Limits of sequences 2

Exercises: reference limits

1. $\lim_{n\to\infty} \sqrt[n]{n}$

2. $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$

Exercises: roots

3. $\lim_{n\to\infty} \frac{(n+4)^{1000}-(n+3)^{1000}}{(n+2)^{1000}-n^{1000}}$

4. $\lim_{n\to\infty} \left(\sqrt[3]{n^{75}+n^{60}}-\sqrt[3]{n^{75}-n^{60}}\right)\cdot\frac{(n^3+n^2)^{20}-(n^2+n)^{30}}{(n+1)^{70}-(n-1)^{70}}$

5. $\lim_{n\to\infty} \frac{\sqrt{n^3}+\sqrt{n}+1}{\sqrt{n+1}-\sqrt{n}}\cdot\frac{(n^4+n)^{50}-(n+1)^{200}}{(n+1)^{202}-n^{202}}$

Bonus exercises

6.* $\lim_{n\to\infty} \left(\frac{1+2+3+\ldots+n}{n+2}-\frac{n}{2}\right)$

7.* $\lim_{n\to\infty} \frac{1^2+2^2+3^2+\ldots+n^2}{n^3}$

8.* $\lim_{n\to\infty} \frac{1^3+2^3+3^3+\ldots+n^3}{n^4}$

9.* $\lim_{n\to\infty} \left(\frac{1}{2}+\frac{3}{2^2}+\frac{5}{2^3}+\ldots+\frac{2n-1}{2^n}\right)$

10.* $\lim_{n\to\infty} \left(\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdot\ldots\cdot\left(1-\frac{1}{n^2}\right)\right)$

Bonus exercises: Sequences given recurrently.

Find the limit of a sequence a_n given recurrently

11. $a_1=0$, $a_{n+1}=\frac{a_n+3}{4}$.

12. $a_1=42$, $a_{n+1}=\frac{a_n-2}{5}$.

13. $a_1=\sqrt{2}$, $a_{n+1}=\sqrt{2+a_n}$.

14. $x_{n+1}=\frac{1}{2}\left(x_n+\frac{S}{x_n}\right)$, $x_1>0$ arbitrary, $S>0$ fixed.

15. $x_{n+1}=\frac{2}{3}x_n+\frac{S}{3x_n^2}$, $x_1>0$, $S>0$ fixed.

16. $x_{n+1}=\frac{1}{3}x_n+\frac{2S}{3x_n^2}$, $x_1>0$, $S>0$ fixed.

Exercises for self practice

17. $\lim_{n\to\infty} \sqrt[n]{2^n+4^n}$

18. $\lim_{n\to\infty} \frac{\sqrt[n]{a^n+b^n}}{\sqrt[n]{a^{2n}+b^{2n}}}$, $a>b>0$

18. $\lim_{n\to\infty} \left(\sqrt{n+1}-\sqrt{n}\right)$

19. $\lim_{n\to\infty} \left(\sqrt[3]{n+1}-\sqrt[3]{n}\right)$

20. $\lim_{n\to\infty} \frac{\sqrt[3]{n^2+7}-\sqrt[3]{n^2+1}}{\sqrt[3]{n^2+6}-\sqrt[3]{n^2}}$

21. $\lim_{n\to\infty} \frac{\sqrt[3]{n^3+n}-\sqrt[3]{n^3+1}}{\sqrt[3]{n^3+2n}-\sqrt[3]{n^3+n}}$

22. $\lim_{n\to\infty} \frac{\sqrt[4]{n+2}-\sqrt[4]{n+1}}{\sqrt[3]{n+3}-\sqrt[3]{n}}$

23. $\lim_{n\to\infty} (-1)^n\sqrt{n}\left(\sqrt{n+1}-\sqrt{n}\right)$

Solutions

4. $\lim\limits_{n\to\infty} \left(\sqrt[3]{n^{75}+n^{60}} - \sqrt[3]{n^{75}-n^{60}}\right) \cdot \dfrac{(n^3+n^2)^{20}-(n^2+n)^{30}}{(n+1)^{70}-(n-1)^{70}} = \frac{-1}{21}$.

5. $\lim\limits_{n\to\infty} \dfrac{\sqrt{n^3}+\sqrt{n}+1}{\sqrt{n+1}-\sqrt{n}} \cdot \dfrac{(n^4+n)^{50}-(n+1)^{200}}{(n+1)^{202}-n^{202}} = \frac{-200}{101}$.

9. 3. Hint: $2k+1 = (2k-1)+2$. Then $S_n = \frac{1}{2} + \left(\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{n-1}}\right) + \frac{1}{2}S_{n-1}$.

10. $\frac{1}{2}$. Hint: $1-\frac{1}{n^2} = \frac{n-1}{n}\cdot\frac{n+1}{n}$.

11. 1. We want to use a theorem about the limit of a monotone sequence.

    We have
$$a_{n+1}-a_n = \frac{3}{4}(1-a_n), \qquad 1-a_{n+1} = \frac{1}{3}(a_{n+1}-a_n),$$
hence $0 < a_n < a_{n+1} < 1$, and there exists a limit a of a_n. Taking the limit in the recurrent relation, we obtain $a = \frac{1}{4}a + \frac{3}{4}$.

