Thus it follows that

$$\frac{B}{(1-x^2)\{P_n(x)\}^2}=\frac{1}{1-x^2}-\frac{T_n(x)}{\{P_n(x)\}^2},$$

or

$$B=\{P_n(x)\}^2+(x^2-1)T_n(x).$$

Let $x=1$, then, since $P_n(1)=1$ and $T_n(1)$ is finite, it follows that $B=1$.

Consequently

$$Q_n(x)=\tfrac{1}{2}P_n(x)\log\frac{x+1}{x-1}-\sum_{r=1}^{N}\frac{2n-4r+3}{(2r-1)(n-r+1)}P_{n-2r+1}(x).$$

In particular,

$$Q_0(x)=\tfrac{1}{2}\log\frac{x+1}{x-1};\quad Q_1(x)=\tfrac{1}{2}x\log\frac{x+1}{x-1}-1;$$

$$Q_2(x)=\tfrac{1}{2}P_2(x)\log\frac{x+1}{x-1}-\tfrac{3}{2}x;\quad Q_3(x)=\tfrac{1}{2}P_3(x)\log\frac{x+1}{x-1}-\tfrac{5}{2}x^2+\tfrac{2}{3}.$$

7·3. The Point at Infinity as an Irregular Singular Point.—Equations whose solutions are irregular at infinity are of frequent occurrence; linear equations with constant coefficients furnish a case in point. To study the behaviour of solutions of such equations for numerically large values of x is therefore a problem of some importance, a problem, however, which cannot be fully treated except with the aid of the theory of functions of a complex variable.*

It is, however, possible to give some rather crude indications of the behaviour of solutions which are irregular at infinity, which, crude as they are, will be found to be not without value in their applications.

Consider the equation of the second order,

$$\frac{d^2y}{dx^2}+p(x)\frac{dy}{dx}+q(x)y=0,$$

in which at least one of the conditions for a regular singularity at infinity, namely,

$$p(x)=O(x^{-1}),\quad q(x)=O(x^{-2})$$

as $x\to\infty$, is violated. It will be supposed that the coefficients $p(x)$ and $q(x)$ can be developed as series of descending powers of x, thus

$$p(x)=p_0x^{\alpha}+\ldots,\quad q(x)=q_0x^{\beta}+\ldots,$$

then since the point at infinity is irregular, one or both of the inequalities

$$\alpha>-1,\quad \beta>-2$$

must be satisfied.

Now consider the possibility of satisfying the equation by a function which, for large values of x, is of the form

$$x^{\sigma}e^{P(x)}v(x),$$

where $P(x)$ is a polynomial in x and $v(x)=O(1)$ as $x\to\infty$. Let λx^{ν} be the leading term in $P(x)$, then on substituting the above expression in the equation and extracting the dominant part of each term it is found that

$$\lambda^2\nu^2x^{2\nu-2}+p_0\lambda\nu x^{\nu+\alpha-1}+q_0x^{\beta}=0.$$

Thus ν is given by

$$\nu=\alpha+1\quad\text{or}\quad 2\nu=\beta+2,$$

whichever furnishes the greater value of ν. Thus 2ν is a positive integer, for simplicity it will be supposed that ν is a positive integer also.

Then a solution of the form

$$y=e^{\lambda x^{\nu}+\mu x^{\nu-1}+\ldots+\varpi x}x^{\sigma}v(x)$$

* See Chaps. XVII.–XIX.

is assumed, where

$$v(x)=1+\frac{a_1}{x}+\frac{a_2}{x^2}+\ldots,$$

and the constants λ, μ, . . . , ϖ, σ, a_1, a_2, . . . determined in succession.

When a solution of this type exists, it is said to be normal and of rank ν. Unfortunately, however, when the series $v(x)$ does not terminate, it diverges in general, and therefore the solution is illusory. Nevertheless it can be shown that the series, though divergent, is *asymptotic,** and therefore is of value in practical computation. It will now be shown, by an application of the process of successive approximation, how it is that the divergent series are of practical value, and an illustration will be taken from the theory of Bessel functions.

