Large time behavior of finite volume schemes for convection-diffusion equations

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Workshop 2020 - Partial differential equations

Joint work with C. Cancès, M. Herda (Lille) and S. Krell (Nice)

Outline of the talk

1. Motivation
2. Linear/nonlinear TPFA schemes
3. Study of the nonlinear schemes

Large time behaviour : from the continuous to the discrete level

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\begin{array}{c}\text{Evolutive equation}\\ \text{or system of equations}\\ \mathbf{F}\end{array} \arrow[r, "t \to \infty"] & \begin{array}{c}\text{Steady-state}\\ \mathbf{F}^{\infty}\end{array} \\
\begin{array}{c}\text{Scheme for the}\\ \text{evolutive problem}\\ \mathbf{F}_{\Delta x, \Delta t}\end{array} \arrow[u, "{\Delta x, \Delta t \to 0}"] & \begin{array}{c}\text{Scheme}\\ \text{for the equilibrium}\\ \mathbf{F}^{\infty}_{\Delta x}\end{array} \arrow[u, "\Delta x \to 0"']
\end{tikzcd}
$$

Large time behaviour : from the continuous to the discrete level

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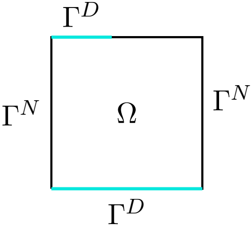
\end{tikzcd}
$$

+ exponential convergence ?

Focus on anisotropic Fokker-Planck equations

$$\begin{cases} \partial_t u + \nabla \cdot \mathbf{J} = 0, \quad \mathbf{J} = \mathbf{\Lambda}(-\nabla u - u\nabla V), \text{ in } \Omega \times \mathbb{R}_+ \\ \mathbf{J} \cdot \mathbf{n} = 0 \text{ on } \Gamma^N \times \mathbb{R}_+ \\ u = u^D \text{ on } \Gamma^D \times \mathbb{R}_+ \\ u(\cdot, 0) = u_0 \geq 0. \end{cases}$$



Examples

- Semiconductor models, corrosion models
  - ➟ $\mathbf{\Lambda} = \mathbf{I}$
  - ➟ coupling with a Poisson equation for V
- Porous media flow
  - ➟ $\mathbf{\Lambda}$ bounded, symmetric and uniformly elliptic
$$\lambda_m |\mathbf{v}|^2 \leq \mathbf{\Lambda}(x)\mathbf{v} \cdot \mathbf{v} \leq \lambda^M |\mathbf{v}|^2$$
  - ➟ $V = gz$

Focus on anisotropic Fokker-Planck equations

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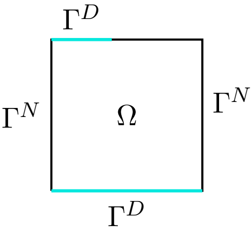

Some references $\mathbf{\Lambda} = \mathbf{I}$

- Carrillo, Toscani, '98
- Arnold, Markowich, Toscani, Unterreiter, '01
- Carrillo et al., '01
- Bodineau, Lebowitz, Mouhot, Villani, '14
- Gajewski, Gröger, '86, '89
- Jüngel, '95

Structural properties

$$\begin{cases} \partial_t u + \text{div}\, \mathbf{J} = 0, \quad \mathbf{J} = -\mathbf{\Lambda}(\nabla u + u\nabla V), \\ u(\cdot, 0) = u_0 \geq 0 \;\; + \text{boundary conditions} \end{cases}$$

- Existence and uniqueness of a nonnegative solution.
- Mass conservation if $\Gamma^D = \emptyset$.
- Existence of a thermal/Gibbs equilibrium :

$$u^\infty = \rho e^{-V} (\Longrightarrow \mathbf{J} = 0)$$

  - if $\Gamma^D = \emptyset$,

$$\rho = \frac{\int_\Omega u_0}{\int_\Omega e^{-V}}, \quad \text{so that} \quad \int_\Omega u^\infty = \int_\Omega u_0.$$

