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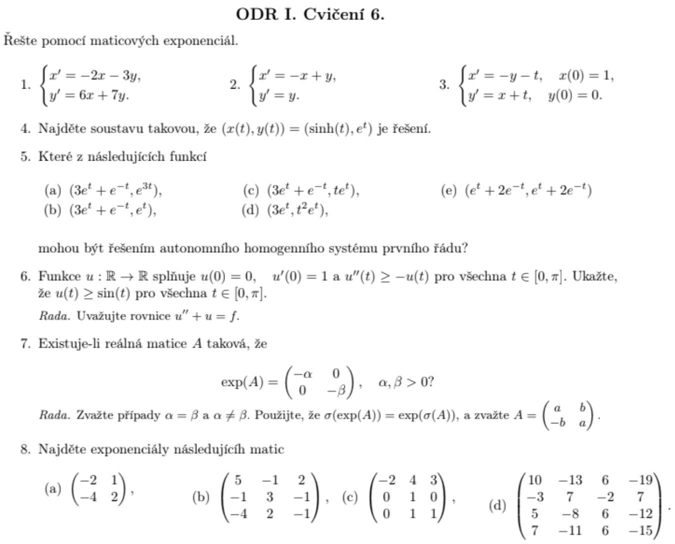

ODR I. Cvičení 6.

Řešte pomocí maticových exponenciál.

1. $\begin{cases} x' = -2x - 3y, \\ y' = 6x + 7y. \end{cases}$ 2. $\begin{cases} x' = -x + y, \\ y' = y. \end{cases}$ 3. $\begin{cases} x' = -y - t, & x(0) = 1, \\ y' = x + t, & y(0) = 0. \end{cases}$

4. Najděte soustavu takovou, že $(x(t), y(t)) = (\sinh(t), e^t)$ je řešení.

5. Které z následujících funkcí

   (a) $(3e^t + e^{-t}, e^{3t})$,
   (b) $(3e^t + e^{-t}, e^t)$,
   (c) $(3e^t + e^{-t}, te^t)$,
   (d) $(3e^t, t^2e^t)$,
   (e) $(e^t + 2e^{-t}, e^t + 2e^{-t})$

   mohou být řešením autonomního homogenního systému prvního řádu?

6. Funkce $u : \mathbb{R} \to \mathbb{R}$ splňuje $u(0) = 0, \quad u'(0) = 1$ a $u''(t) \geq -u(t)$ pro všechna $t \in [0, \pi]$. Ukažte, že $u(t) \geq \sin(t)$ pro všechna $t \in [0, \pi]$.

   Rada. Uvažujte rovnice $u'' + u = f$.

7. Existuje-li reálná matice A taková, že

$$\exp(A) = \begin{pmatrix} -\alpha & 0 \\ 0 & -\beta \end{pmatrix}, \quad \alpha, \beta > 0?$$

   Rada. Zvažte případy $\alpha = \beta$ a $\alpha \neq \beta$. Použijte, že $\sigma(\exp(A)) = \exp(\sigma(A))$, a zvažte $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$.

8. Najděte exponenciály následujícíh matic

   (a) $\begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix}$, (b) $\begin{pmatrix} 5 & -1 & 2 \\ -1 & 3 & -1 \\ -4 & 2 & -1 \end{pmatrix}$, (c) $\begin{pmatrix} -2 & 4 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$, (d) $\begin{pmatrix} 10 & -13 & 6 & -19 \\ -3 & 7 & -2 & 7 \\ 5 & -8 & 6 & -12 \\ 7 & -11 & 6 & -15 \end{pmatrix}$.

1. $\begin{cases} x' = -2x - 3y \\ y' = 6x + 7y \end{cases}$ $\qquad A = \begin{pmatrix} -2 & -3 \\ 6 & 7 \end{pmatrix}$

Najdeme vlastní čísla a vektory matice A.

$\det(A - \lambda I) = \begin{vmatrix} -2-\lambda & -3 \\ 6 & 7-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 4 = (\lambda-1)(\lambda-4)$

$\lambda_1 = 4 \quad (A - 4I)\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = 0, \quad \begin{pmatrix} -6 & -3 \\ 6 & 3 \end{pmatrix}\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = 0; \quad 2u_1 + v_1 = 0 \quad \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\lambda_2 = 1 \quad (A - I)\begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = 0, \quad \begin{pmatrix} -3 & -3 \\ 6 & 6 \end{pmatrix}\begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = 0, \quad u_2 + v_2 = 0 \quad \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Potom $A\cdot\begin{pmatrix}u_1 & u_2\\ v_1 & v_2\end{pmatrix}=\begin{pmatrix}u_1 & u_2\\ v_1 & v_2\end{pmatrix}\begin{pmatrix}4 & 0\\ 0 & 1\end{pmatrix}$

$$A\cdot\begin{pmatrix}1 & 1\\ -2 & -1\end{pmatrix}=\begin{pmatrix}1 & 1\\ -2 & -1\end{pmatrix}\cdot\begin{pmatrix}4 & 0\\ 0 & 1\end{pmatrix}$$

$$A=\begin{pmatrix}1 & 1\\ -2 & -1\end{pmatrix}\cdot\begin{pmatrix}4 & 0\\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix}-1 & -1\\ 2 & 1\end{pmatrix}$$

$$tA=\begin{pmatrix}1 & 1\\ -2 & -1\end{pmatrix}\cdot\begin{pmatrix}4t & 0\\ 0 & t\end{pmatrix}\cdot\begin{pmatrix}-1 & -1\\ 2 & 1\end{pmatrix}$$

