PARTIAL DIFFERENTIAL EQUATIONS 1

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Chapter 0. You should know

a) Function spaces, $\Omega \subseteq \mathbb{R}^d$ open

$L^p(\Omega) := \{u : \Omega \to \mathbb{R},\ u \text{ is measurable},\ \int_\Omega |u|^p dx < \infty\}$

$\|u\|_{L^p(\Omega)} = \|u\|_p = \left(\int_\Omega |u|^p dx\right)^{1/p}$

Hölder inequality: if $u \in L^p(\Omega)$, $v \in L^{p'}(\Omega)$ where $p' = \frac{p}{p-1}$

then $\int_\Omega |u||v| \le \left(\int_\Omega |u|^p\right)^{1/p} \left(\int_\Omega |v|^{p'}\right)^{1/p'}$

$L^p_{loc}(\Omega) := \{u : \Omega \to \mathbb{R},\ \forall K \subset \Omega \text{ compact} : u \in L^p(K)\}$

$C^k(\Omega) := \{u : \Omega \to \mathbb{R},\ \frac{\partial^k u}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}} \text{ is continuous in } \Omega\}$

$C^k(\overline{\Omega}) := \{\ \ - \text{"} -\ \ \overline{\Omega}\}$

$C^{0,\alpha}(\overline{\Omega}) := \{u \in C(\overline{\Omega}),\ \forall x \neq y \in \overline{\Omega} : \frac{|u(x) - u(y)|}{|x-y|^\alpha} \le L\}$ $\alpha \in \langle 0,1 \rangle$, Hölder space

$C^{0,1}(\overline{\Omega})$ = Lipschitz continuous functions

$C^{0,0}(\overline{\Omega}) = C(\overline{\Omega})$

$C^{k,\alpha}(\overline{\Omega}) := \{u \in C^k(\overline{\Omega}),$ all derivatives of k-th order are α-Hölder continuous$\}$

$u \in C^{0,1} \overset{\text{Rademacher theorem}}{\Longrightarrow}$ it has $\frac{\partial u}{\partial x_1}(x)$ for a.e. x in Ω $\frac{\partial u}{\partial x_i} \in L^\infty(\Omega)$

Lipschitz: $u = |x|$

$u = \sqrt{|x|} \in C^{0,1/2}$

Functional analysis

X - **Banach space** = complete linear normed space

$u^n \to u$ in $X \iff \lim_{n\to\infty} \|u^n - u\|_X = 0$ **norm** (or strong) **convergence**

$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p dx\right)^{1/p}$ (Minkowski inequality $\|u+v\|_p \le \|u\|_p + \|v\|_p$)

$\|u\|_{C(\overline{\Omega})} = \max_{x \in \overline{\Omega}} |u(x)| = \|u\|_{L^\infty(\Omega)}$

$\|u\|_{C^{0,\alpha}(\overline{\Omega})} = \|u\|_{C(\overline{\Omega})} + \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x-y|^\alpha}$

X - **Hilbert space** = X is Banach space and we have a scalar product

L^2 - Hilbert space, $(u,v)_{L^2(\Omega)} := \int_\Omega uv\, dx$

separability: X is separable space $\overset{def}{=}$ it has a countable dense subset

$\Leftrightarrow \exists \{v_j\}_{j=1}^{\infty}\ \forall u \in X\ \forall \varepsilon > 0\ \exists v_j : \|u - v_j\|_X < \varepsilon$

L^p are separable for $p \in [1, \infty)$

$C^k(\bar{\Omega})$ are separable

$C^{0,\alpha}(\bar{\Omega})$ are NOT separable $\alpha \in (0, 1]$

reflexivity: We say that X is reflexive if $(X^*)^* \simeq X$

By characterization, all Hilbert spaces are reflexive.

L^p are reflexive for $p \in (1, \infty)$ $\quad (L^p(\Omega))^* = L^{p'}(\Omega)$, $(L^\infty(\Omega))^* \neq L^1(\Omega)$

$(C(\bar{\Omega}))^* = \mathcal{M}(\bar{\Omega})$ - the space of Radon measures

Chapter 1. Introduction motivation

$\Omega \subseteq \mathbb{R}^d$ open, bounded

1. $-\Delta u = f$ in Ω — **LAPLACE EQUATION**

$u = 0$ on $\partial\Omega$ — Laplacian: $\Delta u = \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2}$

2. $\frac{\partial u}{\partial t} - \Delta u = f$ in $(0,T) \times \Omega$ — **HEAT EQUATION**

$u = 0$ on $(0,T) \times \partial\Omega$

$u(0,x) = u_0(x)$ in Ω

Backward HE: $\frac{\partial u}{\partial t} + \Delta u = f$ in $(0,T)\times\Omega$, $u(T,x) = u_T(x)$ in Ω

3. $\frac{\partial^2 u}{\partial t^2} - \Delta u = f$ in $(0,T) \times \Omega$ — **WAVE EQUATION**

$u = 0$ on $(0,T) \times \partial\Omega$

$u(0,x) = u_0(x)$ in Ω

$\frac{\partial u}{\partial t}(0,x) = v_0(x)$ in Ω

WE ALREADY KNOW: **If data** (right hand side, bdary conditions, initial conditions, domain) **are smooth** (C^∞) **and $\Omega = B_1(0)$ then solutions exist and are smooth.**

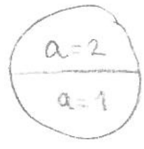
Classical solution. If $f \in C(\bar{\Omega})$ then $u \in C^2(\Omega)$, $u \in C(\bar{\Omega})$, $u = 0$ on $\partial\Omega$

and $-\Delta u(x) = f(x) \quad \forall x \in \Omega$.

It is a too nice notion of solution.

Classical solution does not exist in general:

- $f \notin C(\overline{\Omega})$
- $\Omega =$  $\quad \frac{\partial u}{\partial x_i}(x) \xrightarrow{x \to 0} \infty$ (non convex corner)
- operator is non-smooth, $\quad -\operatorname{div}(a(x)\nabla u(x)) = f(x)$



$$\nabla u(x) = \left(\frac{\partial u}{\partial x_1}(x), \dots, \frac{\partial u}{\partial x_d}(x)\right)$$

$$\operatorname{div} \vec{f}(x) = \sum_{i=1}^{d} \frac{\partial f_i}{\partial x_i}(x)$$

- Hilbert problem: system: $-\operatorname{div}(A(\nabla u)) = f \qquad u = (u_1, \dots, u_v)$

  do we have nice solution for nice data?

  J. Nečas in 70's found counterexample

How to define a notion of "solution"?

Dream: if classical solution exists, then it coinsides with the "solution".

$$-\Delta u = f \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \partial\Omega, \qquad \varphi \in C_0^1(\Omega) = \{v \in C^1(\Omega), \text{ compactly supported}\}$$

$$-\int_\Omega \Delta u \, \varphi \, dx = \int_\Omega f \varphi \, dx \qquad -\int_\Omega \Delta u \, \varphi = -\int_\Omega \operatorname{div}(\nabla u) \varphi \, dx$$

$$\int_\Omega \sum_{i=1}^{d} \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_i} \, dx = \int_\Omega f \varphi \, dx \qquad = \int_\Omega \nabla u \cdot \nabla \varphi \quad (+ \text{ bdary term } \overset{\|}{0})$$

$$\varphi := u: \qquad \int_\Omega \nabla u \cdot \nabla u \, dx = \int_\Omega f u \, dx$$

$$\|\nabla u\|_2^2 \leq \|f\|_2 \|u\|_2$$

Assumption (Poincaré inequality). If $u = 0$ on $\partial\Omega$ then $\|u\|_{L^2(\Omega)} \leq c \|\nabla u\|_{L^2(\Omega)}$ with c independent of u.

$$\int_0^1 |u(x)| \, dx \overset{\wedge}{=} \int_0^1 \left| \int_0^x \frac{\partial u}{\partial s}(s) \, ds \right| dx \leq \int_0^1 |\nabla u|$$

Poincaré $\overset{\Rightarrow}{\uparrow}$ $\|\nabla u\|_{L^2} \leq c \|f\|_2$ (A priori estimate)

$$\|\nabla u\|_2^2 \leq \|f\|_2 \|u_2\| \leq c \|f\|_2 \|\nabla u\|_2$$

$$W^{1,2}(\Omega) := \{u \in L^2(\Omega), \ \frac{\partial u}{\partial x_i} \in L^2(\Omega), \ i = 1, \dots, d\}$$ ← weak derivative

I look for $u \in W_0^{1,2}(\Omega)$ which satisfies $u = 0$ on $\partial\Omega$, $\forall \varphi \in C_0^1(\Omega)$: $\int_\Omega \nabla u \nabla \varphi \, dx = \int_\Omega f \varphi \, dx$

and if $u \in C^2(\Omega) \Rightarrow -\Delta u(x) = f(x)$

Second motivation

$J(u) := \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \int_\Omega f u \, dx$

Find $u \in C^2(\Omega) \cap C(\bar\Omega)$, $u = 0$ on $\partial\Omega$ such that $J(u) \le J(v)$ $\forall v \in C^2(\Omega) \cap C(\bar\Omega)$
$C^1(\Omega)$ enough
$v = 0$ on $\partial\Omega$.

Let such minimizer exist. $J(u) \le J(0) = 0$

The same a priori estimate, $\int_\Omega |\nabla u|^2 dx \le \int_\Omega f u \, dx \overset{\text{Hölder}}{\le} \|f\|_2 \|u\|_2 \overset{\text{Poincaré}}{\le} c \|f\|_2 \|\nabla u\|_2$

$\Rightarrow \|\nabla u\|_2 \le c \|f\|_2$

$\varphi \in C_0^1(\Omega)$ arbitrary and set $v := u - \varepsilon\varphi$, $\varepsilon > 0$:

$$\frac{1}{2} \int_\Omega |\nabla u|^2 dx - \int_\Omega f u \le \frac{1}{2} \int_\Omega |\nabla(u - \varepsilon\varphi)|^2 dx - \int_\Omega f(u - \varepsilon\varphi)$$
$$= \frac{1}{2} \int_\Omega |\nabla u|^2 + \varepsilon^2 |\nabla\varphi|^2 - \varepsilon \int_\Omega \nabla u \cdot \nabla\varphi - \int_\Omega f u + \varepsilon \int_\Omega f \varphi$$

$$\varepsilon \int_\Omega \nabla u \cdot \nabla\varphi \le \varepsilon \int_\Omega f\varphi + \frac{\varepsilon^2}{2} \int_\Omega |\nabla\varphi|^2 \qquad /:\varepsilon$$

$$\int_\Omega \nabla u \cdot \nabla\varphi \le \int_\Omega f\varphi + \frac{\varepsilon}{2} \int_\Omega |\nabla\varphi|^2 \xrightarrow{\varepsilon \to 0+} \int_\Omega f\varphi$$

We got $\forall \varphi \in C_0^1(\Omega)$: $\int_\Omega \nabla u \cdot \nabla\varphi \le \int_\Omega f\varphi$
it is true also for $(-\varphi)$: $-\int_\Omega \nabla u \cdot \nabla\varphi \le -\int_\Omega f\varphi$
$\Rightarrow \int_\Omega \nabla u \cdot \nabla\varphi = \int_\Omega f\varphi$

I did not need the 2nd derivative of u!

Motivation from continuum mechanics

2. step $\quad -\operatorname{div} \vec{j} = f$ in Ω $\qquad$ (Laplace: $\vec{j} = \nabla u$)

1. step $\quad -\int_{\partial V} \vec{j} \cdot n \, ds = \int_V f \, dx \quad \forall V \subseteq \Omega$, balance law

outer normal vector

if something is produced inside the volume, same amount needs to exit through the bdary

Gauss theorem

if $\vec{j} \in C^1 \Rightarrow -\int_V \operatorname{div} \vec{j} = \int_V f \Rightarrow$ 2. step

Motivation for balance laws:

$L^2_{DIV}(\Omega, \mathbb{R}^d) := \{ \vec{j} \in L^2(\Omega, \mathbb{R}^d), \vec{j} = (j_1, \dots, j_d), \int_\Omega j \cdot \nabla\varphi = 0 \ \forall \varphi \in C_0^1(\Omega), \text{if } \vec{j} \in C^1, \int_\Omega \operatorname{div} \vec{j} \varphi = 0 \}$

$L^2_f(\Omega, \mathbb{R}^d) = \{ \vec{j} \in L^2(\Omega, \mathbb{R}^d); \int_\Omega \vec{j} \cdot \nabla\varphi = \int_\Omega f\varphi \quad \forall \varphi \in C_0^1(\Omega) \}$

$\min_{\vec{j} \in L^2_f} \frac{1}{2} \int_\Omega |\vec{j}|^2$ $\qquad$ I will compare it with $\vec{j} + \varepsilon\vec{k}$, $\vec{k} \in L^2_{DIV}(\Omega, \mathbb{R}^d)$

$\frac{1}{2} \int_\Omega |\vec{j}|^2 \le \frac{1}{2} \int_\Omega |\vec{j} + \varepsilon\vec{k}|^2 = \frac{1}{2} \int_\Omega |\vec{j}|^2 + \varepsilon^2 |\vec{k}|^2 + \varepsilon \int_\Omega \vec{j} \cdot \vec{k}$, divide by $\varepsilon > 0$ & $\varepsilon \to 0+$

$0 \le \int_\Omega \vec{j} \, \vec{k} \overset{(-\vec{k})}{\Longrightarrow} \int_\Omega \vec{j} \, \vec{k} = 0 \quad \forall \vec{k} \in L^2_{DIV}(\Omega, \mathbb{R}^d)$

Euler-Lagrange equation

$\vec{j} = \nabla u$? Proof.

$$\int_\Omega |\vec{j} - \nabla u|^2 = \int_\Omega -\vec{j}\cdot(\nabla u - \vec{j}) - \nabla u \cdot (\vec{j} - \nabla u)$$

$\underset{L^2_{DIV}}{\cap}$ $\left(\Leftarrow \quad \begin{array}{l} \int \nabla u \cdot \nabla \varphi = \int f\varphi \\ \int \vec{j}\cdot\nabla\varphi = \int f\varphi \end{array} \right)$

Euler-Lagrange

$$= \quad -\int_\Omega \nabla u (\vec{j} - \nabla u)$$

for smooth functions "=" $\int_\Omega u \operatorname{div}(\vec{j} - \nabla u) = \int_\Omega u(-f+f) = 0$

"=" is rigorous if ∇u can be approximated by smooth fns with compact support

- Let $\forall V \subseteq \Omega$ smooth (almost all) $\quad -\int_{\partial V} \vec{j}\cdot n \, ds = \int_V f \, dx \quad$ ($\vec{j} \in L^1_{loc}(\Omega, \mathbb{R}^d)$, $f \in L^1_{loc}(\Omega)$)

