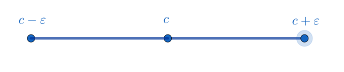
Mathematics I - Functions

24/25

Definition

Let $c \in \mathbb{R}$ and $\varepsilon > 0$. We define

- a neighbourhood of a point c with radius ε by $B(c, \varepsilon) = (c - \varepsilon, c + \varepsilon)$,



Limit of a function

Definition

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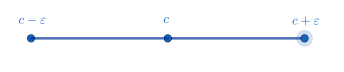
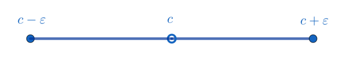


- a punctured neighbourhood of a point c with radius ε by
  $P(c, \varepsilon) = (c - \varepsilon, c + \varepsilon) \setminus \{c\}$.



Definition

We say that $A \in \mathbb{R}$ is a limit of a function f at a point $c \in \mathbb{R}$ if

$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0 \ \exists \delta \in \mathbb{R}, \delta > 0 \ \forall x \in P(c, \delta) : f(x) \in B(A, \varepsilon).$$

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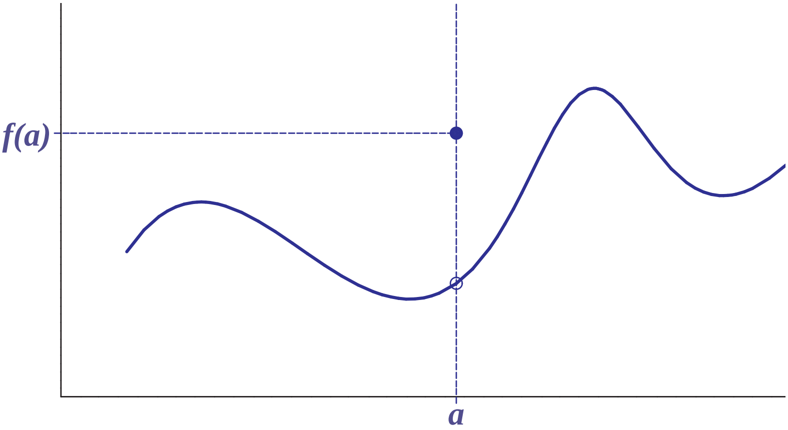
Theorem 1 (uniqueness of a limit)

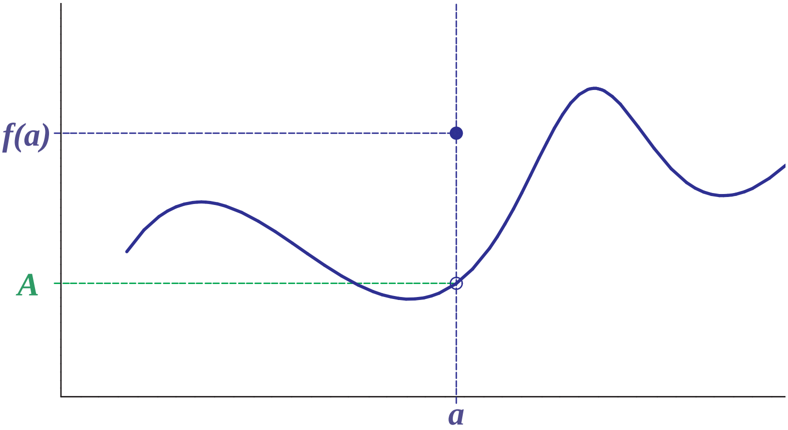
Let f be a function and $c \in \mathbb{R}$. Then f has a most one limit $A \in \mathbb{R}$ at c.

The fact that f has a limit $A \in \mathbb{R}$ at $c \in \mathbb{R}$ is denoted by $\lim\limits_{x \to c} f(x) = A$.

