Mathematics II

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- Functions of several variables

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- Matrix calculus

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- Antiderivative and the Riemann Integral

V.1. $\mathbb{R}^n$ as a linear and metric space

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Definition

The set $\mathbb{R}^n$, $n \in \mathbb{N}$, is the set of all ordered n-tuples of real numbers, i.e.

$$\mathbb{R}^n = \{(x_1, \dots, x_n) : x_1, \dots, x_n \in \mathbb{R}\}.$$

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For $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, $\boldsymbol{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ we set

$$\boldsymbol{x} + \boldsymbol{y} = (x_1 + y_1, \dots, x_n + y_n), \qquad \alpha\boldsymbol{x} = (\alpha x_1, \dots, \alpha x_n).$$

Further, we denote $\boldsymbol{o} = (0, \dots, 0)$ – the origin.

Properties of $+, \cdot$

1. $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}$, for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ (commutativity)
2. $(\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z} = \boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z})$ (associativity)
3. $\exists \boldsymbol{o} \in \mathbb{R}^n \, \forall x \in \mathbb{R}^n : \quad \boldsymbol{x} + \boldsymbol{o} = \boldsymbol{x}$ (existence of neutral element)
4. $\forall x \in \mathbb{R}^n \, \exists (-x) \in \mathbb{R}^n : \quad x + (-x) = \boldsymbol{o}$ (existence of opposite element)
5. $\alpha(\beta \boldsymbol{x}) = (\alpha\beta)\boldsymbol{x}$, for any $\alpha, \beta \in \mathbb{R}$ and $x \in \mathbb{R}^n$

Comment

Properties 1- 4 tells that $\mathbb{R}^n$ is a group with respect to the operation $+$

Definition

The Euclidean metric (distance) on $\mathbb{R}^n$ is the function $\rho\colon \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty)$ defined by

$$\rho(\boldsymbol{x}, \boldsymbol{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}\ .$$

The number $\rho(\boldsymbol{x}, \boldsymbol{y})$ is called the distance of the point $\boldsymbol{x}$ from the point $\boldsymbol{y}$.

Theorem 1 (properties of the Euclidean metric)

The Euclidean metric ρ has the following properties:

(i) $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n\colon \rho(\boldsymbol{x}, \boldsymbol{y}) = 0 \Leftrightarrow \boldsymbol{x} = \boldsymbol{y}$,

Remark

Properties I-III make the pair $(\mathbb{R}^n, \rho)$ a metric space.

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(iii) $\forall \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^n \colon \rho(\boldsymbol{x}, \boldsymbol{y}) \leq \rho(\boldsymbol{x}, \boldsymbol{z}) + \rho(\boldsymbol{z}, \boldsymbol{y})$,
(triangle inequality)

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(iv) $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n, \forall \lambda \in \mathbb{R} \colon \rho(\lambda \boldsymbol{x}, \lambda \boldsymbol{y}) = |\lambda| \rho(\boldsymbol{x}, \boldsymbol{y})$, *(homogeneity)*

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(v) $\forall \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^n \colon \rho(\boldsymbol{x} + \boldsymbol{z}, \boldsymbol{y} + \boldsymbol{z}) = \rho(\boldsymbol{x}, \boldsymbol{y})$. *(translation invariance)*

Remark

Properties I-III make the pair $(\mathbb{R}^n, \rho)$ a metric space.

Definition

Let $\boldsymbol{x}^j \in \mathbb{R}^n$ for each $j \in \mathbb{N}$ and $\boldsymbol{x} \in \mathbb{R}^n$. We say that a sequence $\{\boldsymbol{x}^j\}_{j=1}^{\infty}$ converges to $\boldsymbol{x}$, if

$$\lim_{j \to \infty} \rho(\boldsymbol{x}, \boldsymbol{x}^j) = 0.$$

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The sequence $\{\boldsymbol{y}^j\}_{j=1}^{\infty}$ of points in $\mathbb{R}^n$ is called convergent if there exists $\boldsymbol{y} \in \mathbb{R}^n$ such that $\{\boldsymbol{y}^j\}_{j=1}^{\infty}$ converges to $\boldsymbol{y}$.

