Uniqueness for wave equation

Q: let u be a weak solution to

$$(WE) \quad \partial_{tt} u - \Delta u = 0 \quad \text{in } (0,T) \times \Omega$$
$$u(0) = \partial_t u(0) = 0 \quad \text{in } \Omega$$

Is $u \equiv 0$ in $(0,T) \times \Omega$?

Weak solution: $u \in L^2(0,T; H_0^{1/2}(\Omega)) \cap W^{1/2}(0,T; L^2(\Omega)) \cap W^{2,2}(0,T; (H_0^{1/2}(\Omega))^*)$

$\partial_t u(0) = u(0) = 0$ and for a.a. $t \in (0,T)$ and $\forall w \in H_0^{1/2}(\Omega)$

$$\boxed{\langle \partial_{tt} u, w \rangle + \int_\Omega \nabla u \cdot \nabla w = 0}$$

WRONG PROCEDURE:

Set $w := \partial_t u$ in $\square$

MISTAKE !!!

in $\square$ I me cen use

$w \in H_0^{1/2}(\Omega)$

But $\partial_t u \notin H_0^{1/2}(\Omega)$!!!

$$\langle \partial_{tt} u, \partial_t u \rangle + \int_\Omega \nabla u \cdot \nabla \partial_t u = 0$$

$$\frac{1}{2} \frac{d}{dt} \| \partial_t u \|_2^2 + \frac{1}{2} \frac{d}{dt} \| \nabla u \|_2^2 = 0$$

$$\int_0^t \Rightarrow \quad \| \partial_t u(t) \|_2^2 + \| \nabla u(t) \|_2^2 = \| \partial_t u(0) \|_2^2 + \| \nabla u(0) \|_2^2 = 0$$

$$\Rightarrow \quad u = 0 \quad \text{in } (0,T) \times \Omega$$

CORRECT PROCEDURE:

Fix $s \in (0,T)$ arbitrary. Set $w(t) := \begin{cases} \int_t^s u(\tau)\, d\tau & \text{if } t \le s \\ 0 & \text{if } s < t \le T \end{cases}$

Since w is defined as a Bochner integral, we get that $w \in L^2(0,T; H_0^{1/2}(\Omega))$ and so for a.a. $t \in (0,T)$ $w(t) \in H_0^{1/2}(\Omega)$, so it can be used as a test function in $\square$

$$\Rightarrow \quad \int_0^T \langle \partial_{tt} u, w \rangle \overset{②}{} + \int_0^T \int_\Omega \nabla u \cdot \nabla w \overset{①}{} = 0$$

① $w(t) = 0$ for $t \ge s$ $=$

$$\int_0^s \int_\Omega \nabla u \cdot \nabla w = \int_0^s \int_\Omega \nabla u(t) \cdot 0 \left(\int_t^s \nabla u(\tau)\, d\tau \right) dt$$

$$= -\int_0^s \int_\Omega \frac{\partial}{\partial t} \left(\int_t^s \nabla u(\tau)\, d\tau \right) \cdot \left(\int_t^s \nabla u(\tau)\, d\tau \right) dt$$

$$= -\frac{1}{2} \int_0^s \int_\Omega \frac{\partial}{\partial t} \left| \underbrace{\int_t^s \nabla u(\tau)\, d\tau}_{= \nabla w} \right|^2 dt = -\frac{1}{2} \left(\| \nabla w(s) \|_2^2 - \| \nabla w(0) \|_2^2 \right) = \frac{1}{2} \| \nabla w(0) \|_2^2$$

CORRECT DUALITY + INTEGRATION BY PART

$$(2) = \int_0^s \langle \partial_{tt} u_1 \left(\int_t^s u(\tau) d\tau \right) \rangle \, dt = - \int_0^s \langle \partial_t u_1 , \partial_t \left(\int_t^s u(\tau) d\tau \right) \rangle \, dt$$

$$+ \left[\int_\Omega \partial_t u \cdot \underbrace{\left(\int_t^s u(\tau) d\tau \right)}_{w(t)} \right]_0^s$$

$\partial_t u \in L^2$

$$\uparrow = - \int_0^s \int_\Omega \partial_t u(t) \cdot \underbrace{\left(\partial_t \int_t^s u(\tau) d\tau \right)}_{= -u(t)} dt + \int_\Omega \partial_t u(s) \underbrace{w(s)}_{=0} - \partial_t u(0) \underbrace{w(0)}_{=0}$$

$$= \int_0^s \int_\Omega \partial_t (u(t)) \cdot u(t) = \frac{1}{2} \|u(s)\|_2^2 - \frac{1}{2} \|u(0)\|_2^2 = \frac{1}{2} \|u(s)\|_2^2$$

$$0 = (1) + (2) = \frac{1}{2} \|\nabla w(0)\|_2^2 + \frac{1}{2} \|u(s)\|_2^2 \overset{\text{since arbitrary}}{\Rightarrow} u \equiv 0 \text{ in } (0,T) \times \Omega$$