Then $\int_\Omega \vec{j}\cdot\nabla\varphi \, dx = \int_\Omega f\varphi \, dx$

Proof:

Lemma: Let $\eta : \mathbb{R}^d \to \mathbb{R}$ be Lipschitz and $v \in L^1(\Omega)$. Then

- $v|_{\{x \in \mathbb{R}^d : \eta(x) = r\}}$ is integrable in the sense of $(d-1)$-Hausdorff measure for a.a. $r \in (a, \infty)$
- $\int_{\mathbb{R}^d} v(x) |\nabla \eta(x)| \, dx = \int_{\mathbb{R}} \left(\int_{\{x \in \mathbb{R}^d, \eta(x) = r\}} v(x) \, dS \right) dr$

$V = \{x : \eta(x) > r\}$, $\quad v(x) = \dfrac{\vec{j}\cdot\nabla\eta}{|\nabla\eta|} \qquad \eta \in C_0^\infty(\Omega)$, $\eta \geq 0$

$$-\int_{\partial\{\eta(x)>r\}} \vec{j}\cdot n \, ds = \int_{\{\eta(x)>r\}} f \, dx$$

$$\|$$

$$\int_{\{\eta(x)=r\}} \underbrace{\vec{j}\cdot\frac{\nabla\eta}{|\nabla\eta|}}_{v(x)} ds = \int_{\{\eta(x)>r\}} f \, dx \qquad / \int_0^\infty dr$$

$$\int_0^\infty \left(\int_{\{\eta(x)=r\}} v(x) \, ds \right) dr = \int_0^\infty \left(\int_{\{\eta(x)>r\}} f \, dx \right) dr = \int_0^\infty \int_\Omega f(x) \chi_{[\eta(x)>r]} \, dx \, dr = (*)$$

$\|$ Lemma

$$\int_\Omega v(x) |\nabla\eta(x)| = \int_\Omega \vec{j}\cdot\nabla\eta$$

$$(*) = \int_\Omega \int_0^\infty f(x) \chi_{[\eta(x)>r]} \, dr \, dx = \int_\Omega f(x) \left(\int_0^{\eta(x)} 1 \, dr \right) dx = \int_\Omega f(x) \eta(x) \, dx$$

Chapter 2. Sobolev spaces

Def (multiindex): We say that α is multiindex if $\alpha = (\alpha_1, \dots, \alpha_d)$ and $\alpha_i \in \mathbb{N}_0$. The length of α is $|\alpha| = \alpha_1 + \dots + \alpha_d$.

Notation: if $u \in C^k(\Omega)$ then $\quad D^\alpha u = \dfrac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_d^{\alpha_d}} \qquad \forall |\alpha| \leq k$

Def (weak derivative). Let $u, v_\alpha \in L^1_{loc}(\Omega)$ and α be a multiindex.

We say that v_α is the α-th weak derivative of u iff

$$\forall \varphi \in C_0^\infty(\Omega) \quad \int_\Omega u \, D^\alpha \varphi \, dx = (-1)^{|\alpha|} \int_\Omega v_\alpha \cdot \varphi \, dx$$

Lemma (about the **consistency of the weak derivative**):

- weak derivative is unique
- if classical derivative exists, it is also the weak one.

Proof. • Let v_α^1 and v_α^2 be weak derivatives of u. Then

$$\left.\begin{aligned} \int_\Omega v_\alpha^1 \varphi &= (-1)^{|\alpha|} \int_\Omega u D^\alpha \varphi \\ \int_\Omega v_\alpha^2 \varphi &= (-1)^{|\alpha|} \int_\Omega u D^\alpha \varphi \end{aligned}\right\} \quad \int_\Omega (v_\alpha^1 - v_\alpha^2)\varphi = 0 \quad \forall \varphi \in C_c^\infty(\Omega)$$

Fundamental theorem $\quad v_\alpha^1 = v_\alpha^2$ a.e. in Ω.

• if classical derivative exists, we have IBP (integration by parts)

$$\int_\Omega u D^\alpha \varphi = (-1)^{|\alpha|} \int_\Omega \underbrace{D^\alpha u}_{\text{classical one}} \varphi \qquad \Rightarrow D^\alpha u = v_\alpha$$

Exercise: compute WD of $f(x) := |x|$ on $(-1,1)$
$f(x) := |x|^\alpha$ on $(-1,1)$, $0<\alpha<1$
show that WD of $f(x) := \operatorname{sign} x$ in $(-1,1)$ does not exist.

Notation: $D^\alpha u$ means weak derivative.

10.10.2018 Def. (**Sobolev spaces**) Let $\Omega \subset \mathbb{R}^d$ be open set, $k \in \mathbb{N}$, $p \in [1,\infty]$. We define

$$W^{k,p}(\Omega) := \{u \in L^p(\Omega),\ \forall \alpha, |\alpha| \le k : D^\alpha u \in L^p(\Omega)\}.$$

$$\|u\|_{W^{k,p}(\Omega)} = \begin{cases} \left(\sum_{\alpha, |\alpha| \le k} \|D^\alpha u\|_p^p\right)^{1/p} & \text{for } p \in [1,\infty) \\ \max_{\alpha, |\alpha| \le k} \|D^\alpha u\|_\infty \end{cases}$$

Convention: • $D^0 u = u$

• $\|u\|_{W^{k,p}(\Omega)} = \|u\|_{W^{k,p}} = \|u\|_{k,p}$

• $[\nabla^m u]_{i_1,\dots,i_d} = \dfrac{\partial^m u}{\partial x_1^{i_1} \dots \partial x_d^{i_d}}$

Lemma (**Basic properties of weak derivatives**): Let $u, v \in W^{k,p}(\Omega)$, $k \in \mathbb{N}$ and α be a multiindex, $|\alpha| \le k$,

- $D^\alpha u \in W^{k-|\alpha|,p}(\Omega)$ and $D^\alpha(D^\beta u) = D^\beta(D^\alpha u) = D^{\alpha+\beta} u \quad \forall |\alpha|+|\beta| \le k$
- $\forall \lambda, \mu \in \mathbb{R}$, $\lambda u + \mu v \in W^{k,p}(\Omega)$ and $D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v$
- $\forall \tilde\Omega \subseteq \Omega$ open, we have $u \in W^{k,p}(\tilde\Omega)$
- $\forall \eta \in C^\infty(\Omega)$ then $\eta u \in W^{k,p}(\Omega)$ and $D^\alpha(\eta u) = \sum_{\beta, \beta_i \le \alpha_i} \binom{\alpha}{\beta} D^\beta \eta\, D^{\alpha-\beta} u$, $\binom{\alpha}{\beta} := \prod_{i=1}^d \binom{\alpha_i}{\beta_i}$

1/2 homework: prove the Lemma using the definition of the weak derivative

Exercise: 1. $u(x) = |x|^\alpha$, $\alpha \in (0,1]$ on $(-1,1)$

2. $u(x) = \operatorname{sign} x$

1. Assume that u has weak derivative $v \in L^1_{loc}$, then $\forall \eta \in C_0^\infty(-1,1)$: $\int_{-1}^{1} u \frac{\partial \eta}{\partial x} = -\int_{-1}^{1} v\eta$.

It implies that also $\forall \eta \in C_0^\infty(0,1)$ $\int_0^1 u \frac{\partial \eta}{\partial x} = -\int_0^1 v\eta$

$u = |x|^\alpha \Rightarrow$ if v exists, $v = \begin{cases} \alpha x^{\alpha-1} & \text{on } (0,1) \\ -\alpha |x|^{\alpha-1} & \text{on } (-1,0) \end{cases}$ (follows from $(*)$)

$$\int_0^1 u \frac{\partial \eta}{\partial x} = \int_0^1 |x|^\alpha \frac{\partial \eta}{\partial x} = \int_0^1 x^\alpha \frac{\partial \eta}{\partial x} \overset{IBP}{=} -\int_0^1 \alpha x^{\alpha-1}\eta \qquad (*)$$

We want to show that $v = \alpha |x|^{\alpha-1} \operatorname{sign} x = \frac{\partial u}{\partial x}$ (weak derivative)

$\eta \in C_0^\infty(-1,1)$

$$\int_{-1}^1 v\eta = \int_{-1}^1 \alpha |x|^{\alpha-1} \operatorname{sign} x\, \eta = \int_{-1}^{-\varepsilon} \alpha |x|^{\alpha-1} \operatorname{sign} x\, \eta + \int_{\varepsilon}^1 \alpha |x|^{\alpha-1} \operatorname{sign} x\, \eta + \int_{-\varepsilon}^{\varepsilon} \text{-}\text{"}\text{-}$$
$$= \int_{-1}^{-\varepsilon} \frac{\partial |x|^\alpha}{\partial x}\eta + \int_{\varepsilon}^1 \frac{\partial |x|^\alpha}{\partial x}\eta + \int_{-\varepsilon}^{\varepsilon} \alpha |x|^{\alpha-1}\operatorname{sign} x\, \eta$$
$$= -\int_{-1}^{-\varepsilon} |x|^\alpha \frac{\partial \eta}{\partial x} - \int_{\varepsilon}^1 |x|^\alpha \frac{\partial \eta}{\partial x} + \underbrace{\varepsilon^\alpha \eta(-\varepsilon) - \varepsilon^\alpha \eta(\varepsilon)}_{\downarrow 0} + \underbrace{\int_{-\varepsilon}^{\varepsilon} \alpha |x|^{\alpha-1} \operatorname{sign} x\, \eta}_{\downarrow 0 \ (**)}$$
$$\overset{\varepsilon \to 0+}{\longrightarrow} -\int_{-1}^1 |x|^\alpha \frac{\partial \eta}{\partial x} = -\int_{-1}^1 u \frac{\partial \eta}{\partial x}$$

$$(**) \quad \left| \int_{-\varepsilon}^{\varepsilon} \alpha |x|^{\alpha-1} \operatorname{sign} x\, \eta \right| \le \|\eta\|_\infty \int_{-\varepsilon}^{\varepsilon} \alpha |x|^{\alpha-1} dx = 2\|\eta\|_\infty \int_0^{\varepsilon} \alpha |x|^{\alpha-1} dx = 2\|\eta\|_\infty \varepsilon^\alpha \overset{\varepsilon \to 0+}{\longrightarrow} 0$$

2. We will show that u does not have a weak derivative.

Assume that $v \in L^1_{loc}$ is a weak derivative.

$\forall \eta \in C_0^\infty(0,1)$: $0 = \int_0^1 \operatorname{sign} x \frac{\partial \eta}{\partial x} = -\int_0^1 v\eta \Rightarrow v = 0$ on $(0,1)$

$v = 0$ on $(-1,0)$

the only possibility is $v = 0$

$$0 = \int_{-1}^1 v\eta = -\int_{-1}^1 \operatorname{sign} x \frac{\partial \eta}{\partial x} = \int_{-1}^0 \frac{\partial \eta}{\partial x} - \int_0^1 \frac{\partial \eta}{\partial x} = 2\eta(0)$$

$\Rightarrow \frac{\partial u}{\partial x} = 2\delta_0$ Dirac measure , but this is not a weak derivative.

It is a derivative in the sense of distributions, which is the weakest notion of derivative. The reason for the weak derivative not to exist is that the function $\operatorname{sign} x$ is discontinuous.

Example. $u(x) = \frac{1}{|x|^\alpha}$ in $B_1(0) \subseteq \mathbb{R}^d$, $d \ge 2$ $|x| = \left(\sum_{i=1}^d x_i^2\right)^{1/2}$

$u \overset{?}{\in} L^p(\Omega) \iff p\alpha < d$

Measure theory. $\int_{B_1(0)} \frac{1}{|x|^A} < \infty \iff A < d$

measure of a unit ball

$$\int_{B_1(0)} \frac{1}{|x|^A} = \int_0^1 \left(\int_{\partial B_r} \frac{1}{|x|^A} ds \right) dr = \int_0^1 \int_{\partial B_r} \frac{1}{r^A} ds\, dr = \varkappa_d \int_0^1 r^{d-1-A} dr \begin{cases} < \infty & \text{if } d-1-A > -1 \\ = \infty & \text{otherwise} \end{cases}$$

on the other hand, close to ∞ it is integrable if $A > d$

$u(x) = \frac{1}{|x|^\alpha}$ in $B_1(0) \subset \mathbb{R}^d$, $d \geq 2$

Assume that $v_i \in L^1_{loc}$ is a weak derivative $\frac{\partial u}{\partial x_i}$, then

$$\int_{B_1} u \frac{\partial \eta}{\partial x_i} = -\int_{B_1} v_i \eta \qquad \forall \eta \in C_0^\infty(B_1), \quad \text{also } \forall \eta \in C_0^\infty(B_1 \setminus \{0\}):$$

Outside of $\{0\}$, $\frac{\partial u}{\partial x_i} = \frac{\partial |x|^{-\alpha}}{\partial x_i} = -\alpha \frac{x_i}{|x|^{\alpha+2}}$

- $v_i \in L^p \iff \frac{x_i}{|x|^{\alpha+2}} \in L^p \iff \frac{1}{|x|^{\alpha+1}} \in L^p \iff p(\alpha+1) < d$

  $p < \frac{d}{\alpha+1}$, we need $p \geq 1$ $\Rightarrow \alpha < d-1$
- check that v_i is a weak derivative

$$\int_{B_1(0)} v_i \eta = \int_{B_1(0)\setminus B_\varepsilon(0)} v_i \eta + \int_{B_\varepsilon(0)} \underbrace{v_i \eta}_{\text{integrable function}}$$

$\int_{B_\varepsilon} v_i \eta \to 0$ if $\varepsilon \to 0+$

$$= \int_{B_1 \setminus B_\varepsilon} -\frac{\alpha x_i}{|x|^{\alpha+2}} \eta + \int_{B_\varepsilon} -\text{"}-$$

$$= \int_{B_1 \setminus B_\varepsilon} \frac{\partial |x|^{-\alpha}}{\partial x_i} \eta + \int_{B_\varepsilon} -\text{"}-$$

$n = (n_1, \dots, n_d)$ unit normal outward vecto

$$\stackrel{IBP}{=} -\int_{B_1 \setminus B_\varepsilon} |x|^{-\alpha} \frac{\partial \eta}{\partial x_i} + \int_{B_\varepsilon} -\text{"}- + \int_{\partial(B_1 \setminus B_\varepsilon)} |x|^{-\alpha} \eta \, n_i \, ds \qquad n_i = \frac{-x_i}{|x|}$$

$$\xrightarrow{\varepsilon \to 0+} -\int_{B_1} |x|^{-\alpha} \frac{\partial \eta}{\partial x_i} - \lim_{\varepsilon \to 0+} \int_{\partial B_\varepsilon} |x|^{-\alpha} \eta \frac{x_i}{|x|}$$

$$\left| \int_{\partial B_\varepsilon} |x|^{-\alpha} \eta \frac{x_i}{|x|} \right| \leq \|\eta\|_\infty \varepsilon^{-\alpha} |\partial B_\varepsilon| = \varkappa_d \varepsilon^{d-1-\alpha} \|\eta\|_\infty \xrightarrow{\varepsilon \to 0+} 0 \qquad (d-1-\alpha > 0 \iff \alpha < d-1)$$

$\Rightarrow$ **Sobolev function can even have a blow up!**

! Not only one!