$$e^{tA}=\begin{pmatrix}1 & 1\\ -2 & -1\end{pmatrix}\cdot\begin{pmatrix}e^{4t} & 0\\ 0 & e^{t}\end{pmatrix}\cdot\begin{pmatrix}-1 & -1\\ 2 & 1\end{pmatrix}$$

$$=\begin{pmatrix}e^{4t} & e^{t}\\ -2e^{4t} & -e^{t}\end{pmatrix}\cdot\begin{pmatrix}-1 & -1\\ 2 & 1\end{pmatrix}=$$

$$=\begin{pmatrix}-e^{4t}+2e^{t} & -e^{4t}+e^{t}\\ 2e^{4t}-2e^{t} & 2e^{4t}-e^{t}\end{pmatrix}$$

Řešení původní soustavy s počáteční podmínkou

$$\begin{cases}x(t_0)=x_0\\ y(t_0)=y_0\end{cases}$$

je

$$\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}=e^{A\cdot(t-t_0)}\cdot\begin{pmatrix}x_0\\ y_0\end{pmatrix}=\begin{pmatrix}-e^{4(t-t_0)}+2e^{t-t_0} & -e^{4(t-t_0)}+e^{t-t_0}\\ 2e^{4(t-t_0)}-2e^{t-t_0} & 2e^{4(t-t_0)}-e^{t-t_0}\end{pmatrix}\cdot\begin{pmatrix}x_0\\ y_0\end{pmatrix}$$

2. $\begin{cases}x'=-x+y\\ y'=y\end{cases}$ $\quad\begin{pmatrix}x'\\ y'\end{pmatrix}=\underbrace{\begin{pmatrix}-1 & 1\\ 0 & 1\end{pmatrix}}_{A}\begin{pmatrix}x\\ y\end{pmatrix}$

Vlastní čísla a vektory matice A:

$$\det(A-\lambda I)=\begin{vmatrix}-1-\lambda & 1\\ 0 & 1-\lambda\end{vmatrix}=(\lambda-1)(\lambda+1)$$

$\lambda_1=1.$ $\quad(A-I)\begin{pmatrix}u_1\\ v_1\end{pmatrix}=0,\ \begin{pmatrix}-2 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}u_1\\ v_1\end{pmatrix}=\begin{pmatrix}-2u_1+v_1\\ 0\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$

$$\begin{pmatrix}u_1\\ v_1\end{pmatrix}=\begin{pmatrix}1\\ 2\end{pmatrix}$$

$\lambda_2=-1.$ $\quad(A-I)\begin{pmatrix}u_2\\ v_2\end{pmatrix}=0,\ \begin{pmatrix}0 & 1\\ 0 & 2\end{pmatrix}\begin{pmatrix}u_2\\ v_2\end{pmatrix}=\begin{pmatrix}v_2\\ 2v_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$

$$\begin{pmatrix}u_2\\ v_2\end{pmatrix}=\begin{pmatrix}1\\ 0\end{pmatrix}$$

$$A\begin{pmatrix}1 & 1\\ & 0\end{pmatrix}=\begin{pmatrix}1 & 1\\ 2 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$\underbrace{2 \qquad 0}_{det=-2}$$

$$A=\begin{pmatrix}1 & 1\\ 2 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\cdot\begin{pmatrix}0 & -1\\ -2 & 1\end{pmatrix}\cdot\frac{-1}{2}$$

$$e^{At}=\begin{pmatrix}1 & 1\\ 2 & 0\end{pmatrix}\begin{pmatrix}e^{t} & 0\\ 0 & e^{-t}\end{pmatrix}\cdot\begin{pmatrix}0 & \frac{1}{2}\\ 1 & \frac{-1}{2}\end{pmatrix}$$

$$=\begin{pmatrix}e^{t} & e^{-t}\\ 2e^{t} & 0\end{pmatrix}\cdot\begin{pmatrix}0 & \frac{1}{2}\\ 1 & \frac{-1}{2}\end{pmatrix}$$

$$=\begin{pmatrix}e^{-t} & \frac{e^{t}-e^{-t}}{2}\\ 0 & e^{t}\end{pmatrix}$$

$$\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}=e^{A(t-t_0)}\cdot\begin{pmatrix}x_0\\ y_0\end{pmatrix}$$

$$=\begin{pmatrix}e^{t_0-t} & \frac{e^{t-t_0}-e^{t_0-t}}{2}\\ 0 & e^{t-t_0}\end{pmatrix}\begin{pmatrix}x_0\\ y_0\end{pmatrix}$$

3. $$\begin{cases} x'=-y-t, & x(0)=1,\\ y'=x+t, & y(0)=0. \end{cases}$$

<u>Krok 1</u>

Nejprve řešíme homogenní soustavu

$$\begin{cases} x'=-y\\ y'=x \end{cases}$$

Dosadíme $y=-x'$ do druhé rovnice,

$$x''+x=0,$$

$$\begin{cases} x(t)=A\cdot\cos t+B\cdot\sin t\\ y(t)=A\cdot\sin t-B\cdot\cos t \end{cases}$$