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Remark

The sequence $\{\boldsymbol{x}^j\}_{j=1}^{\infty}$ converges to $\boldsymbol{x} \in \mathbb{R}^n$ if and only if

$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0\ \exists j_0 \in \mathbb{N}\ \forall j \in \mathbb{N}, j \geq j_0 \colon \boldsymbol{x}^j \in U(\boldsymbol{x}, \varepsilon).$$

Theorem 2 (convergence is coordinatewise)

Let $\boldsymbol{x}^j \in \mathbb{R}^n$ for each $j \in \mathbb{N}$ and let $\boldsymbol{x} \in \mathbb{R}^n$. The sequence $\{\boldsymbol{x}^j\}_{j=1}^{\infty}$ converges to $\boldsymbol{x}$ if and only if for each $i \in \{1, \ldots, n\}$ the sequence of real numbers $\{x_i^j\}_{j=1}^{\infty}$ converges to the real number x_i.

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Remark

Theorem 2 says that the convergence in the space $\mathbb{R}^n$ is the same as the "coordinatewise" convergence. It follows that a sequence $\{\boldsymbol{x}^j\}_{j=1}^\infty$ has at most one limit. If it exists, then we denote it by $\lim_{j\to\infty} \boldsymbol{x}^j$. Sometimes we also write simply $\boldsymbol{x}^j \to \boldsymbol{x}$ instead of $\lim_{j\to\infty} \boldsymbol{x}^j = \boldsymbol{x}$.

Definition

Let $\boldsymbol{x} \in \mathbb{R}^n,\ r \in \mathbb{R}, r > 0$. The set $U(\boldsymbol{x}, r)$ defined by

$$U(\boldsymbol{x}, r) = \{\boldsymbol{y} \in \mathbb{R}^n;\ \rho(\boldsymbol{x}, \boldsymbol{y}) < r\}$$

is called an open ball with radius r centred at $\boldsymbol{x}$ or the r-neighbourhood of $\boldsymbol{x}$.

Another notations for $U(\boldsymbol{x}, r)$: $U_r(\boldsymbol{x}), B(\boldsymbol{x}, r)$.

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Exterior of M is $\operatorname{Ext} M = \operatorname{Int}(M^C)$, where $M^C = \mathbb{R}^n \setminus M$ is the complement of M.

Theorem 3 (properties of open sets)

(i) *The empty set and $\mathbb{R}^n$ are open in $\mathbb{R}^n$.*

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(ii) *Let $G_\alpha \subset \mathbb{R}^n$, $\alpha \in A \neq \emptyset$, be open in $\mathbb{R}^n$. Then $\bigcup_{\alpha \in A} G_\alpha$ is open in $\mathbb{R}^n$.*

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(iii) *Let $G_i \subset \mathbb{R}^n$, $i = 1, \dots, m$, be open in $\mathbb{R}^n$. Then $\bigcap_{i=1}^{m} G_i$ is open in $\mathbb{R}^n$.*

Remark

(ii) *A union of an arbitrary system of open sets is an open set.*
(iii) *An intersection of a finitely many open sets is an open set.*

Definition

Let $M \subset \mathbb{R}^n$ and $\boldsymbol{x} \in \mathbb{R}^n$. We say that $\boldsymbol{x}$ is a boundary point of M if for each $r > 0$

$$U(\boldsymbol{x}, r) \cap M \neq \emptyset \quad \text{and} \quad U(\boldsymbol{x}, r) \cap (\mathbb{R}^n \setminus M) \neq \emptyset.$$

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A set $M \subset \mathbb{R}^n$ is said to be closed in $\mathbb{R}^n$ if it contains all its boundary points, i.e. if $\operatorname{bd} M \subset M$, or in other words if $\overline{M} = M$.

Proposition

$x \in \operatorname{bd} M$ iff $x \notin \operatorname{Int} M$ and $x \notin \operatorname{Ext} M$.

Consequence

$\mathbb{R}^n = \operatorname{Int} M \sqcup \operatorname{Ext} M \sqcup \operatorname{bd} M$ (here $A \sqcup B = A \cup B, A \cap B = \emptyset$ is disjoint union).

$\operatorname{bd} M = \operatorname{bd} M^C$.

Theorem 4 (characterisation of closed sets)

Let $M \subset \mathbb{R}^n$. Then the following statements are equivalent:

- (i) *M is closed in $\mathbb{R}^n$.*
- (ii) *$\mathbb{R}^n \setminus M$ is open in $\mathbb{R}^n$.*
- (iii) *Any $\boldsymbol{x} \in \mathbb{R}^n$ which is a limit of a sequence from M belongs to M.*

Theorem 5 (properties of closed sets)

(i) *The empty set and the whole space $\mathbb{R}^n$ are closed in $\mathbb{R}^n$.*

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(iii) *Let $F_i \subset \mathbb{R}^n$, $i = 1, \dots, m$, be closed in $\mathbb{R}^n$. Then $\bigcup_{i=1}^m F_i$ is closed in $\mathbb{R}^n$.*

Remark

(ii) *An intersection of an arbitrary system of closed sets is closed.*
(iii) *A union of finitely many closed sets is closed.*

Lemma 6

(i) *For any set M we have* $\overline{M} = \operatorname{Int} M \sqcup \operatorname{bd} M$.