  - if $\Gamma^D \neq \emptyset$ and the boundary data u^D satisfy a compatibility assumption

$$u^D = \rho e^{-V} \text{ on } \Gamma^D.$$

Study of the large time behaviour : the entropy method

Definition of a relative entropy

$$\Phi \in C^2(\mathbb{R}, \mathbb{R}) \text{ a convex function, } \Phi(1) = \Phi'(1) = 0$$

$$E_\Phi(t) = \int_\Omega u^\infty \Phi\left(\frac{u}{u^\infty}\right) dx$$

Dissipation of the relative entropy

$$\frac{d}{dt} E_\Phi(t) = -I_\Phi(t), \text{ with } I_\Phi(t) \geq 0.$$

Relation between entropy and dissipation

$$I_\Phi(t) \geq \nu E_\Phi(t) \Longrightarrow \frac{d}{dt} E_\Phi(t) \leq -\nu E_\Phi(t)$$

so that $\boxed{E_\Phi(t) \leq e^{-\nu t} E_\Phi(0).}$

Study of the large time behaviour : the entropy method

$$\begin{cases} \partial_t u + \mathsf{div}\ \mathbf{J} = 0, \\ \mathbf{J} = -\mathbf{\Lambda}(\nabla u + u\nabla V), \end{cases} \qquad u^\infty = \rho e^{-V}.$$

The entropy dissipation $\quad E_\Phi(t) = \int_\Omega u^\infty \Phi\left(\frac{u}{u^\infty}\right)$

$$\begin{aligned} \frac{d}{dt}E_\Phi(t) &= \int_\Omega \Phi'\left(\frac{u}{u^\infty}\right)\partial_t u \\ &= \int_\Omega \mathbf{J}\cdot\nabla\Phi'\left(\frac{u}{u^\infty}\right) \end{aligned}$$

Study of the large time behaviour : the entropy method

$$\begin{cases} \partial_t u + \mathsf{div}\ \mathbf{J} = 0, \\ \mathbf{J} = -\boldsymbol{\Lambda} u \nabla \log \dfrac{u}{u^\infty} \end{cases} \qquad u^\infty = \rho e^{-V}.$$

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$$\begin{aligned} \frac{d}{dt} E_\Phi(t) &= \int_\Omega \Phi'\left(\frac{u}{u^\infty}\right) \partial_t u \\ &= \int_\Omega \mathbf{J} \cdot \nabla \Phi'\left(\frac{u}{u^\infty}\right) \\ &= -\int_\Omega \mathbf{\Lambda} u \nabla \log \frac{u}{u^\infty} \cdot \nabla \Phi'\left(\frac{u}{u^\infty}\right) \end{aligned}$$

$$\boxed{I_\Phi(t) = \int_\Omega \mathbf{\Lambda} u \nabla \log \frac{u}{u^\infty} \cdot \nabla \Phi'\left(\frac{u}{u^\infty}\right) \geq 0.}$$

Study of the large time behaviour : the entropy method

$$E_\Phi = \int_\Omega u^\infty \Phi(\frac{u}{u^\infty}) \qquad I_\Phi = \int_\Omega \mathbf{\Lambda} u \nabla \log \frac{u}{u^\infty} \cdot \nabla \Phi'(\frac{u}{u^\infty})$$

Examples of entropies

- Boltzmann-Gibbs entropy : $\Phi_1(s) = s \log s - s + 1$,
- Tsallies entropies : $\Phi_p(s) = \dfrac{s^p - ps}{p-1} + 1$, $p \in (1,2]$

$$I_p (= I_{\Phi_p}) \geq \lambda_m \frac{4}{p} \int_\Omega u^\infty \left| \nabla \left(\frac{u}{u^\infty} \right)^{p/2} \right|^2 = \hat{I}_p.$$

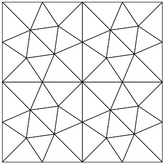
Relations entropy/dissipation $\quad \hat{I}_p \geq \nu E_p$

- $\Gamma^D \neq \emptyset$ and $p \in (1,2]$: Poincaré inequality.
- $\Gamma^D = \emptyset$ and $p = 1$: Log-Sobolev inequality,
- $\Gamma^D = \emptyset$ and $p \in (1,2]$: Beckner inequality,

Adaptation to the discrete level of the entropy method

Isotropic case and "nice" mesh

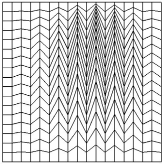
- "standard" linear finite volume schemes (TPFA)



- ❑ C.-H., Herda, '19
- ❑ Filbet, Herda, '17

Anisotropic case and/or almost-general mesh

- "advanced" nonlinear finite volume schemes



- ❑ Cancès, Guichard, '17
- ❑ Cancès, C.-H., Krell, ' 18
- ❑ Cancès, C.-H., Herda, Krell, '20

