Stationary Euler flows near the Kolmogorov and Poiseuille flows

Workshop on Partial differential equations describing far-from-equilibrium open systems

Michele Coti Zelati
joint with T.M. Elgindi and K. Widmayer

September 24, 2020

Imperial College London

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In a 2d domain, consider

$$\begin{cases} \partial_t \boldsymbol{U} + (\boldsymbol{U} \cdot \nabla)\boldsymbol{U} + \nabla P = \nu \Delta \boldsymbol{U}, \\ \nabla \cdot \boldsymbol{U} = 0. \end{cases}$$

- $\boldsymbol{U} = (U_1, U_2)$ is the velocity field of the fluid
- P is the scalar pressure

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- $\nu \geq 0$ is the inverse Reynolds number
  - $\nu = 0$: Inviscid fluid $\rightarrow$ Euler equations
  - $\nu > 0$: Viscous fluid $\rightarrow$ Navier-Stokes equations

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In vorticity formulation $\Omega = \nabla^\perp \cdot \boldsymbol{U} = -\partial_y U_1 + \partial_x U_2$:

$$\begin{cases} \partial_t \Omega + \boldsymbol{U} \cdot \nabla \Omega = \nu \Delta \Omega, \\ \boldsymbol{U} = \nabla^\perp \Psi, \quad \Delta \Psi = \Omega. \end{cases}$$

Main features

- Smooth solutions remain smooth and are global ($\nu \geq 0$)
- All L^p norms are conserved ($\nu = 0$)

What happens as $t \to \infty$?

- In (bounded) domains, all mean-zero solutions decay to 0 ($\nu > 0$)
- For $\nu = 0$, the dynamics can be very complicated: there is no global relaxation mechanism

Vorticity mixing

Mixing can be thought of as a cascading process in which information travels to smaller and smaller spatial scales.

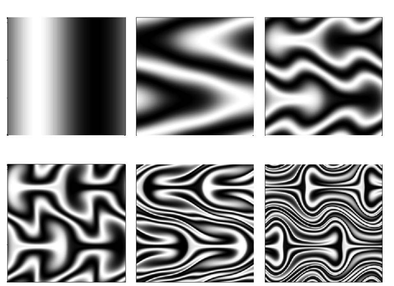

Figure 1: No diffusion (Doering et al.)

Understanding this fundamental process sheds light on:

- Relaxation towards stationary states and coherent structures
- Meta-stable behavior in ocean/atmospheric models
- The derivation of turbulence scaling laws (Kolmogorov, Batchelor)

A conjecture

Longtime behavior for 2D Euler

The generic solution to the 2D Euler equations in vorticity form on $\mathbb{T}^2$ is such that the orbit $\{\Omega(t) : t \in \mathbb{R}\}$ is not precompact in $L^2(\mathbb{T}^2)$.

- All solutions that experience some vorticity mixing as $t \to \infty$ are not precompact (very hard to prove in general!)
- Understand the dynamics near steady states such as shear flows and vortices
- Understand the (local) structure of known steady states

Long-time dynamics

- Bedrossian, Masmoudi '13: sufficiently smooth, non-shear perturbations of the Couette flow $\boldsymbol{U} = (y, 0)$ undergo vorticity mixing and inviscid damping.
- Same for monotonic flows $\boldsymbol{U} = (u(y), 0)$ on $\mathbb{T} \times [-1, 1]$ (Ionescu, Jia '19 and Masmoudi, Zhao '19)
- Same for the point-vortex (Ionescu, Jia '19)

Local structure of steady states

- Lin, Zeng '10: there are steady states near Couette in H^s ($s < 3/2$), with cat's eye structure (i.e. nontrivial x-dependence). All steady states near Couette in H^s ($s > 3/2$) are shears.
- Choffrut, Sverak '12: Neighborhoods of non-degenerate steady states in an annulus can contain only non-degenerate steady states.
- Constantin, Drivas, Ginsberg '20: there are perturbations of non-degenerate Arnold stable steady states that are non-degenerate Arnold stable

