a system of first -order equations,

$$\begin{cases} x' = y, \\ y' = -x - \mu(x^2-1)y. \end{cases}$$

Step 2. Qualitative analysis leads to the following picture:

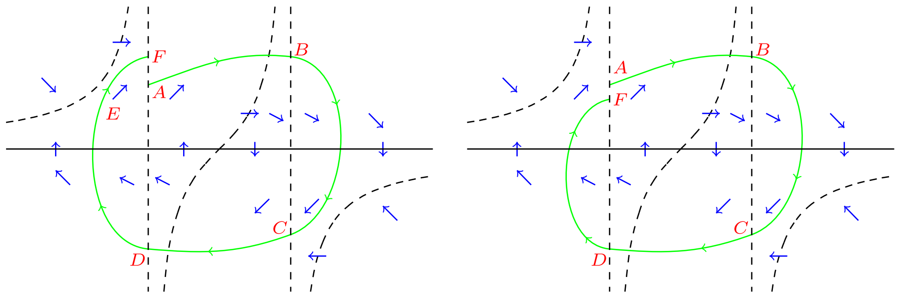

Figure 1: Qualitative analysis leads to the following distrivution of signs

Step 3. The orbital derivative of the function $V = x^2 + y^2$, is $\dot{V} = 2xx' + 2yy' = 2xy + 2y(-x - (x^2-1)y) = 2y^2(1-x^2)$, i.e. the distance from the origin increases inside the strip $-1 < x < 1$ and decreases outside that strip.

Step 4. Linearization or the previous step shows that the origin is repulsive, i.e. a trajectory cannot enter a circle $x^2+y^2 = \varepsilon^2$ of a sufficiently small radius $\varepsilon > 0$ from outside (due to the orbital derivative of the function V, we see that the radius can be taken up to $\varepsilon = 1$).

Step 5. To show the existence of a periodic solution, we want to use Poincaré-Bendixson theorem, and to do so we need to contstruct a trapping region. There is no obvious choice of a trapping region, so we need to make a special construction. To this end, take a trajectory that passes through a point $A(-1, y_A > 0)$. From the qualitative analysis it is clear that the trajectory must subsequently pass through the following points: $B(1, y_B > 0)$, $C(1, y_C < 0)$, $D(-1, y_D < 0)$, $F(-1, y_F > 0)$.

The question is whether y_F bigger than y_A (Figure 1, left) or smaller (Figure 1, right). In the latter case we have a trapping region, in the former we do not.

We want to prove that the situation in Figure 1, on the right, is possible, and we will prove this by contradiction. That is, assume that always $y_F > y_A$.

Note that the trajectory in the domain $x < -1, y > 0$ must lie in the region where $x + \mu(x^2-1)y < 0$, i.e. it is squeezed between the line $x = -1$ and the above mentioned curve. This allows us to estimate y_F from above.

Indeed, taking y_A sufficiently large (for us it sufficies to take $y_A > 1$), we ensure that there is a point $E(x_E, 1)$ between the points D and F.

On the trajectory EF, the y and x increase, and hence y can be represented as an increasing function of x.

Integrating along the trajectory EF, we obtain

$$y_F = y_E + \int_{x_E}^{x_F} \frac{dy}{dx} dx.$$

Using the rule of differentiation of a composed function, we have

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \underbrace{-\mu(x^2-1)}_{\leq 0} \underbrace{-\frac{x}{y}}_{\leq |x|}.$$

On the part EF of trajectory we have $y \geq 1$, $x < -1$ and $x^2 - 1 > 0$, hence the above derivative admits an estimate

$$\frac{dy}{dx} \leq -x.$$

Substituting that in the formula for y_F, we obtain

$$y_F \leq 1 + \int_{x_E}^{-1} -x dx = 1 + \frac{1}{2}\left(x_E^2 - 1\right).$$

It remains to establish a lower bound for x_E (and hence an upper bound for x_E^2). Since the point $E(x_E, 1)$ lies in the region $x + \mu(x^2 - 1)y \leq 0$, then $x_E > x_1$, where x_1 is the negative solution of the quadratic equation $x + \mu(x^2 - 1) = 0$, i.e.