<u>Krok 2</u> Hledame řešení původní soustavy ve tvaru

$$\begin{cases} x(t) = A(t)\cdot\cos t + B(t)\cdot\sin t \\ y(t) = A(t)\cdot\sin t - B(t)\cdot\cos t \end{cases}$$

$$\begin{cases} A'(t)\cos t \cancel{- A(t)\sin t} + B'(t)\sin t \cancel{+ B(t)\cos t} \\ \quad = \cancel{-A(t)\sin t} \cancel{+ B(t)\cos t} - t \\ A'(t)\sin t \cancel{+ A(t)\cos t} - B'(t)\cos t \cancel{+ B(t)\sin t} \\ \quad = \cancel{A(t)\cos t} + \cancel{B(t)\sin t} + t \end{cases}$$

$$\begin{cases} A'(t)\cdot\cos t + B'(t)\cdot\sin t = -t \\ A'(t)\cdot\sin t - B'(t)\cdot\cos t = t \end{cases}$$

$$\begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix}\begin{pmatrix} A'(t) \\ B'(t) \end{pmatrix} = \begin{pmatrix} -t \\ t \end{pmatrix}$$

$$\begin{pmatrix} A'(t) \\ B'(t) \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix}\begin{pmatrix} -t \\ t \end{pmatrix}$$

$$\begin{cases} A'(t) = t(\sin t - \cos t) \\ B'(t) = -t(\sin t + \cos t) \end{cases}$$

$$A(t) = \int t(\sin t - \cos t)\,dt =$$

$$= \int t(-\cos t - \sin t)'\,dt =$$

$$= -t(\cos t + \sin t) + \int(\cos t + \sin t)\,dt$$

$$= -t(\cos t + \sin t) + \sin t - \cos t + C_1$$

$$B(t) = \int -t(\sin t + \cos t)\,dt = \int t(\cos t + \sin t)'\,dt$$

$$B(t) = \int -t(\sin t + \cos t)\,dt = \int -t(-\cos t + \sin t)\,dt$$

$$= -t(-\cos t + \sin t) + \int(-\cos t + \sin t)\,dt =$$

$$= t(\cos t - \sin t) - \sin t - \cos t + C_2$$

Najděme C_1, C_2: $\quad x(0)=1,\ y(0)=0 \implies$

$$x(0) = A(0) = -\cos 0 + C_1 = 1, \qquad C_1 = 2$$

$$y(0) = -B(0) = -(-1 + C_2) = 0, \qquad C_2 = 1.$$

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$$x(t) = -t(\cos^2 t + \cancel{\sin t \cdot \cos t}) + (\cancel{\sin t} - \cos t)\cos t + 2\cos t$$

$$+ t(\cancel{\sin t \cdot \cos t} - \sin^2 t) - (\sin t + \cancel{\cos t})\sin t + \sin t$$

$$= -t - 1 + 2\cos t + \sin t$$

$$y(t) = -t\sin t(\cancel{\cos t} + \sin t) + (\sin t - \cancel{\cos t})\sin t + 2\sin t$$

$$+ t(\cancel{\sin t} - \cos t)\cos t + (\cancel{\sin t} + \cos t)\cos t - \cos t$$

$$= -t + 1 + 2\sin t - \cos t$$

Výsledek: $\begin{cases} x(t) = -t - 1 + 2\cos t + \sin t \\ y(t) = -t + 1 + 2\sin t - \cos t \end{cases}$

4. $\begin{cases} x(t) = \sinh(t) = \dfrac{e^t - e^{-t}}{2} \\ y(t) = e^t \end{cases}$

Máme $\begin{cases} x'(t) = \cosh(t) = \dfrac{e^t + e^{-t}}{2} \\ y'(t) = e^t \end{cases}$

$\lfloor y(0) =$

Chceme vyjádřit x', y' pomocí x, y.

Máme $y'(t) = y(t)$,

$$x'(t) + x(t) = e^{t} = y(t).$$

Dostali jsme

$$\begin{cases} x'(t) = -x(t) + y(t) \\ y'(t) = y(t) \end{cases}$$

5. a) $\begin{cases} x(t) = 3e^{t} + e^{-t} \\ y(t) = e^{3t} \end{cases}$ $\begin{cases} x'(t) = 3e^{t} - e^{-t} \\ y'(t) = 3e^{3t} \end{cases}$

$x' \notin \mathcal{L}in\{x(t), y(t)\}$: kdyby existovali $A, B \in \mathbb{R}$ také, že

$\forall t \in \mathbb{R}$: $x'(t) = A\,x(t) + B\,y(t)$,

$$3e^{t} - e^{-t} = A \cdot (3e^{t} + e^{-t}) + B \cdot e^{3t}$$

$$e^{t}(3 - 3A) = e^{-t}(1 + A) + Be^{3t}$$

$$\emptyset$$

Výsledek: $(x(t), y(t))$ nemůže být řešením soustavy lineárních rovnic.

b) $\begin{cases} x(t) = 3e^{t} + e^{-t} \\ y(t) = e^{t} \end{cases}$ $\begin{cases} x'(t) = 3e^{t} - e^{-t} \\ y'(t) = e^{t} \end{cases}$

x', y' se dá vyjádřit pomocí x, y:

$$\begin{cases} x' + x = 6e^t = 6y \\ y' = y \end{cases} \qquad \begin{cases} x' = -x + 6y \\ y' = y \end{cases}$$

c) $$\begin{cases} x(t) = 3e^t + e^{-t} \\ y(t) = te^t \end{cases} \qquad \begin{cases} x'(t) = 3e^t - e^{-t} \\ y'(t) = (t+1)e^t \end{cases}$$

$y' \notin \mathcal{L}in\{x, y\}$

d) $$\begin{cases} x = 3e^t \\ y = t^2e^t \end{cases} \qquad \begin{cases} x' = 3e^t \\ y' = (t^2+2t)e^t \end{cases}$$

$y' \notin \mathcal{L}in\{x, y\}$

e) $$\begin{cases} x = e^t + 2e^{-t} \\ y = e^t + 2e^{-t} \end{cases} \qquad \begin{cases} x' = e^t - 2e^{-t} \\ y' = e^t - 2e^{-t} \end{cases}$$

$x' \notin \mathcal{L}in\{x, y\} = \mathcal{L}in\{x\}$.