7·31. Asymptotic Development of Solutions.— Consider the linear equation of the second order

$$\frac{d^2y}{dx^2}+p(x)\frac{dy}{dx}+q(x)y=0,$$

in which p and q are real and finite at infinity ; let p and q be developed in the convergent series

$$p(x)=p_0+p_1x^{-1}+p_2x^{-2}+\ldots,$$
$$q(x)=q_0+q_1x^{-1}+q_2x^{-2}+\ldots$$

The substitution $y=e^{\lambda x}v$ transforms the equation into

$$\frac{d^2v}{dx^2}+(2\lambda+p)\frac{dv}{dx}+(\lambda^2+\lambda p+q)v=0\ ;$$

if λ is a root of the equation

$$\lambda^2+\lambda p_0+q_0=0,$$

the constant term in the coefficient of v disappears and the equation takes the form

$$\frac{d^2v}{dx^2}+(\varpi_0+\varpi_1x^{-1}+\ldots)\frac{dv}{dx}+(\rho_1x^{-1}+\rho_2x^{-2}+\ldots)v=0.$$

Now let

$$v=x^\sigma u,$$

then if

$$\varpi_0\sigma+\rho_1=0,$$

the term in x^{-1} in the coefficient of v disappears.

The leading term in the coefficient of $\dfrac{dv}{dx}$ is ϖ_0 and is real if λ is real. It will be supposed that ϖ_0 is negative,† then multiplication of the independent variable by the positive number $(-\varpi_0)^{-1}$ replaces ϖ_0 by -1.

The equation thus becomes

$$\frac{d^2u}{dx^2}+\left\{-1+\frac{a_1}{x}+\frac{a_2}{x^2}+\ldots\right\}\frac{du}{dx}+\left\{\frac{b_2}{x^2}+\frac{b_3}{x^3}+\ldots\right\}u=0\ ;$$

a solution will be found which assumes the value η when $x=+\infty$. Let $u_1=\eta$ and define the sequence of functions (u_n) by the relations

$$\frac{d^2u_2}{dx^2}-\frac{du_2}{dx}=-\left\{\frac{a_1}{x}+\frac{a_2}{x^2}+\ldots\right\}\frac{du_1}{dx}-\left\{\frac{b_2}{x^2}+\frac{b_3}{x^3}+\ldots\right\}u_1,$$

. . . .

$$\frac{d^2u_n}{dx^2}-\frac{du_n}{dx}=-\left\{\frac{a_1}{x}+\frac{a_2}{x^2}+\ldots\right\}\frac{du_{n-1}}{dx}-\left\{\frac{b_2}{x^2}+\frac{b_3}{x^3}+\ldots\right\}u_{n-1},$$

. . . .

* Whittaker and Watson, *Modern Analysis*, Chap. VIII.

† The case in which ϖ_0 is positive and that in which λ is imaginary may be left to the reader. An example of the latter circumstance is given in the following section.

Then *

$$u_n=\eta+\int_x^\infty (e^{x-t}-1)\left\{\frac{a_1}{t}+\frac{a_2}{t^2}+\ \ldots\ \right\}\frac{du_{n-1}(t)}{dt}\,dt$$
$$+\int_x^\infty (e^{x-t}-1)\left\{\frac{b_2}{t^2}+\frac{b_3}{t^3}+\ \ldots\ \right\}u_{n-1}(t)dt$$
$$=\eta+\int_x^\infty e^{x-t}\left\{\frac{\alpha_1}{t}+\frac{\alpha_2}{t^2}+\ \ldots\ \right\}u_{n-1}(t)dt+\int_x^\infty\left\{\frac{\beta_2}{t^2}+\frac{\beta_3}{t^3}+\ \ldots\ \right\}u_{n-1}(t)dt,$$

where α_1, α_2, . . ., β_2, β_3, . . . are expressible in terms of a_1, a_2, . . ., b_2, b_3, ...

It follows that

$$u_n-u_{n-1}=\int_x^\infty e^{x-t}\left\{\frac{\alpha_1}{t}+\frac{\alpha_2}{t^2}+\ \ldots\ \right\}\left\{u_{n-1}(t)-u_{n-2}(t)\right\}dt$$
$$+\int_x^\infty\left\{\frac{\beta_2}{t^2}+\frac{\beta_3}{t^3}+\ \ldots\ \right\}\left\{u_{n-1}(t)-u_{n-2}(t)\right\}dt.$$

Let it be supposed that $|u_{n-1}-u_{n-2}|$ is bounded for $x>a$, and that its upper bound is M_{n-1}. Then $|u_n-u_{n-1}|$ is bounded in the same range and its upper bound M_n satisfies the inequality

$$M_n<\frac{K}{x}M_{n-1},$$

where K is a constant, independent of n. Now M_2 is bounded for sufficiently large values of x; consequently the inequality holds for all values of n. It follows by comparison that the series

$$u=u_1+(u_2-u_1)+\ \ldots\ +(u_n-u_{n-1})+\ \ldots$$

is convergent for sufficiently large values of x. Moreover its sum is a solution of the differential equation in u.

