1. $\int_0^1 x^2 \, d(x^3) = \int_0^1 x^2 \cdot (x^3)' dx = \int_0^1 x^2 \cdot 3x^2 dx = \frac{3}{5} x^5 \Big|_0^1 = \frac{3}{5}$

2. $\int_0^1 x^2 \, d(e^x) = \int_0^1 x^2 (e^x)' dx = \int_0^1 x^2 \cdot e^x \, dx$

Integrujeme per partes:

$$\int_0^1 x^2 d(e^x) = x^2 e^x \Big|_0^1 - \int_0^1 e^x \, d(x^2) =$$

$$= e - \int_0^1 e^x \cdot (x^2)' dx = e - \int_0^1 2x (e^x)' dx = e - 2xe^x \Big|_0^1 + 2\int_0^1 e^x \, dx$$

$$= e - 2e + 2e^x\Big|_0^1 = -e + 2e - 2e^0 = e-2$$

3. $\int_1^e (x+4) \, d(e^x + \log x)$

$$\cancel{= \int_1^e (x+4)(e^x + \log x)' dx = \int_1^e (x+4)\left(e^x + \frac{1}{x}\right) dx = \int_1^e (x+4)(e^x) dx}$$

$$= \int_1^e (x+4) \, d(e^x) + \int_1^e (x+4) \, d(\log x) = \left| \begin{array}{l} \text{v 1.m integrálu: per partes} \\ \text{v 2.ém integrálu: } d(\log x) = (\log x)' dx \end{array} \right|$$

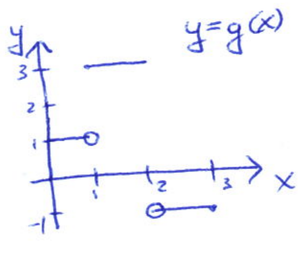
$$= (x+4)e^x \Big|_1^e - \int_1^e e^x \, d(x+4) + \int_1^e (x+4) \cdot \frac{1}{x} \, dx$$

$$= (e+4)e^e - 5e - \int_1^e e^x dx + \int_1^e \left(1 + \frac{4}{x}\right) dx$$

$$= (e+4) \cdot e^e - 5e - e^x \Big|_1^e + (x + 4\log x)\Big|_1^e$$

$$= (e+4) \cdot e^e - 5e - e^e + e + (e+4) - (1)$$

$$= (e+3)e^e - 3e + 3$$



4. $\int_0^3 e^x \, dg(x), \quad g(x) = \begin{cases} 1, & 0 \le x < 1 \\ 3, & 1 \le x \le 2 \\ -1, & 2 < x \le 3 \end{cases}$

Způsob 1: $\int_0^3 e^x dg(x) = \underbrace{\int_0^{1-\varepsilon} e^x dg(x)}_{I_1} + \underbrace{\int_{1-\varepsilon}^{1+\varepsilon} e^x dg(x)}_{I_2} + \underbrace{\int_{1+\varepsilon}^{2-\varepsilon} e^x dg(x)}_{I_3} + \underbrace{\int_{2-\varepsilon}^{2+\varepsilon} e^x dg(x)}_{I_4} + \underbrace{\int_{2+\varepsilon}^{3} e^x dg(x)}_{I_5}$

a) Integrály $I_1, I_3, I_5 = 0$, protože tam g je konstantní, a pak $dg(x) = g'(x)dx = 0dx$

b) $I_2 \le \sup_{x \in [1-\varepsilon, 1+\varepsilon]} e^x \cdot \underbrace{(g(1+\varepsilon) - g(1-\varepsilon))}_{3-1=2}$; $I_2 \ge \inf_{x \in [1-\varepsilon, 1+\varepsilon]} e^x \cdot \underbrace{(g(1+\varepsilon) - g(1-\varepsilon))}_{3-1=2}$

Funkce e^x je spojitá v bodě $x=1$, potom $\exists \lim_{\varepsilon \to 0+} I_2 = 2 \cdot e^1 = 2e$

c) I_4: $\inf_{x \in [2-\varepsilon, 2+\varepsilon]} e^x \cdot \underbrace{(g(2+\varepsilon) - g(2-\varepsilon))}_{-1-3=-4} \le I_4 \le \sup_{x \in [2-\varepsilon, 2+\varepsilon]} e^x \cdot \underbrace{(g(2+\varepsilon) - g(2-\varepsilon))}_{-1-3=-4}$, potom $\lim_{\varepsilon \to 0+} I_4 = -4e^2$

Výsledek: $\boxed{I = 2e - 4e^2}$

4. $\int_0^3 e^x\,dg(x), \quad g(x)=\begin{cases}1, & 0\le x<1\\ 3, & 1\le x\le 2\\ -1, & 2<x\le 3\end{cases}$

Způsob 2: per partes

$$\int_0^3 e^x\,dg(x) = e^x g(x)\Big|_0^3 - \int_0^3 g(x)\,d(e^x)$$

$$= e^3\cdot g(3) - e^0\cdot g(0) - \int_0^3 g(x)e^x\,dx$$

$$= -e^3 - 1 - \int_0^1 1\cdot e^x\,dx - \int_1^2 3\cdot e^x\,dx - \int_2^3 -1\cdot e^x\,dx$$

$$= -e^3 - 1 - e^x\Big|_0^1 - 3e^x\Big|_1^2 + e^x\Big|_2^3$$

$$= -e^3 - 1 - e + 1 - 3e^2 + 3e + e^3 - e^2$$

$$= \boxed{-4e^2 + 2e}$$