<u>Example (discontinuities in a dense set)</u>:

$B_1(0) \subseteq \mathbb{R}^d$, $d \geq 2$, $\{x_i\}_{i=1}^\infty$ dense in $B_1(0)$

$u(x) := \sum_{i=1}^\infty \frac{1}{2^i} |x - x_i|^{-\alpha}$ has discontinuities on a dense set

$\Rightarrow$ SOBOLEV DOES NOT MEAN SMOOTH / GOOD / CONTINUOUS

2/2 Homework: $u(x) := \frac{|x|^{2-d}}{2-d}$ $d \geq 3$ in $B_1(0)$

show that • $u \in W^{1,p}(B_1)$ $(p < d')$

• u is harmonic ($\iff \Delta u = 0$ in $B_1 \setminus \{0\}$) $\approx \varphi(0)$

$u \notin W^{2,1}(B_1)$, $\int_{B_1} u \Delta \varphi \neq 0$ $\forall \varphi \in C_0^\infty(B_1)$

Theorem (**Properties of $W^{k,p}(\Omega)$**). Let $\Omega \subseteq \mathbb{R}^d$ open, $p \in [1,\infty]$, $k \in \mathbb{N}$.

1. $W^{k,p}(\Omega)$ is a Banach space
2. if $p<\infty$ it is a separable space
3. if $p \in (1,\infty)$ it is a reflexive space

Proof: 1. linear normed space $\Leftarrow$ 1/2 homework + Minkowski inequality

complete space: $\{u^n\}_{n=1}^{\infty}$ a Cauchy sequence $\overset{?}{\Rightarrow} \exists u \in W^{k,p}(\Omega)$, $u^n \to u$ in $W^{k,p}(\Omega)$

u^n is Cauchy in $W^{k,p}(\Omega) \Rightarrow \forall \alpha, |\alpha| \le k$ $D^\alpha u^n$ is a Cauchy seq. in $L^p(\Omega)$

L^p is a Banach space $\Rightarrow \forall \alpha, |\alpha|\le k \cdot \exists v_\alpha$, $D^\alpha u^n \to v_\alpha$ in $L^p(\Omega)$ $(*_1)$

$u^n \to u$ in $L^p(\Omega)$ $(*_2)$

it remains to show that $v_\alpha = D^\alpha u$.

$\eta \in C_0^\infty(\Omega)$ $\int_\Omega v_\alpha \eta = \int_\Omega (v_\alpha - D^\alpha u^n)\eta + \int_\Omega D^\alpha u^n \eta$

$= \int_\Omega (v_\alpha - D^\alpha u^n)\eta + (-1)^{|\alpha|}\int_\Omega D^\alpha \eta\, u^n$

$= \int_\Omega (v_\alpha - D^\alpha u^n)\eta + (-1)^{|\alpha|}\int_\Omega (u^n - u) D^\alpha \eta + (-1)^{|\alpha|}\int_\Omega u D^\alpha \eta$

$\left|\int_\Omega (v_\alpha - D^\alpha u^n)\eta\right| \le \|v_\alpha - D^\alpha u^n\|_p \|\eta\|_{p'} \xrightarrow{(*_1)} 0$ as $n\to\infty$

$\left|\int_\Omega (u^n-u) D^\alpha \eta\right| \overset{\text{Hölder}}{\le} \|u^n - u\|_p \|D^\alpha \eta\|_{p'} \xrightarrow{(*_2)} 0$ as $n \to \infty$

2.+3. only for $W^{1,p}(\Omega)$, similarly for $W^{k,p}(\Omega)$

We identify $W^{1,p}(\Omega) \cong X \subseteq \underbrace{L^p(\Omega) \times \ldots \times L^p(\Omega)}_{(d+1) \text{ times}}$,

$u \in W^{1,p}(\Omega) \mapsto U \in X$, $U_0 = u$, $U_1 = D^1 u = \frac{\partial u}{\partial x_1}$, …, $U_d = D^d u = \frac{\partial u}{\partial x_d}$

- X is closed subspace of $L^p(\Omega)\times\ldots\times L^p(\Omega)$

FA $\Rightarrow$ if $L^p(\Omega)\times\ldots\times L^p(\Omega)$ is separable (reflexive) then any closed subspace is separable (reflexive)

② $L^p(\Omega)\times\ldots\times L^p(\Omega)$ is separable for $p<\infty$ $\Rightarrow X$ is separable $\Rightarrow W^{1,p}(\Omega)$ is sep. for $p<\infty$

③ $L^p(\Omega)\times\ldots\times L^p(\Omega)$ is reflexive for $p\in(1,\infty) \Rightarrow X$ is reflexive $\Rightarrow W^{1,p}(\Omega)$ is reflexive for $p \in (1,\infty)$

2.2. Approximation of Sobolev spaces and consequences

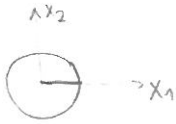
Theorem: Let $\Omega \subseteq \mathbb{R}^d$ open set, $k \in \mathbb{N}$, $p \in [1,\infty)$. Then

- $\overline{C^\infty(\Omega)}^{\|\|_{k,p}} = W^{k,p}(\Omega)$

$\overline{C_0^\infty(\Omega)}^{\|\|_{k,p}} = \{u \in W^{k,p}(\Omega), \exists \{u^n\} \subseteq C_0^\infty(\Omega) \text{ s.t. } \|u^n - u\|_{k,p} \to 0\}$

Proof: summer semester

! WARNING! $\overline{C^\infty(\overline{\Omega})}^{\|\|_{k,p}} \subsetneq W^{k,p}(\Omega)$

Counterexample: $\Omega =$  $u = \begin{cases} 1 & \text{if } x_2 > 0 \text{ or } x_1 < 0 \\ 1 + x_1^2 & \text{if } x_1 > 0 \text{ and } x_2 < 0 \end{cases}$

$\frac{\partial u}{\partial x_1} = \frac{\partial u}{\partial x_2} = 0 \qquad (x_2 > 0) \vee (x_1 < 0)$

$\frac{\partial u}{\partial x_1} = 2x_1, \; \frac{\partial u}{\partial x_2} = 0 \quad (x_2 < 0) \wedge (x_1 > 0)$

$u \in C^1(\Omega), \quad u \notin W^{1,1}(B_1(0))$ Why? $\frac{\partial u}{\partial x_2}$ does not exist.

It has a jump at the line. The problem with the domain is that one cannot say what is outside and what is inside.

Assume that $\frac{\partial u}{\partial x_2}$ exists: $\int_{B_1} \frac{\partial u}{\partial x_2} \eta = -\int_{B_1} u \frac{\partial \eta}{\partial x_2} \qquad \forall \eta \in C_0^\infty(B_1)$

$\Downarrow \; (\Omega \subset B_1)$

$\int_\Omega \frac{\partial u}{\partial x_2} \eta = -\int_\Omega u \frac{\partial \eta}{\partial x_2} \qquad \Leftarrow \forall \eta \in C_0^\infty(\Omega)$

$\Rightarrow \frac{\partial u}{\partial x_2} \equiv 0$ in Ω

$\Rightarrow$ if $\frac{\partial u}{\partial x_2}$ exists in $B_1(0) \Rightarrow \frac{\partial u}{\partial x_2} \equiv 0$ in $B_1(0)$

$0 = -\int_{B_1(0)} u \frac{\partial \eta}{\partial x_2} dx = \int_{\mathsf{a}} + \int_{\mathsf{b}} + \int_{\mathsf{c}}$

$= \int_{\mathsf{a}} \frac{\partial \eta}{\partial x_2} dx + \int_{\mathsf{b}} \frac{\partial \eta}{\partial x_2} dx + \int_{\mathsf{c}} (1 + x_1^2) \frac{\partial \eta}{\partial x_2} dx$

$\overset{IBP}{=} 0 - \int_0^1 \eta(x_1, 0)\, dx_1 + \int_0^1 (1 + x_1^2)\, \eta(x_1, 0)\, dx_1$

$= \int_0^1 x_1^2\, \eta(x_1, 0)\, dx_1 \neq 0$ in general

$u \in C^1(\Omega)$ but $u \notin W^{1,1}(B_1)$

$\exists u^n \in C^\infty(\overline{\Omega}) \quad \|u^n - u\|_{W^{1,1}(\Omega)} \to 0 \Rightarrow u^n$ is Cauchy in $W^{1,1}(\Omega)$ $\quad \overline{\Omega} = \overline{B_1}$

$\Rightarrow \int_{B_1(0)} |u^n - u^m| + |\nabla u^n - \nabla u^m| = \int_\Omega |u^n - u^m| + |\nabla u^n - \nabla u^m|$

RHS Cauchy $\Rightarrow$ LHS Cauchy $\Rightarrow u^n$ Cauchy in $W^{1,1}(B_1)$

$\Rightarrow \exists v: \; u^n \to v$ in $W^{1,1}(B_1)$

$u^n \to u$ in $L^1(\Omega) \Rightarrow v = u$, but $u \notin W^{1,1}(B_1)$!

Notes on the proof of the Theorem.

Extend u by 0 outside Ω $\; \exists \{u^n\} \in C^\infty(\mathbb{R}^d)$

$\forall \tilde{\Omega}$ open $\overline{\tilde{\Omega}} \subseteq \Omega \qquad u^n \to u$ in $W^{k,p}(\tilde{\Omega})$

$u^n := u * \eta_{\frac{1}{n}}$ $\qquad$ **regularizing kernel**: $0 \le \eta \in C_0^\infty(B_1)$ radially symmetric, $\int_{B_1} \eta = 1$

$\eta_\varepsilon(x) := \frac{\eta(\frac{x}{\varepsilon})}{\varepsilon^d}$

$u^n(x) = \int_{\mathbb{R}^d} u(y)\, \eta_{\frac{1}{n}}(x - y)\, dy = \int_{\mathbb{R}^d} u(x+z)\, \eta_{1/n}(z)\, dz \overset{n = \varepsilon^{-1}}{=} \int_{\mathbb{R}^d} u(x+z) \frac{\eta(\frac{z}{\varepsilon})}{\varepsilon^d} dz$

$= \int_{\mathbb{R}^d} u(x + \varepsilon z)\, \eta(z)\, dz \xrightarrow{x\text{-Lebesgue}} u(x)$ $\qquad$ **x- Lebesgue point of u**

point wise (a.e.) convergence $\Leftrightarrow \fint_{B_\varepsilon(x)} |u(y) - u(x)|\, dy \xrightarrow{\varepsilon \to 0^+} 0$

I am averaging u

$u \in L^1_{loc}(\Omega) \Rightarrow$ a.e. x is Leb. point of u

Properties of mollification: • if $u \in L^p(\Omega) \Rightarrow u^n = u * \varphi_{\frac{1}{n}} \to u$ in $L^p(\Omega)$

• $u^n \in C^\infty(\mathbb{R}^d)$

• $\tilde{\Omega} \subseteq \overline{\tilde{\Omega}} \subseteq \Omega$ $\exists n_0 \in \mathbb{N}\ \forall n > n_0 : D^\alpha u^n = (D^\alpha u) * \eta_{\frac{1}{n}}$ in $\tilde{\Omega}$

$\varphi \in C_0^\infty(\tilde{\Omega})$ $\int_{\mathbb{R}^d} D^\alpha u^n \varphi = \dots$

$((D^\alpha u) * \varphi_{1/n})(x) = \int_{\mathbb{R}^d} D^\alpha u(y)\, \eta_{1/n}(x-y)\, dy = \int_{B_{1/n}(x)} D^\alpha u(y)\, \eta_{1/n}(x-y)\, dy$

take n large enough s.t. $B_{\frac{1}{n}}(x) \subseteq \Omega \ \forall x \in \tilde{\Omega}$

$= \int_\Omega D^\alpha u(y)\, \eta_{1/n}(x-y)\, dy = (-1)^{|\alpha|} \int_\Omega u(y)\, D_y^\alpha \eta_{1/n}(x-y)\, dy$

$= (-1)^{|\alpha|} (-1)^{|\alpha|} \int_\Omega u(y)\, D_x^\alpha \eta_{1/n}(x-y)\, dy$

$D^\alpha u^n(x) = D_x^\alpha \int_{\mathbb{R}^d} u(y)\, \eta_{1/n}(x-y)\, dy \underset{\text{for } n \text{ large}}{=} \int_{\cancel{\mathbb{R}^d}\ \Omega} u(y)\, D_x^\alpha \eta_{1/n}(x-y)\, dy$

$\Rightarrow D^\alpha(u^n) = (D^\alpha u)^n$ in the classical sense

$\Rightarrow$ show that they are equal also in the weak sense

$\Rightarrow u^n \to u$ in $L^p(\Omega)$

$D^\alpha u^n \to D^\alpha u$ in $L^p(\tilde{\Omega})$ $\Leftrightarrow$ $u^n \to u$ in $W^{k,p}(\tilde{\Omega})$

! WARNING! Never say it is true for $p = \infty$!

