Asymptotic behaviour of solutions to abstract wave equations with damping

Tomáš Bárta

Charles University, Prague

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Goal of the talk

Introduce a method that allows to

- prove convergence to an equilibrium
- estimate the rate of convergence

for various evolutionary problems.

The method is based on

- existence of a Lyapunov function and
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Definition

Let V be a Banach space. We say that a C^1 function $E : V \to \mathbb{R}$ satisfies the Łojasiewicz inequality on a neighborhood B of $\varphi \in V$, if $\exists\, \theta \in (0, \frac{1}{2}]$, $C > 0$ s.t. $\forall u \in B$

$$|E(u) - E(\varphi)|^{1-\theta} \le C\|E'(u)\|_{V'}. \qquad \text{(LI)}$$

- θ ... Łojasiewicz exponent (important constant)
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- interesting case: $E'(\varphi) = 0$

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- $E(u)$ quadratic form $\ldots \theta = \frac{1}{2}$.

$\ldots \sim E(u)$ not too flat around φ

- $E(u_1, u_2) = |u_1|^p \ldots \theta = \frac{1}{p}$.
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Assumptions

- $u : [0, +\infty) \to V$ is a global solution to our differential equation
- u has relatively compact range, i.e. $\{u(t) : t \geq 0\}$ is relatively compact
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Gradient systems in $\mathbb{R}^n$

$$\dot{u} + \nabla E(u) = 0 \tag{GS}$$

Theorem (Łojasiewicz '63; Haraux, Jendoubi '01)

If E satisfies (LI) with θ, then $\lim_{t\to+\infty} u(t) = \varphi$.

Moreover,
$$\|u(t)-\varphi\| \le \begin{cases} Ce^{-ct} & \text{if } \theta = \frac{1}{2}, \\ C(t+1)^{-\frac{\theta}{1-2\theta}} & \text{if } \theta < \frac{1}{2}. \end{cases}$$

Counterexample (Palis, De Mello '82) if E is exponentially flat

. . . not a unique limit point

Gradient-like systems in $\mathbb{R}^n$

$$\dot{u} + F(u) = 0 \tag{GLS}$$

gradient-like system = system with a strict Lyapunov function

$$\exists E : \mathbb{R}^n \to \mathbb{R} \qquad F(u) \neq 0 \quad \Rightarrow \quad \langle F(u), E'(u) \rangle > 0.$$

Theorem (Lageman '97; Haraux, Jendoubi '01)

If E satisfies (LI) *and* (AC), *then* $\lim_{t \to +\infty} u(t) = \varphi$.

Moreover, if (C) $$\|u(t) - \varphi\| \leq \begin{cases} Ce^{-ct} & \text{if } \theta = \frac{1}{2}, \\ C(t+1)^{-\frac{\theta}{1-2\theta}} & \text{if } \theta < \frac{1}{2}. \end{cases}$$

$$\langle F(u), E'(u) \rangle \geq \alpha \|F(u)\| \|E'(u)\| \tag{AC}$$

$$\|F(u)\| \geq C \|E'(u)\| \tag{C}$$

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$$-\frac{d}{dt}\mathcal{E}(u(t)) = -\mathcal{E}'(u(t))\,u'(t) = \mathcal{E}'(u(t))\,F(u(t))$$

WLOG $\mathcal{E}(\varphi)=0$

$$\geq \alpha\,\|\mathcal{E}'(u(t))\|\cdot\|F(u(t))\| \quad \text{(AC)}$$

(L1)

$$\geq \alpha\cdot c\;\mathcal{E}(u(t))^{1-\theta}\,\|u'(t)\| \qquad /:\mathcal{E}^{1-\theta}$$

$$\|u'(t)\| \leq \frac{-1}{\alpha c}\,\frac{d}{dt}\,\mathcal{E}(u(t))^{\theta} \qquad /\int$$

$$\Rightarrow u'\in L^1 \Rightarrow \lim_{t\to\infty} u(t) \text{ EXISTS}$$

$$\|u(t)-\varphi\| \leq \int_t^\infty \|u'(s)\|\,ds \leq -\frac{1}{\alpha c}\left(\mathcal{E}(\varphi)^\theta - \mathcal{E}(u(t))^\theta\right) = \frac{1}{\alpha c}\,\mathcal{E}(u(t))^\theta$$

(C)

$$\geq \alpha C\,\|\mathcal{E}'(u(t))\|^2 \overset{(L1)}{\geq} \alpha\tilde{C}\;\mathcal{E}(u(t))^{2-2\theta}$$

$$-\frac{d}{dt}\left(\mathcal{E}(u(t))\right)^{2\theta-1} \geq \alpha\tilde{C}$$

$$\mathcal{E}(u(t))^{2\theta-1} \geq C(t+1)$$

$$\mathcal{E}(u(t)) \leq C(t+1)^{\frac{-1}{1-2\theta}}$$

$$\|u(t)-\varphi\| \leq C(t+1)^{-\frac{\theta}{1-2\theta}}$$

Abstract wave equation

$$\ddot{u} + g(\dot{u}) + E'(u) = 0 \qquad \text{(AWE)}$$

- $V \hookrightarrow H \hookrightarrow V'$ Hilbert spaces
- $E \in C^1(V, \mathbb{R})$, Łojasiewicz inequality ... potential,
- $g : V \to V'$, $\langle g(v), v\rangle_{V',V} \geq 0$ ... damping

