Limits of sequences 1

Definition

Let $a \in \mathbb{R}$ and let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. We say, that a is the limit of the sequence $\{a_n\}_{n=1}^{\infty}$ (and write $\lim_{n\to\infty} a_n = a$), if

$$\forall \varepsilon\ \exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}\ n \geq n_0 :\ |a_n - a| < \varepsilon.$$

Theorem 1 (Arithmetics of limits)**.** *Let $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Then*

1. $\lim_{n\to\infty} (a_n + b_n) = a + b.$
2. $\lim_{n\to\infty} (a_n - b_n) = a - b.$
3. *Let* $\in \mathbb{R}$. *Then* $\lim_{n\to\infty} (c \cdot a_n) = c \cdot a.$
4. $\lim_{n\to\infty} (a_n \cdot b_n) = a \cdot b.$
5. *If* $b \neq 0$, *then* $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}.$

Theorem 2 (two policemen)**.** *Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$ be three sequences. Let*

1. $\exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}$ *such that* $n > n_0 :\ a_n \leq b_n \leq c_n,$ *and*
2. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = a.$

Then $\exists \lim_{n\to\infty} c_n = a.$

Theorem 3 (Binomial formula)**.** *For all $n \in \mathbb{N}$ and for all $a, b \in \mathbb{R}$ the following is true:*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}, \qquad \text{where } \binom{n}{k} = \frac{n!}{k!(n-k)!}, \qquad n! = 1 \cdot 2 \cdot \ldots \cdot n, \quad 0! = 1.$$

Known limits

1. $\lim_{n\to\infty} c = c;$
2. $\lim_{n\to\infty} \frac{1}{n} = 0;$
3. $\lim_{n\to\infty} \sqrt[n]{n} = 1;$
4. $\lim_{n\to\infty} (-1)^n$ does not exist.

Exercises

1. $\lim_{n\to\infty} \frac{2n^2+n-3}{n^3+1}$
2. $\lim_{n\to\infty} \frac{2n^5+3n-2}{n^5-3n^3+1}$
3. $\lim_{n\to\infty} \frac{2n^3+6n}{n^3-7n+7}$
4. $\lim_{n\to\infty} \left(\frac{1+2+3+\ldots+n}{n+2} - \frac{n}{2}\right)$
5. $\lim_{n\to\infty} \frac{1^2+2^2+3^2+\ldots+n^2}{n^3}$
6. $\lim_{n\to\infty} \frac{1^3+2^3+3^3+\ldots+n^3}{n^4}$
7. $\lim_{n\to\infty} \frac{(n+4)^{100}-(n+3)^{100}}{(n+2)^{100}-n^{100}}$
8. $\lim_{n\to\infty} \frac{n}{2^n}$

Exercises

9. $\lim_{n\to\infty} \sqrt[n]{2^n + 4^n}$
10. $\lim_{n\to\infty} \dfrac{2^n}{n!}$
11. $\lim_{n\to\infty} \dfrac{3^n + n^5}{n^6 + n!}$
12. $\lim_{n\to\infty} \dfrac{\sqrt[n]{a^n + b^n}}{\sqrt[n]{a^{2n} + b^{2n}}},\ a > b > 0$
13. $\lim_{n\to\infty} \left(\sqrt{n+1} - \sqrt{n}\right)$
14. $\lim_{n\to\infty} \left(\sqrt[3]{n^{75} + n^{60}} - \sqrt[3]{n^{75} - n^{60}}\right) \cdot \dfrac{(n^3+n^2)^{20} - (n^2+n)^{30}}{(n+1)^{70} - (n-1)^{70}}$
15. $\lim_{n\to\infty} \dfrac{\sqrt{n^3} + \sqrt{n} + 1}{\sqrt{n+1} - \sqrt{n}} \cdot \dfrac{(n^4+n)^{50} - (n+1)^{200}}{(n+1)^{202} - n^{202}}$

Exercises

16. $\lim_{n\to\infty} \left(\sqrt[3]{n+1} - \sqrt[3]{n}\right)$

17. $\lim_{n\to\infty} \frac{\sqrt[3]{n^2+7} - \sqrt[3]{n^2+1}}{\sqrt[3]{n^2+6} - \sqrt[3]{n^2}}$

18. $\lim_{n\to\infty} \frac{\sqrt[3]{n^3+n} - \sqrt[3]{n^3+1}}{\sqrt[3]{n^3+2n} - \sqrt[3]{n^3+n}}$