Local vs global degeneracies

Write Euler near a shear $(u(y), 0)$:

$$u\partial_x\omega - u''\Delta^{-1}\partial_x\omega + \boldsymbol{u} \cdot \nabla\omega = 0$$

- Local degeneracy: u has a (simple) critical point
- Global degeneracy: The kernel of the linear operator

  $$\mathcal{L}_u = u\partial_x - u''\Delta^{-1}\partial_x$$

  is "big" (does not only contain shears)

Question: what is the role of degeneracies in the local structure of steady states?

Examples

- Couette: $v(y) = y$, on $\mathbb{T} \times [-1, 1]$, is non-degenerate
- Poiseuille: $v(y) = y^2$, on $\mathbb{T} \times [-1, 1]$, is locally degenerate but the kernel of

  $$\mathcal{L}_P = y^2 \partial_x - 2\Delta^{-1} \partial_x$$

  only contains shears
- Kolmogorov: $v(y) = \sin y$, on $\mathbb{T}^2$ is both locally and globally degenerate, since the kernel of

  $$\mathcal{L}_K = \sin y (1 + \Delta^{-1}) \partial_x$$

  contains also $\{\sin x, \cos x\}$. This does not happen on a rectangular torus $\mathbb{T}^2_\delta := [0, 2\pi\delta] \times [0, 2\pi]$, $\delta > 0$ with $\delta \notin \mathbb{N}$.

The Kolmogorov flow

Steady Euler flows

Any steady Euler flows $\boldsymbol{U} = \nabla^{\perp}\Psi$ satisfies

$$\nabla^{\perp}\Psi \cdot \nabla\Delta\Psi = 0.$$

Hence, if

$$\Delta\Psi = F(\Psi), \qquad F \in C^1,$$

then Ψ is a steady solution. Kolmogorov flow is $\boldsymbol{U}_K = (\sin y, 0)$, hence $\Psi_K := \cos(y)$, and

$$\Delta\Psi_K = F_K(\Psi_K), \qquad F_K(z) = -z.$$

Structures near Kolmogorov

Structures near Kolmogorov [CZ, Elgindi, Widmayer '20]

There exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \leq \varepsilon_0$ there exist analytic functions $\Psi_\varepsilon \in C^\omega(\mathbb{T}^2)$ and $F_\varepsilon \in C^\omega(\mathbb{R})$ satisfying

$$\Delta \Psi_\varepsilon = F_\varepsilon(\Psi_\varepsilon) \tag{1}$$

and

$$\|\cos(y) - \Psi_\varepsilon\|_{C^\omega(\mathbb{T}^2)} = O(\varepsilon), \tag{2}$$

with

$$\langle \Psi_\varepsilon, \cos(x)\cos(4y)\rangle = -\varepsilon^2 \frac{\pi^2}{128} + O(\varepsilon^3). \tag{3}$$

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- F_ε is a polynomial of degree 5, so if $\Psi_\varepsilon \in H^2$ then, by elliptic regularity, it is analytic.
- There are families of non-trivial (i.e. not in the kernel of $\mathcal{L}_K$), non-shear and stationary solutions $U_\varepsilon := \nabla^\perp \Psi_\varepsilon : \mathbb{T}^2 \to \mathbb{R}^2$ of the incompressible Euler equations.