Remarks (Consequences):

• if Ω is open and connected then $u = \text{const.} \Leftrightarrow \frac{\partial u}{\partial x_i} = 0$ in Ω $\forall i = 1, \dots, d$

• $W^{1,1}(I)$, I is interval, $W^{1,1}(I) \hookrightarrow C(I)$

$X \hookrightarrow Y \Leftrightarrow (X \subset Y) \wedge (\exists c : \|u\|_Y \le c \|u\|_X)$

17.10.2018

$u^n := u * \eta_{\frac{1}{n}} \in C^\infty$ $u^n \to u$ in $W^{1,p}$

$D^\alpha u^n = (D^\alpha u)^n := D^\alpha u * \eta_{\frac{1}{n}}$

1) • if $\Omega \subseteq \mathbb{R}^d$ is ~~simply~~ connected open domain, then $\forall i\ \frac{\partial u}{\partial x_i} = 0 \Leftrightarrow u = \text{const.}$ in Ω

2) • $W^{1,1}(I) \hookrightarrow C(\bar{I})$, I interval

Proof: 1) $\varepsilon > 0$, $\Omega_\varepsilon := \{x \in \Omega, B_\varepsilon(x) \subseteq \Omega\} \Rightarrow \operatorname{dist}(x, \partial\Omega) \geq \varepsilon$; "$\Leftarrow$" is clear

"$\Rightarrow$" Ω_ε connected ($\Leftarrow \Omega$ open & connected)

$u_\varepsilon = u * \eta_\varepsilon$, $\forall x \in \Omega_\varepsilon \Rightarrow \frac{\partial u_\varepsilon}{\partial x_i} = \underbrace{\frac{\partial u}{\partial x_i}}_{=0} * \eta_\varepsilon = 0$

$\Rightarrow \frac{\partial u_\varepsilon}{\partial x_i} = 0$ in Ω_ε $\Rightarrow u_\varepsilon = \text{const}$ in Ω_ε $(\text{const} = \text{const}(\varepsilon))$

$x \in \Omega_\varepsilon$, $r > 0$, $B_r(x) \subseteq \Omega_\varepsilon$

$$\text{const} = \fint_{B_r(x)} u_\varepsilon(y)\,dy = \frac{1}{|B_r(x)|}\int_{B_r(x)} u_\varepsilon(y)\,dy = \frac{1}{|B_r(x)|}\int_{B_r(x)}\int_{\mathbb{R}^d} u(z)\,\eta_\varepsilon(z-y)\,dz\,dy$$

$$= \frac{1}{|B_r(x)|}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u(z)\,\eta_\varepsilon(z-y)\,\chi_{\{y \in B_r(x)\}}\,dz\,dy = \fint_{B_r(x)} u(y)\,dy$$

$\Rightarrow u = \text{const}$ in $\Omega_\varepsilon \xrightarrow{\varepsilon \to 0+} u = \text{const}$ in Ω

2, $W^{1,1}(\Omega) \hookrightarrow C(\bar{I})$ means: a) $u \in W^{1,1} \Rightarrow u \in C(\bar{I})$

b) $\exists K > 0\ \forall u \quad \|u\|_{C(\bar{I})} \le K\|u\|_{W^{1,1}(I)}$

For simplicity: $I = (0,1)$

Define auxiliary $v(x) := \int_0^x \frac{\partial u(y)}{\partial y}\,dy$, goal is to prove $\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x}$ in $(0,1)$

$\varphi \in C_0^\infty(0,1)$, $\quad \int_0^1 v(x)\frac{\partial \varphi}{\partial x}(x)\,dx = \int_0^1\int_0^x \frac{\partial u}{\partial y}(y)\,dy\,\frac{\partial \varphi}{\partial x}(x)\,dx$

$$\overset{\text{Fubini}}{=} \int_0^1\int_0^1 \frac{\partial u(y)}{\partial y}\frac{\partial \varphi(x)}{\partial x}\chi_{0<y<x}\,dy\,dx \overset{\text{Fubini}}{=} \int_0^1\int_0^1 \text{-- " --}\,dx\,dy$$

$$= \int_0^1\int_y^1 \frac{\partial u(y)}{\partial y}\underbrace{\frac{\partial \varphi(x)}{\partial x}}_{\in C_0^\infty}\,dx\,dy \overset{\text{IBP}}{=} -\int_0^1 \frac{\partial u(y)}{\partial y}\varphi(y)\,dy \overset{\text{def. of WD}}{\Longleftrightarrow} \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y}$$

From 1, : $\frac{\partial(u-v)}{\partial x} = 0$ in $(0,1)$ $\Rightarrow u = v + C$, $C = \text{const.}$

u- continuous:

$$|u(x_1) - u(x_2)| = |v(x_1) \overset{+C}{} - v(x_2) \overset{-C}{}| \overset{\text{def of } v}{=} \left|\int_0^{x_1}\frac{\partial u}{\partial x} - \int_0^{x_2}\frac{\partial u}{\partial x}\right| = \left|\int_{x_1}^{x_2}\frac{\partial u}{\partial x}\right| \le \int_{x_1}^{x_2}\left|\frac{\partial u}{\partial x}\right|$$

$x_1 \to x_2 \Rightarrow \int_{x_1}^{x_2}\left|\frac{\partial u}{\partial x}\right| \to 0 \Rightarrow u$ is continuous

Proof of 2, b) :

$$|v(x)| \le \left|\int_0^{\overset{x \in (0,1)}{x}} \frac{\partial u}{\partial y}\right| \le \int_0^1\left|\frac{\partial u}{\partial x}\right| \le \|u\|_{1,1}$$

$$|C| = |u(x) - v(x)| = \left|\int_0^1 u(x) - v(x)\,dx\right| \le \int_0^1 |u(x)| + |v(x)|\,dx \le \|u\|_1 + \|v\|_1 \le 2\|u\|_{1,1}$$

Difficult homework: $W^{d,1}((-1,1)^d) \hookrightarrow C(\overline{(-1,1)^d})$

2.3. Characterization of "being Sobolev"

Theorem: Let $\Omega \subseteq \mathbb{R}^d$, $p \in [1,\infty]$, $\delta > 0$, $\Omega_\delta := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \delta\}$

Then $\forall u \in W^{1,p}(\Omega)$: $\quad \| \Delta_i^h u \|_{L^p(\Omega_\delta)} \leq \| \frac{\partial u}{\partial x_i} \|_{L^p(\Omega)}$, $\quad \forall h, i, \delta$

where $\quad \Delta_i^h u(x) = \frac{1}{h}\left(u(x+he_i) - u(x)\right)$

e_i is the unit vector in the i-th direction

Also, if $\forall h, i, \delta$ $\quad \|\Delta_i^h u\|_{L^p(\Omega_\delta)} \leq c_i \quad$ (for $p>1$)

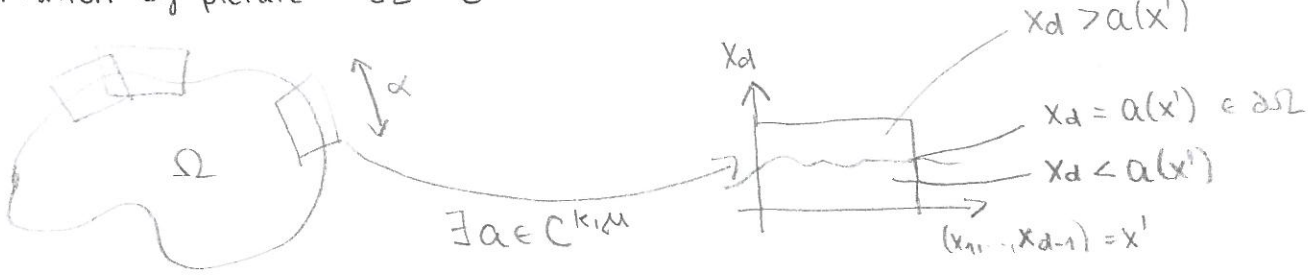
then $\frac{\partial u}{\partial x_i}$ exist $\forall i$ and $\quad \| \frac{\partial u}{\partial x_i} \|_{L^p(\Omega)} \leq c_i$.

Counterexample for $p=1$: $\quad u = \operatorname{sign} x$ in $(-1,1)$, $\quad u \notin W^{1,1}(-1,1)$

$$\int_{-1}^{1-h} |\Delta_h u| = \int_{-1}^{1-h} \frac{1}{h} |u(x+h) - u(x)| = \int_{-1}^{-h} \text{-''-} + \int_{-h}^{0} \text{-''-} + \int_{0}^{1-h} \text{-''-} = 0 + \int_{-h}^{0} \frac{2}{h} + 0 = 2$$

Remark for $p=1$: $\quad \|\Delta_i^h u\|_1 \leq c \implies u \in BV$

"Definition" by picture: $\Omega \in C^{k,\mu}$



Definition (Ω is of class $C^{k,\mu}$, $\Omega \in C^{k,\mu}$): Let $\Omega \subseteq \mathbb{R}^d$ open bounded set.

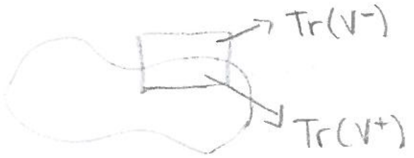
We say that $\Omega \in C^{k,\mu}$ ($\partial\Omega \in C^{k,\mu}$) iff:

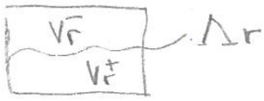
- there exist M coordinate systems $\quad x = (x_{r_1}, \dots, x_{r_d}) = (x_r', x_{r_d})$ and functions $a_r : \Delta_r \to \mathbb{R}$ where $\Delta_r := \{x_r' \in \mathbb{R}^{d-1}, |x_{r_i}| \leq \alpha\}$ s.t. $a_r \in C^{k,\mu}(\Delta_r)$
- denoting T_r the orthogonal transformation from (x_r', x_{r_d}) to (x', x_d), then $\forall x \in \partial\Omega \ \exists r \in \{1, \dots, M\}$ such that $x = T_r(x_r', a(x_{r_d}))$
- $\exists \beta > 0$, if we define $V_r^+ := \{(x_r', x_{r_d}) \in \mathbb{R}^d, x_r' \in \Delta_r : a(x_r') < x_{r_d} < a(x_r') + \beta\}$

$V_r^- := \{(x_r', x_{r_d}) \in \mathbb{R}^d, x_r' \in \Delta_r : a(x_r') - \beta < x_{r_d} < a(x_r')\}$

$\Lambda_r := \{(x_r', x_{r_d}) \in \mathbb{R}^d, x_r' \in \Delta_r : a(x_r') = x_{r_d}\}$

Then $T_r(V_r^+) \subset \Omega$, $T_r(V_r^-) \subset \mathbb{R}^d \setminus \overline{\Omega}$, $T_r(\Lambda_r) \subseteq \partial\Omega$ and $\bigcup_{r=1}^{M} T_r(\Lambda_r) = \partial\Omega$.





Theorem (density of smooth functions): Let $\Omega \in C^0$. Then $W^{k,p}(\Omega) = \overline{C^\infty(\bar\Omega)}^{\|\ \|_{k,p}}$, $p \in [1,\infty)$

Theorem (extension of Sobolev functions). Let $\Omega \in C^{0,1}$ (Ω is Lipschitz) and $k \in \mathbb{N}$, $p \in [1,\infty]$

Then there exists a continuous linear operator $E : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^d)$ such that:

1. $\|Eu\|_{W^{k,p}(\mathbb{R}^d)} \le c\,\|Eu\|_{W^{k,p}(\Omega)}$ (c is independent of u!)
2. $Eu = u$ a.e. in Ω.

Theorem (Trace theorem). Let $\Omega \in C^{0,1}$, $p \in [1,\infty]$ Then there exists a continuous linear operator $Tr : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ such that:

1. $\|Tr\,u\|_{L^p(\partial\Omega)} \le c\,\|u\|_{1,p}$
2. $\forall u \in W^{1,p}(\Omega) \cap C(\bar\Omega)$: $Tr\,u|_{\partial\Omega} = u|_{\partial\Omega}$.

Definition: $W_0^{k,p}(\Omega) := \overline{\{u \in C_0^\infty(\Omega)\}}^{\|\ \|_{k,p}}$

Theorem: Let $\Omega \in C^{0,1}$. Then $W_0^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega),\ Tr\,u = 0\}$.

Theorem (Embeddings): Let $\Omega \in C^{0,1}$ and let $p \in [1,\infty]$. Then

1. if $p<d$, then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $q \le \frac{dp}{d-p}$,
2. if $p=d$, then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $q < \infty$,
3. if $p>d$, then $W^{1,p}(\Omega) \hookrightarrow C^{0,1-\frac{d}{p}}(\bar\Omega)$.

Moreover,

1. — " — $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ for all $q < \frac{dp}{d-p}$,
2. — " — $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ for all $q < \infty$,
3. — " — $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow C^{0,\alpha}(\bar\Omega)$ for all $\alpha < 1 - \frac{d}{p}$.

$X \hookrightarrow\hookrightarrow Y \iff X \subseteq Y$ and if $A \subseteq X$ is bounded in X then it is precompact in Y.

$X \hookrightarrow\hookrightarrow Y \Rightarrow X \subseteq Y$ and if $\{u^n\}_{n=1}^\infty$ $\exists c$ $\|u^n\|_{1,p} \le c$

$\Rightarrow \exists\, u^{n_j}$ a subsequence and u such that

$u^{n_j} \to u$ in 1) L^q, $q < \frac{dp}{d-p}$

2) L^q, $q < \infty$

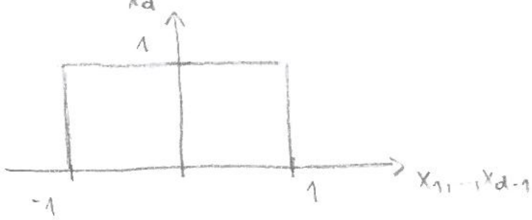
3) $C^{0,\alpha}$, $\alpha < 1 - \frac{d}{p}$

Remarks:

Extension. Why? – easier mollification

– for boundary regularity of $\Delta u = f$

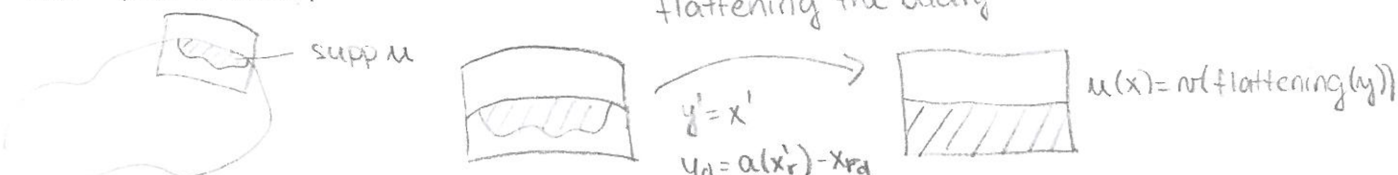
How? – flat domain $(-1,1)^{d-1} \times (0,1)$



$u \in W^{1,p}((-1,1)^{d-1} \times (0,1))$

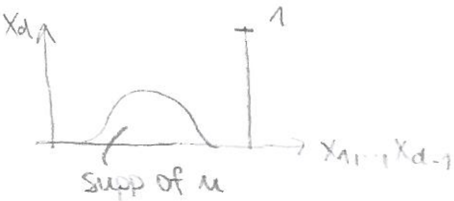
for $x_d < 0$, $u(x) := u(x_1, \ldots, x_{d-1}, -x_d)$