Now

$$u_2-u_1=\int_x^\infty e^{x-t}\left\{\frac{\alpha_1}{t}+\frac{\alpha_2}{t^2}+\ \ldots\ \right\}\eta dt+\int_x^\infty\left\{\frac{\beta_2}{t^2}+\frac{\beta_3}{t^3}+\ \ldots\ \right\}\eta dt$$
$$=\frac{A^1{}_1}{x}+\frac{A^1{}_2}{x^2}+\ \ldots\ +\frac{A^1{}_{m-1}}{x^{m-1}}+\frac{A^1{}_m+\epsilon_1}{x^m},$$

where $\epsilon_1\to 0$ as $x\to\infty$.

Similarly

$$u_3-u_2=\frac{A^2{}_2}{x^2}+\ \ldots\ +\frac{A^2{}_{m-1}}{x^{m-1}}+\frac{A^2{}_m+\epsilon_2}{x^m},$$

and finally, if $m>n$,

$$u_n-u_{n-1}=\frac{A^{n-1}{}_{n-1}}{x^{n-1}}+\ \ldots\ +\frac{A^{n-1}{}_{m-1}}{x^{m-1}}+\frac{A^{n-1}{}_m+\epsilon_{n-1}}{x^m},$$

where $\epsilon_{n-1}\to 0$ as $x\to\infty$.

Consequently,

$$u_1+(u_2-u_1)+\ \ldots\ +(u_n-u_{n-1})$$
$$=\eta+\frac{C_1}{x}+\frac{C_2}{x^2}+\ \ldots\ +\frac{C_{m-1}}{x^{m-1}}+\frac{C_m+\epsilon}{x^m},$$

* The solution of

$$\frac{d^2u}{dx^2}-\frac{du}{dx}=-f(x)$$

which reduces to η when $x=+\infty$ is

$$u=\eta+\int_x^\infty(e^{x-t}-1)f(t)dt,$$

provided that the integral exists.

where $\epsilon \to 0$ as $x \to \infty$. On the other hand

$$|(u_{n+1}-u_n)+(u_{n+2}-u_{n+1})+\ .\ .\ .\ | < M_n\left(\frac{K}{x}+\frac{K^2}{x^2}+\ .\ .\ .\ \right)<\frac{H}{x^n},$$

where H is a constant, for sufficiently large values of x.

It follows that

$$u=\eta+\frac{C_1}{x}+\frac{C_2}{x^2}+\ .\ .\ .\ +\frac{C_{n-1}}{x^{n-1}}+\frac{C_n+\gamma_n}{x^n},$$

where $\gamma_n \to 0$ as $x \to \infty$.

Consequently the given differential equation admits of a solution of the form

$$y=e^{\lambda x}x^{\sigma}\left\{\eta+\frac{C_1}{x}+\frac{C_2}{x^2}+\ .\ .\ .\ +\frac{C_{n-1}}{x^{n-1}}+\frac{C_n+\gamma_n}{x^n}\right\}.$$

The series $\sum C_r x^{-r}$ may terminate, in which case the representation is exact. But when the series does not terminate, it in general diverges.* Nevertheless if m is fixed, and S_m denotes the sum of the series

$$e^{\lambda x}x^{\sigma}\left\{\eta+\frac{C_1}{x}+\ .\ .\ .\ +\frac{C_m}{x^m}\right\},$$

then if ϵ is arbitrarily small,

$$|x^m(y-S_m)|<\epsilon$$

for sufficiently large values of $|x|$. Consequently the series furnishes an asymptotic representation of the solution, and the sign of equality is replaced by the sign of asymptotic equivalence, thus :

$$y \sim e^{\lambda x}x^{\sigma}\left\{\eta+\frac{C_1}{x}+\ .\ .\ .\ +\frac{C_n}{x^n}+\ .\ .\ .\ \right\}.$$

7·32. The Bessel Equation.—When n is not an integer, the Bessel equation †

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0$$

is satisfied by the two distinct solutions

$$y_1=J_n(x), \quad y_2=J_{-n}(x),$$

where

$$J_n(x)=\frac{x^n}{2^n\Gamma(n+1)}\left\{1-\frac{x^2}{2^2\,.\,1\,!\,.\,(n+1)}+\frac{x^4}{2^4\,.\,2\,!\,.\,(n+1)(n+2)}-\ .\ .\ .\ \right\}.$$