Outline of the talk

1. Motivation
2. Linear/nonlinear TPFA schemes
3. Study of the nonlinear schemes

TPFA schemes for the Fokker-Planck equation

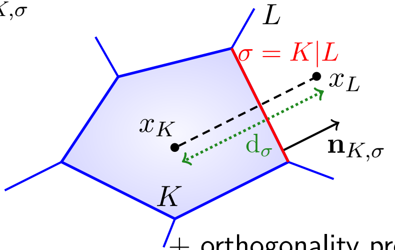
From the equation... $(\mathbf{\Lambda} = \mathbf{I})$

$$\begin{cases} \partial_t u + \nabla \cdot \mathbf{J} = 0, \quad \mathbf{J} = -\nabla u - u \nabla V, \\ u(\cdot, 0) = u_0 \geq 0 \ \ + \text{Neumann boundary conditions} \end{cases}$$

... to the scheme

$$\begin{cases} \mathrm{m}_K \dfrac{u_K^{n+1} - u_K^n}{\Delta t} + \displaystyle\sum_{\sigma \in \mathcal{E}_K^{int}} \mathcal{F}_{K,\sigma}^{n+1} = 0 \\ \mathcal{F}_{K,\sigma} \approx \displaystyle\int_\sigma \mathbf{J} \cdot \mathbf{n}_{K,\sigma} \end{cases}$$

- $\mathcal{T}$: control volumes, $K \in \mathcal{T}$
- $\mathcal{E}$: edges, $\sigma \in \mathcal{E}$
- $\mathcal{P}$: points, $(x_K)_{K \in \mathcal{T}}$
- Δt : time step

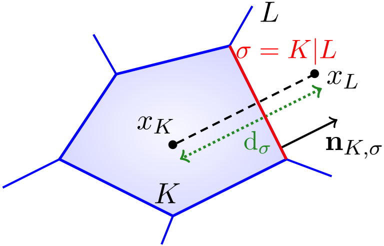


+ orthogonality property

Linear numerical fluxes

$$\mathcal{F}_{K,\sigma} \approx \int_\sigma (-\nabla u - u\nabla V)\cdot \mathbf{n}_{K,\sigma}$$

$$\mathrm{m}_\sigma \frac{u_K - u_L}{\mathrm{d}_\sigma} \approx \int_\sigma -\nabla u \cdot \mathbf{n}_{K,\sigma}$$



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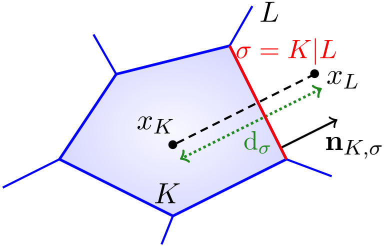

Generic form $\quad \sigma = K|L \in \mathcal{E}^{int}$

$$\mathcal{F}_{K,\sigma} = \tau_\sigma\Big(B(V_L - V_K)u_K - B(V_K - V_L)u_L\Big),\ \tau_\sigma = \frac{\mathrm{m}_\sigma}{\mathrm{d}_\sigma}$$

with $B(0) = 1,\ B(x) > 0$ and $B(x) - B(-x) = -x\ \forall x \in \mathbb{R}$

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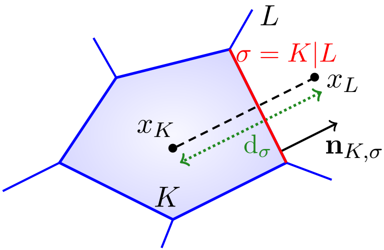

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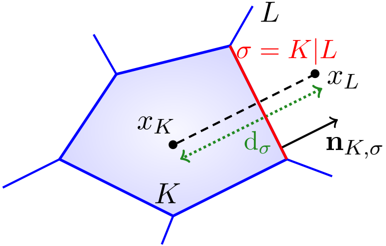
Classical examples

$$B_{up}(s) = 1 + s^-, \quad B_{ce}(s) = 1 - \frac{s}{2}, \quad B_{sg}(s) = \frac{s}{e^s - 1}$$

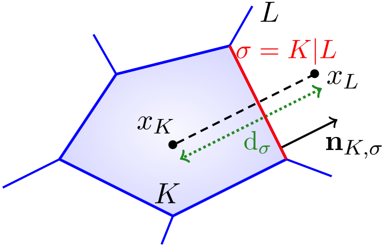
Nonlinear numerical fluxes

$$\mathcal{F}_{K,\sigma} \approx \int_{\sigma} (-\nabla u - u\nabla V) \cdot \mathbf{n}_{K,\sigma}$$