– non-flat domain



and go back

Trace.



$$\|u\|^p_{L^p(\partial\Omega)} = \int_{-1}^{1} |u(x',0)|^p \, dx' = \int_{-1}^{1} \left| \int_0^1 \frac{\partial u}{\partial x_d} \, dx_d \right|^p dx' \leq \int_{-1}^{1} \int_0^1 \left| \frac{\partial u}{\partial x} \right|^p dx' \, dx_d$$

I'm going inside, need to know the direction

for the non-flat bdary, I would do similar thing as before

Consequence of the Trace theorem: Let $\Omega \in C^{0,1}$. Then $\forall u \in W^{1,p}(\Omega)$ and $v \in W^{1,p'}(\Omega)$

we have **integration by parts**:

$$\int_\Omega u \frac{\partial v}{\partial x_i} \, dx = -\int_\Omega v \frac{\partial u}{\partial x_i} \, dx + \int_{\partial\Omega} u v \Big|_{\substack{u = \mathrm{Tr}\, u \\ v = \mathrm{Tr}\, v}} n_i \, ds$$

Easy homework: show that $\forall u, v \in W^{1,1}(a,b)$:

$$\int_a^b \frac{\partial u}{\partial x} v = u(b)v(b) - u(a)v(a) - \int_a^b u \frac{\partial v}{\partial x}$$

(prove IBP for Sobolev functions on real line)

Embeddings: • $u=|x|^{-\alpha}$, $\quad u\in L^q(B_1) \iff q\alpha<d \iff q<\frac{d}{\alpha}$

$\nabla u \sim |x|^{-\alpha-1} \quad u\in W^{1,p}(B_1) \iff p(\alpha+1)<d \iff p<\frac{d}{\alpha+1}$

if $p=\frac{d}{\alpha+1}$, $\quad q<\frac{dp}{d-p}$

$$1-\frac{d}{p} = 1-(\alpha+1) = -\alpha$$

(α corresponds to the optimal choice)

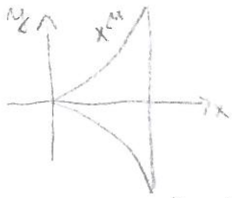
• take $u\in C_0^\infty(\mathbb{R}^d)$, maybe $\|u\|_q \le \|\nabla u\|_p \cdot C \quad \forall u$

if it is true $\forall u$ then also for $u_\lambda(x) := u(\lambda x)$

$$\lambda^{-\frac{d}{q}} \int_{\mathbb{R}^d} |u(y)|^q \underset{y=\lambda x}{=} \left(\int_{\mathbb{R}^d} |u(\lambda x)|^q\right)^{1/q} = \|u_\lambda\|_q \le C\,\|\nabla u_\lambda\|_p$$

$$= \left(\int |\nabla_x u(\lambda x)|^p\right)^{1/p} = \left(\int |\nabla u(\lambda x)|^p \lambda^p\right)^{1/p} = \lambda^{1-\frac{d}{p}}\left(\int|\nabla u|^p\right)^{1/p}$$

$$\Rightarrow -\frac{d}{q} = 1-\frac{d}{p} \iff q=\frac{dp}{d-p}$$



• necessity of $\Omega\in C^{0,1}$ for embedding:

take $\Omega_\mu := \{(x,y)\in\mathbb{R}^2,\ (x,y)\in(0,1)\times(-1,1),\ |y|<x^\mu\}$

$$u=x^\alpha,\quad \left.\begin{aligned}\int_{\Omega_\mu}|u|^q &= \int_0^1\int_{-x^\mu}^{x^\mu} x^{\alpha q}\,dy\,dx\\ \int_{\Omega_\mu}\left|\frac{\partial u}{\partial x}\right|^p &= \int_0^1\int_{-x^\mu}^{x^\mu}|\alpha x^{\alpha-1}|^p\,dy\,dx\end{aligned}\right\} \Rightarrow \left(u\in W^{1,p} \not\Rightarrow u\in L^{\frac{dp}{d-p}}\right)$$

Theorem (Poincaré): Let $\Omega\in C^{0,1}$ and $p\in[1,\infty]$. Let $\Omega_1,\Omega_2\subseteq\Omega$, $|\Omega_i|>0$ and $\Gamma_1,\Gamma_2\subseteq\partial\Omega$, $|\Gamma_i|_{d-1}>0$. Let $\alpha_1,\alpha_2\ge0$ and $\beta_1,\beta_2\ge0$ and at least one of $\alpha_1,\alpha_2,\beta_1,\beta_2>0$.

Then there exist $c_1,c_2>0$ such that $\forall u\in W^{1,p}(\Omega)$

$$c_1\|u\|_{1,p}^p \le \|\nabla u\|_p^p + \alpha_1\int_{\Omega_1}|u|^p + \alpha_2\left|\int_{\Omega_2}u\right|^p + \beta_1\int_{\Gamma_1}|u|^p+\beta_2\left|\int_{\Gamma_2}u\right|^p \le c_2\|u\|_{1,p}^p$$

$\left(\|u\|_{1,p}^p = \|u\|_p^p+\|\nabla u\|_p^p\right)$

Proof (of the first (the only difficult) inequality). By contradiction.

$\forall n\ \exists u^n\in W^{1,p}(\Omega)\quad n\|u^n\|_{1,p}^p > \|\nabla u^n\|_p^p + -\|-\qquad u^n\neq0,\ /\cdot\frac{1}{\|u^n\|_{1,p}}$

$v^n := \frac{u^n}{\|u^n\|_{1,p}}$, $\quad \|v^n\|_{1,p}=1$, $\quad \cancel{n\|v^n\|_{1,p}^p > \|\nabla v^n\|_p^p + \alpha_1\int_{\Omega_1}|v^n|^p + \alpha_2\left|\int_{\Omega_2}v^n\right|^p}$

$$n\|v^n\|_{1,p}^p > \|\nabla v^n\|_p^p + \alpha_1\int_{\Omega_1}|v^n|^p + \alpha_2\left|\int_{\Omega_2}v^n\right|^p + \beta_1\int_{\Gamma_1}|v^n|^p + \beta_2\left|\int_{\Gamma_2}v^n\right|^p$$

$$1 > \frac{1}{n}\left(-\|-\right)$$

$$n\to0 \Rightarrow \|\nabla v^n\|_p\to0 \Rightarrow \|v^n\|_p^p = \|v^n\|_p^p+\|\nabla v^n\|_p^p - \|\nabla v^n\|_p^p = 1-\|\nabla v^n\|_p^p \to 1$$

compact embedding:

$v^n \to v$ in $L^p(\Omega)$ $\quad(\|v\|_p = 1)$

$\nabla v^n \to 0$ in $L^p(\Omega)$

$v^n \to v$ in $W^{1,p}(\Omega)$ & $\nabla v = 0 \Rightarrow v = \text{const.}$ & $v \not\equiv 0$

contradiction: we will show that $v \equiv 0$.

from the rest of inequality we know that also

$$\alpha_1 \int_{\Omega_1} |v^n|^p \to 0, \quad \alpha_2 \Big|\int_{\Omega_2} v^n\Big|^p \to 0, \quad \beta_1 \int_{\Gamma_1} |v^n|^p \to 0, \quad \beta_2 \Big|\int_{\Gamma_2} v^n\Big|^p \to 0$$

$\downarrow$ Trace theorem

$$\alpha_1 \int_{\Omega_1} |v|^p = 0 \qquad \alpha_2 \Big|\int_{\Omega_2} v\Big|^p = 0 \qquad \beta_1 \int_{\Gamma_1} |v|^p = 0 \qquad \beta_2 \Big|\int_{\Gamma_2} v\Big|^p = 0$$

and at least one of $\alpha_i, \beta_i \neq 0$

but $v = \text{const} \neq 0$ - CONTRADICTION!

Chapter 3. Linear elliptic PDEs

Prototype

$$-\Delta u = f \quad \text{in } \Omega \qquad /\varphi;\ \varphi|_{\partial\Omega} = 0,\ /\int$$
$$u = 0 \quad \text{on } \partial\Omega$$

$$\int_\Omega \Delta u \, \varphi = \int_\Omega f \varphi$$

$$\int_\Omega \nabla u \nabla \varphi = \int_\Omega f \varphi \qquad \text{weak formulation}$$

$\forall f \in L^2(\Omega)\ \exists!\ u \in W_0^{1,2}(\Omega)\ \forall \varphi \in W_0^{1,2}(\Omega) \quad \int_\Omega \nabla u \nabla \varphi = \int_\Omega f\varphi$

$W_0^{1,2}(\Omega) = \{\varphi \in W^{1,2}(\Omega),\ Tr\,\varphi = 0\}$

one of the applications of the Poincaré inequality. $c_1 \|\nabla u\|_2 \le \|u\|_{W_0^{1,2}(\Omega)} \le c_2 \|\nabla u\|_2$

scalar product on $W_0^{1,2}(\Omega)$: $(u,\varphi)_{W_0^{1,2}(\Omega)} := \int_\Omega \nabla u \cdot \nabla \varphi$

if $f \in L^2(\Omega)$ $\exists F \in (W_0^{1,2}(\Omega))^*$: $\langle F, \varphi \rangle_{W_0^{1,2}(\Omega)} := \int_\Omega f\varphi$

Riesz representation theorem: $\forall F \in (W_0^{1,2}(\Omega))^*\ \exists!\ u \in W_0^{1,2}(\Omega)\ \forall \varphi \in W_0^{1,2}(\Omega)\ (u,\varphi)_{W_0^{1,2}(\Omega)} = \langle F, \varphi\rangle_{W}$

TOGETHER: $\forall f \in L^2(\Omega)\ \exists!\ u \in W_0^{1,2}(\Omega)\ \forall \varphi \in W_0^{1,2}(\Omega)$: $\int_\Omega \nabla u \cdot \nabla \varphi = \int_\Omega f\varphi$

Reality: $$Lu := -\sum_{i,j=1}^d \frac{\partial}{\partial x_i}\Big(a_{ij} \frac{\partial u}{\partial x_j}\Big) + bu + \sum_{i=1}^d c_i \frac{\partial u}{\partial x_i} + \sum_{i=1}^d \frac{\partial}{\partial x_i}(d_i u)$$

$a_{ij}, b, c_i, d_i : \Omega \to \mathbb{R}$

$(A)_{ij} = a_{ij}$, $\vec{c} = (c_1, \dots, c_d)$, $\vec{d} = (d_1, \dots, d_d)$

$$Lu = -\operatorname{div}(A\nabla u) + bu + \vec{c}\cdot\nabla u + \operatorname{div}(\vec{d}u)$$

Definition: Let $a_{ij}, b, c_i, d_i \in L^\infty(\Omega)$, where $\Omega \subseteq \mathbb{R}^d$ is bounded. We say that L is **elliptic** if $\exists c_1 > 0$ such that $\forall \xi \in \mathbb{R}^d$ and a.a. $x \in \Omega$:

$$A\xi . \xi \geq c_1 |\xi|^2 \qquad (A\xi . \xi) = \sum_{i,j=1}^{d} a_{ij}(x) \xi_i \xi_j$$

PROBLEM:

$$Lu = f \quad \text{in } \Omega$$

$$u = u_0 \quad \text{on } \Gamma_1 \qquad \textbf{Dirichlet bc}$$

$$(A\nabla u - \vec{d}u).\nu = g \quad \text{on } \Gamma_2 \qquad \textbf{Neuman bc}$$

$$(A\nabla u - \vec{d}u)\nu + \sigma u = g \quad \text{on } \Gamma_3 \qquad \textbf{Newton bc}$$

$\vec{\nu}$ - unit outward normal vector

DATA: f, u_0, σ, g

$\Gamma_1, \Gamma_2, \Gamma_3 \subseteq \partial\Omega$, $\Gamma_i \cap \Gamma_j = \emptyset \quad \forall i \neq j$, $|\Gamma_1 \cup \Gamma_2 \cup \Gamma_3| = |\partial\Omega|$

3.2. Notion of a weak solution

Take φ which is zero on Γ_1!

$$\Rightarrow \quad \int_\Omega Lu\varphi = \int_\Omega f\varphi$$

$$LHS = \int_\Omega -\operatorname{div}(A\nabla u)\varphi + bu\varphi + \vec{c}\nabla u\varphi + \operatorname{div}(\vec{d}u)\varphi$$

$$= \int_\Omega A\nabla u . \nabla\varphi - u\vec{d}.\nabla\varphi + \vec{c}.\nabla u\varphi + bu\varphi + \int_{\partial\Omega} -A\nabla u.\vec{\nu}\varphi + u\varphi\vec{d}.\vec{\nu}$$

$$\int_{\partial\Omega} -A\nabla u . \vec{\nu}\varphi + u\varphi\vec{d}.\vec{\nu} \overset{\varphi = 0 \text{ on } \Gamma_1}{=} \int_{\Gamma_2} + \int_{\Gamma_3} = -\underbrace{\int_{\Gamma_2} g\varphi}_{\text{Neuman}} - \underbrace{\int_{\Gamma_3} (g - \sigma u)\varphi}_{\text{Newton}}$$

$$\underbrace{\int_\Omega A\nabla u.\nabla\varphi - u\vec{d}.\nabla\varphi + \vec{c}.\nabla u\varphi + bu\varphi + \int_{\Gamma_3} \sigma u\varphi}_{=: B_{L,\sigma}(u,\varphi)} = \int_\Omega f\varphi + \int_{\Gamma_2 \cup \Gamma_3} g\varphi$$

I have just proved a lemma:

Lemma: If u is a classical solution, then $\forall \varphi \in C^1(\bar{\Omega}), \varphi = 0$ on Γ_1: $B_{L,\sigma}(u,\varphi) = \int_\Omega f\varphi + \int_{\Gamma_2 \cup \Gamma_3} g\varphi$ (*)

Lemma: If $u \in C^2(\bar{\Omega})$ and $A, b, \vec{c}, \vec{d}$ are smooth and (*) holds $\forall \varphi \in C^1, \varphi|_{\Gamma_1} = 0$ and $u = u_0$ on Γ_1, then u is a classical solution.

