

Each step must be **carefully** justified. If you use some lemma or theorem do **not** forget to check that all assumptions are satisfied.

Given name and family name: _____

Question	1	2	Score
Maximum points	100	100	200
Points			

- [100] 1. Decide which of the following statements is correct. Prove the correct statement. Show the counterexample for the incorrect statement.

- 1) Let $\Omega := \mathbb{R}^3 \setminus \overline{B_1(0)}$. Then $W^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$ for all $p \in [1, 6]$.
- 2) Let $\Omega \subset \mathbb{R}^3$ be an open bounded set. Then $W_0^{1,4}(\Omega) \hookrightarrow \hookrightarrow C^{0,\alpha}(\overline{\Omega})$ for all $\alpha \in [0, \frac{1}{4})$.

Solution:

The correct statement is 2).

To prove 2), we first find $R > 0$ such that $\Omega \subset B_R(0)$ - this is possible since Ω is bounded. Because

$$W_0^{1,4}(\Omega) := \overline{\mathcal{C}_0^\infty(\Omega)}^{\|\cdot\|_{1,4}},$$

we can trivially arbitrary $u \in W_0^{1,4}(\Omega)$ by zero outside Ω so that $u \in W^{1,4}(B_R)$. The ball B_R is the set with Lipschitz boundary and for the rest of the proof as statement 2), we can use the lecture.

Statement 1) is incorrect because Ω is **unbounded**. Define $f(x) := |x|^{-3}$ (Recall $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$). Function f is smooth in Ω and we also have $|\nabla f| \leq C|x|^{-4}$. We also have

$$\int_{\Omega} |f|^2 + |\nabla f|^2 \leq C \int_{\Omega} |x|^{-6} + |x|^{-8} < \infty,$$

so $f \in W^{1,2}(\Omega)$. However,

$$\int_{\Omega} |f| = \int_{\Omega} |x|^{-3} = \infty,$$

so $f \notin L^1(\Omega)$.

- [100] 2. Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz set. Consider the problem: for given $p, q \in (1, \infty)$ and given $\mathbf{f} = (f_1, \dots, f_L) : \Omega \rightarrow \mathbb{R}^L$, where $L \in \mathbb{N}$, find $\mathbf{u} = (u_1, \dots, u_L) : \Omega \rightarrow \mathbb{R}^L$ solving for all $\nu = 1, \dots, L$

$$-\sum_{i=1}^d \frac{\partial}{\partial x_i} \left(|\nabla \mathbf{u}|^{p-2} \frac{\partial u_\nu}{\partial x_i} \right) + |\mathbf{u}|^{q-2} u_\nu = f_\nu \quad \text{in } \Omega,$$

$$\sum_{i=1}^d |\nabla \mathbf{u}|^{p-2} \frac{\partial u_\nu}{\partial x_i} n_i + u_\nu = g_\nu \quad \text{on } \partial\Omega,$$

where

$$|\nabla \mathbf{u}|^2 := \sum_{i=1}^d \sum_{\nu=1}^L \left| \frac{\partial u_\nu}{\partial x_i} \right|^2,$$

$$|\mathbf{u}|^2 := \sum_{\nu=1}^L (u_\nu)^2$$

and $\mathbf{n} = (n_1, \dots, n_d)$ denotes the outward unit normal vector on $\partial\Omega$.

For $\mathbf{f} \in L^{q'}(\Omega; \mathbb{R}^L)$ and $\mathbf{g} \in L^2(\partial\Omega; \mathbb{R}^L)$ define the notion of a weak solution. Check that for given data such a definition is meaningful. Prove the existence and uniqueness of the weak solution. (*Help: As a leading function space consider $V := W_0^{1,p}(\Omega; \mathbb{R}^L) \cap L^q(\Omega; \mathbb{R}^L) \cap L^2(\partial\Omega; \mathbb{R}^L)$*)

Solution:

First, we recall some facts proved in the lectures. The mapping $A : \mathbb{R}^{d \times L} \rightarrow \mathbb{R}^{d \times L}$ defined as

$$A(\eta) := |\eta|^{p-2} \eta,$$

is strictly monotone p -coercive and has $(p-1)$ -growth. Moreover, the mapping $F_1 : \mathbb{R}^{d \times L} \rightarrow \mathbb{R}$ defined as

$$F_1(\eta) := \frac{|\eta|^p}{p},$$

is strictly convex, p -coercive and has p -growth and fulfills

$$\frac{\partial F_1(\eta)}{\partial \eta} = A(\eta).$$

Similarly, the mapping $B : \mathbb{R}^L \rightarrow \mathbb{R}^L$ defined as

$$B(\mathbf{u}) := |\mathbf{u}|^{q-2} \mathbf{u},$$

is strictly monotone, q -coercive and has $(q-1)$ -growth. Moreover, the mapping $F_2 : \mathbb{R}^L \rightarrow \mathbb{R}$ defined as

$$F_2(\mathbf{u}) := \frac{|\mathbf{u}|^q}{q},$$

is strictly convex, q -coercive, has q -growth and fulfills

$$\frac{\partial F_2(\mathbf{u})}{\partial \mathbf{u}} = B(\mathbf{u}).$$

The problem can be written in the form

$$\begin{aligned} -\operatorname{div} A(\nabla \mathbf{u}) + B(\mathbf{u}) &= \mathbf{f} && \text{in } \Omega, \\ \mathbf{u} &= 0 && \text{on } \partial\Omega. \end{aligned} \quad (*)$$

Formal derivation of the notion of weak solution: Let $\mathbf{v} \in V$ be arbitrary and take the scalar product of (*) and \mathbf{v} , i.e., multiply the ν -th equation in (*) by v_ν and sum the result over $\nu = 1, \dots, L$. Finally integrate over Ω to get

$$-\int_{\Omega} \operatorname{div} A(\nabla \mathbf{u}) \cdot \mathbf{v} \, dx + \int_{\Omega} B(\mathbf{u}) \cdot \mathbf{v} \, dx = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, dx.$$

For the first integral use integration by parts and the fact that $\mathbf{v} = 0$ on $\partial\Omega$ (so the boundary term formally vanishes) to get the weak formulation

$$\int_{\Omega} A(\nabla \mathbf{u}) \cdot \nabla \mathbf{v} \, dx + \int_{\Omega} B(\mathbf{u}) \cdot \mathbf{v} \, dx = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, dx \quad \text{for all } \mathbf{v} \in V. \quad (**)$$

We check that all integrals are well defined for $\mathbf{u}, \mathbf{v} \in V$. Using the Hölder inequality we have

$$\begin{aligned} \left| \int_{\Omega} A(\nabla \mathbf{u}) \cdot \nabla \mathbf{v} \, dx \right| &\leq \int_{\Omega} |A(\nabla \mathbf{u})| |\nabla \mathbf{v}| \, dx \leq \int_{\Omega} |\nabla \mathbf{u}|^{p-1} |\nabla \mathbf{v}| \, dx \leq \|\mathbf{u}\|_p^{p-1} \|\nabla \mathbf{v}\|_p < \infty, \\ \left| \int_{\Omega} B(\mathbf{u}) \cdot \mathbf{v} \, dx \right| &\leq \int_{\Omega} |B(\mathbf{u})| |\mathbf{v}| \, dx \leq \int_{\Omega} |\mathbf{u}|^{q-1} |\mathbf{v}| \, dx \leq \|\mathbf{u}\|_q^{q-1} \|\mathbf{v}\|_q < \infty, \\ \left| \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, dx \right| &\leq \int_{\Omega} |\mathbf{f}| |\mathbf{v}| \, dx \leq \|\mathbf{f}\|_{q'} \|\mathbf{v}\|_q < \infty \end{aligned}$$

so (**) is meaningful.

