12. $\displaystyle\int_0^1 \underbrace{\log(\operatorname{arctg} x)\cdot\frac{\pi-2\arcsin x}{(e^{1-x}-1)^2}}_{f(x)}\,dx$ vyšetřete konvergence

1) Funkce $f(\cdot)$ je spojitá na $(0,1)$.

2) Pro $x\to 0$, $\operatorname{arctg} x = x(1+\omega(x))$, kde $\omega(x)\to 0$ pro $x\to 0$

potom $\log(\operatorname{arctg} x) = \log\left(x\cdot\frac{\operatorname{arctg} x}{x}\right) = \log x + \log\frac{\operatorname{arctg} x}{x} =$

$= \log x\Big[\underbrace{1+\underbrace{\frac{1}{\log x}}_{\to 0}\cdot\underbrace{\log\frac{\operatorname{arctg} x}{x}}_{\to 0}}_{\to 0}\Big] = \log x(1+\omega_1(x))$, kde $\omega_1(x)\to 0$ pro $x\to 0$

Dále, $\pi - 2\arcsin x \to \pi$ pro $x\to 0$,

$e^{1-x}-1 \to e-1 \neq 0$.

Takže srovnáme f s funkcí $g(x)=|\log x|$. Máme, že $\int_0^1 |\log x|\,dx$ K,

a $\exists \lim\limits_{x\to 0}\frac{f(x)}{g(x)} = \lim\limits_{x\to 0}\frac{\log x(1+\omega_1(x))\cdot\pi}{|\log x|\cdot(e-1)^2} = \frac{-\pi}{(e-1)^2}\neq 0$, Potom $\int_0^{1/2} f(x)\,dx$ K

3) Pro $x\to 1$, a) $\operatorname{arctg} x\to\frac{\pi}{4}$, $\log(\operatorname{arctg} x)\to\log\left(\frac{\pi}{4}\right)\neq 0$

b) $e^{1-x}-1\to e^0-1=0$. Udělejme substituci $y=1-x$, $y\to 0$,

potom $e^{1-x}-1 = e^y-1 = 1+y+y\,\omega_3(y)-1 = y(1+\omega_3(y))$, kde $\omega_3(y)\to 0$ pro $y\to 0$.

$(e^{1-x}-1)^2 = y^2(1+\omega_4(y))$, $\omega_4(y)\to 0$ pro $y\to 0$.

c) Označme $\varphi = \frac{\pi}{2}-\arcsin x$. $\arcsin x\to\frac{\pi}{2}$ pro $x\to 1$, pak $\varphi\to 0$.

$\frac{\pi}{2}-\varphi = \arcsin x$,

$\sin\left(\frac{\pi}{2}-\varphi\right) = \sin(\arcsin x)$

$\cos\varphi = x$

$1-\frac{\varphi^2}{2}+\omega_4(\varphi)\cdot\varphi^2 = 1-y$; $y = \frac{\varphi^2}{2}(1+\omega_5(\varphi))$

$\varphi = \sqrt{2y}\,(1+\omega_6(y))$, $y\to 0$.

Takže funkci f srovnáme s $\frac{\sqrt{2y}}{y^2}$

Máme, $\int_0^{1/2}\frac{\sqrt{2y}}{y^2}\,dy = \sqrt{2}\int_0^{1/2}\frac{dy}{y^{3/2}}$ D, potom $|x=1-y|$ $\sqrt{2}\int_{1/2}^1\frac{dx}{(1-x)^{3/2}}$ diverguje

Označme $g(x)=\frac{1}{(1-x)^{3/2}}$. $\int_{1/2}^1 g(x)\,dx$ D, a $\exists\lim\limits_{x\to 1-}\frac{f(x)}{g(x)} = \lim\limits_{x\to 1-}\frac{\log\left(\frac{\pi}{4}\right)\cdot 2\cdot\varphi(1-x)^{3/2}}{y^2(1+\omega_4(y))} = |y=1-x|$

$= \lim\limits_{y\to 0+}\frac{\log\frac{\pi}{4}\cdot 2\cdot\sqrt{2y}\cdot y^{3/2}}{y^2(1+\omega_4(y))} = 2\sqrt{2}\log\frac{\pi}{4}\neq 0$.

Potom $\int_{1/2}^1 f(x)\,dx$ D, a pak $\int_0^1 f(x)\,dx$ D.