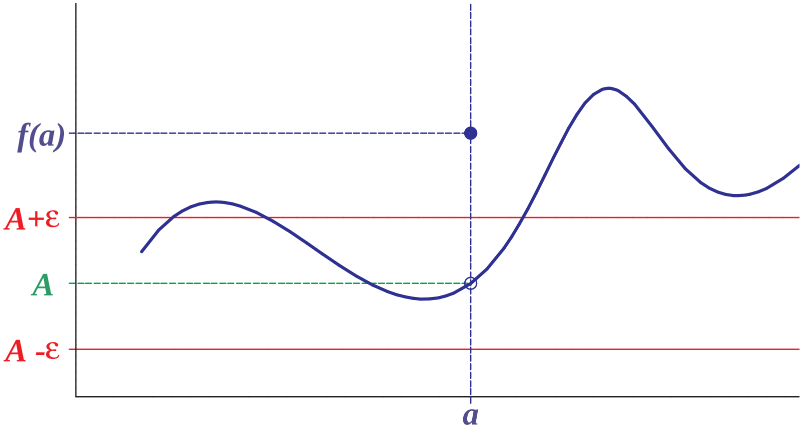
https://www.geogebra.org/m/tCnmrWg2

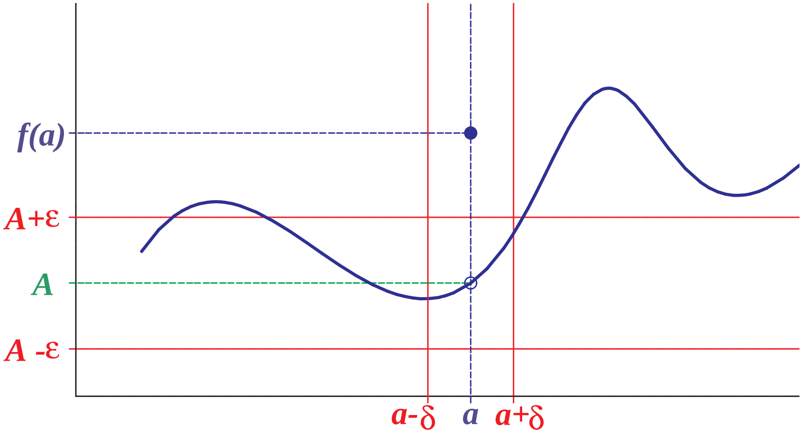
https://www.geogebra.org/m/wfdvtRTb

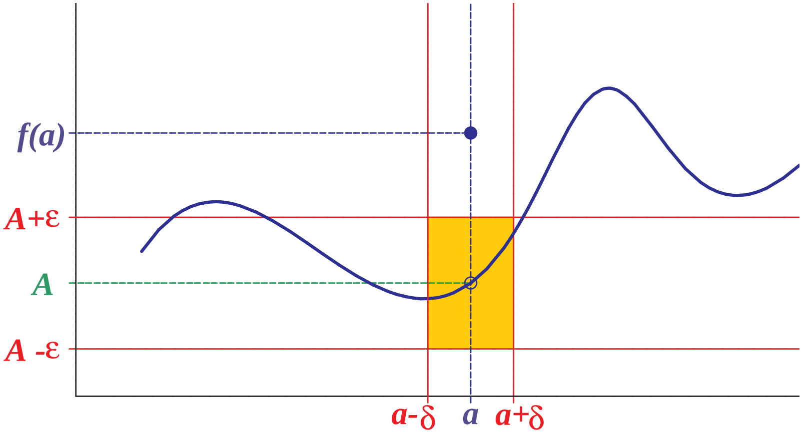

f(a)
a

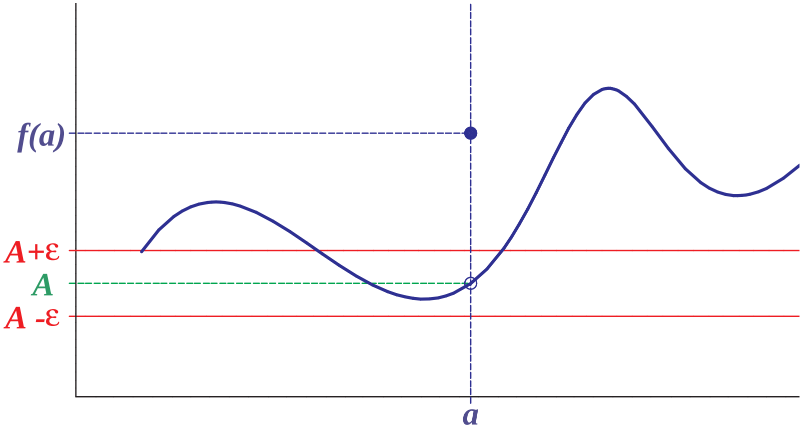


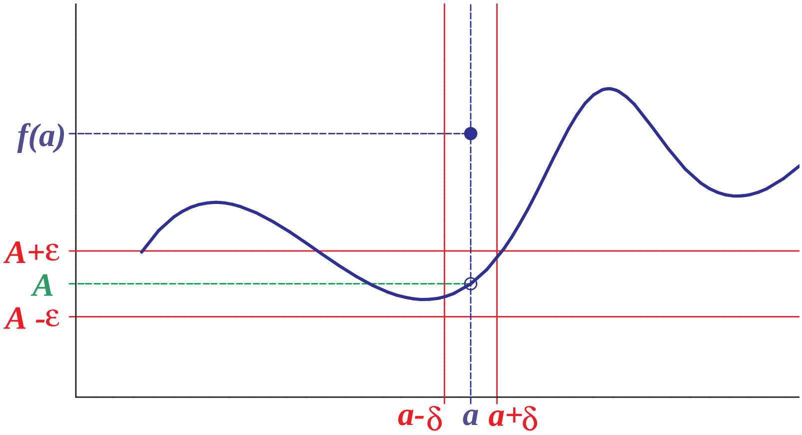
f(a)
A
a

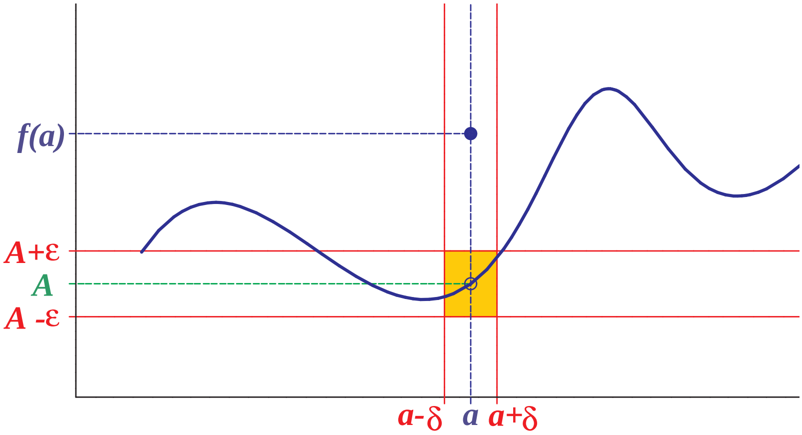

f(a)
A+ε
A
A -ε
a


f(a)
A+ε
A
A -ε
a-δ
a
a+δ

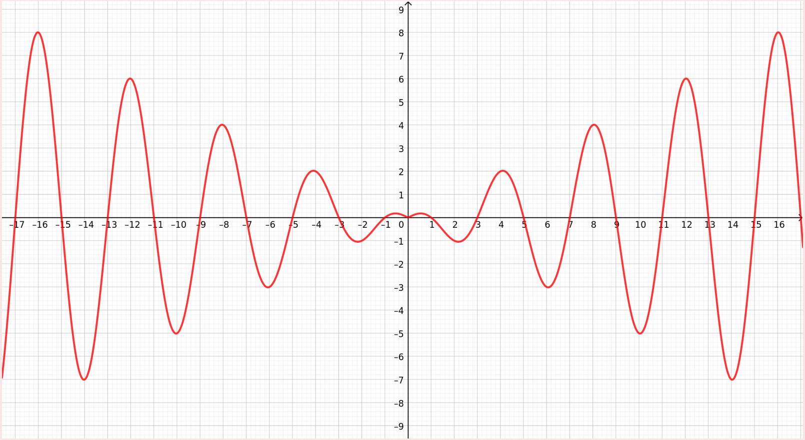

f(a)
A+ε
A
A -ε
a-δ
a
a+δ


f(a)
A+ε
A
A -ε
a


f(a)
A+ε
A
A -ε
a-δ
a
a+δ


f(a)
A+ε
A
A -ε
a-δ
a
a+δ

Exercise



`https://www.geogebra.org/calculator/xjkuxemz`

Exercise

Find $\lim_{x \to 0} f(x)$

A -3 B 0 C 5 D 7 E ∞

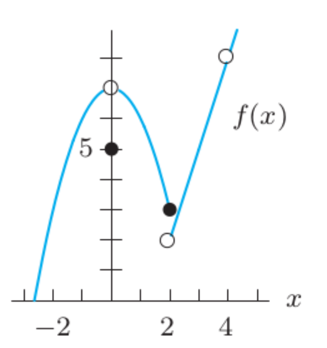


Figure: Calculus: Single and Multivariable, Hughes-Hallet

Exercise

Find $\lim_{x\to 2} f(x)$

A ∞

B 3

C 2

D 0

E does not exist

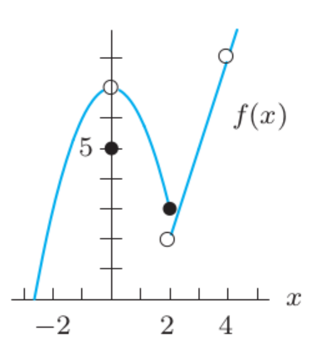


Figure: Calculus: Single and Multivariable, Hughes-Hallet

Exercise

Find $\lim_{x \to 4} f(x)$

A 4

B 8

C 0

D ∞

E does not exists

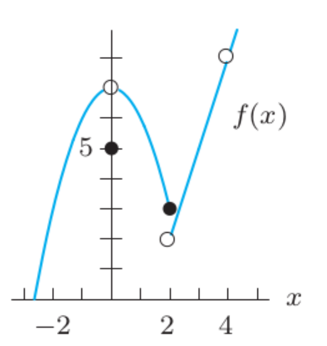


Figure: Calculus: Single and Multivariable, Hughes-Hallet

Definition

Let $\varepsilon > 0$. A neighbourhood and a punctured neighbourhood of $+\infty$ (resp. $-\infty$) is defined as follows:

$$P(+\infty, \varepsilon) = B(+\infty, \varepsilon) = (1/\varepsilon, +\infty),$$
$$P(-\infty, \varepsilon) = B(-\infty, \varepsilon) = (-\infty, -1/\varepsilon).$$

Example

$$P\left(+\infty, \frac{1}{10}\right) = B\left(+\infty, \frac{1}{10}\right) = (10, +\infty)\,,$$
$$P\left(-\infty, \frac{1}{200}\right) = B\left(-\infty, \frac{1}{200}\right) = (-\infty, -200)\,.$$

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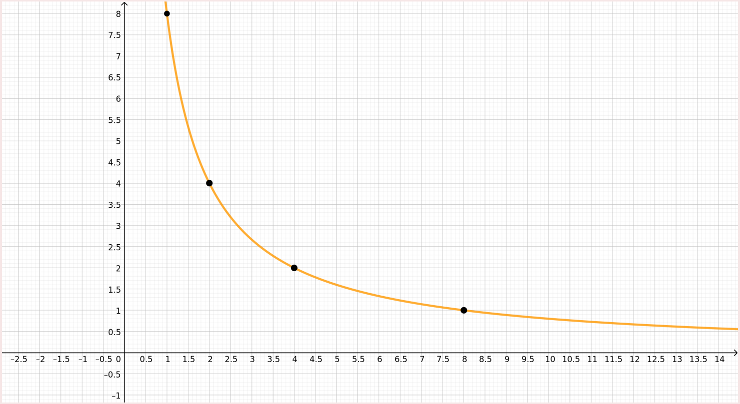
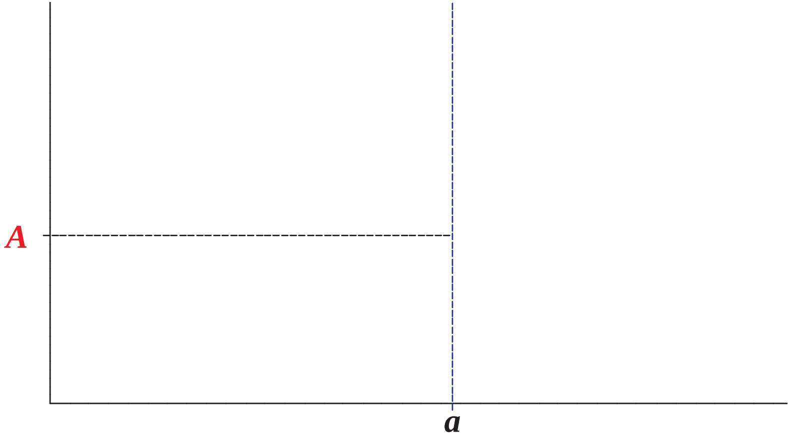
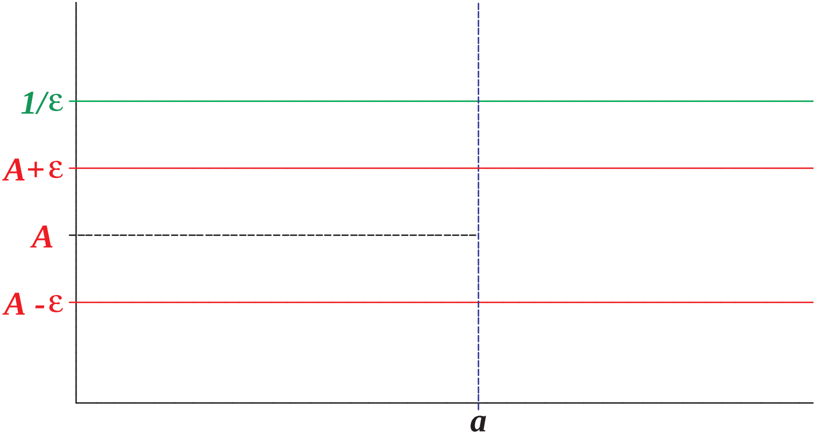
Theorem 1 holds also for $c \in \mathbb{R}^*$, $A \in \mathbb{R}^*$, so we can again use the notation $\lim_{x \to c} f(x) = A$.