When n is an integer these two solutions cease to be independent. The second solution, when n is an integer, is of the logarithmic type.‡

Now consider solutions appropriate to the irregular singularity at infinity.§ The substitution

$$y=x^{-\frac{1}{2}}u$$

removes the second term from the equation, which becomes

$$\frac{d^2u}{dx^2}+\left\{1+\frac{\frac{1}{4}-n^2}{x^2}\right\}u=0.$$

* This can be verified by considering the simple equation

$$\frac{d^2y}{dx^2}-\frac{dy}{dx}+\frac{\beta}{x^2}y=0.$$

† Bessel, *Abh. Akad. Wiss. Berlin*, 1824, p. 34. An account of the early history of this and allied equations is given by Watson, *Bessel Functions*, Chap. I.

‡ This solution will be given explicitly in a later section (§ 16·82).

§ For a complete discussion of the problem, see Watson, *Bessel Functions*, Chap. VII.

For large values of $|x|$ this equation becomes effectively $u''+u=0$, which suggests the substitution *

$$u=e^{ix}v.$$

The equation now becomes

$$\frac{d^2v}{dx^2}+2i\frac{dv}{dx}+\frac{\frac{1}{4}-n^2}{x^2}v=0.$$

This equation is formally satisfied by a series of descending powers of x, namely

$$1-\frac{\frac{1}{4}-n^2}{2x}i-\frac{(\frac{1}{4}-n^2)(\frac{9}{4}-n^2)}{2^2.2!.x^2}+\frac{(\frac{1}{4}-n^2)(\frac{9}{4}-n^2)(\frac{25}{4}-n^2)}{2^3.3!.x^3}i$$
$$+\frac{(\frac{1}{4}-n^2)(\frac{9}{4}-n^2)(\frac{25}{4}-n^2)(\frac{49}{4}-n^2)}{2^4.4!.x^4}-\dots.$$

This series is divergent for all values of x, but it is of asymptotic type. In fact, if $|x|$ is large, the earlier terms diminish rapidly with increasing rank, and as will be seen later the series furnishes a valuable method for computing $J_n(x)$ when x is large.

By combining the series obtained with that obtained by changing i into $-i$ two asymptotic relations are obtained, namely

$$y_1 \sim x^{-\frac{1}{2}}(U\cos x+V\sin x),$$
$$y_2 \sim x^{-\frac{1}{2}}(U\sin x-V\cos x),$$

where U and V stand respectively for the even and odd series

$$1-\frac{(\frac{1}{4}-n^2)(\frac{9}{4}-n^2)}{2^2.2!.x^2}+\frac{(\frac{1}{4}-n^2)(\frac{9}{4}-n^2)(\frac{25}{4}-n^2)(\frac{49}{4}-n^2)}{2^4.4!x^4}-\dots$$

and

$$\frac{\frac{1}{4}-n^2}{2x}-\frac{(\frac{1}{4}-n^2)(\frac{9}{4}-n^2)(\frac{25}{4}-n^2)}{2^3.3!.x^3}+\dots.$$

The connection between the function $J_0(x)$ and the corresponding asymptotic series may be derived from the relation,†

$$\pi J_0(x)=\int_0^\pi \cos(x\cos\theta)d\theta.$$

Let

$$J_0(x)=Ay_1+By_2,$$

then as $x\to\infty$

$$\lim x^{\frac{1}{2}}J_0(x)=A\cos x+B\sin x,$$
$$\lim x^{\frac{1}{2}}J_0'(x)=-A\sin x+B\cos x.$$

Thus

$$\begin{aligned}A&=\lim x^{\frac{1}{2}}\{J_0(x)\cos x-J_0'(x)\sin x\}\\&=\lim\frac{x^{\frac{1}{2}}}{\pi}\int_0^\pi\{\cos x\cos(x\cos\theta)+\sin x\cos\theta\sin(x\cos\theta)\}d\theta\\&=\lim\frac{x^{\frac{1}{2}}}{\pi}\int_0^\pi\cos(2x\sin^2\tfrac{1}{2}\theta)\cos^2\tfrac{1}{2}\theta d\theta\\&\quad+\lim\frac{x^{\frac{1}{2}}}{\pi}\int_0^\pi\cos(2x\cos^2\tfrac{1}{2}\theta)\sin^2\tfrac{1}{2}\theta d\theta.\end{aligned}$$

Let

$$\sqrt{(2x)}\sin\tfrac{1}{2}\theta=\phi,$$

* For an alternative method of procedure when $n=0$, see Stokes, *Trans. Camb. Phil. Soc.* 9 (1850), p. 182 ; [*Math. and Phys. Papers*, 2, p. 350].