19. $\lim_{n\to\infty} \frac{\sqrt[4]{n+2} - \sqrt[4]{n+1}}{\sqrt[3]{n+3} - \sqrt[3]{n}} \cdot \sqrt[12]{n}$

20. $\lim_{n\to\infty} \frac{\sqrt[3]{n^2+n} - \sqrt[3]{n^2+2n}}{\sqrt[3]{n^2+3n} - \sqrt[3]{n^2-5n}}$

21. $\lim_{n\to\infty} (-1)^n \sqrt{n} \left(\sqrt{n+1} - \sqrt{n}\right)$

Exercises

22. $\lim_{n\to\infty} \left(\sqrt[3]{n^4 + \sqrt[3]{n}} - \sqrt[3]{n^4}\right) \left(\lfloor \sqrt[3]{n+1} \rfloor + \lfloor 2\sqrt[3]{n-1} \rfloor + \ldots + \lfloor n\sqrt[3]{n+(-1)^{n+1}} \rfloor\right).$

23.
$$\lim_{n\to\infty} \frac{n\sqrt{n}\sqrt[n]{(n+1)^n + n^{n+1}}}{\lfloor \sqrt{n} \rfloor + \lfloor 2\sqrt{n} \rfloor + \ldots + \lfloor n\sqrt{n} \rfloor}.$$

24. $\lim_{n\to\infty} \left(\frac{1}{2} + \frac{3}{2^2} + \frac{5}{2^3} + \ldots + \frac{2n-1}{2^n}\right)$

25. $\lim_{n\to\infty} \left(\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right) \cdot \ldots \cdot \left(1 - \frac{1}{n^2}\right)\right)$

Sequences given recurrently.

Find the limit of a sequence a_n given recurrently

26. $a_1 = 0$, $a_{n+1} = \frac{a_n+3}{4}$.

27. $a_1 = 42$, $a_{n+1} = \frac{a_n-2}{5}$.

28. $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2 + a_n}$.

29. $a_1 > 0$ arbitrary, $a_{n+1} = \frac{1}{2}\left(a_n + \frac{1}{a_n}\right)$

Solutions 9 -15

9. $\lim_{n\to\infty} \sqrt[n]{2^n+4^n}$. Here we will use the Theorem about two policiemen. The idea is that 2^n is negligible compared to 4^n, so the limit will be 4. It remains to find two cops. For every $n \in \mathbb{N}$, it holds that

$$\sqrt[n]{4^n} \leq \sqrt[n]{2^n+4^n} \leq \sqrt[n]{2\cdot 4^n}.$$

Apparently $\lim_{n\to\infty} \sqrt[n]{4^n} = \lim_{n\to\infty} 4 = 4$. And thanks to Theorem of Arithmetics of limits,

$$\lim_{n\to\infty} \sqrt[n]{2\cdot 4^n} = \left(\lim_{n\to\infty} \sqrt[n]{2}\right)\left(\lim_{n\to\infty} \sqrt[n]{4^n}\right) = 1\cdot 4 = 4,$$

where we used the known limit. So we have two cops, they both go to the same constant, and thus

$$\lim_{n\to\infty} \sqrt[n]{2^n+4^n} = 4.$$

10. $\lim_{n\to\infty} \frac{2^n}{n!} = 0.$

11. $\lim_{n\to\infty} \frac{3^n+n^5}{n^6+n!} = 0.$

12. $\lim_{n\to\infty} \frac{\sqrt[n]{a^n+b^n}}{\sqrt[n]{a^{2n}+b^{2n}}}$, $a > b > 0$. Here, similar to exercise 9, we use two cops. It holds for every $n \in \mathbb{N}$ that

$$\frac{a}{a^2\sqrt[n]{2}} = \frac{\sqrt[n]{a^n}}{\sqrt[n]{2a^{2n}}} \leq \frac{\sqrt[n]{a^n+b^n}}{\sqrt[n]{a^{2n}+b^{2n}}} \leq \frac{\sqrt[n]{2a^n}}{\sqrt[n]{a^{2n}}} = \frac{a\sqrt[n]{2}}{a^2}.$$

Since $\lim_{n\to\infty} \frac{a}{a^2\sqrt[n]{2}} = \lim_{n\to\infty} \frac{a\sqrt[n]{2}}{a^2} = \frac{1}{a}$, then also $\lim_{n\to\infty} \frac{\sqrt[n]{a^n+b^n}}{\sqrt[n]{a^{2n}+b^{2n}}} = \frac{1}{a}$.