Nonlinear numerical fluxes

$$\mathcal{F}_{K,\sigma} \approx \int_\sigma -u\nabla(\log u + V)\cdot \mathbf{n}_{K,\sigma}$$



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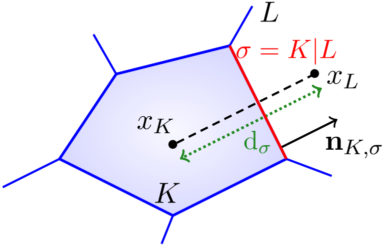

Generic form $\sigma = K|L \in \mathcal{E}^{int}$

$$\mathcal{F}_{K,\sigma} = \tau_\sigma \bar{u}_\sigma \Big(\log u_K + V_K - \log u_L - V_L\Big), \ \tau_\sigma = \frac{\mathrm{m}_\sigma}{\mathrm{d}_\sigma}$$

with $\bar{u}_\sigma = r(u_K, u_L)$

Examples of r functions

$$r(x,y) = \frac{x+y}{2}, \quad r(x,y) = \frac{x-y}{\log x - \log y}$$

Preservation of the thermal equilibrium

- At the continuous level : $u^\infty = \rho e^{-V} \Longrightarrow \mathbf{J} = 0.$
- At the discrete level : $u_K^\infty = \rho e^{-V_K} \Longrightarrow \mathcal{F}_{K,\sigma} = 0\,?$

Linear fluxes

$$\mathcal{F}_{K,\sigma} = \tau_\sigma \Big(B(V_L - V_K) u_K - B(V_K - V_L) u_L \Big),$$

with $B(0) = 1,\ B(x) > 0$ and $B(x) - B(-x) = -x \ \forall x \in \mathbb{R}$

➠ YES iff $B = B_{sg}$.

Nonlinear fluxes

$$\mathcal{F}_{K,\sigma} = \tau_\sigma \bar{u}_\sigma \Big(\log u_K + V_K - \log u_L - V_L \Big),$$

➠ YES.

Structural properties

Linear fluxes

- Linear system whose matrix is an M-matrix.
  - ➠ Existence and uniqueness of a solution to the scheme.
  - ➠ Positivity of the numerical solution.
- Mass conservation when $\Gamma^D = \emptyset$.
- Discrete entropy method
  - ➠ Uniform bounds on the discrete solution.
  - ➠ Exponential decay towards the associate steady-state.

Structural properties

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- Linear system whose matrix is an M-matrix.
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Nonlinear fluxes

- Nonlinear system of equations at each time step
- Mass conservation when $\Gamma^D = \emptyset$.
- All the properties (existence, positivity, bounds, large time) will be obtained as a consequence of the entropy method.

Outline of the talk

Nonlinear schemes for the FP equation ($\Gamma^D = \emptyset$)

$$\begin{cases} \mathrm{m}_K \dfrac{u_K^{n+1} - u_K^n}{\Delta t} + \displaystyle\sum_{\sigma \in \mathcal{E}_K^{int}} \mathcal{F}_{K,\sigma}^{n+1} = 0, \\ \mathcal{F}_{K,\sigma}^{n+1} = \tau_\sigma \underbrace{r(u_K^{n+1}, u_L^{n+1})}_{\bar{u}_\sigma^{n+1}} \Big(\log u_K^{n+1} + V_K - \log u_L^{n+1} - V_L \Big). \end{cases}$$

Hypotheses on r

- r increasing with respect to both variables
- $r(x,x) = x$ and $r(x,y) = r(y,x)$
- $r(\lambda x, \lambda y) = \lambda r(x,y)$
- $\dfrac{x-y}{\log x - \log y} \le r(x,y) \le \max(x,y)$

Entropy-entropy dissipation property

Discrete relative Φ-entropy $$E_\Phi = \int_\Omega u^\infty \Phi\left(\frac{u}{u^\infty}\right)$$

$$\mathbb{E}_\Phi^n = \sum_{K\in\mathcal{T}} \mathrm{m}_K u_K^\infty \Phi\left(\frac{u_K^n}{u_K^\infty}\right)$$

Discrete dissipation $$I_\Phi = \int_\Omega u \nabla \log \frac{u}{u^\infty} \cdot \nabla \Phi'\left(\frac{u}{u^\infty}\right)$$

$$\mathbb{I}_\Phi^n = \sum_{\sigma\in\mathcal{E}^{int}} \tau_\sigma \bar{u}_\sigma^n \left(\log \frac{u_L^n}{u_L^\infty} - \log \frac{u_K^n}{u_K^\infty}\right)\left(\Phi'\left(\frac{u_L^n}{u_L^\infty}\right) - \Phi'\left(\frac{u_K^n}{u_K^\infty}\right)\right)$$