The general strategy

To find a larger class of solutions near Kolmogorov, we make the ansatz

$$\Psi_\varepsilon = \Psi_K + \varepsilon\psi, \qquad F_\varepsilon = F_K + \varepsilon f,$$

which yields a nonlinear elliptic equation for ψ, with f to be determined as well,

$$\Delta\psi + \psi = f(\Psi_K + \varepsilon\psi).$$

GOAL

Find (f, ψ), with ψ even in x and y separately, such that

$$\Delta\psi + \psi = f(\cos(y) + \varepsilon\cos(x) + \varepsilon\psi), \qquad \text{with } \psi \perp \ker(\Delta + 1),$$

with f as a quintic polynomial (with coefficients $A, B \in \mathbb{R}$ to be determined as functionals of ψ and $\varepsilon > 0$)

$$f(A, B; s) = As + Bs^3 + \frac{1}{5}s^5.$$

The resulting steady state

- Ψ_ε can be computed to have the expansion

$$\Psi_\varepsilon = \cos(y) + \varepsilon \left[\cos(x) + c_0 \cos(3y) - c_1 \cos(5y)\right] + \varepsilon^2 \left[-c_2 \cos(x)\cos(4y) - \frac{1}{32} b_1 \cos(3y) - c_3 \cos(7y) + c_4 \cos(9y)\right] + O(\varepsilon^3).$$

- Many such families $(\Psi_\varepsilon)_\varepsilon$ exist. Modify the functions F_ε by adding polynomials with coefficients of order ε^2.

Explicitly

This amounts to solve

$$\begin{aligned}\Delta\psi + \psi = & A\cos(y) + B\cos^3(y) + \frac{1}{5}\cos^5(y) \\ & + \varepsilon\psi\Big(A + 3B\cos^2(y) + \cos^4(y)\Big) \\ & + \varepsilon\cos(x)\Big(A + 3B\cos^2(y) + \cos^4(y)\Big) \\ & + R(B, \psi, \varepsilon; x, y),\end{aligned}$$

with $R(B, \psi, \varepsilon; x, y) = O(\varepsilon^2)$.

Solvability conditions (SC)

$$\langle f(A, B; \cos(y) + \varepsilon\cos(x) + \varepsilon\psi), \cos(x)\rangle = 0$$
$$\langle f(A, B; \cos(y) + \varepsilon\cos(x) + \varepsilon\psi), \cos(y)\rangle = 0.$$

The contraction set-up

The space is

$$X := \Big\{ \psi \in H^2 : \psi(-x,y) = \psi(x,-y) = \psi(x,y), \quad \psi \perp \cos(y), \cos(x),$$
$$\big|\langle \psi, \cos^2(y)\cos(x)\rangle\big| + \big|\langle \psi, \cos^4(y)\cos(x)\rangle\big| \leq \frac{1}{100}, \quad \|\psi\|_{H^2} \leq 10 \Big\}.$$

The coefficients

For $\psi \in X$ and ε small, for $0 \leq \varepsilon \leq \varepsilon_1$, **(SC)** inductively define $(a_j(\psi))_{j\geq 0}, (b_j(\psi))_{j\geq 0} \subset \mathbb{R}$ such that

$$A(\psi;\varepsilon) := \sum_{j\geq 0} a_j(\psi)\varepsilon^j, \qquad B(\psi;\varepsilon) := \sum_{j\geq 0} b_j(\psi)\varepsilon^j$$

are well-defined, uniformly bounded for $\psi \in X$, and satisfy **(SC)**. The maps $\psi \mapsto a_j(\psi), \psi \mapsto b_j(\psi)$ are Lipschitz on L^2 and the maps $\psi \mapsto a_0(\psi), \psi \mapsto b_0(\psi)$, are Lipschitz on $\dot{H}^2$ with constant $\tilde{L}_0 \leq \frac{1}{4\pi}$.