Proof of 2nd lemma:

1. Step $\varphi \in C_0^\infty(\Omega)$ and use it in (*): $B_{L,\sigma}(u,\varphi) = \int_\Omega f\varphi + \int_{\Gamma_2 \cup \Gamma_3} g\varphi$

φ compactly supported: $\int_\Omega Lu\varphi = \int_\Omega A\nabla u.\nabla\varphi - \vec{d}.\nabla\varphi u + \vec{c}.\nabla u\varphi + bu\varphi = \int_\Omega f\varphi$

fundamental theorem $\Rightarrow$ $Lu = f$ a.e. in Ω

2. step $\varphi \in C^1(\bar{\Omega})$, $\varphi|_{\Gamma_1} = 0$

$$\int_{\Omega} A\nabla u\cdot\nabla\varphi - \vec{a}\cdot\nabla\varphi u + \vec{c}\cdot\nabla u\varphi + bu\varphi + \int_{\Gamma_3}\sigma u\varphi = \int_{\Omega} f\varphi + \int_{\Gamma_2\cup\Gamma_3} g\varphi$$

$$\int_{\Omega} Lu\varphi + \int_{\partial\Omega} A\nabla u\cdot\vec{\nu}\varphi - u\vec{a}\cdot\vec{\nu}\varphi + \int_{\Gamma_3}\sigma u\varphi$$

from the 1st step: $\int_{\Gamma_2\cup\Gamma_3} A\nabla u\cdot\vec{\nu}\varphi - u\vec{a}\cdot\vec{\nu}\varphi + \int_{\Gamma_3}\sigma u\varphi = \int_{\Gamma_2\cup\Gamma_3} g\varphi$

φ is arbitrary $\Rightarrow$ $(A\nabla u - \vec{a}u)\cdot\vec{\nu} = g$ on Γ_2

$$(A\nabla u - \vec{a}u)\vec{\nu} + \sigma u = g \quad \text{on } \Gamma_3$$

Definition (**Weak solution**). Let $\Omega \subseteq \mathbb{R}^d$ Lipschitz, L be an elliptic operator, $u_0 \in W^{1,2}(\Omega)$, $f \in (W^{1,2}(\Omega))^*$, $g \in L^2(\Gamma_2 \cup \Gamma_3)$. We say that $u \in W^{1,2}(\Omega)$ is a weak solution iff

- $Tr\, u = Tr\, u_0$ on Γ_1 and
- $B_{L\sigma}(u,\varphi) = \langle f,\varphi\rangle + \int_{\Gamma_2\cup\Gamma_3} g\varphi \quad \forall \varphi \in V$,

where $V := \{\varphi \in W^{1,2}(\Omega),\ Tr\,\varphi = 0 \text{ on } \Gamma_1\}$

Remarks: - Why $f \in V^*$?? consider $f = \operatorname{div}\vec{N}$, $\vec{N} \in C_0(\Omega)$, but $\vec{N}$ is not Sobolev

$$\langle f,\varphi\rangle \sim \int_{\Omega} f\varphi \sim \int_{\Omega} \operatorname{div}\vec{N}\varphi = -\int_{\Omega} \vec{N}\nabla\varphi$$

$$\langle f,\varphi\rangle := -\int_{\Omega} \vec{N}\cdot\nabla\varphi \quad \Rightarrow \quad f \in V^*$$

- Why we assume $\exists u_0 \in W^{1,2}(\Omega)$ and not $u_0 \in L^2(\Gamma_1)$

Q. $Tr\ W^{1,2}(\Omega) \xrightarrow{\text{on}} X = W^{\frac{1}{2},2}(\partial\Omega)$, $u_0 \in W^{\frac{1}{2},2}(\partial\Omega)$ correct formulation

In PDE theory, we do not need the sharp characterization.

Example:
$$\underbrace{-\frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2}}_{-\operatorname{div}(\nabla u)} + \underbrace{\frac{1}{10}(\operatorname{sign} x_1)\frac{\partial^2 u}{\partial x_1\partial x_2}}_{} = 1 \quad \text{in } (-1,1)^2$$

$u = 0$ on $\partial(-1,1)^2$

CORRECT WAY: $\frac{\partial}{\partial x_2}\left(\operatorname{sign} x_1 \frac{\partial u}{\partial x_1}\right)$

NEVER DO THAT: $\cancel{\frac{\partial}{\partial x_1}\left(\operatorname{sign} x_1 \frac{\partial u}{\partial x_2}\right) - \delta_0 \frac{\partial u}{\partial x_2}}$

Dirac $\Rightarrow$ this has to be continuous !

$\Rightarrow$ the weak formulation:

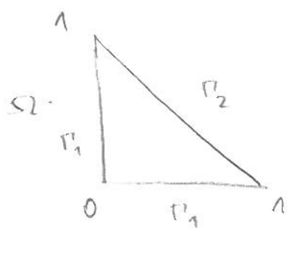
Let $\varphi \in W_0^{1,2}(\Omega)$, look for $u \in W_0^{1,2}(\Omega)$ such that

$$\int_{\Omega} \nabla u\cdot\nabla\varphi - \frac{1}{10}\operatorname{sign} x_1 \frac{\partial u}{\partial x_1}\cdot\frac{\partial\varphi}{\partial x_2} = \int_{\Omega}\varphi$$

Homework:
$$-\frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} + \frac{1}{2}\frac{\partial^2 u}{\partial x_1\partial x_2} = f \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \Gamma_1$$

$$2\frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} = 0 \quad \text{on } \Gamma_2$$



find weak formulation.

3.3 Existence of solution for "coercive" operators

"coercive" ~ for which we can get nice estimates

Problem: We have V - Hilbert space, we want to find u such that $v := u - u_0 \in V$

$$B_{L,\sigma}(u,\varphi) = \langle F,\varphi\rangle_V \quad \forall \varphi \in V$$

$$\Longleftrightarrow \quad B_{L,\sigma}(v,\varphi) = \langle F,\varphi\rangle_V + B_{L,\sigma}(u_0,\varphi) =: \langle \tilde{F},\varphi\rangle_V$$

Definition (**elliptic form**): Let $B: V \times V \to \mathbb{R}$ bilinear and V be a Hilbert space, $c_1, c_2 > 0$.

We say that B is **elliptic** if it is:

1) V-bounded $\Leftrightarrow$ $|B(u,\varphi)| \le c_2 \|u\|_V \|\varphi\|_V$

2) V-coercive $\Leftrightarrow$ $B(u,u) \ge c_1 \|u\|_V^2$

Theorem (**Lax-Milgram**): Let B be a bilinear elliptic form. Then

$$\forall F \in V^* \; \exists! u \in V \; \forall \varphi \in V: B(u,\varphi) = \langle F,\varphi\rangle$$

Definition: Let $\mathcal{B}: V \to V^*$. We say that $\mathcal{B}$ is

1) **Lipschitz** $\Leftrightarrow$ $\forall u,v \in V: \; \|\mathcal{B}(u) - \mathcal{B}(v)\|_{V^*} \le c_2 \|u-v\|_V \quad , c_2 > 0$

2) **uniformly monotone** $\Leftrightarrow$ $\forall u,v \in V: \; \langle \mathcal{B}(u) - \mathcal{B}(v), u-v\rangle_V \ge c_1 \|u-v\|_V^2, \; c_1 > 0$

Theorem (**non-linear Lax-Milgram**): Let $\mathcal{B}$ be Lipschitz continuous and uniformly monotone. Then

$$\forall F \in V^* \; \exists! u \in V \; \forall \varphi \in V: \; \langle \mathcal{B}(u),\varphi\rangle = \langle F,\varphi\rangle$$

Proof of L-M using the nL-M:

Define $\mathcal{B}(u): V \to V^*$: $\langle \mathcal{B}(u),\varphi\rangle := B(u,\varphi)$

1) $\mathcal{B}(u)$ is Lipschitz:

$$\|\mathcal{B}(u) - \mathcal{B}(v)\|_{V^*} = \sup_{\varphi \in V, \|\varphi\|_V \le 1} \langle \mathcal{B}(u) - \mathcal{B}(v),\varphi\rangle \overset{\text{def of } \mathcal{B}}{=} \sup_{\|\varphi\|_V \le 1} [B(u,\varphi) - B(v,\varphi)]$$

$$\overset{\text{linearity of } B}{=} \sup_{\|\varphi\| \le 1} B(u-v,\varphi) \le \sup_{\|\varphi\|_V \le 1} c_2 \|u-v\|_V \|\varphi\|_V = c_2 \|u-v\|_V$$

2) $\mathcal{B}(u)$ is uniformly monotone:

$$\langle \mathcal{B}(u) - \mathcal{B}(v), u-v\rangle \overset{\text{def of } \mathcal{B}}{=} B(u-v,u-v) \overset{\text{V-ellipticity of } B}{\ge} c_1 \|u-v\|_V^2$$

$\Rightarrow \mathcal{B}$ satisfies assumptions of nL-M $\Rightarrow$ $\forall F \in V^* \; \exists! u \in V \; \forall \varphi \in V: B(u,\varphi) = \langle \mathcal{B}(u),\varphi\rangle = \langle F,\varphi\rangle$.

Proof of nL-M:

Uniqueness: $u,v \in V$, $\langle \mathcal{B}(u),\varphi\rangle = \langle F,\varphi\rangle$, $\langle \mathcal{B}(v),\varphi\rangle = \langle F,\varphi\rangle \quad \forall \varphi \in V$

$\Rightarrow \langle \mathcal{B}(u) - \mathcal{B}(v),\varphi\rangle = 0 \quad \forall \varphi \in V$, $\varphi := u - v$ $\Rightarrow$ $\langle \mathcal{B}(u) - \mathcal{B}(v), u-v\rangle = 0 \overset{\text{monotonicity}}{\ge} c_1 \|u-v\|_V^2 \Rightarrow u = v$

Existence: $\langle B(u),\varphi\rangle = \langle F,\varphi\rangle \quad \forall\varphi$

$$\Leftrightarrow \quad (u,\varphi)_V = (u,\varphi)_V - \varepsilon\left(\langle B(u),\varphi\rangle - \langle F,\varphi\rangle\right) \quad ,\varepsilon>0 \quad \forall\varphi$$

Define a problem for $v\in V$. Find $u\in V$ s.t.

$$(u,\varphi)_V = (v,\varphi)_V - \varepsilon\left(\langle B(v),\varphi\rangle - \langle F,\varphi\rangle\right) \quad \forall\varphi \qquad (*)$$

Define $M: V\to V$, $v\mapsto u$, u solve $(*)$. If M has a fixed point $(M(v)=v)$, then I find a solution to the original problem.

1. M is well-defined. For given $v\in V$, define $\tilde F\in V^*$: $\langle \tilde F,\varphi\rangle := (v,\varphi)_V - \varepsilon(\langle B(v),\varphi\rangle - \langle F,\varphi\rangle)$
   $\langle \tilde F,\varphi\rangle$ linear in φ, Riesz tells me that $\forall \tilde F\in V^*\ \exists!\, u\in V\ \forall\varphi\in V$: $(u,\varphi)_V = \langle\tilde F,\varphi\rangle$
2. M has a fixed point. We show that $\exists\,\delta>0\ \forall u,v\in V$

$$\|M(u)-M(v)\|_V \le (1-\delta)\,\|u-v\|_V \quad (M \text{ is a contraction}) \overset{\text{Banach}}{\Rightarrow} M \text{ has a fixed point}$$

$\bar u = M(u)$, $\bar v = M(v)$

$$(\bar u,\varphi)_V = (u,\varphi)_V - \varepsilon(\langle B(u),\varphi\rangle - \langle F,\varphi\rangle)$$

$$(\bar v,\varphi)_V = (v,\varphi)_V - \varepsilon(\langle B(v),\varphi\rangle - \langle F,\varphi\rangle)$$

$$\Rightarrow (\bar u-\bar v,\varphi)_V = (u-v,\varphi)_V - \varepsilon\langle B(u)-B(v),\varphi\rangle$$

Riesz $\Rightarrow \exists\, w_1, w_2$: $(w_1,\varphi)_V = \langle B(u),\varphi\rangle$, $(w_2,\varphi) = \langle B(v),\varphi\rangle_V$

$$\Rightarrow (\bar u-\bar v,\varphi)_V = (u-v-\varepsilon(w_1-w_2),\varphi)_V$$

$$\|M(u)-M(v)\|_V^2 = \|\bar u-\bar v\|_V^2 = \|u-v-\varepsilon(w_1-w_2)\|_V^2$$

$$= \|u-v\|_V^2 - 2\varepsilon(u-v, w_1-w_2) + \varepsilon^2\|w_1-w_2\|_V^2 \qquad (G)$$

$$(u-v, w_1-w_2) = \langle B(u)-B(v), u-v\rangle \ge c_1\|u-v\|_V^2 \qquad !$$

$$\|w_1-w_2\|_V^2 = (w_1-w_2, w_1-w_2)_V = \langle B(u)-B(v), w_1-w_2\rangle \le \|B(u)-B(v)\|_{V^*}\|w_1-w_2\|$$

$$\Rightarrow \|w_1-w_2\|_V^2 \le \|B(u)-B(v)\|_{V^*}^2 \overset{B \text{ Lipschitz}}{\le} c_2\|u-v\|_V^2 \qquad !$$

$! + !$ in (G)

$$\|M(u)-M(v)\|_V^2 \le \|u-v\|_V^2 - 2\varepsilon c_1\|u-v\|_V^2 + \varepsilon^2 c_2\|u-v\|_V^2 = (1-2\varepsilon c_1+\varepsilon^2 c_2)\|u-v\|_V^2$$

$\delta := 2\varepsilon c_1 - \varepsilon^2 c_2$, $0<\varepsilon$ sufficiently small: $\quad = (1-\delta)\|u-v\|_V^2$

How to apply L-M:

We want to find u, $u-u_0 \in V$, $B_{L,\sigma}(u,\varphi) = \langle f,\varphi\rangle + \int_{\Gamma_2\cup\Gamma_3} g\varphi$

$w := u - u_0$, $B_{L,\sigma}(w,\varphi) = \langle f,\varphi\rangle + \int_{\Gamma_2\cup\Gamma_3} g\varphi - B_{L,\sigma}(u_0,\varphi)$

Theorem: If $B_{L,\sigma}$ is bilinear, V-bounded and V-elliptic. Then there exists a unique WS u.