Uniqueness of the weak solution: Let $\mathbf{u}_1, \mathbf{u}_2 \in V$ be two weak solutions to (**). Subtracting the weak formulation for \mathbf{u}_2 from that one for \mathbf{u}_1 we deduce that

$$\int_{\Omega} (A(\nabla \mathbf{u}_1) - A(\nabla \mathbf{u}_2)) \cdot \nabla \mathbf{v} \, dx + \int_{\Omega} (B(\mathbf{u}_1) - B(\mathbf{u}_2)) \cdot \mathbf{v} \, dx = 0 \quad \text{for all } \mathbf{v} \in V.$$

Hence, setting $\mathbf{v} := \mathbf{u}_1 - \mathbf{u}_2$ (which is a possible choice since $\mathbf{u}_1 - \mathbf{u}_2 \in V$), we have

$$\int_{\Omega} (A(\nabla \mathbf{u}_1) - A(\nabla \mathbf{u}_2)) \cdot (\nabla \mathbf{u}_1 - \nabla \mathbf{u}_2) \, dx + \int_{\Omega} (B(\mathbf{u}_1) - B(\mathbf{u}_2)) \cdot (\mathbf{u}_1 - \mathbf{u}_2) \, dx = 0.$$

Using the monotonicity of A and of B , it directly follows that

$$\begin{aligned} (A(\nabla \mathbf{u}_1) - A(\nabla \mathbf{u}_2)) \cdot (\nabla \mathbf{u}_1 - \nabla \mathbf{u}_2) &\equiv 0 && \text{a.e. in } \Omega, \\ (B(\mathbf{u}_1) - B(\mathbf{u}_2)) \cdot (\mathbf{u}_1 - \mathbf{u}_2) &\equiv 0 && \text{a.e. in } \Omega. \end{aligned}$$

Consequently, using the strict monotonicity of B we get that $\mathbf{u}_1 = \mathbf{u}_2$ almost everywhere in Ω .

Existence of a weak solution via variational approach: Look for $\mathbf{u} \in V$ solving

$$\int_{\Omega} F_1(\nabla \mathbf{u}) + F_2(\mathbf{u}) - \mathbf{f} \cdot \mathbf{u} \, dx \leq \int_{\Omega} F_1(\nabla \mathbf{w}) + F_2(\mathbf{w}) - \mathbf{f} \cdot \mathbf{w} \, dx, \quad \text{for all } \mathbf{w} \in V. \quad (***)$$

First, assume that such \mathbf{u} exists. Then setting $\mathbf{w} := \mathbf{u} + t\mathbf{v}$ with $t > 0$ we have

$$0 \leq \int_{\Omega} \frac{F_1(\nabla \mathbf{u} + t\nabla \mathbf{v}) - F_1(\nabla \mathbf{u})}{t} + \frac{F_2(\mathbf{u} + t\mathbf{v}) - F_2(\mathbf{u})}{t} - \mathbf{f} \cdot \mathbf{v} \, dx, \text{ for all } \mathbf{v} \in V.$$

Thus, letting $t \rightarrow 0_+$, using the properties of F_1 , F_2 , A and B and using the results proved in the lectures we get

$$0 \leq \int_{\Omega} A(\nabla \mathbf{u}) \cdot \nabla \mathbf{v} + B(\mathbf{u}) \cdot \mathbf{v} - \mathbf{f} \cdot \mathbf{v} \, dx, \text{ for all } \mathbf{v} \in V.$$

Since we can set also $-\mathbf{v}$ above, we directly have (**).

To prove the existence of \mathbf{u} solving (**), we first define

$$I := \inf_{\mathbf{w} \in V} \int_{\Omega} F_1(\nabla \mathbf{w}) + F_2(\mathbf{w}) - \mathbf{f} \cdot \mathbf{w} \, dx \leq 0,$$

where the last inequality follows from the fact that $0 \in V$. From the definition of \inf there must exist a sequence $\{\mathbf{u}^n\}_{n=1}^{\infty}$ such that

$$I = \lim_{n \rightarrow \infty} \int_{\Omega} F_1(\nabla \mathbf{u}^n) + F_2(\mathbf{u}^n) - \mathbf{f} \cdot \mathbf{u}^n \, dx \leq 0.$$

Consequently, there must exist n_0 such that for all $n \geq n_0$

$$\int_{\Omega} F_1(\nabla \mathbf{u}^n) + F_2(\mathbf{u}^n) - \mathbf{f} \cdot \mathbf{u}^n \, dx \leq 1,$$

from which it follows that

$$\frac{\|\nabla \mathbf{u}^n\|_p^p}{p} + \frac{\|\nabla \mathbf{u}^n\|_q^q}{q} \leq 1 + \int_{\Omega} \mathbf{f} \cdot \mathbf{u}^n \, dx \leq 1 + \|\mathbf{f}\|_{q'} \|\mathbf{u}^n\|_q \leq \frac{\|\mathbf{u}^n\|_q^q}{2q} + 1 + 2^{q'} \|\mathbf{f}\|_{q'}^{q'}.$$