13. $\lim_{n\to\infty} \left(\sqrt{n+1}-\sqrt{n}\right)$. In this type of example, we will use a trick that we will call standard procedure from now on. Namely, we will use the relation $(a-b)(a+b) = (a^2-b^2)$, and we multiply our sequence by the so-called "smart one":

$$\begin{aligned}\lim_{n\to\infty} \left(\sqrt{n+1}-\sqrt{n}\right) &= \lim_{n\to\infty} \left(\sqrt{n+1}-\sqrt{n}\right)\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} \\ &= \lim_{n\to\infty} \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty} \frac{1}{\sqrt{n}}\frac{1}{\sqrt{1+\frac{1}{n}}+\sqrt{1}} \\ &= 0\cdot\frac{1}{1+1} = 0\cdot\frac{1}{2} = 0.\end{aligned}$$

We used (several times) Theorem of arithmetics of limits, theorem that the limit of (square) roots is the (square) root of the limit.

14. $\lim_{n\to\infty} \left(\sqrt[3]{n^{75}+n^{60}} - \sqrt[3]{n^{75}-n^{60}}\right)\cdot\frac{(n^3+n^2)^{20}-(n^2+n)^{30}}{(n+1)^{70}-(n-1)^{70}} = \frac{-1}{21}.$

15. $\lim_{n\to\infty} \frac{\sqrt{n^3}+\sqrt{n}+1}{\sqrt{n+1}-\sqrt{n}}\cdot\frac{(n^4+n)^{50}-(n+1)^{200}}{(n+1)^{202}-n^{202}} = \frac{-200}{101}.$

Solutions 16-21

16. $\lim\limits_{n\to\infty}\left(\sqrt[3]{n+1}-\sqrt[3]{n}\right)$. Here, as in previous cases, we need to transform the expression using the formula $a^3-b^3=(a-b)(\sqrt[3]{a^2}+\sqrt[3]{ab}+\sqrt[3]{b^2})$. Then

$$\lim_{n\to\infty}\left(\sqrt[3]{n+1}-\sqrt[3]{n}\right)=\lim_{n\to\infty}\frac{n+1-n}{\left(\sqrt[3]{n+1}\right)+\sqrt[3]{n+1}\sqrt[3]{n}+\left(\sqrt[3]{n}\right)^2}$$
$$=\lim_{n\to\infty}\frac{1}{n^{\frac{2}{3}}}\frac{1}{\cdot\left(\left(\sqrt[3]{1+\frac{1}{n}}\right)+\sqrt[3]{1+\frac{1}{n}}\sqrt[3]{n}+\left(\sqrt[3]{1}\right)^2\right)}=0\cdot\frac{1}{\cdot\left(\left(\sqrt[3]{1+0}\right)+\sqrt[3]{1+0}\sqrt[3]{n}+\left(\sqrt[3]{1}\right)^2\right)}=0.$$

We used (several times) Theorem of arithmetics of limits, theorem that the limit of (third) roots is the (third) root of the limit.

17. $\lim\limits_{n\to\infty}\dfrac{\sqrt[3]{n^2+7}-\sqrt[3]{n^2+1}}{\sqrt[3]{n^2+6}-\sqrt[3]{n^2}}$. We need to transform the expression, as in the previous example, both in numerator and denominator. The answer is 1.

18. $\lim\limits_{n\to\infty}\dfrac{\sqrt[3]{n^3+n}-\sqrt[3]{n^3+1}}{\sqrt[3]{n^3+2n}-\sqrt[3]{n^3+n}}$. Again, we need to transform the expression, as in the previous example, both in numerator and denominator. The answer is 1.

19. $\lim\limits_{n\to\infty}\dfrac{\sqrt[4]{n+2}-\sqrt[4]{n+1}}{\sqrt[3]{n+3}-\sqrt[3]{n}}$. Here it is just as simple, although a little more laborious. In the numerator we use the formula $a^4-b^4=(a-b)(a^3+a^2b+ab^2+b^3)$, and in the denominator we use the formula $a^3-b^3=(a-b)(a^2+ab+b^2)$. After substituting and cancelling the largest factor, we get the result 0.