6. $u : \mathbb{R} \to \mathbb{R}$ $\qquad u(0) = 0$

$u'(0) = 1$

$u''(t) + u(t) \geq 0 \quad \forall t \in [0, \pi]$.

Ukažte, že $u(t) \geq \sin t \quad \forall t \in [0, \pi]$.

Důkaz: označíme $u''(t)+u(t)=f(t)$

a řešíme

$$\begin{cases} u''(t)+u(t)=f(t) \\ u(0)=0,\ u'(0)=1 \end{cases}$$

Krok 1. Homogenní rovnice $u''+u=0$

$$u(t)=A(t)\cos t + B(t)\sin t.$$

Krok 2. Přepíšeme diferenciální rovnice druhého řádu jako soustavu rovnic prvního řádu a použijeme variaci konstant.

$$v=u', \quad \begin{cases} u'=v \\ v'=-u+f \end{cases}$$

$$\begin{pmatrix} u' \\ v' \end{pmatrix}=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}+\begin{pmatrix} 0 \\ f(t) \end{pmatrix}$$

$$\begin{cases} u(t)=A(t)\cdot\cos t + B(t)\cdot\sin t \\ v(t)=-A(t)\cdot\sin t + B(t)\cdot\cos t \end{cases}$$

$$\begin{cases} u'=A'\cdot\cos t + B'\cdot\sin t - A\cdot\sin t + B\cdot\cos t \\ v'=-A'\cdot\sin t + B'\cdot\cos t - A\cdot\cos t - B\sin t \end{cases}$$

$$\begin{cases} A'(t)\cdot\cos t + B'(t)\cdot\sin t = 0 \\ -A'(t)\cdot\sin t + B'(t)\cdot\cos t = f(t) \end{cases}$$

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} A'(t) \\ B'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ f(t) \end{pmatrix}$$

$$\begin{pmatrix} A'(t) \\ B'(t) \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}\begin{pmatrix} 0 \\ f(t) \end{pmatrix} = \begin{pmatrix} -f(t)\cdot\sin t \\ f(t)\cdot\cos t \end{pmatrix}$$

$$\begin{cases} A(t) = -\int_0^t f(s)\cdot\sin s\, ds + A(0) \\ B(t) = \int_0^t f(s)\cdot\cos s\, ds + B(0) \end{cases}$$

Takže $u(t) = A(t)\cdot\cos t + B(t)\cdot\sin t$, $u(0) = A(0) = 0$

$u'(t) = -A(t)\cdot\sin t + B(t)\cdot\cos t$, $u'(0) = B(0) = 1.$

$$u(t) = -\int_0^t f(s)\cdot\sin s\, ds\cdot\cos t + \int_0^t f(s)\cdot\cos s\, ds\cdot\sin t + \sin t$$

$$= \int_0^t \underbrace{f(s)}_{\geq 0}\cdot\underbrace{\sin(t-s)}_{\substack{\geq 0 \\ \text{pro } 0\leq s\leq t\leq\pi}} ds + \sin t \geq \sin t.$$

Z A (−d 0)

T. $e = \begin{pmatrix} & \\ 0 & -\beta \end{pmatrix}$, $\alpha, \beta > 0$.

Máme $sp(e^{A}) = e^{(sp A)}$

Podíváme se na vlastní čísla λ_1, λ_2 matice A.

Rozlišujeme několik případů.

1) $\lambda_1, \lambda_2 \in \mathbb{R} \Rightarrow \{e^{\lambda_1}, e^{\lambda_2}\} \subset (0, +\infty)$ $\emptyset$

2) $\lambda_1 = a + ib$
$\lambda_2 = a - ib$, $a, b \in \mathbb{R}$

$$e^{\lambda_1} = e^{a}(\cos b + i \sin b)$$
$$e^{\lambda_2} = e^{a}(\cos b - i \sin b)$$

Abychom bylo $e^{\lambda_1} < 0$, $e^{\lambda_2} < 0$ potřebujeme aby $\sin b = 0$, $\cos b = -1$, pak $e^{\lambda_1} = e^{\lambda_2} = -e^{a}$, takže $\lambda_1 = \lambda_2$.

Příklad: $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, $\det(A - \lambda I) = (\lambda - a)^2 + b^2$

vlastní čísla jsou $\lambda_{1,2} = a \pm ib$

Najdeme e^{A}. Máme

$$(A - \lambda_1)\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} -ib & b \\ -b & -ib \end{pmatrix}\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$-iu_1 + v_1 = 0$

$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix}$

$$(A - \lambda_2)\begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} ib & b \\ -b & ib \end{pmatrix}\begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$iu_2 + v_2 = 0$

$\begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} i \\ 1 \end{pmatrix}$