Finally, using the Poincaré inequality (note that \mathbf{u} has zero trace) we get that

$$\|\mathbf{u}\|_V \leq C(\Omega, \mathbf{f}).$$

Due to the reflexivity, we can extract a non-relabeled subsequence and find $\mathbf{u} \in V$ such that

$$\mathbf{u}^n \rightharpoonup \mathbf{u} \text{ weakly in } V.$$

In particular, we have that

$$\begin{aligned} \nabla \mathbf{u}^n &\rightharpoonup \nabla \mathbf{u} \quad \text{weakly in } L^p(\Omega; \mathbb{R}^{d \times L}), \\ \mathbf{u} &\rightharpoonup \mathbf{u} \quad \text{weakly in } L^q(\Omega; \mathbb{R}^L). \end{aligned}$$

Finally, we use the weak-lower semicontinuity of convex functionals (proved in the lectures and here applied to F_1 since it has p -growth and A has $(p-1)$ -growth and to F_2 since it has q growth and B has $(q-1)$ growth) to get

$$I = \lim_{n \rightarrow \infty} \int_{\Omega} F_1(\nabla \mathbf{u}^n) + F_2(\mathbf{u}^n) - \mathbf{f} \cdot \mathbf{u}^n \, dx \geq \int_{\Omega} F_1(\nabla \mathbf{u}) + F_2(\mathbf{u}) - \mathbf{f} \cdot \mathbf{u} \, dx \geq I,$$

where the last inequality follows from the definition of I . Hence, we see that \mathbf{u} satisfies (***)).

Existence via monotone operator theory: Since V is a separable space there exists a countable set $\{\mathbf{w}_i\}_{i=1}^\infty$ which is dense in V . We look for the Galerkin approximation \mathbf{u}^n being of the form

$$\mathbf{u}^n := \sum_{i=1}^n c_i^n \mathbf{w}_i$$

and solving

$$\int_{\Omega} A(\nabla \mathbf{u}^n) \cdot \nabla \mathbf{w}_i \, dx + \int_{\Omega} B(\mathbf{u}^n) \cdot \mathbf{w}_i \, dx = \int_{\Omega} \mathbf{f} \cdot \mathbf{w}_i \, dx \quad \text{for all } i = 1, \dots, n. \quad (**_n)$$

Due to the continuity of A and B and their coercivity, such \mathbf{u}^n exists (see the lecture). Then multiplying the i -th equation in $(**_n)$ by c_i^n and summing the result over $i = 1, \dots, n$ we get

$$\int_{\Omega} A(\nabla \mathbf{u}^n) \cdot \nabla \mathbf{u}^n \, dx + \int_{\Omega} B(\mathbf{u}^n) \cdot \mathbf{u}^n \, dx = \int_{\Omega} \mathbf{f} \cdot \mathbf{u}^n \, dx. \quad (\text{En})$$

From this we get

$$\|\mathbf{u}^n\|_V \leq C$$

and consequently, using the $(p-1)$ -growth of A and $(q-1)$ -growth of B also that

$$\|A(\nabla \mathbf{u}^n)\|_{L^{p'}(\Omega; \mathbb{R}^{d \times L})} + \|B(\mathbf{u}^n)\|_{L^{q'}(\Omega; \mathbb{R}^L)} \leq C$$

and due to the reflexivity of Lebesgue spaces (here we use that $p, q \in (1, \infty)$) we have

$$\begin{aligned} \mathbf{u}^n &\rightharpoonup \mathbf{u} \quad \text{weakly in } V, \\ A(\nabla \mathbf{u}^n) &\rightharpoonup \bar{A} \quad \text{weakly in } L^{p'}(\Omega; \mathbb{R}^{d \times L}), \\ B(\mathbf{u}^n) &\rightharpoonup \bar{B} \quad \text{weakly in } L^{q'}(\Omega; \mathbb{R}^L). \end{aligned}$$