(ii) *If $M \subset \mathbb{R}^n$ is closed, then* $M = \operatorname{Int} M \sqcup \operatorname{bd} M$.

Theorem 7

Let $M \subset \mathbb{R}^n$. Then the following holds:

(i) *The set* $\operatorname{Int} M$ *is open in* $\mathbb{R}^n$ *(i.e.* $\operatorname{Int}(\operatorname{Int} M) = \operatorname{Int} M$*).*

(ii) *The set* $\overline{M}$ *is closed in* $\mathbb{R}^n$ *(i.e.* $\overline{\overline{M}} = \overline{M}$*).*

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Theorem 7

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(i) *The set $\operatorname{Int} M$ is open in $\mathbb{R}^n$ (i.e. $\operatorname{Int}(\operatorname{Int} M) = \operatorname{Int} M$).*

(ii) *The set $\overline{M}$ is closed in $\mathbb{R}^n$ (i.e. $\overline{\overline{M}} = \overline{M}$).*

Remark

We have $\operatorname{Int} M \subset M \subset \overline{M}$. Moreover, the set $\operatorname{Int} M$ is the largest open set contained in M in the following sense: If G is a set open in $\mathbb{R}^n$ and satisfying $G \subset M$, then $G \subset \operatorname{Int} M$.

Lemma 6

(i) *For any set M we have* $\overline{M} = \operatorname{Int} M \sqcup \operatorname{bd} M$.

(ii) *If* $M \subset \mathbb{R}^n$ *is closed, then* $M = \operatorname{Int} M \sqcup \operatorname{bd} M$.

Theorem 7

Let $M \subset \mathbb{R}^n$. *Then the following holds:*

(i) *The set* $\operatorname{Int} M$ *is open in* $\mathbb{R}^n$ *(i.e.* $\operatorname{Int}(\operatorname{Int} M) = \operatorname{Int} M$*).*

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Remark

We have $\operatorname{Int} M \subset M \subset \overline{M}$. Moreover, the set $\operatorname{Int} M$ is the largest open set contained in M in the following sense: If G is a set open in $\mathbb{R}^n$ and satisfying $G \subset M$, then $G \subset \operatorname{Int} M$. Similarly $\overline{M}$ is the smallest closed set containing M.

Lemma 8 (monotonicity of taking interior and closure)

If $A \subset B \subset \mathbb{R}^n$, then

(i) $\operatorname{Int} A \subset \operatorname{Int} B$,

(ii) $\overline{A} \subset \overline{B}$.

Exercise 1

Let $A \sqcup B = \mathbb{R}^n$. Then

(a) $\operatorname{Int} A = \mathbb{R}^n \setminus \overline{B}$, (b) $\overline{A} = \mathbb{R}^n \setminus \operatorname{Int} B$.

Exercise 2

Let $A, B \subset \mathbb{R}^2$. Prove that (a) $\overline{\operatorname{bd} A} = \operatorname{bd} A$;
(b) $\overline{A} = \operatorname{Int} A \Longrightarrow ((A = \mathbb{R}^2) \vee (A = \emptyset))$; (c) disprove the equality $\overline{A \cap B} = \overline{A} \cap \overline{B}$, change the equality sign to a correct inclusion and prove the latter.

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Theorem 9

A set $M \subset \mathbb{R}^n$ is bounded if and only if its closure $\overline{M}$ is bounded.

Definition

We say that a point $\boldsymbol{a} \in \mathbb{R}^n$ is an accumulation point (or condensation point) of a function $f : D_f \to \mathbb{R}$, $D_f \subset \mathbb{R}^n$, if every neighborhood of $\boldsymbol{a}$ contains at least one point of D_f, not equal to $\boldsymbol{a}$. The point $\boldsymbol{a}$ might be or might not be a point of D_f.

Definition

We say that a function f of n variables has a limit at a point $\boldsymbol{a} \in \mathbb{R}^n$ equal to $A \in \mathbb{R}^*$ if $\boldsymbol{a}$ is an accumulation point of D_f and

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall \boldsymbol{x} \in U(\boldsymbol{a}, \delta) \cap D_f \setminus \{\boldsymbol{a}\} \colon f(\boldsymbol{x}) \in U(A, \varepsilon).$$

Remark

- Each function has at a given point at most one limit. We write $\lim_{\boldsymbol{x}\to\boldsymbol{a}} f(\boldsymbol{x}) = A$.