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$$A\begin{pmatrix}u_1 & u_2\\ v_1 & v_2\end{pmatrix}=\begin{pmatrix}u_1 & u_2\\ v_1 & v_2\end{pmatrix}\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}$$

$$A\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}=\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}\begin{pmatrix}a+ib & 0\\ 0 & a-ib\end{pmatrix}$$

$$A=\frac{1}{2}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}\begin{pmatrix}a+ib & 0\\ 0 & a-ib\end{pmatrix}\begin{pmatrix}1 & -i\\ -i & 1\end{pmatrix}$$

$$e^A=\frac{1}{2}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}\begin{pmatrix}e^{a+ib} & 0\\ 0 & e^{a-ib}\end{pmatrix}\begin{pmatrix}1 & -i\\ -i & 1\end{pmatrix}$$

$$=\frac{1}{2}\begin{pmatrix}e^{a+ib} & ie^{a-ib}\\ ie^{a+ib} & e^{a-ib}\end{pmatrix}\begin{pmatrix}1 & -i\\ -i & 1\end{pmatrix}$$

$$=\frac{1}{2}\begin{bmatrix}e^{a+ib}+e^{a-ib} & -i(e^{a+ib}-e^{a-ib})\\ i(e^{a+ib}-e^{a-ib}) & e^{a+ib}+e^{a-ib}\end{bmatrix}$$

Pro $b=\pi$ dostáváme

$$e^A=\begin{pmatrix}-e^a & 0\\ 0 & -e^a\end{pmatrix}$$

8.a) $A=\begin{pmatrix}-2 & 1\\ -4 & 2\end{pmatrix}$

$$\det(A-\lambda I)=\begin{vmatrix}-2-\lambda & 1\\ -4 & 2-\lambda\end{vmatrix}=\lambda^2$$

$$(A-0\cdot I)\begin{pmatrix}u\\ v\end{pmatrix}=0,\ \begin{pmatrix}-2 & 1\\ -4 & 2\end{pmatrix}\begin{pmatrix}u_1\\ v_1\end{pmatrix}=\begin{pmatrix}-2u_1+v_1\\ -4u_1+2v_1\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\quad v_1=2u_1$$

$$\begin{pmatrix}u_1\\ v_1\end{pmatrix}=\begin{pmatrix}1\\ 2\end{pmatrix}$$

Najdeme sdružený vektor

$$(A-0\cdot I)\begin{pmatrix}u_2\\ v_2\end{pmatrix}=\begin{pmatrix}u_1\\ v_1\end{pmatrix},\quad \left(\begin{array}{cc|c}-2 & 1 & 1\\ -4 & 2 & 2\end{array}\right)\sim\left(\begin{array}{cc|c}-2 & 1 & 1\end{array}\right),$$

$-2u_2+v_2=1$ $\quad\left(u_2\right)=\left(0\right)$

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$$A\begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} = \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$A\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

$$e^A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix}$$

8b)

$$A = \begin{pmatrix} 5 & -1 & 2 \\ -1 & 3 & -1 \\ -4 & 2 & -1 \end{pmatrix}$$

Najdeme vlastní čísla a vlastní vektory.

$$\begin{vmatrix} 5-\lambda & -1 & 2 \\ -1 & 3-\lambda & -1 \\ -4 & 2 & -1-\lambda \end{vmatrix} = \left(R_3 \to R_3 + (-4)R_2\right) = \begin{vmatrix} 5-\lambda & -1 & 2 \\ -1 & 3-\lambda & -1 \\ 0 & -10+4\lambda & 3-\lambda \end{vmatrix}$$

$$= \left| R_1 \to R_1 + (5-\lambda)R_2 \right| = \begin{vmatrix} 0 & (3-\lambda)(5-\lambda)-1 & \lambda-3 \\ -1 & 3-\lambda & -1 \\ 0 & -10+4\lambda & 3-\lambda \end{vmatrix} =$$

$$= \begin{vmatrix} (\lambda-3)(\lambda-5)-1 & \lambda-3 \\ 4\lambda-10 & 3-\lambda \end{vmatrix} = (\lambda-3)\begin{vmatrix} (\lambda-3)(\lambda-5)-1 & 1 \\ 4\lambda-10 & -1 \end{vmatrix} =$$

$$= (\lambda-3)\left[-(\lambda^2-8\lambda+14)-4\lambda+10\right] = (\lambda-3)(-\lambda^2+4\lambda-4) = -(\lambda-3)(\lambda-2)^2$$

$\lambda_1 = 3, \quad \lambda_2 = 2$

1) $(A-3I)\begin{pmatrix} u_1 \\ v_1 \\ w_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 2 & -1 & 2 \\ -1 & 0 & -1 \\ -4 & 2 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & 2 \\ 4 & -2 & 4 \end{pmatrix} \sim$

$$\sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \quad \begin{cases} u_1 + w_1 = 0 \\ v_1 = 0 \end{cases}$$

$$\begin{pmatrix} v_1 \\ w_1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

2) $\lambda_2 = 2$

$$(A-2I)\begin{pmatrix} u_2 \\ v_2 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 3 & -1 & 2 \\ -1 & 1 & -1 \\ -4 & 2 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 \\ 3 & -1 & 2 \\ -4 & 2 & -3 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & -2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{-1}{2} \end{pmatrix}$$

$$\begin{cases} u_2 + \frac{1}{2} w_2 = 0 \\ v_2 - \frac{1}{2} w_2 = 0 \end{cases} \qquad \begin{pmatrix} u_2 \\ v_2 \\ w_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