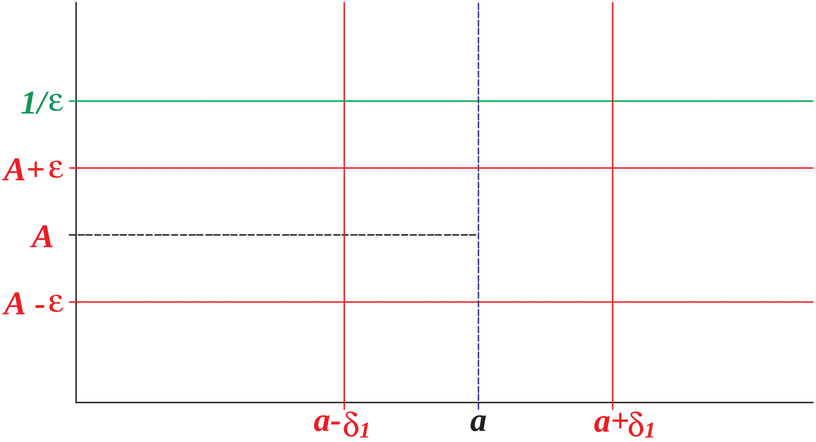
Exercise

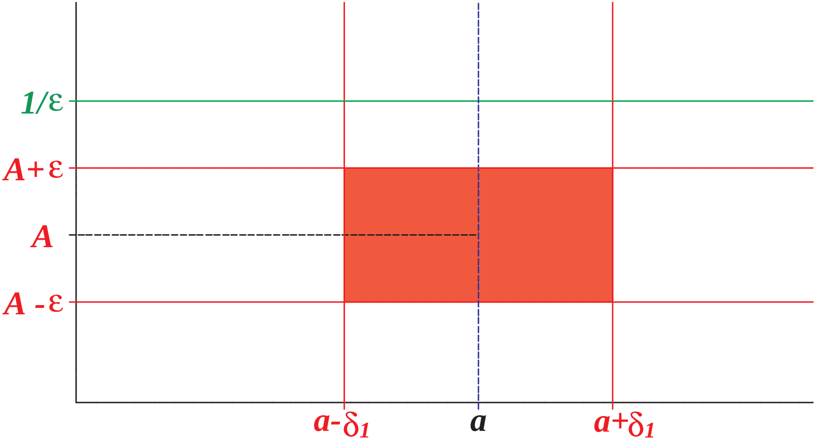


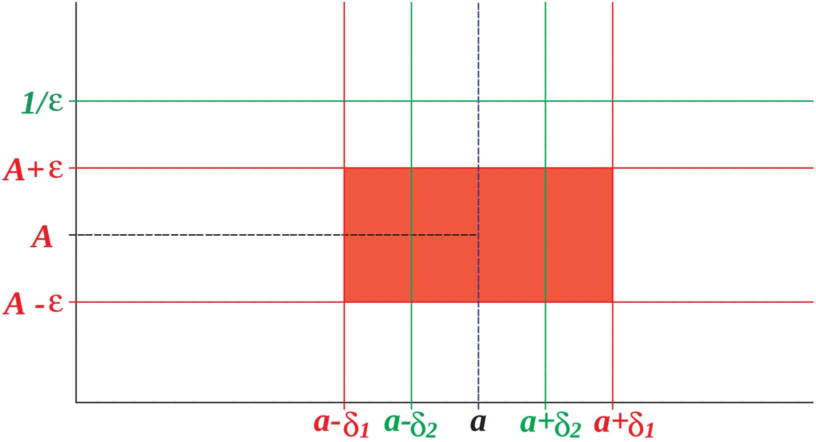
`https://www.geogebra.org/calculator/xjkuxemz`