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- For limits of functions of several variables one can prove similar theorems as for limits of functions of one variable (arithmetics, the sandwich theorem, ...).

Theorem 10

Let $r, s \in \mathbb{N}$, $\boldsymbol{a} \in \mathbb{R}^s$, and let $\varphi_1, \dots, \varphi_r$ be functions of s variables such that $\lim_{\boldsymbol{x}\to\boldsymbol{a}} \varphi_j(\boldsymbol{x}) = b_j$, $j = 1, \dots, r$. Denote $\boldsymbol{b} = (b_1, \dots, b_r)$. Let f be a function of r variables which is continuous at the point $\boldsymbol{b}$. If we define a compound function F of s variables by

$$F(\boldsymbol{x}) = f(\varphi_1(\boldsymbol{x}), \varphi_2(\boldsymbol{x}), \dots, \varphi_r(\boldsymbol{x})),$$

then $\lim_{\boldsymbol{x}\to\boldsymbol{a}} F(\boldsymbol{x}) = f(\boldsymbol{b})$.

V.2. Continuous functions of several variables

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Definition

Let $M \subset \mathbb{R}^n$, $\boldsymbol{x} \in M$, and $f\colon M \to \mathbb{R}$. We say that f is **continuous at $\boldsymbol{x}$ with respect to M**, if

$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0\ \exists \delta \in \mathbb{R}, \delta > 0\ \forall \boldsymbol{y} \in U(\boldsymbol{x}, \delta) \cap M\colon f(\boldsymbol{y}) \in U(f(\boldsymbol{x}), \varepsilon).$$

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We say that f is continuous at the point $\boldsymbol{x}$ if it is continuous at $\boldsymbol{x}$ with respect to a neighbourhood of $\boldsymbol{x}$, i.e.

$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0 \, \exists \delta \in \mathbb{R}, \delta > 0 \, \forall \boldsymbol{y} \in U(\boldsymbol{x}, \delta) \colon f(\boldsymbol{y}) \in U(f(\boldsymbol{x}), \varepsilon).$$

Remark

If $\boldsymbol{a}$ is an accumulation point of D_f, then the function f is continuous at $\boldsymbol{a}$ if and only if $\lim_{\boldsymbol{x} \to \boldsymbol{a}} f(\boldsymbol{x}) = f(\boldsymbol{a})$.

Other neighborhoods

In the definition of continuity, the circle $U(x, \delta)$ can be changed with the cube $A(x, \delta)$, where
$A(x, r) = \{y = (y_1, \dots, y_n) \in \mathbb{R}^n : \quad |y_j - x_j| < r\}$.
We have $U(x, r) \subset A(x, r) \subset U(x, r\sqrt{n})$.

Theorem 11 (Heine)

Let $M \subset \mathbb{R}^n$, $\boldsymbol{x} \in M$, and $f\colon M \to \mathbb{R}$. Then the following are equivalent.

(i) *The function f is continuous at $\boldsymbol{x}$ with respect to M.*

(ii) $\lim\limits_{j\to\infty} f(\boldsymbol{x}^j) = f(\boldsymbol{x})$ *for each sequence* $\{\boldsymbol{x}^j\}_{j=1}^{\infty}$ *such that* $\boldsymbol{x}^j \in M$ *for* $j \in \mathbb{N}$ *and* $\lim\limits_{j\to\infty} \boldsymbol{x}^j = \boldsymbol{x}$.

Examples

1. $f(x,y) = \begin{cases} \frac{x}{y}, & y \neq 0, \\ 0, & y = 0. \end{cases}$

   $M_1 = \mathbb{R}^2; \quad M_2 = \mathbb{R} \times \{0\}; \quad M_3 = \{0\} \times \mathbb{R};$

   $M_4 = \{(x,x) : \ x \in \mathbb{R}\}.$
2. $f(x,y) = \frac{xy}{x^2+y^2}.$
3. $f(x,y) = \frac{x^2-y^2}{x^2+y^2}; \qquad f(x,y) = \frac{x^2y}{x^2+y^2}$

Repeated limits

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) \neq \lim_{x\to x_0} \lim_{y\to y_0} f(x,y) \neq \lim_{y\to y_0} \lim_{x\to x_0} f(x,y).$$

Example

$f(x,y) = \frac{x-y+x^2+y^2}{x+y}.$