$$x_{1,2} = \frac{-1}{2\mu} \pm \sqrt{\frac{1}{4\mu^2} + 1}, \quad x_1 = \frac{-1}{2\mu} - \sqrt{\frac{1}{4\mu^2} + 1}.$$

Then $x_E^2 < x_1^2$, and the final estimate for y_F reads

$$y_F \leq 1 + \frac{1}{2}(x_1^2 - 1) = 1 + \frac{-x_1}{2\mu}.$$

Further interesting things about the van der Pol oscillator.
Strongly nonlinear regime $\mu \to +\infty$. For large positive μ, one can use Liénard substitution: the original equation can be written in the form

$$\Bigg(\underbrace{x' + \mu(\frac{x^3}{3} - x)}_{=:\tilde{w}}\Bigg)' + x = 0, \qquad \text{i.e.} \quad \begin{cases} x' = -\mu F(x) + \tilde{w}, \\ \tilde{w}' = -x, \end{cases} \qquad \text{where } F(x) \equiv \frac{x^3}{3} - x,$$

or, making another substitution $\tilde{w} = \mu w$, as

$$\begin{cases} x' = \mu(w - F(x)), \\ w' = \frac{-x}{\mu}. \end{cases}$$

For large μ, heuristic/qualitative analysis considerations allow to conclude that trajectories "converge" to a limiting cycle, where travelling along some pieces of trajectories is fast (takes $o(1)$ time), and is slow on another pieces (takes $\mathcal{O}(\mu)$ time). The overall period of oscillations is of the order μ.

Physically this represents a *relaxation* oscillator, since tensions in the system, which build up slowly, are released fast (earthquakes).

Weakly nonlinear regime $\mu \to +0$. For small positive μ, arguing heuristically that the periodic trajectory takes the form $x(t) = (A + \mathcal{O}(\mu))(\cos(t - t_0) + \mathcal{O}(\mu))$, one can find that $|A| = 2$.

Oscillations in an electric circuit.
Damped oscillations in an electric circuit. Take an electric circuit consisting of an *electromotive force* $E(t)$, *resistor with resistance* R, *capacitor with capacitance* C and *inductor with inductance* L, connected in series. The laws that connect the voltage drop and the electric current in resistor, inductor and capacitor are

$$V_R(t) = RI_R(t), \qquad \frac{dV_C(t)}{dt} = \frac{1}{C}I_C(t), \qquad \frac{dI_L(t)}{dt} = \frac{1}{L}V_L(t).$$

Since the current passing through all components has the same value, adding up voltage drops we obtain the following equation:

$$E(t) = R \cdot I(t) + \frac{1}{C}\int_0^t I(t)ds + L\frac{dI(t)}{dt}.$$

Assuming that the electromotive force is constant (battery), $E(t) = const$, and differentiating, we obtain damped oscillations

$$LI''(t) + RI'(t) + \frac{1}{C}I(t) = 0.$$

Here the term R is responsible for damping.

Van der Pol oscillations. Changing the resistor to a semiconductor, which has the following law connecting voltage and current:

$$V_{sc}(t) = I_{sc}(t)(I_{sc}^2(t) - a^2),$$

leads to the equation

$$LI'' + (I(I^2 - a^2))' + \frac{1}{C}I = 0,$$

which, after the substitutions $I = \frac{a}{\sqrt{3}}\tilde{x}$, and $x(t) = \tilde{x}(t\sqrt{c})$, takes the form of the van der Pol oscillator

$$x'' + \underbrace{\frac{a^2\sqrt{C}}{L}}_{=:\mu}(x^2 - 1)x' + x = 0.$$

Physically, the semiconductor works as a resistor when the current is large ($|x| \in (1, 2)$), but instead of dispersing the energy it slowly accumulates it. When the current drops behind a certain value ($|x| < 1$), the semiconductor quickly pumps energy back into the system.