5) $\begin{cases} x' = xy^2 + x(x^2+y^2)^2 \\ y' = -x^2y + x(x^2+y^2)^2 \end{cases}$

$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$

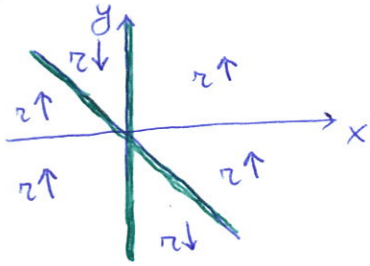
$x' = r'\cos\varphi - r\sin\varphi\cdot\varphi' = r^3\cos\varphi\sin^2\varphi + r^5\cos\varphi \quad \Big| \cdot\cos\varphi \; + \; \Big| \cdot -\sin\varphi$

$y' = r'\sin\varphi + r\cos\varphi\cdot\varphi' = -r^3\cos^2\varphi\sin\varphi + r^5\cos\varphi \quad \Big| \cdot\sin\varphi \quad \Big| \cdot\cos\varphi$

$\begin{cases} r' = r^5\cos\varphi(\cos\varphi+\sin\varphi) \\ r\varphi' = -r^3\sin\varphi\cos\varphi + r^5\cos\varphi(\cos\varphi-\sin\varphi) \end{cases}$

$\begin{cases} r' = r^5\cos\varphi(\cos\varphi+\sin\varphi) \\ \varphi' = -r^2\sin\varphi\cos\varphi + r^4\cos\varphi(\cos\varphi-\sin\varphi) \\ \quad = r^2\cos\varphi\left[-\sin\varphi + r^2(\cos\varphi-\sin\varphi)\right] \end{cases}$

Nakřeslime oblasti kde $r\uparrow,\downarrow$, a $\varphi\uparrow,\downarrow$. Udělame zvlášt pro $r$, zvlášt pro $\varphi$.

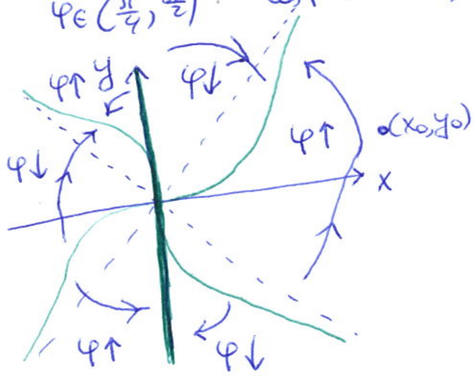



Pro $\varphi$, nakřeslime křivku

$r^2(\cos\varphi - \sin\varphi) = \sin\varphi$

$\varphi\in(0,\frac{\pi}{4})$: $\cos\varphi-\sin\varphi>0$, $\sin\varphi>0$

$\varphi\in(\frac{\pi}{4},\frac{\pi}{2})$: $\cos\varphi-\sin\varphi<0$, $\sin\varphi>0$ — nejsou řešení




Vezmeme $(x_0,y_0)$: $x_0>0, y_0>0$, a $-\sin\varphi_0 + r_0^2(\cos\varphi_0-\sin\varphi_0)>0$, kde $\begin{cases} x_0 = r_0\cos\varphi_0 \\ y_0 = r_0\sin\varphi_0 \end{cases}$

Potom pro odpovídající řešení, $\varphi\uparrow$, $\varphi<\frac{\pi}{4}$, potom $\varphi_0\le\varphi<\frac{\pi}{4}$,

$\cos\varphi\cdot(\cos\varphi+\sin\varphi) \ge \cos\frac{\pi}{4}\cdot(\sin\varphi_0+\cos\frac{\pi}{4}) = a>0.$

Potom $r' = r^5\cos\varphi(\cos\varphi+\sin\varphi)\ge a r^5$.

Dokažeme, že $r$ nezůstane omezením, a pak 0-é řešení není stabilní.