A
a

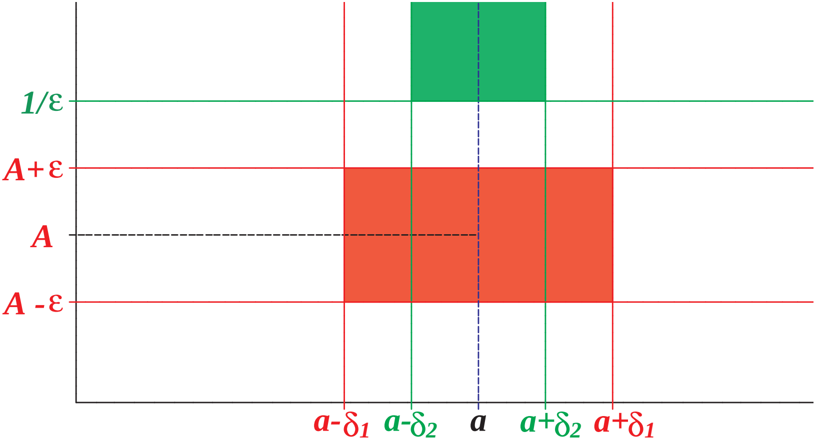

1/ε
A+ε
A
A -ε
a

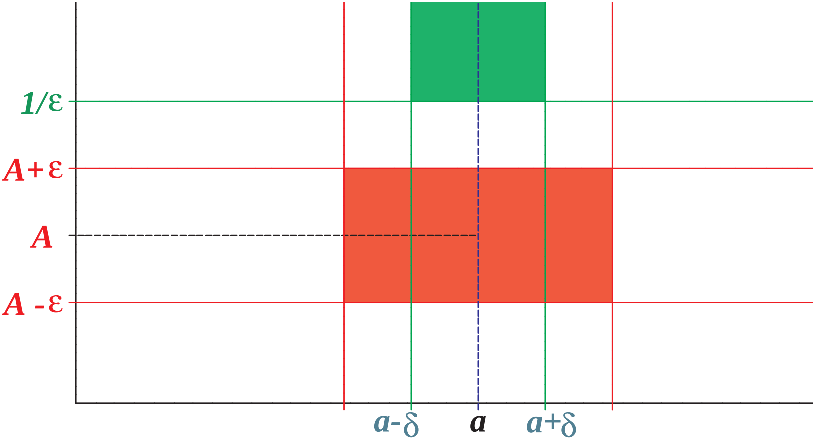

1/ε
A+ε
A
A -ε
a-δ1
a
a+δ1

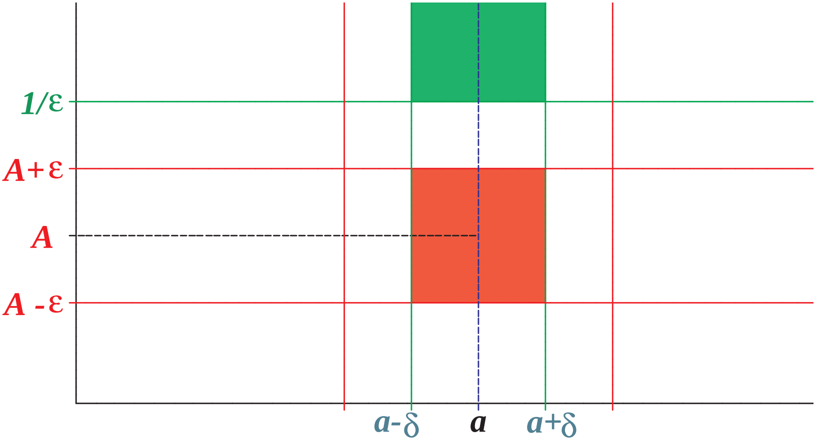

1/ε
A+ε
A
A -ε
a-δ1
a
a+δ1

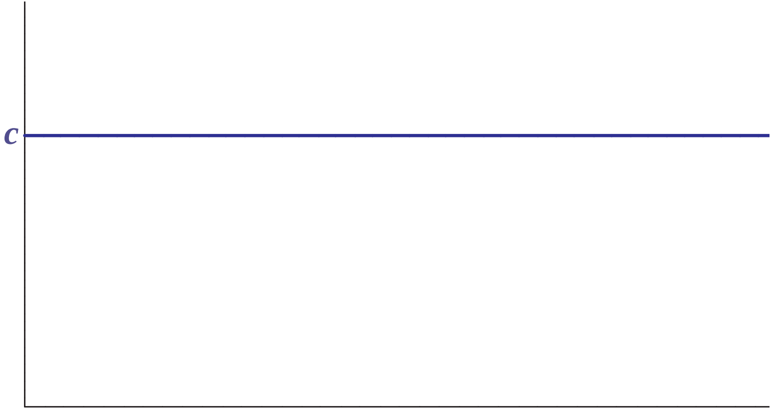


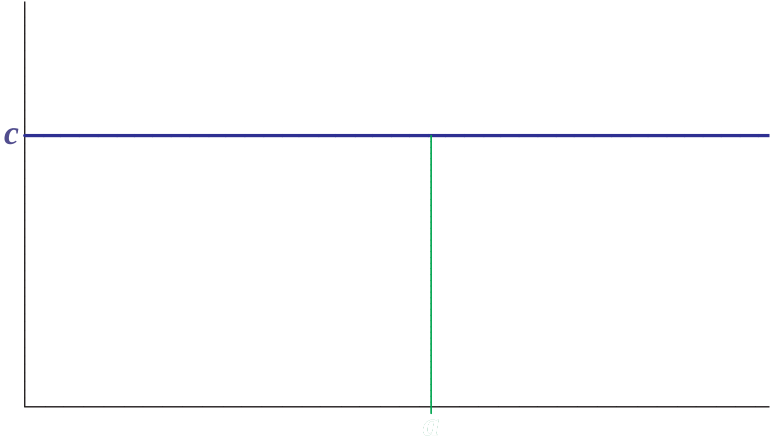
1/ε
A+ε
A
A -ε
a-δ1
a-δ2
a
a+δ2
a+δ1

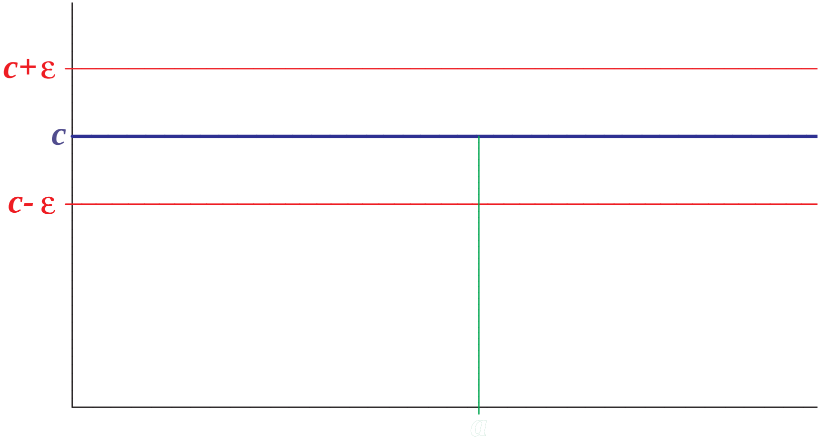

1/ε
A+ε
A
A -ε
a-δ1
a-δ2
a
a+δ2
a+δ1

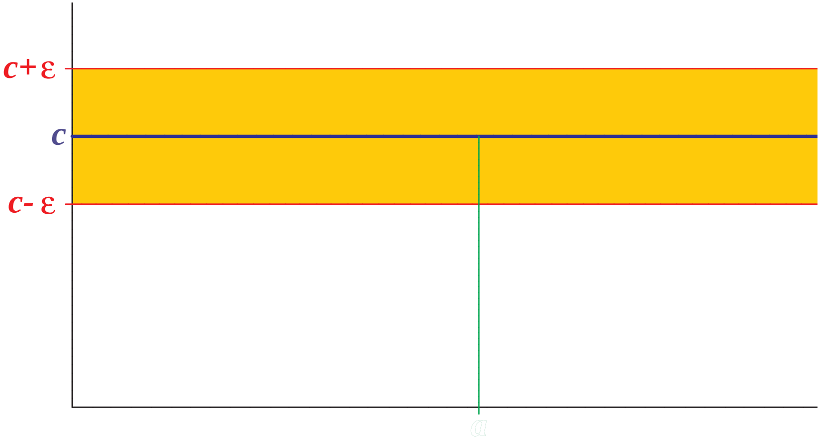

1/ε
A+ε
A
A -ε
a-δ
a
a+δ

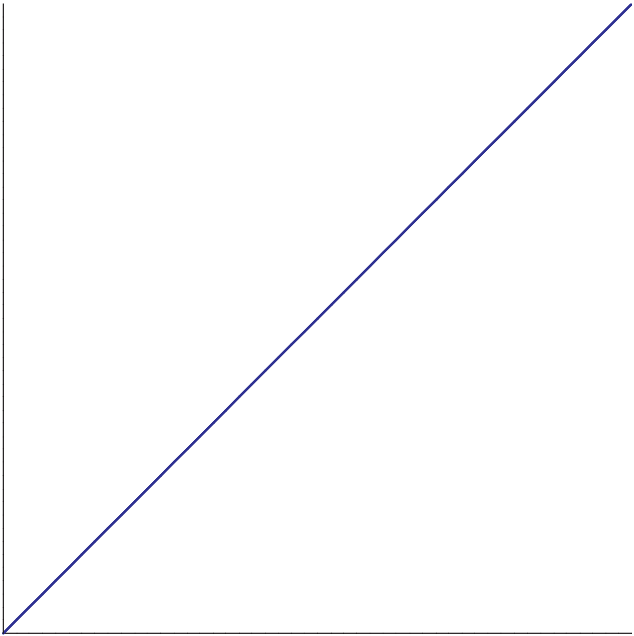

1/ε
A+ε
A
A -ε
a-δ
a
a+δ

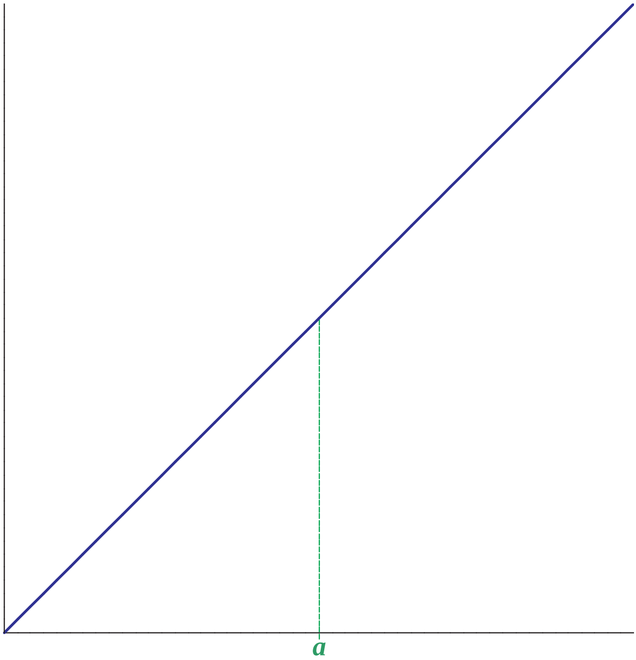

c

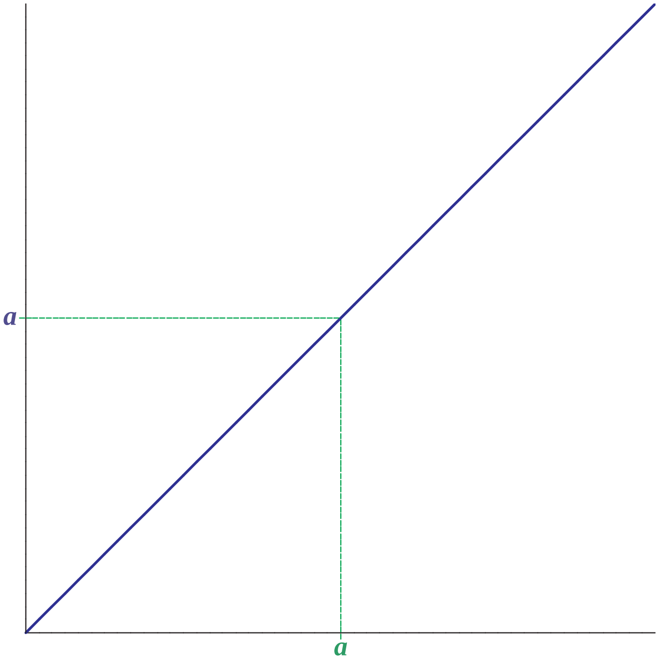

c
a


c+ ε
c
c- ε
a

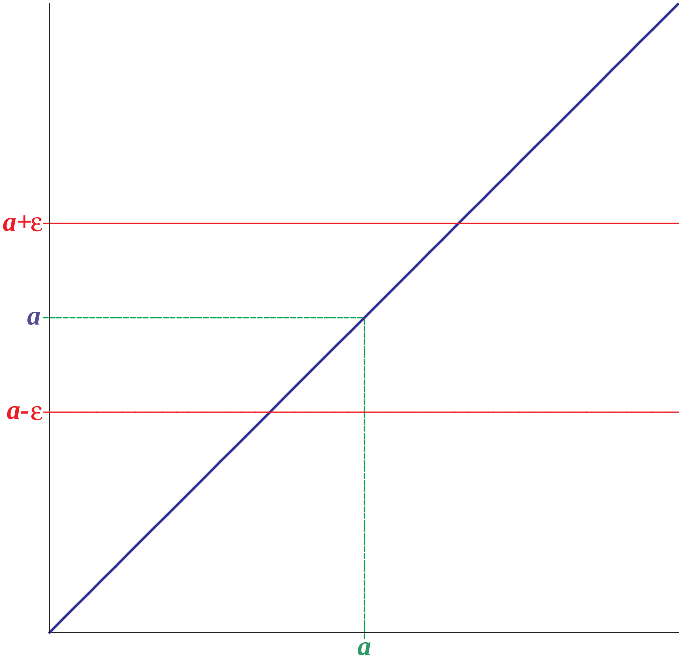

c+ε
c
c- ε
a



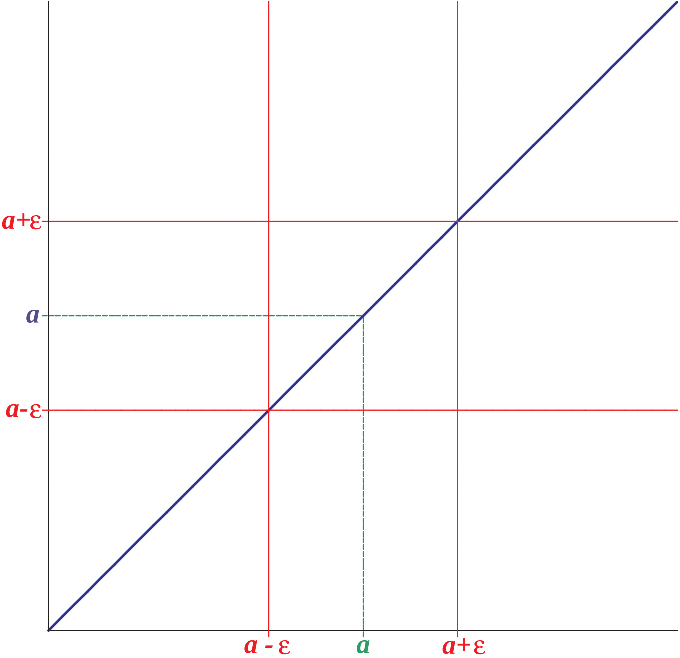

a

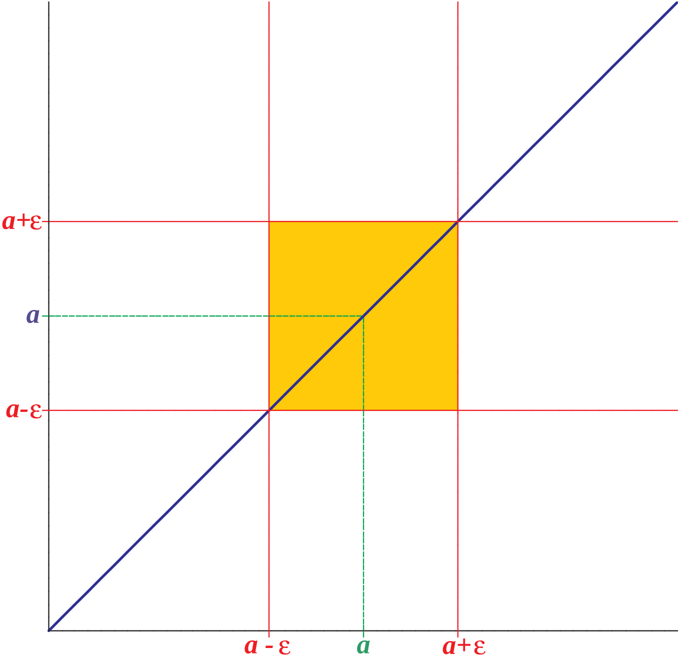


a
a



a+ε
a
a-ε
a


a+ε
a
a-ε
a - ε
a
a+ε


$a+\varepsilon$
a
$a-\varepsilon$
$a - \varepsilon$
a
$a+\varepsilon$

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- a right neighbourhood and right punctured neighbourhood of $-\infty$ by $B^+(-\infty, \varepsilon) = P^+(-\infty, \varepsilon) = (-\infty, -1/\varepsilon)$.

Definition