Najdeme sdružený vektor

$$(A-2I)\begin{pmatrix} u_3 \\ v_3 \\ w_3 \end{pmatrix} = \begin{pmatrix} u_2 \\ v_2 \\ w_2 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 3 & -1 & 2 & -1 \\ -1 & 1 & -1 & 1 \\ -4 & 2 & -3 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & -1 \\ 3 & -1 & 2 & -1 \\ -4 & 2 & -3 & 2 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & -1 \\ 0 & 2 & -1 & 2 \\ 0 & -2 & 1 & -2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & -1 \\ 0 & 1 & \frac{-1}{2} & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{-1}{2} & 1 \end{array}\right)$$

$$\begin{cases} u_3 + \frac{1}{2} w_3 = 0 \\ v_3 - \frac{1}{2} w_3 = 1 \end{cases} \qquad \begin{pmatrix} u_3 \\ v_3 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$A\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} = \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

$$A\underbrace{\begin{pmatrix}1&-1&0\\0&1&1\\-1&2&0\end{pmatrix}}_{C}=\underbrace{\begin{pmatrix}1&-1&0\\0&1&1\\-1&2&0\end{pmatrix}}_{C}\begin{pmatrix}3&0&0\\0&2&1\\0&0&2\end{pmatrix}$$

Najdeme C^{-1}:

$$\left(\begin{array}{ccc|ccc}1&-1&0&1&0&0\\0&1&1&0&1&0\\-1&2&0&0&0&1\end{array}\right)\sim\left(\begin{array}{ccc|ccc}1&-1&0&1&0&0\\0&1&1&0&1&0\\0&1&0&1&0&1\end{array}\right)\sim$$

$$\sim\left(\begin{array}{ccc|ccc}1&0&0&2&0&1\\0&1&0&1&0&1\\0&1&1&0&1&0\end{array}\right)\sim\left(\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix}\right|\underbrace{\left.\begin{matrix}2&0&1\\1&0&1\\-1&1&-1\end{matrix}\right)}_{C^{-1}}$$

Pak

$$A=\begin{pmatrix}1&-1&0\\0&1&1\\-1&2&0\end{pmatrix}\begin{pmatrix}3&0&0\\0&2&1\\0&0&2\end{pmatrix}\begin{pmatrix}2&0&1\\1&0&1\\-1&1&-1\end{pmatrix}$$

$$e^A=\begin{pmatrix}1&-1&0\\0&1&1\\-1&2&0\end{pmatrix}\begin{pmatrix}e^3&0&0\\0&e^2&e^2\\0&0&e^2\end{pmatrix}\begin{pmatrix}2&0&1\\1&0&1\\-1&1&-1\end{pmatrix}$$

$$=\begin{pmatrix}e^3&-e^2&-e^2\\0&e^2&2e^2\\-e^3&2e^2&2e^2\end{pmatrix}\begin{pmatrix}2&0&1\\1&0&1\\-1&1&-1\end{pmatrix}=$$

$$=\begin{pmatrix}2e^3&-e^2&e^3\\-e^2&2e^2&-e^2\\-2e^3&2e^2&-e^3\end{pmatrix}$$

8c) $A=\begin{pmatrix}-2&4&3\\0&1&0\\0&1&1\end{pmatrix}$

$$\begin{vmatrix} -2-\lambda & 4 & 3 \\ 0 & 1-\lambda & 0 \\ 0 & 1 & 1-\lambda \end{vmatrix} = (-2-\lambda)\begin{vmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{vmatrix} = -(\lambda+2)\cdot(\lambda-1)^2$$

1) $\lambda_1 = -2$

$$\begin{pmatrix} 0 & 4 & 3 \\ 0 & 3 & 0 \\ 0 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 4 & 3 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$v_1 = 0$
$w_1 = 0$

$$\begin{pmatrix} u_1 \\ v_1 \\ w_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad A\begin{pmatrix} u_1 \\ v_1 \\ w_1 \end{pmatrix} = -2\begin{pmatrix} u_1 \\ v_1 \\ w_1 \end{pmatrix}$$

2) $\lambda_2 = 1$

$$\begin{pmatrix} -3 & 4 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

$u_2 - w_2 = 0$
$v_2 = 0$

$$\begin{pmatrix} u_2 \\ v_2 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

3) Sdružený vektor k $\begin{pmatrix} u_2 \\ v_2 \\ w_2 \end{pmatrix}$:

$$(A-I)\begin{pmatrix} u_3 \\ v_3 \\ w_3 \end{pmatrix} = \begin{pmatrix} u_2 \\ v_2 \\ w_2 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} -3 & 4 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} -3 & 0 & 3 & -3 \\ 0 & 1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 \end{array}\right)$$

$u_3 - w_3 = 1$
$v_3 = 1$

$$\begin{pmatrix} u_3 \\ v_3 \\ w_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Teď!

$$A\underbrace{\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}}\begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}_{C} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Najdeme C^{-1}

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right)$$

$$\underbrace{\begin{matrix} 1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{matrix}}_{C^{-1}}$$

Pak

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$e^A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} e^{-2} & 0 & 0 \\ 0 & e & e \\ 0 & 0 & e \end{pmatrix}\begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} e^{-2} & e & 2e \\ 0 & 0 & e \\ 0 & e & e \end{pmatrix}\begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} e^{-2} & -e^{-2}+2e & -e^{-2}+e \\ 0 & e & 0 \\ 0 & e & e \end{pmatrix}$$