Example 1.

$$u_{tt} + g(u_t) - \Delta u + f(u, x) = 0,$$

- $H = L^2(\Omega)$, $V = H_0^1(\Omega)$,
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Example 2.

$$\ddot{u} + \alpha\dot{u} + u = 0,$$

- $H = V = \mathbb{R}^n$,
- $E(u) = \frac{1}{2}u^2$.

Lyapunov function

$$\ddot{u} + g(\dot{u}) + E'(u) = 0 \tag{AWE}$$

$$\mathcal{E}(u,v) = \frac{1}{2}\|v\|^2 + E(u) \quad \text{is a Lyapunov function}$$

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(u(t),v(t)) &= E'(u)v + \langle v, \dot{v}\rangle \\ &= E'(u)v + \langle v, -E'(u) - g(\dot{u})\rangle \\ &= -\langle g(v), v\rangle \\ &\leq 0\end{aligned}$$

(AWE) with linear damping

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$$\mathcal{E}(u,v) = \frac{1}{2}\|v\|^2 + E(u) + \varepsilon\|E'(u)\|\|v\|$$

is a strict Lyapunov function satisfying (AC)

Theorem (Haraux, Jendoubi '01)

If E satisfies (LI), then $\lim_{t\to+\infty} u(t) = \varphi$.

Moreover,

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smaller than linear damping = no angle condition

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small damping $\Rightarrow$ no angle condition $\Rightarrow$ worse decay estimates

One needs a better Lyapunov function

$$\mathcal{E}(u,v) = E(u) + \frac{1}{2}\|v\|^2 + \varepsilon G(E'(u), v)$$

Chergui '08 in $\mathbb{R}^n$ If E satisfies (LI) with θ, $g(v) = \|v\|^\alpha v$, $\alpha < \frac{\theta}{1-\theta}$, then

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Further results for (AWE)

Ben Hassen, Haraux '13: similar result in Hilbert spaces, $g(v) \sim \|v\|^\alpha v$.

B. and Fašangová '16, B. '16 the same estimate for more general g

- $g(v) \geq C$ for $\|v\| \geq K$ and
- $g(v) \geq c\|v\|^\alpha v$ for $\|v\| \leq \varepsilon$
- or $\langle g(v), v\rangle \geq c\|v\|^{\alpha+2}$ for $\|v\| \leq \varepsilon$
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Integrodifferential equations

Zacher '09: convergence to equilibrium for an IDE in $\mathbb{R}^n$

Yassine '17: decay for exponentially decaying kernels for

$$u_{tt} - \Delta u + f(x,u) + \int_0^t k(s)\Delta u(t-s)ds = g$$

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OPTIMAL if

- (AC) holds and
- "the reverse inequality in (LI) holds"

OPTIMAL for (AWE) with linear damping

NOT OPTIMAL for "small damping"

Haraux '11: non-optimality for $\ddot{u} + |\dot{u}|^\alpha \dot{u} + |u|^\beta u = 0$

For $E(u_1, u_2) = |u_1|^p$ we have $\|E'\| \sim E^{1-\frac{1}{p}}$, OK

For $E(u_1, u_2) = |u_1|^p + |u_2|^q$, $p > q$ we have

$$CE(u)^{1-\frac{1}{q}} \geq \|E'(u)\| \geq cE(u)^{1-\frac{1}{p}}$$

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$$CE(u)^{1-\frac{1}{q}} \geq \|E'(u)\| \geq cE(u)^{1-\frac{1}{p}}$$

Optimality of decay estimates

OPTIMAL if

- (AC) holds and
- "the reverse inequality in (LI) holds"

OPTIMAL for (AWE) with linear damping

NOT OPTIMAL for "small damping"

Haraux '11: non-optimality for $\ddot{u} + |\dot{u}|^\alpha \dot{u} + |u|^\beta u = 0$

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Fast and slow solutions

$$\ddot{u} + \alpha\dot{u} + \beta u = 0, \quad \alpha,\ \beta > 0 \qquad u = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

- either complex roots with the same real part
  oscilations, all solutions have the same speed of convergence
- or real roots with different real parts
  no oscilations, two speeds of convergence

Haraux & co. '11, '14, '17

$$\ddot{u} + |\dot{u}|^\alpha \dot{u} + |u|^\beta u = 0, \qquad \alpha,\ \beta > 0$$

SIMILAR SITUATION

B. 2018

$$E(u_1, u_2, \ldots, u_n) = |u_1|^{p_1} + \cdots + |u_n|^{p_n}$$

up to n speeds of convergence.

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THANK YOU FOR YOUR ATTENTION!