Limits of sequences 1

Definition

Let $a \in \mathbb{R}$ and let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. We say, that a is the limit of the sequence $\{a_n\}_{n=1}^{\infty}$ (and write $\lim\limits_{n\to\infty} a_n = a$), if

$$\forall \varepsilon \; \exists n_0 \in \mathbb{N} \; \forall n \in \mathbb{N} \; n \geq n_0 : \; |a_n - a| < \varepsilon.$$

Theorem 1 (Arithmetics of limits)**.** *Let* $\lim\limits_{n\to\infty} a_n = a$ *and* $\lim\limits_{n\to\infty} b_n = b$*. Then*

1. $\lim\limits_{n\to\infty} (a_n + b_n) = a + b.$
2. $\lim\limits_{n\to\infty} (a_n - b_n) = a - b.$
3. *Let* $\in \mathbb{R}$*. Then* $\lim\limits_{n\to\infty} (c \cdot a_n) = c \cdot a.$
4. $\lim\limits_{n\to\infty} (a_n \cdot b_n) = a \cdot b.$
5. *If* $b \neq 0$*, then* $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}.$

Theorem 2 (two policemen)**.** *Let* $\{a_n\}_{n=1}^{\infty}$*,* $\{b_n\}_{n=1}^{\infty}$*,* $\{c_n\}_{n=1}^{\infty}$ *be three sequences. Let*

1. $\exists n_0 \in \mathbb{N} \; \forall n \in \mathbb{N}$ *such that* $n > n_0 : \; a_n \leq b_n \leq c_n$*, and*
2. $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = a.$

Then $\exists \lim\limits_{n\to\infty} c_n = a.$

Theorem 3 (Binomial formula)**.** *For all* $n \in \mathbb{N}$ *and for all* $a, b \in \mathbb{R}$ *the following is true:*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}, \qquad \text{where } \binom{n}{k} = \frac{n!}{k!(n-k)!}, \qquad n! = 1 \cdot 2 \cdot \ldots \cdot n, \quad 0! = 1.$$

Some known limits

1. $\lim\limits_{n\to\infty} c = c;$
2. $\lim\limits_{n\to\infty} \frac{1}{n} = 0;$
3. $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1;$
4. $\lim\limits_{n\to\infty} (-1)^n$ does not exist.

Scale of growth of sequences

$$\log_a n \quad << \quad n^b \quad << \quad c^n \quad << \quad n! \quad << \quad n^n, \qquad a > 0, b > 0, c > 1$$

Exercises

1. $\lim\limits_{n\to\infty} \dfrac{n}{2^n}$
2. $\lim\limits_{n\to\infty} \dfrac{2n^2 + \sin(n!) - n - 3 + 2n^3}{n^3 + 1}$
3. $\lim\limits_{n\to\infty} \dfrac{2n^5 + 2^n - 3n - 2}{n^5 - 3n^3 + 1 - 2^n}$
4. $\lim\limits_{n\to\infty} \dfrac{2n^3 + 6n + \cos(n^3 - n)}{n^3 - 7n + 7}$
5. $\lim\limits_{n\to\infty} \dfrac{\sqrt{3n-4} - \sqrt{3n-1}}{\sqrt{2n+2} - \sqrt{2n-1}}$
6. $\lim\limits_{n\to\infty} \dfrac{\sqrt{2n-3} - \sqrt{2n+3}}{\sqrt[3]{3n+4} - \sqrt[3]{3n-1}}$
7. $\lim\limits_{n\to\infty} \dfrac{(n+4)^{10} - (n+3)^{10}}{(n+2)^{10} - n^{10}}$
8. $\lim\limits_{n\to\infty} \left(\dfrac{1+2+3+\ldots+n}{n+2} - \dfrac{n}{2} \right)$
9.* $\lim\limits_{n\to\infty} \dfrac{1^2 + 2^2 + 3^2 + \ldots + n^2}{n^3}$
10.* $\lim\limits_{n\to\infty} \dfrac{1^3 + 2^3 + 3^3 + \ldots + n^3}{n^4}$