8d) $$A = \begin{pmatrix} 10 & -13 & 6 & -19 \\ -3 & 7 & -2 & 7 \\ 5 & -8 & 6 & -12 \\ 7 & -11 & 6 & -15 \end{pmatrix}$$

$$\det(A-\lambda I) = \begin{vmatrix} 10-\lambda & -13 & 6 & -19 \\ -3 & 7-\lambda & -2 & 7 \\ 5 & -8 & 6-\lambda & -12 \\ 7 & -11 & 6 & -15-\lambda \end{vmatrix} \sim (R_4 \to R_4 + R_2 - R_3)$$

$$= \begin{vmatrix} 10-\lambda & -13 & 6 & -19 \\ -3 & 7-\lambda & -2 & 7 \\ 5 & -8 & 6-\lambda & -12 \\ -1 & 4-\lambda & \lambda-2 & 4-\lambda \end{vmatrix} = \begin{vmatrix} 0 & (10-\lambda)(4-\lambda) -13 & (10-\lambda)(\lambda-2) +6 & (10-\lambda)(4-\lambda) -19 \\ 0 & 2\lambda-5 & -3\lambda+4 & 3\lambda-5 \\ 0 & 12-5\lambda & 4\lambda-4 & 8-5\lambda \\ -1 & 4-\lambda & \lambda-2 & 4-\lambda \end{vmatrix} =$$

$$= \begin{vmatrix} \lambda^2-14\lambda+27 & -\lambda^2+12\lambda-14 & \lambda^2-14\lambda+21 \\ 2\lambda-5 & -3\lambda+4 & 3\lambda-5 \\ 12-5\lambda & 4\lambda-4 & 8-5\lambda \end{vmatrix} = \begin{pmatrix} S_1 \to S_1 - S_3 \\ S_2 \to S_2 + S_3 \end{pmatrix}$$

$$= \begin{vmatrix} 6 & -2\lambda+7 & \lambda^2-14\lambda+21 \\ -\lambda & -1 & 3\lambda-5 \\ 4 & 4-\lambda & 8-5\lambda \end{vmatrix} = \begin{vmatrix} S_1 \to S_1 + (-\lambda) S_2 \\ S_3 \to S_3 + (3\lambda-5) S_2 \end{vmatrix}$$

$$= \begin{vmatrix} 6+\lambda(2\lambda-7) & 2\lambda-33 & (3\lambda-5)(-2\lambda+7) + \lambda^2-14\lambda+21 \\ 0 & -1 & 0 \\ 4+\lambda(\lambda-4) & 4-\lambda & (3\lambda-5)(4-\lambda) + 8-5\lambda \end{vmatrix}$$

$$= - \begin{vmatrix} 2\lambda^2-7\lambda+6 & -5\lambda^2+17\lambda-14 \\ \lambda^2-4\lambda+4 & -3\lambda^2+12\lambda-12 \end{vmatrix} =$$

$$= -3(\lambda^2-4\lambda+4)$$

$$= -(\lambda^2-4\lambda+4) \begin{vmatrix} 2\lambda^2-7\lambda+6 & -5\lambda^2+17\lambda-14 \\ 1 & -3 \end{vmatrix}$$

$$= -(\lambda-2)^2 (-6\lambda^2+21\lambda-18 + 5\lambda^2-17\lambda+14)$$

$= -(\lambda-2)(-\lambda^2+4\lambda-4) = (\lambda-2)$

Najdeme vlastní vektory odpovídající vlastnímu číslu $\lambda=2$.

$$\begin{pmatrix} 8 & -13 & 6 & -19 \\ -3 & 5 & -2 & 7 \\ 5 & -8 & 4 & -12 \\ 7 & -11 & 6 & -17 \end{pmatrix} \sim \begin{pmatrix} R_3 \to R_1+R_2-R_3 \\ R_1 \to R_1 - R_4 \\ R_2 \to R_2 + R_3 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 0 & -2 \\ 2 & -3 & 2 & -5 \\ 0 & 0 & 0 & 0 \\ 7 & -11 & 6 & -17 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -2 & 0 & -2 \\ 0 & 1 & 2 & -1 \\ 0 & 3 & 6 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 4 & -4 \\ 0 & 1 & 2 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 4 & -4 \\ 0 & 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ p \end{pmatrix} = 0$$

$$\varphi_1 = \begin{pmatrix} -4 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \quad \varphi_2 = \begin{pmatrix} 4 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

Takže máme dva lineárně nezávislých vektorů φ_1, φ_2, odpovídajících vlastnímu číslu $\lambda=2$.

To znamená, že buď máme dve Jordanovy buňky 2×2, nebo jednu buňku 1×1 a jednu 3×3. Abychom to zjistily, najdeme sdružený vektor odpovídající nějaké linearně kombinaci $\alpha\varphi_1+\beta\varphi_2$.

$$(A-2I)\,\psi = \alpha\varphi_1 + \beta\varphi_2$$

$$\left(\begin{array}{cccc|c} 8 & -13 & 6 & -19 & -4\alpha+4\beta \\ -3 & 5 & -2 & 7 & -2\alpha+\beta \\ 5 & -8 & 4 & -12 & \alpha \\ 7 & -11 & 6 & -17 & \beta \end{array}\right) \sim$$

$$\sim\left(\begin{array}{cccc|c} 1 & -2 & 0 & -2 & -4\alpha+5\beta \\ 0 & 0 & 0 & 0 & -7\alpha+5\beta \\ 2 & -3 & 2 & -5 & -\alpha+\beta \\ 7 & -11 & 6 & -17 & \beta \end{array}\right)$$

$$\sim\left(\begin{array}{cccc|c} 1 & -2 & 0 & -2 & -4\alpha+3\beta \\ 0 & 1 & 2 & -1 & 7\alpha-5\beta \\ 0 & 3 & 6 & -3 & 28\alpha-20\beta \\ 0 & 0 & 0 & 0 & -7\alpha+5\beta \end{array}\right)\sim$$

$$\sim\left(\begin{array}{cccc|c} 1 & 0 & 4 & -4 & 10\alpha-7\beta \\ 0 & 1 & 2 & -1 & 7\alpha-5\beta \\ 0 & 0 & 0 & 0 & 7\alpha-5\beta \end{array}\right)$$

Takže $7\alpha=5\beta$ a pak existuje jenom jeden vlastní vektor ke kterému existuje sdružený vektor. To znamená, že máme buňky 1x1 a 3x3.