Discrete entropy-entropy dissipation property

$$\boxed{\frac{\mathbb{E}_\Phi^{n+1} - \mathbb{E}_\Phi^n}{\Delta t} + \mathbb{I}_\Phi^{n+1} \le 0 \quad \forall n \ge 0.}$$

First consequences

- Decay of the relative Φ-entropy : $\mathbb{E}_\Phi^n \leq \mathbb{E}_\Phi^0$.
- Uniform bounds :

$$m = \min(1, \min_{K\in\mathcal{T}} \frac{u_K^0}{u_K^\infty}) \leq \frac{u_K^n}{u_K^\infty} \leq \max(1, \max_{K\in\mathcal{T}} \frac{u_K^0}{u_K^\infty}) = M$$

obtained with $\Phi(x) = (x-M)^+$ and $\Phi(x) = (x-m)^-$.

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obtained with $\Phi(x) = (x-M)^+$ and $\Phi(x) = (x-m)^-$.

- Control of the entropy dissipation :

$$\sum_{n=0}^{N} \Delta t \; \mathbb{I}_1^{n+1} \leq \mathbb{E}_1^0.$$

- Existence of a positive solution to the scheme

via a Leray-Schauder's fixed point theorem.

Towards the exponential decay : the continuous level

$$E_1 = \int_\Omega u^\infty \Phi_1(\frac{u}{u^\infty}) = \int_\Omega u \log \frac{u}{u^\infty} \qquad (\int_\Omega u = \int_\Omega u^\infty)$$

$$I_1 = \int_\Omega u \left|\nabla \log \frac{u}{u^\infty}\right|^2 = 4 \int_\Omega u^\infty \left|\nabla \sqrt{\frac{u}{u^\infty}}\right|^2$$

Log-Sobolev inequality $\quad \mu$ probability measure

$$\int_\Omega f^2 \log \frac{f^2}{\|f\|^2_{L^2(\Omega,d\mu)}} d\mu \leq C_{LS} \int_\Omega |\nabla f|^2 \, d\mu.$$

Towards the exponential decay : the continuous level

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Log-Sobolev inequality μ probability measure

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Application with $d\mu = \frac{u^\infty}{\int_\Omega u^\infty} dx$ and $f = \sqrt{\frac{u}{u^\infty}}$

$$\boxed{E_1 \leq C_{LS} I_1}$$

Towards the exponential decay : the discrete level

$$\mathbb{E}_1^n = \sum_{K\in\mathcal{T}} \mathrm{m}(K) u_K^\infty \Phi_1\left(\frac{u_K^n}{u_K^\infty}\right) = \sum_{K\in\mathcal{T}} \mathrm{m}(K) u_K^n \log \frac{u_K^n}{u_K^\infty}$$

$$\mathbb{I}_1^n = \sum_{\sigma\in\mathcal{E}^{int}} \tau_\sigma r(u_K^n, u_L^n) \left(\log \frac{u_L^n}{u_L^\infty} - \log \frac{u_K^n}{u_K^\infty}\right)^2$$

$$\widehat{\mathbb{I}}_1^n = 4 \sum_{\sigma\in\mathcal{E}^{int}} \tau_\sigma \min(u_K^\infty, u_L^\infty) \left(\sqrt{\frac{u_L^n}{u_L^\infty}} - \sqrt{\frac{u_K^n}{u_K^\infty}}\right)^2$$

Step 1 $\boxed{\widehat{\mathbb{I}}_1^n \leq \mathbb{I}_1^n}$

- $4(\sqrt{x} - \sqrt{y})^2 \leq (x-y)(\log x - \log y) \leq r(x,y)(\log x - \log y)^2$
- $\min(u_K^\infty, u_L^\infty)\, r\left(\frac{u_K^n}{u_K^\infty}, \frac{u_L^n}{u_L^\infty}\right) \leq r(u_K^n, u_L^n)$

Step 2 $\boxed{\mathbb{E}_1^n \leq C\widehat{\mathbb{I}}_1^n}$

Discrete Log-Sobolev inequality

- EYMARD, GALLOUËT, HERBIN, '00
- BESSEMOULIN-CHATARD, JÜNGEL, '14

Theorem

- $(\mathcal{T}, \mathcal{E}, \mathcal{P})$ a mesh of Ω, with a regularity $\zeta > 0$
- $(\mu_K)_{K\in\mathcal{T}}$ given with $\mu_K \geq 0$ and $\displaystyle\sum_{K\in\mathcal{T}} \mathrm{m}(K)\mu_K = 1$
- $\displaystyle\mu^\infty = \sup_{K\in\mathcal{T}} \mu_K$