Let $A \in \mathbb{R}^*$, $c \in \mathbb{R} \cup \{-\infty\}$. We say that a function f has a limit from the right at c equal to $A \in \mathbb{R}^*$ (denoted by $\lim\limits_{x \to c+} f(x) = A$) if

$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0 \; \exists \delta \in \mathbb{R}, \delta > 0 \; \forall x \in P^+(c, \delta) : f(x) \in B(A, \varepsilon).$$

Analogously we define the notion of limit from the left at $c \in \mathbb{R} \cup \{+\infty\}$ and we use the notation $\lim\limits_{x \to c-} f(x)$.

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Let $A \in \mathbb{R}^*$, $c \in \mathbb{R} \cup \{-\infty\}$. We say that a function f has a limit from the right at c equal to $A \in \mathbb{R}^*$ (denoted by $\lim\limits_{x \to c+} f(x) = A$) if

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Remark

Let $c \in \mathbb{R}$, $A \in \mathbb{R}^*$. Then

$$\lim_{x \to c} f(x) = A \Leftrightarrow \left(\lim_{x \to c+} f(x) = A \;\;\&\;\; \lim_{x \to c-} f(x) = A \right).$$

Exercise

Find $\lim_{x\to 2-} f(x)$.
Find $\lim_{x\to 2+} f(x)$.

A 0
B 1
C 2
D 3
E does not exist

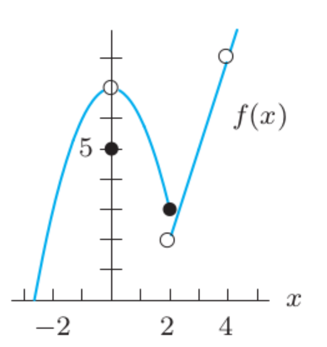
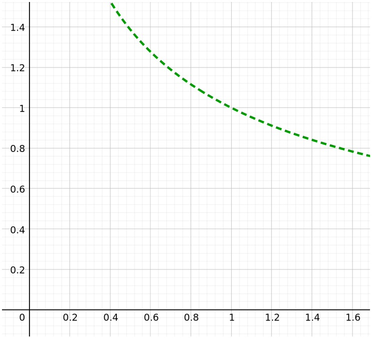


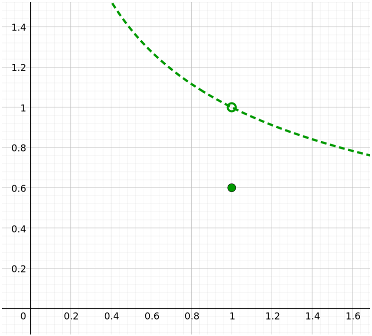
Figure: Calculus: Single and Multivariable, Hughes-Hallet