The contraction set-up

The map $K_\varepsilon : X \to H^2$

We look for a fixed point of

$$\psi \mapsto \left[(x,y) \mapsto (1+\Delta)^{-1} f(A(\psi;\varepsilon), B(\psi,\varepsilon); \cos(y) + \varepsilon\cos(x) + \varepsilon\psi)\right]$$

The contraction property boils down to

$$\begin{aligned}
&\left\|(a_0(\psi_1) - a_0(\psi_2))\cos(y) + (b_0(\psi_1) - b_0(\psi_2))\cos^3(y)\right\|_{L^2} \\
&\leq \frac{1}{4\pi}\left[\|\cos(y)\|_{L^2} + \left\|\cos^3(y)\right\|_{L^2}\right]\|\psi_1 - \psi_2\|_{\dot{H}^2} \\
&= \frac{\sqrt{2}}{4}\left[1 + \sqrt{\frac{5}{8}}\right]\|\psi_1 - \psi_2\|_{\dot{H}^2} \leq \frac{2}{3}\|\psi_1 - \psi_2\|_{H^2},
\end{aligned}$$

This shows that

$$\|K_\varepsilon(\psi_1) - K_\varepsilon(\psi_2)\|_{H^2} \leq \left(\frac{2}{3} + O(\varepsilon)\right)\|\psi_1 - \psi_2\|_{H^2},$$

and for $\varepsilon > 0$ sufficiently small we thus obtain a contraction.

Non-triviality of steady states

Recall that $\Psi_\varepsilon = \cos(y) + \varepsilon\cos(x) + \varepsilon\psi_\varepsilon$ and

$$\begin{aligned}\Delta\psi_\varepsilon + \psi_\varepsilon = &-\frac{1}{48}\cos(3y) + \frac{1}{80}\cos(5y)\\ &+ \varepsilon(\psi_\varepsilon|_{\varepsilon=0} + \cos(x))\frac{1}{8}\cos(4y) + \varepsilon\cos(y)\left[a_1 + \frac{3}{4}b_1\right]\\ &+ \varepsilon\cos(3y)\left[\frac{1}{4}b_1\right]\\ &+ O(\varepsilon^2).\end{aligned}$$

Hence

$$\begin{aligned}\Psi_\varepsilon = &\cos(y) + \varepsilon\left[\cos(x) + c_0\cos(3y) - c_1\cos(5y)\right]\\ &+ \varepsilon^2\left[-c_2\cos(x)\cos(4y) - \frac{1}{32}b_1\cos(3y) - c_3\cos(7y) + c_4\cos(9y)\right]\\ &+ O(\varepsilon^3).\end{aligned}$$

Remarks and consequences

Inviscid damping

$$\begin{cases} \partial_t \omega + \mathcal{L}_K \omega = -\boldsymbol{u} \cdot \nabla \omega, \\ \boldsymbol{u} = \nabla^{\perp} \psi, \quad \Delta \psi = \omega. \end{cases}$$

- Wei, Zhang, Zhao '17: there is linear inviscid damping, namely, linearly all modes away from the kernel of $\mathcal{L}_K$ decay.
- CZ, Elgindi, Widmayer '20: the result cannot be extended perturbatively at the nonlinear level, no matter the regularity. The dynamics near Kolmogorov on $\mathbb{T}^2$ is much richer.

Obstructions on the Square Torus

Not all directions are good! There are elements of $\ker \mathcal{L}_K$ which cannot arise as projections of stationary states.

Obstructions on the Torus

If for some $\ell \in \mathbb{N}$, $\ell \geq 2$,

$$\frac{\mathbb{P}_K(\Omega_* - \cos(y))}{\|\mathbb{P}_K(\Omega_* - \cos(y))\|_{L^2}} = \sin(\ell y) + \cos(x),$$

then there exists $\varepsilon_0 > 0$ small so that if $\|\Omega_* - \cos(y)\|_{H^6} = \varepsilon < \varepsilon_0$, then Ω_* is not a stationary solution to the $2d$ Euler equations.

Rigidity on Rectangular Tori

Rigidity near Kolmogorov on a rectangular torus

Consider the stationary solution $U_K(x, y) = (\sin(y), 0)$ on $\mathbb{T}^2_\delta$, $\delta > 0$ with $\delta \notin \mathbb{N}$. There exists $\varepsilon_0 > 0$ (depending on δ) such that if $U : \mathbb{T}^2_\delta \to \mathbb{R}^2$ is a further stationary solution to the Euler equations with

$$\|U - U_K\|_{H^3} \leq \varepsilon_0,$$

then $U = U(y)$ is necessarily a shear flow.