Then, for all $(f_K)_{K\in\mathcal{T}}$ with $f_K > 0$, one has

$$\sum_{K\in\mathcal{T}} \mathrm{m}_K f_K^2 \log \frac{f_K^2}{\displaystyle\sum_{L\in\mathcal{T}} \mathrm{m}_L f_L^2 \mu_L} \mu_K \leq \frac{C_{LS}}{\zeta^2}\sqrt{\mu^\infty} \sum_{\sigma\in\mathcal{E}^{int}} \tau_\sigma |f_K - f_L|^2$$

Discrete Log-Sobolev inequality : application

$$\sum_{K\in\mathcal{T}} \mathrm{m}_K f_K^2 \log \frac{f_K^2}{\sum\limits_{L\in\mathcal{T}} \mathrm{m}_L f_L^2 \mu_L} \mu_K \leq \frac{C_{LS}}{\zeta^2}\sqrt{\mu^\infty} \sum_{\sigma\in\mathcal{E}^{int}} \tau_\sigma |f_K - f_L|^2$$

- $(u_K^n)_{K\in\mathcal{T}}$ and (u_K^∞) verify :

$$u_K^n > 0 \text{ and } u_K^\infty > 0 \quad \forall K \in \mathcal{T},$$
$$\sum_{K\in\mathcal{T}} \mathrm{m}_K u_K^n = \sum_{K\in\mathcal{T}} \mathrm{m}_K u_K^\infty = M^1, \quad M^\infty = \max_{K\in\mathcal{T}} u_K^\infty.$$

- With $\left(f_K = \sqrt{\dfrac{u_K^n}{u_K^\infty}}\right)_{K\in\mathcal{T}}$ and $\mu_K = \dfrac{u_K^\infty}{M^1}$, we get :

$$\boxed{\sum_{K\in\mathcal{T}} \mathrm{m}_K u_K^n \log \frac{u_K^n}{u_K^\infty} \leq \frac{C_{LS}}{\zeta^2}\sqrt{M^1 M^\infty} \sum_{\sigma\in\mathcal{E}^{int}} \tau_\sigma \left| \sqrt{\frac{u_K^n}{u_K^\infty}} - \sqrt{\frac{u_L^n}{u_L^\infty}} \right|^2}$$

Final result

Step 0

$$\frac{\mathbb{E}_1^{n+1} - \mathbb{E}_1^n}{\Delta t} + \mathbb{I}_1^{n+1} \leq 0 \quad \forall n \geq 0.$$

Step 1 $\quad \widehat{\mathbb{I}}_1^n \leq \mathbb{I}_1^n$

Step 2 $\quad \mathbb{E}_1^n \leq \dfrac{1}{\nu} \widehat{\mathbb{I}}_1^n$, so that

$$\mathbb{E}_1^n \leq (1 + \nu \Delta t)^{-n} \mathbb{E}_1^0.$$

Theorem

For any $k > 0$, if $\Delta t \leq k$, one has

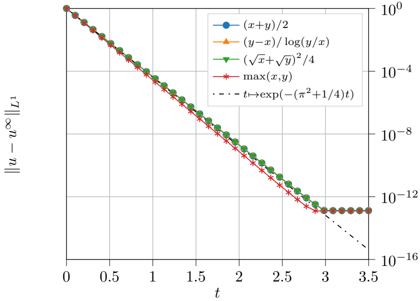
$$\mathbb{E}_1^n \leq e^{-\tilde{\nu} t^n} \mathbb{E}_1^0 \quad \forall n \geq 0,$$

with $\tilde{\nu} = \log(1 + \nu k)/k$.

Conclusion

TPFA schemes

- Exponential decay of $\mathbb{E}_p^n$ for $p \in [1, 2]$ if $\Gamma^D = \emptyset$.
- Exponential decay of $\mathbb{E}_p^n$ for $p \in (1, 2]$ if $\Gamma^D \neq \emptyset$.



- What happens for a general velocity field ? for a general steady-state ?

Conclusion

DDFV schemes (anisotropic case + general meshes)

- Discrete Log-Sobolev inequality adapted to the DDFV reconstruction.
- Exponential decay of the discrete Boltzmann entropy.