Definition

We say that a function f is continuous at a point $c \in \mathbb{R}$ if

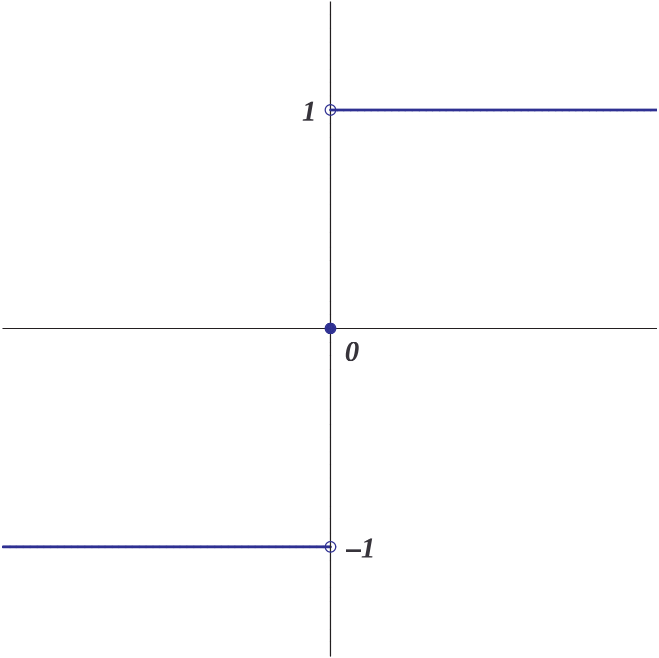
$$\lim_{x \to c} f(x) = f(c).$$

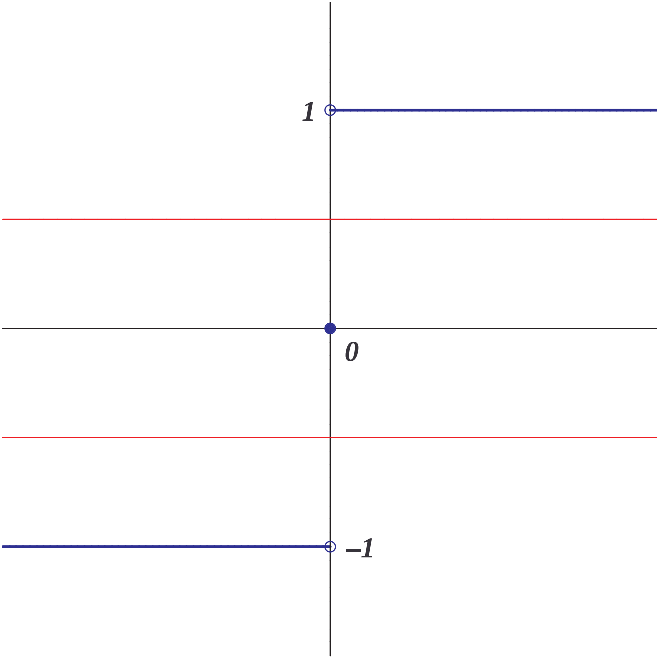


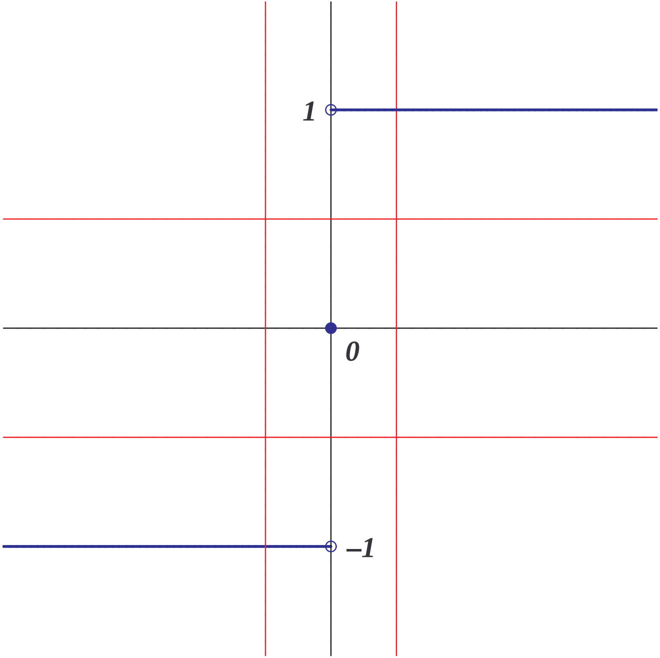


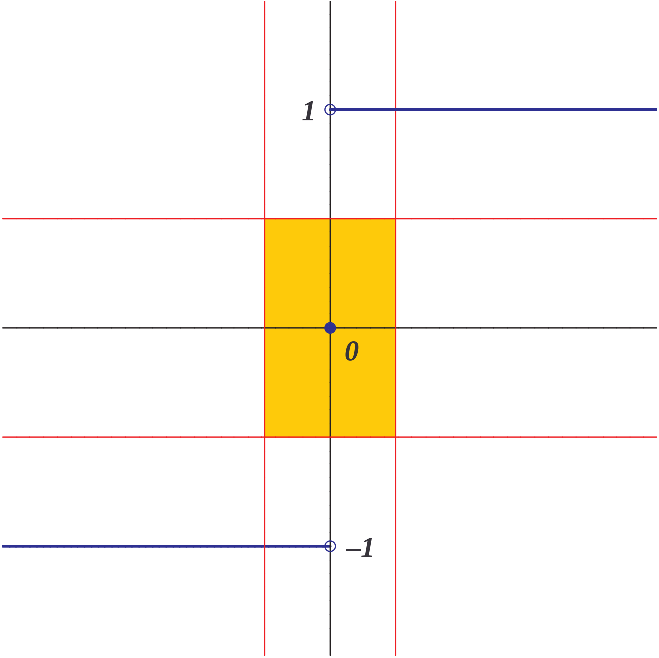
Definition

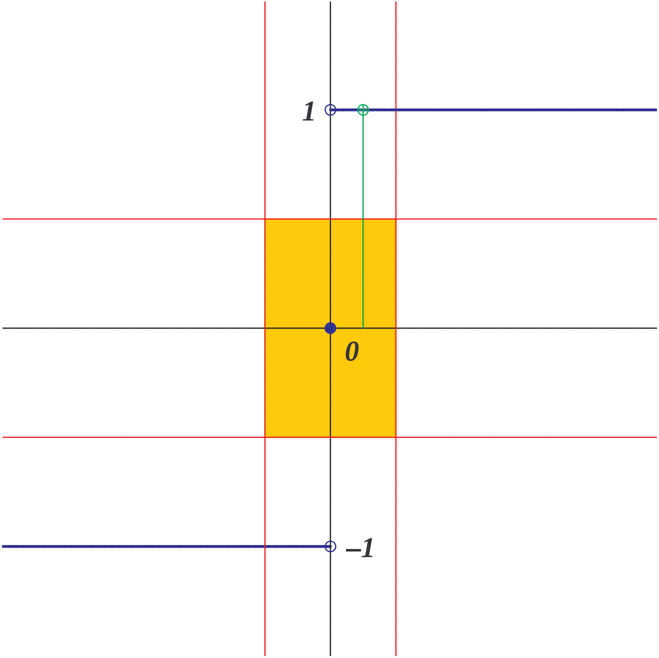
Let $c \in \mathbb{R}$. We say that a function f is continuous at c from the right (from the left, resp.) if $\lim_{x\to c+} f(x) = f(c)$ ($\lim_{x\to c-} f(x) = f(c)$, resp.).


1
0
–1


1
0
–1


1
0
−1


1
0
–1


1
0
–1

Theorem 2

Let f has a finite limit at $c \in \mathbb{R}^*$. Then there exists $\delta > 0$ such that f is bounded on $P(c,\delta)$.