Proof. u_1, u_2 are solutions $\Rightarrow B_{L,\sigma}(u_1-u_2,\varphi) = 0 \quad \forall \varphi \in V$

$u_1 - u_0 \in V$, $u_2 - u_0 \in V \Rightarrow u_1 - u_2 \in V$

$\varphi := u_1 - u_2$: $B_{L,\sigma}(u_1-u_2, u_1-u_2) = 0 \geq c_1 \|u_1-u_2\|_V^2$

$\Rightarrow u_1 = u_2$

Existence: $\langle \tilde{F}, \varphi\rangle_V = \langle f,\varphi\rangle_V + \int_{\Gamma_2\cup\Gamma_3} g\varphi - B_{L,\sigma}(u_0,\varphi)$

claim $\tilde{F} \in V^*$ (all objetcs on the RHS exist and are well-defined)

$\tilde{F}$ is well-defined: $\cdot \langle f,\varphi\rangle_V$ OK

$\cdot \left|\int_{\Gamma_2\cup\Gamma_3} g\varphi\right| \leq \|g\|_{L^2(\Gamma_2\cup\Gamma_3)} \|\varphi\|_{L^2(\Gamma_2\cup\Gamma_3)} \overset{\text{trace}}{\leq} c\, \|g\|_{L^2(\Gamma_2\cup\Gamma_3)} \|\varphi\|_{1,2}$

$\cdot\ |B_{L,\sigma}(u_0,\varphi)| = \left|\int_\Omega A\nabla u_0 \nabla\varphi + b u_0\varphi + \vec{c}\nabla u_0 \varphi + \vec{d}\nabla\varphi u_0 + \int_{\Gamma_3}\sigma u_0\varphi\right|$

$\overset{\text{Hölder}}{\leq} \|A\|_\infty \|\nabla u_0\|_2 \|\nabla\varphi\|_2 + \|b\|_\infty \|u_0\|_2 \|\varphi\|_2 + \|\vec{c}\|_\infty \|\nabla u_0\|_2 \|\varphi\|_2$

$+ \|\vec{d}\|_\infty \|u_0\|_2 \|\nabla\varphi\|_2 + \|\sigma\|_\infty \|u\|_{L^2(\partial\Omega)} \|\varphi\|_{L^2(\partial\Omega)}$

$\leq c(A, b, \vec{c}, \vec{d}, \sigma, \Omega) \|u_0\|_{1,2} \|\varphi\|_{1,2}$

$\Rightarrow B_{L,\sigma}$ is V-bdd, $\tilde{F} \in V^*$ is well defined

+ assumption V-ellipticity $\overset{\text{L-M}}{\Rightarrow} \exists!\ u$ a solution

3.4 Examples of V-elliptic operators

1. Neuman problem : $\Gamma_1, \Gamma_3 = \emptyset$ $(\partial\Omega = \Gamma_2)$ and $b = c_i = d_i = 0$

$-\mathrm{div}(A\nabla u) = f$ in Ω

$A\nabla u \cdot \vec{\nu} = g$ on $\partial\Omega$

Weak formulation: $\int_\Omega A\nabla u \cdot \nabla\varphi = B_{L,\sigma}(u,\varphi) = \langle f,\varphi\rangle + \int_{\partial\Omega} g\varphi \quad \forall \varphi \in W^{1,2}(\Omega)$

Does the sol. exist? Not in general. $\varphi = 1$ is a possible choice,

then necessary $\langle f, 1\rangle + \int_{\partial\Omega} g = 0$

What is then the good space, where I should look for solution?

$V := \{u \in W^{1,2}(\Omega);\ \int_\Omega u = 0\}$

Theorem: Let f, g satisfy $\langle f,1\rangle + \int_{\partial\Omega} g = 0$ and $A\xi\cdot\xi \geq c_1|\xi|^2$. Then $\exists!\ u \in V$: $\int_\Omega A\nabla u \cdot \nabla\varphi = \langle f,\varphi\rangle + \int_{\partial\Omega} g\varphi \quad \forall\varphi$.

Proof (hints): $B(u,\varphi) = \int_\Omega A\nabla u.\nabla\varphi$

$B(u,u) \geq c_1 \|\nabla u\|_2^2 \overset{\text{Poincaré}}{\geq} K\|u\|_{1,2}^2$

2. Existence of a solution if $\Gamma_1 \neq \emptyset$ or $\Gamma_3 \neq \emptyset$

31.10.2018

- it is enough to show that $B_{L,\sigma}(u,v)$ is V-coercive / elliptic

$\Leftrightarrow \quad \exists K \quad B_{L,\sigma}(u,u) \geq K\|u\|_V^2 \quad \forall u \in V$

$$B_{L,\sigma}(u,u) = \int_\Omega A\nabla u.\nabla u + bu^2 + \vec{c}\,\nabla u\, u - \vec{d}.\nabla u\, u + \int_{\Gamma_3}\sigma u^2$$

$$\overset{\text{ellipticity}}{\geq} \int_\Omega c_1|\nabla u|^2 - \underbrace{(|\vec{c}|+|\vec{d}|)\,|\nabla u|\,|u|}_{} + bu^2 + \int_{\Gamma_3}\sigma u^2$$

$$\left(\sqrt{c_1}|\nabla u| \cdot \frac{|u|(|\vec{c}|+|\vec{d}|)}{\sqrt{c_1}}\right) \overset{\text{Young ineq}}{\leq} \frac{1}{2}\left(\sqrt{c_1}|\nabla u|\right)^2 + \frac{1}{2}\left(\frac{|u|(|\vec{c}|+|\vec{d}|)}{\sqrt{c_1}}\right)^2 = \frac{c_1}{2}|\nabla u|^2 + \frac{|u|^2(|\vec{c}|+|\vec{d}|)^2}{2c_1}$$

$$\geq \int_\Omega \frac{c_1}{2}|\nabla u|^2 + u^2\left(b - \frac{(|\vec{c}|+|\vec{d}|)^2}{2c_1}\right) + \int_{\Gamma_3}\sigma u^2 \quad =: (*)$$

$a_+ = \max\{0,a\}, \quad a_- = \min\{0,a\}$

Case 1. $|\Gamma_1| > 0 \Rightarrow$ Poincaré, $\exists c_P \quad \|\nabla u\|_2^2 \geq c_P\|u\|_V^2$

$$(*) \geq \frac{c_1}{2} c_P\|u\|_V^2 - \|u\|_2^2 \left\|\left(b - \frac{(|\vec{c}|+|\vec{d}|)^2}{2c_1}\right)_-\right\|_\infty - \|u\|_{L^2(\Gamma_3)}^2 \|\sigma_-\|_\infty$$

$$\geq \frac{c_1}{2} c_P\|u\|_V^2 - \|u\|_V^2 \quad '' \quad - c(\text{trace})\|u\|_V^2\|\sigma_-\|_\infty$$

$$= \Big[\underbrace{\frac{c_1}{2}c_P - \Big\|\Big(b - \frac{(|\vec{c}|+|\vec{d}|)^2}{2c_1}\Big)_-\Big\|_\infty - c(\text{trace})\|\sigma_-\|_\infty}_{\text{number}}\Big]\|u\|_V^2$$

$\exists\, \varepsilon > 0$ (depending on Ω and c_1 such that if $b - \frac{(|\vec{c}|+|\vec{d}|)^2}{2c_1} \geq -\varepsilon$ in Ω

and $\sigma \geq -\varepsilon$ on $\Gamma_3 \quad \Rightarrow \quad$ Number > 0

Case 2. $|\Gamma_1| = 0$ and $|\Gamma_3| \neq 0$

$$B_{L,\sigma}(u,u) \geq \int_\Omega c_1|\nabla u|^2 + |u|^2\left(b - \frac{(|\vec{c}|+|\vec{d}|)^2}{2c_1}\right) + \int_{\Gamma_3}\sigma u^2$$

Subcase 2a. $\quad b - \frac{(|\vec{c}|+|\vec{d}|)^2}{2c_1} \geq 0$ in Ω and $\int_\Omega b - \frac{(|\vec{c}|+|\vec{d}|)^2}{2c_1} > 0$

$$(*) \overset{\text{Poincaré}}{\geq} c_P(c_1, b, \vec{c}, \vec{d})\|u\|_V^2 - \|\sigma_-\|_\infty\|u\|_{L^2(\Gamma_3)} \geq (c_P - c(\text{trace})\|\sigma_-\|_\infty)\|u\|_V^2$$

$\Rightarrow \exists\, \varepsilon > 0$ if $\sigma \geq -\varepsilon \Rightarrow B_{L,\sigma}$ is elliptic

Subcase 2b. $\quad \sigma \geq 0$ on Γ_3 and $\int_{\Gamma_3}\sigma > 0$

$$(*) \overset{\text{Poincaré}}{\geq} c_P\|u\|_V^2 - \|u\|_V^2\left\|\left(b - \frac{(|\vec{c}|+|\vec{d}|)^2}{2c_1}\right)_-\right\|_\infty$$

$\Rightarrow \exists\, \varepsilon > 0$ if $b - \frac{(|\vec{c}|+|\vec{d}|)^2}{2c_1} \geq -\varepsilon \Rightarrow B_{L,\sigma}$ is elliptic

Easy homework: $\vec{d}$ and $\vec{c} \in C^1$, $\operatorname{div}(\vec{d}-\vec{c}) \geq 0$, $\Gamma_1 = \partial\Omega$ and $b \geq 0$

$\overset{?}{\Rightarrow} \exists!\, u$ a solution

You should see that $\operatorname{div}(\vec{d}-\vec{c}) + 2b \geq 0$ is enough.

3.5. Existence via Fredholm alternative

Lemma (FA): Let H be a Hilbert space and $K: H \to H$ be linear compact operator.

(F1) $N(I-K)$ has finite dimension (N.. kernel: $u \in N(I-K) \iff (I-K)u = 0$)

(F2) $R(I-K)$ is closed (R.. range: $u \in R(I-K) \iff \exists w \in H: (I-K)w = u$)

(F3) $R(I-K) = (N(I-K^*))^\perp$ ($u \in R(I-K)$, $w \in N(I-K^*)^\perp \iff (u,w)_H = 0$)

(F4) $N(I-K) = \{0\} \iff R(I-K) = H$

(F5) $\dim(N(I-K)) = \dim(N(I-K^*)) < \infty$

(F6) Spectrum of K is at most countable and if it is infinite then zero is the only attracting point

Theorem (FA-PDE). Let $\Omega \in C^{0,1}$, $u_0 = 0$ and $\Gamma_1 = \partial\Omega$ and L be an elliptic Operator. Then

1. either $\forall f \in L^2(\Omega)\ \exists!\, u \in W_0^{1,2}(\Omega): Lu = f$ in Ω or $\exists u \neq 0: Lu = 0$

2. denote $N_L := \{u \in V,\ Lu = 0\}$ $\quad B_L(u,\varphi) = 0\ \forall \varphi \in V$

$N_{L^*} := \{\varphi \in V,\ L^*\varphi = 0\}$

Then N_L and N_{L^*} are closed subspaces of V and $\dim N_L = \dim N_{L^*} < \infty$.

3. for $f \in L^2$: $(\exists u \in V.\ Lu = f) \iff (\forall \varphi \in N_{L^*}: \int_\Omega f\varphi = 0)$

Here $Lu = -\sum_{i,j=1}^d \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^d c_i \frac{\partial u}{\partial x_i} + \frac{\partial}{\partial x_i}(d_i u) + bu$

$L^*\varphi = -\sum_{i,j=1}^d \frac{\partial}{\partial x_i}\left(a_{ji}\frac{\partial \varphi}{\partial x_j}\right) - \left(\sum_{i=1}^d \frac{\partial}{\partial x_i}(c_i\varphi) + d_i\frac{\partial \varphi}{\partial x_i}\right) + b\varphi$

$Lu = f \overset{\text{weak}}{\iff} \forall \varphi\ B_L(u,\varphi) = \int_\Omega f\varphi \quad \left(B_L(u,\varphi) = \int_\Omega A\nabla u\cdot\nabla\varphi + \vec{c}\cdot\nabla u\varphi + bu\varphi - \vec{d}u\cdot\nabla\varphi\right)$

$L^*\varphi = g \iff \forall u\ B_{L^*}(\varphi,u) = \int_\Omega gu \quad \left(B_{L^*}(\varphi,u) = \int_\Omega A^T\nabla\varphi\cdot\nabla u + \vec{c}\cdot\nabla u\varphi + b\varphi u - \vec{d}\cdot\nabla\varphi u\right)$

$\Rightarrow B_L(u,\varphi) = B_{L^*}(\varphi,u)$!!!

Formally $B_L(u,\varphi) = B_{L^*}(\varphi,u)$ "$\iff$" $\langle Lu,\varphi\rangle_{(W_0^{1,2}(\Omega))^*} = \langle u, L^*\varphi\rangle_{(W_0^{1,2}(\Omega))^*}$

Proof of FA-PDE:

Step 1. To find a proper compact operator.

From L-M I know $\exists \gamma \geq 0$: $\forall f \in L^2(\Omega)\ \exists!\, u \in V.\ Lu + \gamma u = f$ $\quad (B_{L_\gamma}(u,\varphi) = B_L(u,\varphi) + \gamma\int_\Omega u\varphi$

$\left(\gamma + b + \frac{|\vec{c}| + |\vec{d}|}{2c_1}\right) \geq -\varepsilon$

$L_\gamma^{-1}: L^2(\Omega) \to L^2(\Omega)$, $f \mapsto u$, where u solve $Lu + \gamma u = f$, $\quad \|u\|_{1,2} \leq c\|f\|_2$

but I know $L_\gamma^{-1}: L^2(\Omega) \overset{\text{bdd}}{\to} W_0^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$, therefore L_γ^{-1} is compact in the same way: L_γ^{*-1}, $f \mapsto \varphi$, $L^*\varphi + \gamma\varphi = f$

Step 2. u, φ solve $Lu = f$ and equivalent notions:

$Lu + \gamma u = f + \gamma u$

$$\left(\begin{array}{ll} B_L(u,\varphi) = \int_\Omega f\varphi & \forall \varphi \in V \\ B_{L,\gamma}(u,\varphi) = \int_\Omega (f+\gamma u)\varphi & \forall \varphi \in V \end{array} \right)$$

now $\quad u = L_\gamma^{-1}(f + \gamma u)$

$u - \gamma L_\gamma^{-1} u = \gamma L_\gamma^{-1}\left(\frac{f}{\gamma}\right)$; define $K := \gamma L_\gamma^{-1}$; $K^* := \gamma L_\gamma^{*-1}$

$(I-K)u = K\left(\frac{f}{\gamma}\right)$ (I do not yet know that K, K^* are adjoint)

Step 3. Recall what we want in 1. $(\forall f \ \exists! u, Lu = f) \Leftrightarrow$ (there is no $u \neq 0$, $Lu = 0$)

if $\exists \tilde{u} \neq 0$, $L\tilde{u} = 0$, then if u solves $Lu = f$, then also $L(u+\tilde{u}) = f$, nonuniqueness!