Then for fix i it is easy to let $n \rightarrow \infty$ in $(**_n)$ to observe

$$\int_{\Omega} \bar{A} \cdot \nabla \mathbf{w}_i \, dx + \int_{\Omega} \bar{B} \cdot \mathbf{w}_i \, dx = \int_{\Omega} \mathbf{f} \cdot \mathbf{w}_i \, dx \quad \text{for all } i = 1, \dots, \infty.$$

and due to the density of $\{\mathbf{w}_i\}$ in V also that

$$\int_{\Omega} \bar{A} \cdot \nabla \mathbf{w} \, dx + \int_{\Omega} \bar{B} \cdot \mathbf{w} \, dx = \int_{\Omega} \mathbf{f} \cdot \mathbf{w} \, dx \quad \text{for all } \mathbf{w} \in V. \quad (****)$$

Hence to prove (**) it remains to show that the left hand side of (****) is equal to the left hand side of (**). To prove it, we first set $\mathbf{w} := \mathbf{u}$ in (****) to get

$$\int_{\Omega} \bar{A} \cdot \nabla \mathbf{u} \, dx + \int_{\Omega} \bar{B} \cdot \mathbf{u} \, dx = \int_{\Omega} \mathbf{f} \cdot \mathbf{u} \, dx. \quad (\text{E})$$

Then taking the limit in (En) and using (E) we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{\Omega} A(\nabla \mathbf{u}^n) \cdot \nabla \mathbf{u}^n \, dx + \int_{\Omega} B(\mathbf{u}^n) \cdot \mathbf{u}^n \, dx &= \lim_{n \rightarrow \infty} \int_{\Omega} \mathbf{f} \cdot \mathbf{u}^n \, dx \\ &= \int_{\Omega} \mathbf{f} \cdot \mathbf{u} \, dx = \int_{\Omega} \bar{A} \cdot \nabla \mathbf{u} \, dx + \int_{\Omega} \bar{B} \cdot \mathbf{u} \, dx. \end{aligned} \quad (\text{EE})$$

Finally, for any $\mathbf{w} \in V$, we can use the monotonicity of the operators, (EE) and weak convergence results established above so that we have

$$\begin{aligned}
0 &\leq \lim_{n \rightarrow \infty} \int_{\Omega} (A(\nabla \mathbf{u}^n) - A(\nabla \mathbf{w})) \cdot (\nabla \mathbf{u}^n - \nabla \mathbf{w}) + (B(\mathbf{u}^n) - B(\mathbf{w})) \cdot (\mathbf{u}^n - \mathbf{w}) \, dx \\
&= \lim_{n \rightarrow \infty} \int_{\Omega} A(\nabla \mathbf{u}^n) \cdot \nabla \mathbf{u}^n \, dx + \int_{\Omega} B(\mathbf{u}^n) \cdot \mathbf{u}^n \, dx \\
&\quad - \lim_{n \rightarrow \infty} \int_{\Omega} A(\nabla \mathbf{w}) \cdot (\nabla \mathbf{u}^n - \nabla \mathbf{w}) + B(\mathbf{w}) \cdot (\mathbf{u}^n - \mathbf{w}) \, dx \\
&\quad - \lim_{n \rightarrow \infty} \int_{\Omega} A(\nabla \mathbf{u}^n) \cdot \nabla \mathbf{w} + B(\mathbf{u}^n) \mathbf{w} \, dx \\
&= \int_{\Omega} \bar{A} \cdot \nabla \mathbf{u} \, dx + \int_{\Omega} \bar{B} \cdot \mathbf{u} \, dx \\
&\quad - \int_{\Omega} A(\nabla \mathbf{w}) \cdot (\nabla \mathbf{u} - \nabla \mathbf{w}) + B(\mathbf{w}) \cdot (\mathbf{u} - \mathbf{w}) \, dx \\
&\quad - \int_{\Omega} A(\nabla \mathbf{u}) \cdot \nabla \mathbf{w} + B(\mathbf{u}) \mathbf{w} \, dx \\
&= \int_{\Omega} (\bar{A} - A(\nabla \mathbf{w})) \cdot (\nabla \mathbf{u} - \nabla \mathbf{w}) + (\bar{B} - B(\mathbf{w})) \cdot (\mathbf{u} - \mathbf{w}) \, dx.
\end{aligned}$$

The Minty trick then finishes the proof.