Vezmeme $\alpha=5, \beta=7$, pak vlastní vektor odpovídající buňce 3x3 je

$\psi := 5\varphi_1 + 7\varphi_2 = \begin{pmatrix} 8 \\ -3 \\ 5 \\ 7 \end{pmatrix}$, a sdružený vektor k ψ řeší soustavu

$$\left(\begin{array}{cccc|c} 1 & 0 & 4 & -4 & 1 \\ 0 & 1 & 2 & -1 & 0 \end{array}\right), \quad \psi_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \gamma\varphi_1 + \delta\varphi_2$$

Teď najdeme sdružený vektor k ψ_2,

$$\left(\begin{array}{cccc|c} 8 & -13 & 6 & -19 & -4\gamma+4\delta+1 \\ -3 & 5 & -2 & 7 & -2\gamma+\delta \end{array}\right.$$

$$\left(\begin{array}{cccc|c} 5 & -8 & 4 & -12 & \gamma \\ 7 & -11 & 6 & -17 & \delta \end{array}\right) \sim$$

$$\sim \left(\begin{array}{cccc|c} 1 & -2 & 0 & -2 & -4\gamma+3\delta+1 \\ 0 & 0 & 0 & 0 & -7\gamma+5\delta+1 \\ 2 & -3 & 2 & -5 & -\gamma+\delta \\ 7 & -11 & 6 & -17 & \delta \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & -2 & 0 & -2 & -4\gamma+3\delta+1 \\ 0 & 1 & 2 & -1 & 7\gamma-5\delta-2 \\ 0 & 3 & 6 & -3 & 28\gamma-20\delta-7 \\ 0 & 0 & 0 & 0 & -7\gamma+5\delta+1 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 4 & -4 & 10\gamma-7\delta-3 \\ 0 & 1 & 2 & -1 & 7\gamma-5\delta-2 \\ 0 & 0 & 0 & 0 & 7\gamma-5\delta-1 \end{array}\right)$$

Zvolime $\gamma=-2$, $\delta=-3$: $\left(\begin{array}{cccc|c} 1 & 0 & 4 & -4 & -2 \\ 0 & 1 & 2 & -1 & -1 \end{array}\right)$

a $\varphi_3 = \begin{pmatrix} -2 \\ -1 \\ 0 \\ 0 \end{pmatrix}$

Takže máme $\varphi_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + (-2)\cdot\begin{pmatrix} -4 \\ -2 \\ 1 \\ 0 \end{pmatrix} + (-3)\begin{pmatrix} 4 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ -2 \\ -3 \end{pmatrix}$

$$A\underbrace{\begin{pmatrix} -4 & 8 & -3 & -2 \\ -2 & -3 & 1 & -1 \\ 1 & 5 & -2 & 0 \\ 0 & 7 & -3 & 0 \end{pmatrix}}_{C} = \begin{pmatrix} -4 & 8 & -3 & -2 \\ -2 & -3 & 1 & -1 \\ 1 & 5 & -2 & 0 \\ 0 & 7 & -3 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Máme $A = C\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}C^{-1}$

$$e^{A} = C\begin{pmatrix} e^2 & 0 & 0 & 0 \\ 0 & e^2 & e^2 & \frac{1}{2}e^2 \\ 0 & 0 & e^2 & e^2 \\ 0 & 0 & 0 & e^2 \end{pmatrix}C^{-1}$$

Najdeme C^{-1}

$$\left(\begin{array}{cccc|cccc} -4 & 8 & -3 & -2 & 1 & 0 & 0 & 0 \\ -2 & -3 & 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 5 & -2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 7 & -3 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \sim$$

$$\left(\begin{array}{cccc|cccc} 1 & 5 & -2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 7 & -3 & -1 & 0 & 1 & 2 & 0 \\ 0 & 28 & -11 & -2 & 1 & 0 & 4 & 0 \\ 0 & 7 & -3 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 5 & -2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 7 & -3 & 0 & 0 & 0 & 0 & 1 \\ 0 & 28 & -11 & -2 & 1 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & -2 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 5 & -2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 7 & -3 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 & 1 & 0 & 4 & -4 \\ 0 & 0 & 0 & 1 & 0 & -1 & -2 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 5 & -2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 7 & -3 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & -2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 & -1 & -2 & 1 \end{array}\right)$$

$$\left(\begin{array}{cccc|cccc} 1 & 5 & 0 & 0 & 2 & -4 & 1 & -4 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|cccc} 0 & 7 & 0 & 0 & 3 & -6 & 0 & -5 \\ 0 & 0 & 1 & 0 & 1 & -2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 & -1 & -2 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{-1}{7} & \frac{2}{7} & 1 & \frac{-3}{7} \\ 0 & 1 & 0 & 0 & \frac{3}{7} & \frac{-6}{7} & 0 & \frac{-5}{7} \\ 0 & 0 & 1 & 0 & 1 & -2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 & -1 & -2 & 1 \end{array}\right)$$