Theorem 3 (arithmetics of limits)

Let $c \in \mathbb{R}^*$, $\lim_{x\to c} f(x) = A \in \mathbb{R}^*$ *and* $\lim_{x\to c} g(x) = B \in \mathbb{R}^*$. *Then*

(i) $\lim_{x\to c}(f(x) + g(x)) = A + B$ *if the expression* $A + B$ *is defined,*

(ii) $\lim_{x\to c} f(x)g(x) = AB$ *if the expression* AB *is defined,*

(iii) $\lim_{x\to c} f(x)/g(x) = A/B$ *if the expression* A/B *is defined.*

Exercise

Find $\lim_{x\to 1+} f(x) + 2g(x)$

A 13

B 9

C 8

D 6

E 3

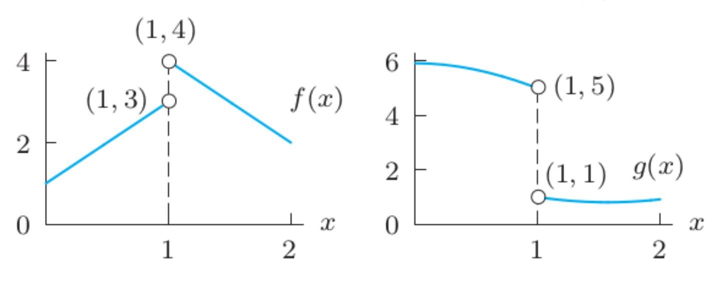


Figure: Calculus: Single and Multivariable, Hughes-Hallet

Exercise

Find $\lim_{x\to 1-} f(x)g(x)$

A 20

B 15

C 4

D 3

E does not exist

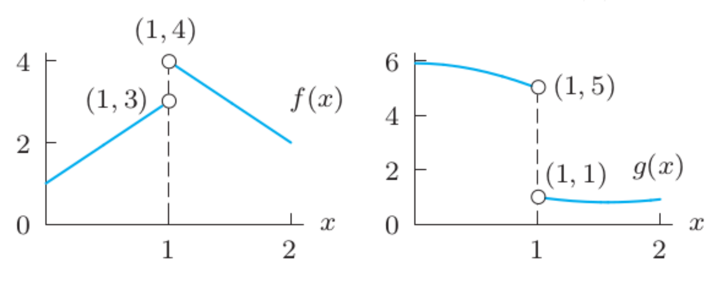


Figure: Calculus: Single and Multivariable, Hughes-Hallet

Corollary

Suppose that the functions f and g are continuous at $c \in \mathbb{R}$. Then also the functions $f + g$ and fg are continuous at c. If moreover $g(c) \neq 0$, then also the function f/g is continuous at c.

Corollary

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Exercise

Which functions are continuous at $\mathbb{R}$?

- A $x^3 + \sin(4-x)$
- B $\frac{e^x}{2+x}$
- C $\frac{2+x}{e^x}$
- D $\cos(e^{\sqrt[3]{x}})$
- E $\ln(2+x^2)$

Theorem 4

Let $c \in \mathbb{R}^*$, $\lim_{x \to c} g(x) = 0$, $\lim_{x \to c} f(x) = A \in \mathbb{R}^*$ *and* $A > 0$. *If there exists* $\eta > 0$ *such that the function g is positive on* $P(c, \eta)$, *then* $\lim_{x \to c} \big(f(x)/g(x)\big) = +\infty$.

Theorem 5 (limits and inequalities)

Suppose that $c \in \mathbb{R}^*$ *and* $\lim_{x\to c} f(x)$, $\lim_{x\to c} g(x)$ *exist.*
(i) If $\lim_{x\to c} f(x) > \lim_{x\to c} g(x)$, *then there exists* $\delta > 0$ *such that*

$$\forall x \in P(c, \delta) : f(x) > g(x).$$

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$$\forall x \in P(c,\delta) : f(x) > g(x).$$

(ii) If there exists $\delta > 0$ *such that* $\forall x \in P(c,\delta) : f(x) \leq g(x)$, *then*

$$\lim_{x\to c} f(x) \leq \lim_{x\to c} g(x).$$

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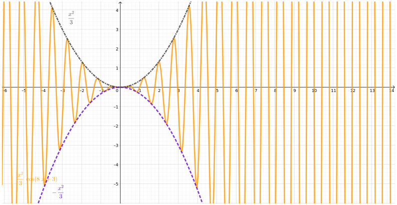
(iii) (two policemen/sandwich theorem) Suppose that there exists $\eta > 0$ *such that*

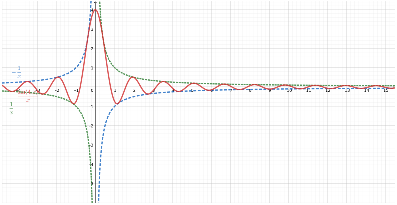
$$\forall x \in P(c,\eta) : f(x) \le h(x) \le g(x).$$

If moreover $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = A \in \mathbb{R}^*$, *then the limit* $\lim_{x\to c} h(x)$ *also exists and equals* A.

$\frac{x^2}{3}\cos(8x+3)$

$\frac{\sin(4x)}{x}$





https://www.geogebra.org/calculator/dvqdpqag

Corollary

Let $c \in \mathbb{R}^*$, $\lim_{x\to c} f(x) = 0$ *and suppose there exists* $\eta > 0$ *such that* g *is bounded on* $P(c,\eta)$*. Then* $\lim_{x\to c}\big(f(x)g(x)\big) = 0$.

Example

$$\lim_{x\to 0}(\sin x)(\operatorname{sgn} x)$$

Theorem 6 (limit of a composition)

Let $c, A, B \in \mathbb{R}^*$, $\lim_{x \to c} g(x) = A$, $\lim_{y \to A} f(y) = B$ *and at least one of the following conditions is satisfied:*

(I) $\exists \eta \in \mathbb{R}, \eta > 0 \ \forall x \in P(c, \eta) \colon g(x) \neq A$,

(C) *the function* f *is continuous at* A.

Then

$$\lim_{x \to c} f\big(g(x)\big) = B.$$

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Then

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Corollary

Suppose that the function g *is continuous at* $c \in \mathbb{R}$ *and the function* f *is continuous at* $g(c)$. *Then the function* $f \circ g$ *is continuous at* c.

Proof

Step 1.: definition of lim of f:
$\forall \varepsilon > 0\ \exists \delta > 0\ \forall y \in \mathring{U}_\delta(A): \quad f(y) \in U_\varepsilon(B).$
Step 2: definition of lim of g. We use it for δ from the definition of $\lim f$. For that δ, there exists $\sigma > 0$ such that ... :
$\exists \sigma > 0\ \forall x \in \mathring{U}_\sigma(c): \quad g(x) \in U_\delta(A).$

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We want to combine these definitions.
if $g(x) \in \mathring{U}_\delta(A)$ then for $y := g(x)$ we have
$f(y) = f(g(x)) \in U_\varepsilon(B).$

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The problem is the mismatch between $\mathring{U}_\delta(A)$ and $U_\delta(A)$.

Can we guarantee that $g(x) \in \mathring{U}_\delta(A)$ in the definition of $\lim g$?

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The problem is the mismatch between $\mathring{U}_\delta(A)$ and $U_\delta(A)$.

Can we guarantee that $g(x) \in \mathring{U}_\delta(A)$ in the definition of $\lim g$?

Yes, our condition (I) guarantees that $g(x) \neq A$ for sufficiently small σ. (if needed, σ can be decreased to satisfy both condition (I) and definition of lim g).

What about the (C) case? In that case we do not need to exclude A in the definition of $\lim f$.
$\forall \varepsilon > 0\ \exists \delta > 0\ \forall y \in U_\delta(A): \quad f(y) \in U_\varepsilon(B).$

Why in the definition of a limit we require punctured neighborhoods?

Why in the definition of a limit we require punctured neighborhoods?
Function may not be defined exactly at the point of interest.

Remarkable limits

$$\lim_{x\to 0}\frac{\sin x}{x}=1, \qquad \lim_{x\to 0}\frac{\log(1+x)}{x}=1.$$

https://math.stackexchange.com/questions/98998/why-x-tanx-while-0x-frac-pi2

Exercise

$$\lim_{x \to \infty} \ln\left(\frac{x-1}{x+2}\right)$$

A 0 B 1 C $\ln 1$ D $-\frac{1}{2}$ E ∞

Exercise

$$\lim_{x\to\infty} \ln\left(\frac{x-1}{x+2}\right)$$

A 0 B 1 C $\ln 1$ D $-\frac{1}{2}$ E ∞

Exercise

$$\lim_{x\to-\infty} \cos\frac{1}{x}$$

A 0

B 1

C π

D $-\infty$

E does not exist

Exercise

$$\lim_{x\to\infty} \ln\left(\frac{x-1}{x+2}\right)$$

A 0 B 1 C $\ln 1$ D $-\frac{1}{2}$ E ∞

Exercise

$$\lim_{x\to-\infty} \cos\frac{1}{x}$$

A 0
B 1
C π
D $-\infty$
E does not exist

Exercise

$$\lim_{x\to 0} \arctan\frac{1}{x^2}$$

A 0 B 1 C $\frac{\pi}{2}$ D $-\frac{\pi}{4}$ E ∞

Theorem 7 (Heine)

Let $c \in \mathbb{R}^*$, $A \in \mathbb{R}^*$ and the function f satisfies $\lim_{x\to c} f(x) = A$. If the sequence $\{x_n\}$ satisfies $x_n \in D_f$, $x_n \neq c$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = c$, then $\lim_{n\to\infty} f(x_n) = A$.