17. $\lim\limits_{n\to\infty} \sqrt[n]{2^n+4^n}$. Here we will use the Theorem about two policiemen. The idea is that 2^n is negligible compared to 4^n, so the limit will be 4. It remains to find two cops. For every $n\in\mathbb{N}$, it holds that
$$\sqrt[n]{4^n} \le \sqrt[n]{2^n+4^n} \le \sqrt[n]{2\cdot 4^n}.$$
Apparently $\lim\limits_{n\to\infty}\sqrt[n]{4^n} = \lim\limits_{n\to\infty} 4 = 4$. And thanks to Theorem of Arithmetics of limits,
$$\lim_{n\to\infty}\sqrt[n]{2\cdot 4^n} = \left(\lim_{n\to\infty}\sqrt[n]{2}\right)\left(\lim_{n\to\infty}\sqrt[n]{4^n}\right) = 1\cdot 4 = 4,$$
where we used the known limit. So we have two cops, they both go to the same constant, and thus
$$\lim_{n\to\infty}\sqrt[n]{2^n+4^n} = 4.$$

18. $\lim\limits_{n\to\infty} \dfrac{\sqrt[n]{a^n+b^n}}{\sqrt[n]{a^{2n}+b^{2n}}}$, $a>b>0$. Here, similar to exercise 17, we use two cops. It holds for every $n\in\mathbb{N}$ that
$$\frac{a}{a^2\sqrt[n]{2}} = \frac{\sqrt[n]{a^n}}{\sqrt[n]{2a^{2n}}} \le \frac{\sqrt[n]{a^n+b^n}}{\sqrt[n]{a^{2n}+b^{2n}}} \le \frac{\sqrt[n]{2a^n}}{\sqrt[n]{a^{2n}}} = \frac{a\sqrt[n]{2}}{a^2}.$$
Since $\lim\limits_{n\to\infty}\frac{a}{a^2\sqrt[n]{2}} = \lim\limits_{n\to\infty}\frac{a\sqrt[n]{2}}{a^2} = \frac{1}{a}$, then also $\lim\limits_{n\to\infty}\dfrac{\sqrt[n]{a^n+b^n}}{\sqrt[n]{a^{2n}+b^{2n}}} = \frac{1}{a}$.

19. $\lim\limits_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)$. In this type of example, we will use a trick that we will call standard procedure from now on. Namely, we will use the relation $(a-b)(a+b) = (a^2-b^2)$, and we multiply our sequence by the so-called "smart one":
$$\begin{aligned}\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right) &= \lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} \\ &= \lim_{n\to\infty}\frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty}\frac{1}{\sqrt{n}}\frac{1}{\sqrt{1+\frac{1}{n}}+\sqrt{1}} \\ &= 0\cdot\frac{1}{1+1} = 0\cdot\frac{1}{2} = 0.\end{aligned}$$
We used (several times) Theorem of arithmetics of limits, theorem that the limit of (square) roots is the (square) root of the limit.

20. $\lim_{n\to\infty}\left(\sqrt[3]{n+1}-\sqrt[3]{n}\right)$. Here, as in previous cases, we need to transform the expression using the formula $a^3-b^3=(a-b)(\sqrt[3]{a^2}+\sqrt[3]{ab}+\sqrt[3]{b^2})$. Then

$$\lim_{n\to\infty}\left(\sqrt[3]{n+1}-\sqrt[3]{n}\right)=\lim_{n\to\infty}\frac{n+1-n}{\left(\sqrt[3]{n+1}\right)+\sqrt[3]{n+1}\,\sqrt[3]{n}+\left(\sqrt[3]{n}\right)^2}$$

$$=\lim_{n\to\infty}\frac{1}{n^{\frac{2}{3}}}\frac{1}{\cdot\left(\left(\sqrt[3]{1+\frac{1}{n}}\right)+\sqrt[3]{1+\frac{1}{n}}\,\sqrt[3]{n}+\left(\sqrt[3]{1}\right)^2\right)}=0\cdot\frac{1}{\cdot\left(\left(\sqrt[3]{1+0}\right)+\sqrt[3]{1+0}\,\sqrt[3]{n}+\left(\sqrt[3]{1}\right)^2\right)}=0.$$

We used (several times) Theorem of arithmetics of limits, theorem that the limit of (third) roots is the (third) root of the limit.

21. $\lim_{n\to\infty}\frac{\sqrt[3]{n^2+7}-\sqrt[3]{n^2+1}}{\sqrt[3]{n^2+6}-\sqrt[3]{n^2}}$. We need to transform the expression, as in the previous example, both in numerator and denominator. The answer is 1.

22. $\lim_{n\to\infty}\frac{\sqrt[3]{n^3+n}-\sqrt[3]{n^3+1}}{\sqrt[3]{n^3+2n}-\sqrt[3]{n^3+n}}$. Again, we need to transform the expression, as in the previous example, both in numerator and denominator. The answer is 1.

23. $\lim_{n\to\infty}\frac{\sqrt[4]{n+2}-\sqrt[4]{n+1}}{\sqrt[3]{n+3}-\sqrt[3]{n}}$. Here it is just as simple, although a little more laborious. In the numerator we use the formula $a^4-b^4=(a-b)(a^3+a^2b+ab^2+b^3)$, and in the denominator we use the formula $a^3-b^3=(a-b)(a^2+ab+b^2)$. After substituting and cancelling the largest factor, we get the result 0.

24. $\lim_{n\to\infty}(-1)^n\sqrt{n}\left(\sqrt{n+1}-\sqrt{n}\right)$.

This limit does not exist. First, observe that $\lim_{n\to\infty}\sqrt{n}\left(\sqrt{n+1}-\sqrt{n}\right)=\frac{1}{2}$. Now we can select two subsequences: with n odd and n even, and they have different limits, namely $-\frac{1}{2}$ and $\frac{1}{2}$.