Sporem: kdyby $r(t)\le C$ pro všechny $t\ge0$,

pak $r'(t)\ge a\, r^5(t)$, $\quad \frac{r'(t)}{r^5(t)}\ge a$, $\quad \int_{t_0}^{t_1}\frac{r'(t)dt}{r^5(t)}\ge a(t_1-t_0)$

$\Leftrightarrow \quad \int_{r_0}^{r_1}\frac{dr}{r^5}\ge a(t_1-t_0);\quad \frac{1}{4r_0^5}-\frac{1}{4r_1^5}\ge a(t_1-t_0),\ \forall t_1\ge t_0$

PS $\to+\infty$ pro $t_1\to+\infty$

LS $\le\frac{1}{4r_0^5}$

~~Potom~~ Přicházíme ke sporu, což znamená, že $r(t)$ nezůstane omezením. $\Rightarrow$ nestabilita.

2) $x'' - 3x' + 2x \geq 1,\quad t \geq 0$
$x(0) = \frac{1}{2},\ x'(0) = 1$ $\Bigg\} \Longrightarrow\ x(t) \geq e^{2t} - e^{t} + \frac{1}{2},\quad t \geq 0.$

---

$x'' - 3x' + 2x = f(t)$, pro nějakou $f$, $f \geq 1, t \geq 0$

$A = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$, $\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -2 & 3-\lambda \end{vmatrix} = (\lambda - 3)\lambda + 2$

$\begin{cases} x' = y \\ y' = -2x + 3y + f(t) \end{cases}$ $= \lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1)$

$$e^{At} = \begin{pmatrix} -e^{2t} + 2e^{t} & e^{2t} - e^{t} \\ -2e^{2t} + 2e^{t} & 2e^{2t} - e^{t} \end{pmatrix}$$

variace konstant:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{At}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \int_0^t e^{A(t-s)}\begin{pmatrix} 0 \\ f(s) \end{pmatrix} ds$$

$$x(t) = \overset{\frac{1}{2}}{\overset{\shortparallel}{x_0}}(-e^{2t} + 2e^{t}) + \overset{1}{\overset{\shortparallel}{y_0}}(e^{2t} - e^{t}) + \int_0^t \underbrace{f(s)}_{\geq 1}\underbrace{\left(e^{2(t-s)} - e^{(t-s)}\right)}_{\geq 0} ds$$

máme, že $e^{2(t-s)} \geq e^{t-s}$ pro $0 \leq s \leq t$, pak můžeme odhadnout:

$$x(t) \geq \tfrac{1}{2}(-e^{2t} + 2e^{t}) + (e^{2t} - e^{t}) + \int_0^t \left(e^{2(t-s)} - e^{t-s}\right) ds = \left| \tilde{s} = t - s \right|$$

$$= +\tfrac{1}{2}e^{2t} \quad + \quad \int_0^t \left(e^{2s} - e^{s}\right) ds$$

$$= +\tfrac{1}{2}e^{2t} + \frac{e^{2t}}{2} - \frac{1}{2} - e^{t} + 1$$

$$= e^{2t} - e^{t} + \tfrac{1}{2},\quad t \geq 0$$

---

3) $x' = \frac{\lambda}{x} + t,\ x(0) = \mu.$ $\Phi(t, \mu, \lambda)$ - řešící fce. Najít $\frac{\partial \Phi}{\partial \mu}(t, \mu = 1, \lambda = 0)$, $\frac{\partial \Phi}{\partial \lambda}(t, \mu = 1, \lambda = 0)$, approximovat $x(3)$, kde $x' = \frac{1}{5x} + t,\ x(0) = \frac{6}{5}$

a) $\lambda = 0, \mu = 1$: $\begin{cases} x' = t \\ x(0) = 1 \end{cases}$ $x(t) = \frac{t^2}{2} + 1$; $\boxed{\Phi(t, \mu = 1, \lambda = 0) = \frac{t^2}{2} + 1}$

b) rovnice ve variacích pro $\frac{\partial \Phi}{\partial \mu}$:

$\begin{cases} u' = \left.\frac{-\lambda}{x^2(t)}\right|_{\lambda = 0} u \\ u(0) = 1 \end{cases}$ $\begin{cases} u' = 0 \\ u(0) = 1 \end{cases}$ $u(t) = 1$; $\boxed{\frac{\partial \Phi}{\partial \mu}(t, \mu = 1, \lambda = 0) = 1}$

c) rovnice ve variacích pro $\frac{\partial \Phi}{\partial \lambda}$

pomocná rovnice pro $\lambda$:

$\begin{cases} x' = \frac{y}{x} + t \\ y' = 0 \end{cases}$ $\begin{cases} x(0) = \mu \\ y(0) = \lambda \end{cases}$

rovnice ve variacích: $\begin{cases} \begin{pmatrix} p' \\ q' \end{pmatrix} = \left.\begin{pmatrix} \frac{-y}{x^2} & \frac{1}{x} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}\right|_{\substack{\lambda = 0 \\ \mu = 1}} \\ \begin{pmatrix} p(0) \\ q(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{cases}$

$y(t) = \lambda = 0$
$x(t) = \frac{t^2}{2} + 1$

$\begin{matrix} q' = 0 \\ q(0) = 1 \end{matrix}$ : $q(t) \equiv 1$, $p'(t) = \frac{-\overset{=0}{y(t)}}{x^2(t)} p(t) + \frac{1}{x(t)} \overset{1}{\overset{\shortparallel}{q(t)}}$
$= \frac{1}{x(t)}$

takže, $\begin{cases} p'(t) = \frac{1}{x(t)} \\ p(0) = 0 \end{cases}$ $p'(t) = \frac{1}{\frac{t^2}{2} + 1}$; $p(t) = \int_0^t \frac{dt}{\frac{t^2}{2} + 1} + \overset{0}{\overset{\shortparallel}{p(0)}} = \sqrt{2}\,\mathrm{arctg}\,\frac{t}{\sqrt{2}}$; $\boxed{\frac{\partial \Phi}{\partial \lambda}(t, 1, 0) = \sqrt{2}\,\mathrm{arctg}\,\frac{t}{\sqrt{2}}}$

d) $x'(t) = \frac{1}{5x} + t,\ x(0) = \frac{6}{5}$; $\lambda = \frac{1}{5}, \mu = \frac{6}{5}, t = 3$; $x(3) \approx \Phi(3, 1, 0) + \frac{\partial \Phi}{\partial \mu}(3, 1, 0)\cdot\left(\frac{6}{5} - 1\right) + \frac{\partial \Phi}{\partial \lambda}(3, 1, 0)\cdot\frac{1}{5}$

$= \left(\frac{9}{2} + 1\right) + 1\cdot\frac{1}{5} + \sqrt{2}\,\mathrm{arctg}\,\frac{3}{\sqrt{2}}\cdot\frac{1}{5} = \frac{11}{2} + \frac{1}{5} + \frac{\sqrt{2}}{5}\mathrm{arctg}\,\frac{3}{\sqrt{2}}$

$= \frac{57}{10} + \frac{\sqrt{2}}{5}\mathrm{arctg}\,\frac{3}{\sqrt{2}}$

4) $x'' + ax' + x^2(5x^2+4x) = 0$, $a \geq 0$. Vyšetřete stabilitu.

$$\begin{cases} x' = y \\ y' = -ay - x^2(5x^2+4x) \end{cases}$$

Hledáme Ljapunovskou funkci ve tvaru

$$V(x,y) = b x^{2n} + c y^{2m}$$

$$\frac{1}{2}\frac{d}{dt} V(x(t),y(t)) = bn x^{2n-1} + cm y^{2m-1}(-ay - x^2(5x^2+4x))$$

$$= bn x^{2n-1} - acm y^{2m} - cm y^{2m-1} x^2(4x+5x^2)$$

najdeme $b, c, n, m$ tak, aby $bn x^{2n-1} - cm y^{2m-1} \cdot x^2 \cdot 4x = 0$, tj.

$n = 2$, $m = 1$

$2b - 4c = 0$

$b = 2$

$c = 1$

$$V(x,y) = 2x^4 + y^2$$

$$\frac{1}{2}\frac{d}{dt} V(x(t),y(t)) = -ay^2 - 5x^4$$

Pro $a > 0$ máme asymptotickou stabilitu, protože $ay^2 + 5x^4$ je poz. def.

Pro $a = 0$ máme ~~jen~~ stabilitu. Nevíme jestli máme asymptotickou stabilitu nebo ne.