Example

$$\lim_{n\to\infty} \ln\left(\frac{n+\sin n}{n-\cos n}\right)$$

$$\lim_{n\to\infty} \cos\left(\sin\left(\frac{\pi}{2}\frac{1}{n^2}\right)\right)$$

Theorem 8 (limit of a monotone function)

Let $a, b \in \mathbb{R}^*$, $a < b$. Suppose that f is a function monotone on an interval (a, b). Then the limits $\lim_{x\to a+} f(x)$ and $\lim_{x\to b-} f(x)$ exist. Moreover,

- *if f is non-decreasing on (a, b), then* $\lim_{x\to a+} f(x) = \inf f\big((a, b)\big)$ *and* $\lim_{x\to b-} f(x) = \sup f\big((a, b)\big)$*;*
- *if f is non-increasing on (a, b), then* $\lim_{x\to a+} f(x) = \sup f\big((a, b)\big)$ *and* $\lim_{x\to b-} f(x) = \inf f\big((a, b)\big)$.

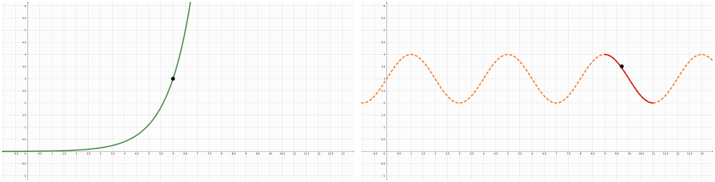


Figure: `https://www.geogebra.org/calculator/bfutkyne`

Definition

Let $J \subset \mathbb{R}$ be a non-degenerate interval (i.e. it contains infinitely many points). A function $f : J \to \mathbb{R}$ is continuous on the interval J if

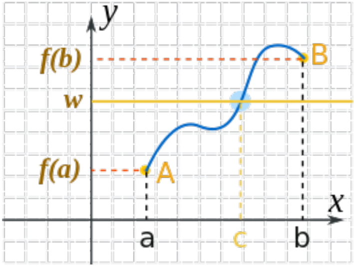
- f is continuous at every inner point J,
- f is continuous from the right at the left endpoint of J if this point belongs to J,
- f is continuous from the left at the right endpoint of J if this point belongs to J.

Theorem 9 (continuity of the compound function on an interval)

Let I and J be intervals, $g\colon I \to J$, $f\colon J \to \mathbb{R}$, let g be continuous on I and let f be continuous on J. Then the function $f \circ g$ is continuous on I.

Theorem 10 (Bolzano, intermediate value theorem)

Let f be a function continuous on an interval $[a,b]$ and suppose that $f(a) < f(b)$. Then for each $C \in (f(a), f(b))$ there exists $\xi \in (a,b)$ satisfying $f(\xi) = C$.



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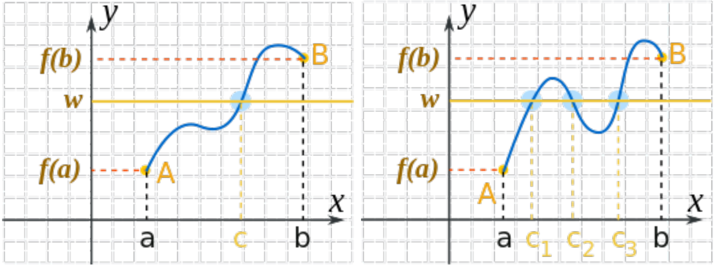

Figure: https://www.mathsisfun.com/algebra/intermediate-value-theorem.html

Theorem 11 (Bolzano, intermediate value theorem)

Let f be a function continuous on an interval $[a, b]$ and suppose that $f(a) < f(b)$. Then for each $C \in (f(a), f(b))$ there exists $\xi \in (a, b)$ satisfying $f(\xi) = C$.

Exercise

Is there a solution between 0 and 2?

- $x^5 - 2x - 1 = 0$
- $x^3 - 4x^2 + 4x + 1 = 0$
- $5x^3 - 15x^2 + 10x + 1 = 0$

`https://www.geogebra.org/calculator/pqbtmk54`

Theorem 12 (an image of an interval under a continuous function)

Let J be an interval and let $f : J \to \mathbb{R}$ be a function continuous on J. Then $f(J)$ is an interval.

Exercise

Find the image of the interval $(-1, 2]$ under the functions

- x^2
- $\operatorname{sgn} x$

Definition

Let $M \subset \mathbb{R}$, $x \in M$ and a function f is defined at least on M (i.e. $M \subset D_f$). We say that f attains its maximum (resp. minimum) on M at $x \in M$ if

$$\forall y \in M : f(y) \leq f(x) \quad (\text{resp. } \forall y \in M : f(y) \geq f(x)).$$

The point x is called the point of maximum (resp. minimum) of the function f on M. The symbol $\max_M f$ (resp. $\min_M f$) denotes the maximal (resp. minimal) value of f on M (if such a value exists). The points of maxima or minima are collectively called the points of extrema.

Definition

Let $M \subset \mathbb{R}$, $x \in M$ and a function f is defined at least on M (i.e. $M \subset D_f$). We say that the function f has at x

- a local maximum with respect to M if there exists $\delta > 0$ such that $\forall y \in B(x, \delta) \cap M : f(y) \leq f(x)$,
- a local minimum with respect to M if there exists $\delta > 0$ such that $\forall y \in B(x, \delta) \cap M : f(y) \geq f(x)$,
- a strict local maximum with respect to M if there exists $\delta > 0$ such that $\forall y \in P(x, \delta) \cap M : f(y) < f(x)$,
- a strict local minimum with respect to M if there exists $\delta > 0$ such that $\forall y \in P(x, \delta) \cap M : f(y) > f(x)$.

The points of local maxima or minima are collectively called the points of local extrema.

Exercise

Find local extrema:

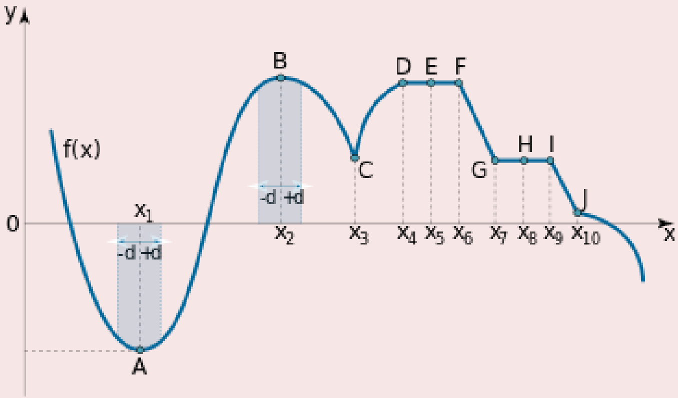

Figure: https://math24.net/local-extrema-functions.html

Theorem 13 (boundedness of a continuous function)

Let f be a function continuous on an interval $[a, b]$. Then f is bounded on $[a, b]$.

Theorem 14 (extrema of continuous functions)

Let f be a function continuous on an interval $[a, b]$. Then f attains its maximum and minimum on $[a, b]$.

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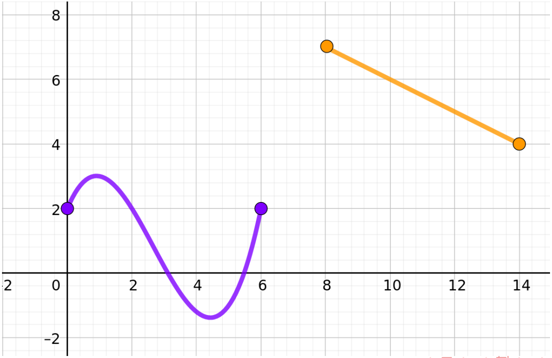

Proof of existence of extrema

Let $M = \sup\limits_{[a,b]} f$. Then
$\forall \varepsilon > 0 \; \exists x = x(\varepsilon) \in [a,b] : \quad f(x) > M - \varepsilon.$
Choose consequently $\varepsilon = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$. Then we obtain a sequence $x_n \in [a,b]$ such that $f(x_n) > M - \frac{1}{n}$ (and also $f(x_n) \le M$). Hence, $\exists \lim\limits_{n\to\infty} f(x_n) = M$.
Next, every bounded sequence has a convergent subsequence (Bolzano-Weierstrass). This subsequence x_{n_k} will converge to some point $c \in [a,b]$, and $f(x_0) = \lim\limits_{k\to\infty} f(x_{n_k}) = M$.

Theorem 15 (continuity of an inverse function)

Let f be a continuous function that is increasing (resp. decreasing) on an interval J. Then the function f^{-1} is continuous and increasing (resp. decreasing) on the interval $f(J)$.

Corollary 16

Functions nth root, exponential, general power, arcsin, arccos, arctg, arccotg are continuous on their domains.