† An equivalent relation will be established in the following chapter, § 8·22.

then

$$\lim \frac{x^{\frac{1}{2}}}{\pi}\int_0^{\pi} \cos\,(2x\sin^2\tfrac{1}{2}\theta)\cos^2\tfrac{1}{2}\theta d\theta = \lim \frac{2^{\frac{1}{2}}}{\pi}\int_0^{\sqrt{(2x)}} \left(1-\frac{\phi^2}{2x}\right)^{\frac{1}{2}} \cos\phi^2 d\phi$$

$$= \frac{2^{\frac{1}{2}}}{\pi}\int_0^{\infty} \cos\phi^2 d\phi = \tfrac{1}{2}\pi^{-\frac{1}{2}}.$$

The second integral has the same limit and therefore

$$A = \pi^{-\frac{1}{2}}.$$

Similarly $B=\pi^{-\frac{1}{2}}$, and thus

$$J_0(x) \sim \left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\left\{\left(1-\frac{1^2.3^2}{2^6.2!.x^2}+\frac{1^2.3^2.5^2.7^2}{2^{12}.4!.x^4}-\ldots\right)\cos\,(x-\tfrac{1}{4}\pi)\right.$$

$$\left.+\left(\frac{1^2}{2^3.x^3}-\frac{1^2.3^2.5^2}{2^9.3!.x^3}+\ldots\right)\sin\,(x-\tfrac{1}{4}\pi).\right.$$

7·321. Use of the Asymptotic Series in Numerical Calculations.—The value of the asymptotic series may be illustrated by computing particular values of $J_0(x)$. If the ascending series

$$J_0(x) = 1-\frac{x^2}{2^2}+\frac{x^4}{2^6}-\frac{x^6}{2^8.3^2}+\frac{x^8}{2^{10}.3^2.4^2}-\frac{x^{10}}{2^{12}.3^2.4^2.5^2}+\ldots$$

is used to evaluate $J_0(2)$, and the last term taken is that in x^{16}, the value

$$J_0(2) = 0{\cdot}223\,890\,779\,14$$

correct to eleven places is obtained. But if $x=6$, and terms up to and including that in x^{20} are taken, the value obtained is

$$J_0(6) = 0{\cdot}15067,$$

which is correct to four places only; in fact the last term used has the value 0·00026 which affects the fourth decimal place. Thus for even comparatively small values of x the ascending series is useless for practical calculations.

Now consider the asymptotic representation of $J_0(6)$; it is found that

$$J_0(6) = \frac{1}{\sqrt{(3\pi)}}\{(\sin 6+\cos 6)U+(\sin 6-\cos 6)V\},$$

where

$$U = 1-\frac{1^2.3^2}{2^6.2!.6^2}+\frac{1^2.3^2.5^2.7^2}{2^{12}.4!.6^4}-\frac{1^2.3^2.5^2.7^2.9^2.11^2}{2^{18}.6!.6^6}+\ldots$$

$$= 1-0{\cdot}00195+0{\cdot}00009-0{\cdot}00001+\ldots$$

$$= 0{\cdot}99812,$$

and

$$V = \frac{1^2}{2^3.6}-\frac{1^2.3^2.5^2}{2^9.3!.6^3}+\frac{1^2.3^2.5^2.7^2.9^2}{2^{15}.5!.6^5}-\ldots$$

$$= 0{\cdot}02083-0{\cdot}00034+0{\cdot}00003$$

$$= 0{\cdot}02052.$$

Since $2\pi-6=0{\cdot}28318$, it is found from Burrau's tables that

$$\sin 6 = -0{\cdot}27941, \quad \cos 6 = 0{\cdot}96017,$$

and therefore

$$J_0(6) = 0{\cdot}23033\,(0{\cdot}67948-0{\cdot}02544)$$

$$= 0{\cdot}15064,$$

correct to five places of decimals. Thus by the use of the asymptotic series a more correct result is obtained with far less labour than in using the convergent ascending series.

7·322. The Large Zeros of the Bessel Functions.—It may be proved, as in § 7·32, that

$$J_n(x) \sim \left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\{U_n\cos\,(x-\tfrac{1}{2}n\pi-\tfrac{1}{4}\pi)+V_n\sin\,(x-\tfrac{1}{2}n\pi-\tfrac{1}{4}\pi)\},$$