Rigidity near Poiseuille flow

Near Poiseuille flow, even any nearby travelling wave solution must simply be a shear flow.

Rigidity near Poiseuille

Let $s > 5$, and consider the $2d$ Euler equations on $\mathbb{T} \times [-1, 1]$

$$\partial_t U + U \cdot \nabla U + \nabla P = 0, \qquad \nabla \cdot U = 0, \qquad U_2(x, \pm 1) = 0.$$

There exists $\varepsilon_0 > 0$ such that if $U(x - ct, y)$, with $c \in \mathbb{R}$, is any traveling wave solution that satisfies

$$\|\Omega + 2y\|_{H^s} \leq \varepsilon_0, \quad \text{where } U = \nabla^\perp \Psi, \quad \Delta \Psi = \Omega,$$

then it follows that $U \equiv (U_1, 0)$, that is, U is necessarily a shear flow.

Enhanced Dissipation near Bar States on $\mathbb{T}^2$

The linearization of the Navier-Stokes equations near the bar states $\Omega_{bar} = -\mathrm{e}^{-\nu t}\cos(y)$ is then given by

$$\partial_t f + \mathrm{e}^{-\nu t}\mathcal{L}_K f = \nu \Delta f.$$

Ibrahim, Maekawa and Masmoudi '17 and Wei, Zhang, Zhao '17 showed that

$$\|\mathbb{P}_{\mathcal{D}} f(t)\|_{L^2} \lesssim \mathrm{e}^{-c_1\nu^{1/2}t}\|\mathbb{P}_{\mathcal{D}} f(0)\|_{L^2}, \qquad \forall t \leq \frac{\mathcal{T}}{\nu}, \qquad \mathcal{D} := (\ker \mathcal{L}_K)^{\perp}.$$

Typical nonlinear transition threshold

At the nonlinear level, there exists $\gamma \geq 0$ such that if

$$\|\mathbb{P}_{\mathcal{D}}\omega^{in}\|_X \lesssim \nu^{\gamma} \qquad \Rightarrow \qquad \|\mathbb{P}_{\mathcal{D}}\omega(t)\|_{L^2} \lesssim \mathrm{e}^{-c_1\nu^{1/2}t}\left\|\mathbb{P}_{\mathcal{D}}\omega^{in}\right\|_{L^2}$$

- True for rectangular tori (Wei, Zhang, Zhao '17)
- True for Poiseuille flow (CZ, Elgindi, Widmayer '19)

No Threshold near Bar States on $\mathbb{T}^2$

No nonlinear threshold

For any $\nu > 0$ there exists $0 < \varepsilon_0 \ll \nu$ with the following property: let $0 < \varepsilon \leq \varepsilon_0$ and let $\Omega_\varepsilon = \Delta\Psi_\varepsilon$ be the vorticity of the stationary Euler flow found before. Then $\mathbb{P}_\mathcal{D}\Omega_\varepsilon$ is not dissipated at an enhanced rate: i.e. the solution Ω^ν of the initial value problem

$$\begin{cases} \partial_t \Omega^\nu + U^\nu \cdot \nabla \Omega^\nu = \nu \Delta \Omega^\nu, \\ \Omega^\nu(0) = \Omega_\varepsilon, \end{cases}$$

on $\mathbb{T}^2$ satisfies for all $t \in [\frac{1}{2\nu}, \frac{1}{\nu}]$ the lower bound

$$\left\| \mathbb{P}_\mathcal{D} \Omega^\nu(t) \right\|_{L^2} \gtrsim \left\| \mathbb{P}_\mathcal{D} \Omega_\varepsilon \right\|_{L^2}.$$

THANK YOU