20. $\frac{-1}{8}$

21. $\lim\limits_{n\to\infty}(-1)^n\sqrt{n}\left(\sqrt{n+1}-\sqrt{n}\right)$.

    This limit does not exist. First, observe that $\lim\limits_{n\to\infty}\sqrt{n}\left(\sqrt{n+1}-\sqrt{n}\right)=\frac{1}{2}$. Now we can select two subsequences: with n odd and n even, and they have different limits.

Solutions 22-26

22. $\frac{1}{6}$. Hint: První závorka:

$$\left(\sqrt[3]{n^4+\sqrt[3]{n}}-\sqrt[3]{n^4}\right)=\frac{\sqrt[3]{n}}{\sqrt[3]{(n^4+\sqrt[3]{n})^2}+\sqrt[3]{n^4+\sqrt[3]{n}}\sqrt[3]{n^4}+\sqrt[3]{n^8}}=\frac{a_n}{3\sqrt[3]{n^7}}$$

kde a_n je nějaká posloupnost která konverguje k 1, takže $\lim_{n\to\infty} a_n = 1$.

Teď odhadneme druhou závorku zdola a shora, a potom použijeme VODP.

Odhad shora: $\lfloor x\rfloor \geq x$, a $\sqrt[3]{n-1} < \sqrt[3]{n+1}$, a pak

$$\left(\lfloor\sqrt[3]{n+1}\rfloor+\lfloor 2\sqrt[3]{n-1}\rfloor+\ldots+\lfloor n\sqrt[3]{n+(-1)^{n+1}}\rfloor\right) \leq (1+2+\ldots+n)\sqrt[3]{n+1} = \frac{n(n+1)}{2}\cdot\sqrt[3]{n+1}.$$

Odhad zdola: $\lfloor x\rfloor > x-1$, a $\sqrt[3]{n+1} > \sqrt[3]{n-1}$, a pak

$$\left(\lfloor\sqrt[3]{n+1}\rfloor+\lfloor 2\sqrt[3]{n-1}\rfloor+\ldots+\lfloor n\sqrt[3]{n+(-1)^{n+1}}\rfloor\right) \geq (1+2+\ldots+n)\sqrt[3]{n+1}-n = \frac{n(n+1)}{2}\cdot\sqrt[3]{n+1}-n.$$

23. 2. Návod: V jmenovateli odhadneme pomoci $x-1<\lfloor x\rfloor\leq x$, s tím abych použit VODP:

$$\lfloor\sqrt{n}\rfloor+\lfloor 2\sqrt{n}\rfloor+\ldots+\lfloor n\sqrt{n}\rfloor \leq \sqrt{n}\,(1+2+\ldots+n)=\frac{n(n+1)}{2}\sqrt{n}$$

$$\lfloor\sqrt{n}\rfloor+\lfloor 2\sqrt{n}\rfloor+\ldots+\lfloor n\sqrt{n}\rfloor \geq \left(\sqrt{n}-1\right)+\left(2\sqrt{n}-1\right)+\ldots+\left(n\sqrt{n}-1\right)=(1+2+\ldots+n)\sqrt{n}-n=\frac{n(n+1)}{2}\sqrt{n}-n$$

V čitateli upravíme n-ou odmocninu: vytkneme $(n+1)^n$:

$$\sqrt[n]{(n+1)^n+n^{n+1}}=(n+1)\underbrace{\sqrt[n]{1+n\frac{n^n}{(n+1)^n}}}_{\leq\sqrt[n]{1+n}\;\to\;1}.$$

24. 3. Hint: $2k+1=(2k-1)+2$. Then $S_n=\frac{1}{2}+\left(\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{n-1}}\right)+\frac{1}{2}S_{n-1}$.

25. $\frac{1}{2}$. Hint: $1-\frac{1}{n^2}=\frac{n-1}{n}\cdot\frac{n+1}{n}$.

26. 1. We want to use a theorem about the limit of a monotone sequence.

We have

$$a_{n+1}-a_n=\frac{3}{4}(1-a_n),\qquad 1-a_{n+1}=\frac{1}{3}(a_{n+1}-a_n),$$

hence $0<a_n<a_{n+1}<1$, and there exists a limit a of a_n. Taking the limit in the recurrent relation, we obtain $a=\frac{1}{4}a+\frac{3}{4}$.