(u,f) solve $Lu = f \iff (I-K)u = K\left(\frac{f}{\gamma}\right)$

(F4) $\Rightarrow$ if $N(I-K) = \{0\}$ then $\forall b \in L^2 \ \exists! u \in L^2 : (I-K)u = b$

$(K(0) = 0 \ \wedge \ Lu = 0) \Rightarrow ((I-K)u = K\left(\frac{0}{\gamma}\right) = 0)$

if there is no $u \neq 0$ $Lu = 0 \Rightarrow$ there is no $u \neq 0$ $(I-K)u = 0$

$\Rightarrow b := K\left(\frac{f}{\gamma}\right)$ $\exists! u \in L^2(\Omega)$ $(I-K)u = K\left(\frac{f}{\gamma}\right)$

$u = \underbrace{K\left(\frac{f}{\gamma}\right)}_{\in W_0^{1,2}} + \underbrace{Ku}_{\in W_0^{1,2}}$ $\Rightarrow u \in W_0^{1,2}(\Omega)$ - go back: $(I-K)u = K\left(\frac{f}{\gamma}\right) \iff Lu = f$

Step 4. Proof of 2. $[N_L := \{u, Lu = 0\} \wedge (Lu = 0 \iff (I-K)u = 0)] \Rightarrow N_L = N(I-K) \overset{(F5)}{\Rightarrow} 2.$

The same way for $N_{L^*} = N(I-K^*) \overset{(F5)}{\Rightarrow} 2.$

? $\dim N_{L^*} = \dim N_L$ we need that K^* is adjoint to K!

K^* is adjoint to K: $\forall u, \varphi \in L^2$: $\int_\Omega Ku\varphi = \int_\Omega u K^*\varphi$

$\gamma \int_\Omega Ku\varphi = \gamma \int_\Omega u K^*\varphi \iff \gamma \int_\Omega v\varphi = \gamma \int_\Omega u w$ (*)

$\exists! v$, $v = \gamma L_\gamma^{-1} u \iff L_\gamma v = \gamma u \iff B_L(v,\psi) + \gamma \int_\Omega v\psi = \int_\Omega \gamma u \psi \quad \forall \psi$

$\exists! w$, $w = \gamma L_\gamma^{*-1} \varphi \iff L_\gamma^* w = \gamma\varphi \iff B_{L^*}(w,\tilde{\psi}) + \gamma \int_\Omega w\tilde{\psi} = \int_\Omega \gamma \varphi \tilde{\psi} \quad \forall \tilde{\psi}$

set $\tilde{\psi} = v$, $\psi = w$ & plug into (*)

$\tilde{\psi} = v \Rightarrow \int_\Omega \gamma \varphi v = B_{L^*}(w,v) + \gamma \int_\Omega w v$

$\psi = w \Rightarrow \int_\Omega \gamma u w = B_L(v,w) + \gamma \int_\Omega v w$

from (*): $B_{L^*}(w,v) + \gamma \int_\Omega w v = B_L(v,w) + \gamma \int_\Omega v w$, but this is true

because $B_{L^*}(w,v) = B_L(v,w) \Rightarrow K^*$ is adj. to $K \Rightarrow \dim N_{L^*} = \dim N_L < \infty$.

Step 5. The proof of 3. $(f \in L^2 \; u \in V \; Lu = f) \iff \forall \varphi \in N_{L^*} \; \int_\Omega f\varphi = 0$

we know (u, f) solve $Lu = f$

$$\iff (I-K)u = K\left(\tfrac{f}{\gamma}\right) \iff K\left(\tfrac{f}{\gamma}\right) \in R(I-K) \overset{(F3)}{\iff} \int_\Omega K\left(\tfrac{f}{\gamma}\right)\cdot\varphi = 0 \quad \forall \varphi \in N(I-K^*) \; (\varphi = K^*\varphi)$$

$R(I-K) = (N(I-K^*))^\perp$

K^* is adjoint

$$\iff \int_\Omega f\, K^*(\varphi) = 0 \iff \int_\Omega f\varphi = 0$$

but we also know $N(I-K^*) = N_{L^*}$!

Difficult homework:

$$Lu = -\operatorname{div}(A\nabla u) + bu + \vec{c}\cdot\nabla u \quad , \quad b \geq \tfrac{1}{10}$$

goal: $\forall f \in L^2 \; \exists!\, u \in W_0^{1,2} : Lu = f$

equivalent problem: $u \in W_0^{1,2}$, u is WS to $Lu = 0 \Rightarrow u = 0$ in Ω (u is WS to $Lu = 0 \iff B_L(u,\varphi) = 0 \; \forall \varphi \in W_0^{1,2}$)

Hint (u-CS): think that u is a classical solution. If u has a minimum at $x_0 \in \Omega$,

then $\nabla u(x_0) = 0$ and $\nabla^2 u(x_0)$ positively semidefinite

$$\underbrace{-\operatorname{div} A(\nabla u)} + \vec{c}\cdot\nabla u + bu = 0 \quad \text{at } x_0$$

$$-\sum \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) = \sum \frac{\partial a_{ij}}{\partial x_i}\frac{\partial u}{\partial x_j} + \sum a_{ij}\frac{\partial u}{\partial x_i \partial x_j}$$

$$b(x_0)u(x_0) \geq -\underset{-\,\shortparallel\,-}{A\cdot\nabla^2 u(x_0)} + b(x_0)u(x_0) = 0 \Rightarrow \left.\begin{array}{l} u \geq 0 \text{ everywhere} \\ u \leq 0 \text{ everywhere}\end{array}\right\} \Rightarrow u \equiv 0$$

Hint (d=1):

$d=1$. $B_L(u,\varphi) = 0 \quad \forall \varphi \in W_0^{1,2}(\Omega)$

$$\varphi := |u|^{p-2}u \quad , \; p \geq 2 \quad \Rightarrow \quad \nabla\varphi = (p-1)\nabla u\, |u|^{p-2}$$

$$\int_\Omega A\nabla u\cdot\nabla(|u|^{p-2}u) + \vec{c}\cdot\nabla u\, |u|^{p-2}u + b|u|^p = 0$$

$$(p-1)\int_\Omega |\nabla u|^2|u|^{p-2} - \|\vec{c}\|_\infty \int_\Omega |\nabla u|\,|u|^{\frac{p-2}{2}}|u|^{\frac{p}{2}} + b|u|^p \leq 0$$

$$\text{Young: } \int_\Omega \frac{|\nabla u|\,|u|^{\frac{p-2}{2}}\,\|\vec{c}\|_\infty\,|u|^{\frac{p}{2}}\,b^{\frac{1}{2}}}{b^{1/2}} \leq \int_\Omega \frac{b|u|^p}{2} + \frac{|\nabla u|^2|u|^{p-2}\|\vec{c}\|_\infty^2}{2b}$$

$$\int_\Omega |u|^{p-2}|\nabla u|^2 \underbrace{\left(p-1-\frac{\|\vec{c}\|_\infty}{2b}\right)}_{p >>>>>> 1,\; \geq 0} + \frac{b}{2}|u|^p \leq 0$$

Definition (**real spectrum of L**). Let $\Gamma_1 = \partial\Omega$. We say that $\lambda \in \mathbb{R}$ belongs to the spectrum of $L \iff$ there $\exists\, u \neq 0 : Lu = \lambda u$, $u|_{\partial\Omega} = 0$.

Denote $\Sigma := \{\lambda \in \mathbb{R},\ \exists\, u \neq 0,\ Lu = \lambda u\}$

Theorem. Let $\Omega \in C^{0,1}$, L be an elliptic operator and $\Gamma_1 = \partial\Omega$. Then.

1. Σ is at most countable and if infinite ($\Sigma = \{\lambda_k\}_{k=1}^\infty$) then $\lambda_k \to \infty$
2. $(\lambda \notin \Sigma) \iff (\forall f \in L^2 \; \exists!\, u \;\; Lu = f + \lambda u)$
3. $\forall \lambda \notin \Sigma \; \exists c > 0 \; \forall f \in L^2 \; \exists!\, u \;\; Lu = f + \lambda u$, $\|u\|_{1,2} \leq c\|f\|_2$

Proof: $L_\gamma u = Lu + \gamma u$, $K = \gamma L_\gamma^{-1}$ (γ was found before)

if $\lambda \in \Sigma \Rightarrow \lambda \geq -\gamma$

$Lu = f + \lambda u \iff (I-K)u = K\left(\frac{f+\lambda u}{\gamma}\right) \iff \left(I - \left(1+\frac{\lambda}{\gamma}\right)K\right)u = K\left(\frac{f}{\gamma}\right)$

if $\lambda \in \Sigma \Rightarrow Lu = \lambda u + 0 \Rightarrow \left(I - \left(1+\frac{\lambda}{\gamma}\right)K\right)u = 0 \Rightarrow Ku = \frac{\gamma}{\gamma+\lambda}u$

if $\lambda \in \Sigma \Rightarrow \frac{\gamma}{\gamma+\lambda} \in$ spectrum of K

(F6) $\Rightarrow$ K has at most countable spectrum $\Rightarrow$ Σ is at most countable

(F6) if spectrum is infinite the only attracting point is zero $\Rightarrow \frac{\gamma}{\gamma+\lambda_k} \xrightarrow{k\to\infty} 0 \iff \lambda_k \to \infty$

Proof of 2. .. still the same

Proof of 3. $\forall \lambda \notin \Sigma \ \exists C > 0 \ \forall f \in L^2(\Omega) \ \exists u \in W_0^{1,2}(\Omega)$ $Lu = f + \lambda u$ $\|u\|_{1,2} \leq C\|f\|_2$

Contradiction: $\forall n \ \exists u^n, f^n$. $Lu^n = f^n + \lambda u^n$, $\|u^n\|_{1,2} > n\|f^n\|_2$

linearity: I can choose u^n such that $\|u^n\|_{1,2} = 1 \Rightarrow \|f^n\|_2 \leq \frac{1}{n}$

$\Rightarrow$ • $f^n \to 0$ in L^2

• $u^{n_k} \to$ (compact embedding)

$c_1 \|\nabla u^n - \nabla u^m\|_2^2 =: c_1 \|w^{n,m}\|_2^2 \leq \int_\Omega A \nabla w^{n,m} \cdot \nabla w^{n,m} = B_L(w^{n,m}, w^{n,m}) - \int_\Omega b (w^{n,m})^2$
$- \int_\Omega \vec{c} \cdot \nabla w^{n,m} w^{n,m} + \vec{d} \cdot \nabla w^{n,m} w^{n,m}$

$= \int_\Omega (f^n + \lambda u^n) w^{n,m} - (f^m + \lambda u^m) w^{n,m} - \int_\Omega b (w^{n,m})^2 - \int_\Omega \vec{c} \cdot \nabla w^{n,m} w^{n,m} + \vec{d} \cdot \nabla w^{n,m} w^{n,m}$

$\overset{\text{Hölder}}{\leq} \|w^{n,m}\|_2 \left(\|f^n + \lambda u^n\|_2 + \|f^m + \lambda u^m\|_2 + \|b\|_\infty \|w^{n,m}\|_2 + (\|\vec{c}\|_\infty + \|\vec{d}\|_\infty) \|\nabla w^{n,m}\|_2 \right)$

$\leq$ constand $\|w^{n,m}\|_2$ = constand $\|u^n - u^m\|_2$

u^{n_k} is Cauchy $\Rightarrow \nabla u^{n_k}$ is Cauchy

$\Rightarrow \exists \tilde{u} \in W_0^{1,2}(\Omega)$: $u^{n_k} \to \tilde{u}$ in $W_0^{1,2}(\Omega)$, but $u^{n_k} \to u$ in $L^2(\Omega)$

$\Rightarrow \|u^{n_k}\|_{1,2} = 1 \Rightarrow \|u\|_{1,2} = 1$

$\exists \lambda$ $Lu^n = f^n + \lambda u^n$ $\|u^n\|_{1,2} \geq n\|f^n\|_2$, $\|u^n\|_{1,2} = 1 \Rightarrow \|f^n\|_2 \leq \frac{1}{n}$ 7.11.2018

(*) $u^n \to u$ in $W_0^{1,2}(\Omega) \Rightarrow \|u\|_{1,2} = 1$

Step 3b. "$u=0$"

$B_L(u^n, \varphi) = \int_\Omega (f^n + \lambda u^n)\varphi$ $\forall \varphi \in V$

RHS: $\lim_{n\to\infty} \int_\Omega (f^n + \lambda u^n)\varphi = \lim_{n\to\infty} \int_\Omega (f^n + \lambda(u^n - u))\varphi + \lambda \int_\Omega u\varphi$

$\left|\int_\Omega (f^n + \lambda(u^n-u))\varphi\right| \overset{\text{Hölder}}{\leq} \|\varphi\|_2 \left(\underbrace{\|f^n\|_2}_{\leq \frac{1}{n}} + \lambda \underbrace{\|u^n - u\|_2}_{\to 0 \ (*)} \right) \to 0 \quad n \to \infty$

$= \int_\Omega \lambda u \varphi$

LHS $\lim_{n\to\infty} B_L(u^n,\varphi) = \lim_{n\to\infty} B_L(u^n-u,\varphi) + B_L(u,\varphi)$

$|B_L(u^n-u,\varphi)| \overset{V\text{-bdd}}{\leq} c\,\|u^n-u\|_V \|\varphi\|_V \xrightarrow{(*)} 0$

$= \lim_{n\to\infty} B_L(u,\varphi)$

$\Rightarrow \forall \varphi\in V \quad B_L(u,\varphi) = \lambda \int_\Omega u\varphi$

But $\lambda \notin \Sigma \overset{!}{\Rightarrow} u = 0 \;\wedge\; \|u\|_V = 1 \quad \to$ contradiction!

3.6 Variational approach - minimizers

Assumption: the bilinear form $B_{L\sigma}(u,v)$ is symmetric $\Leftrightarrow$ $B_{L\sigma}(u,v) = B_{L\sigma}(v,u)$

Theorem (equivalence of weak solutions and minimizers): Let $B_{L\sigma}: V\times V \to \mathbb{R}$ be bilinear, V-bounded and V-elliptic. Assume that $B_{L\sigma}$ is symmetric. Let $f\in V^*$ and $g\in L^2(\Gamma_2\cup\Gamma_3)$; $u_0\in W^{1,2}(\Omega)$. Then the following is equivalent

1. $u-u_0\in V$ and $B_{L\sigma}(u,\varphi) = \langle f,\varphi\rangle + \int_{\Gamma_2\cup\Gamma_3} g\varphi$

   "u is a WS of $Lu=f$ in Ω, $u=u_0$ on Γ_1"

2. $u-u_0\in V \quad \forall v\in W^{1,2}(\Omega),\ v-u_0\in V:\ \frac{1}{2}B_{L\sigma}(u,u) - \langle f,u\rangle - \int_{\Gamma_2\cup\Gamma_3} gu \leq \frac{1}{2}B_{L\sigma}(v,v) - \langle f,v\rangle - \int_{\Gamma_2\cup\Gamma_3} gv$

   "u minimizes this problem"

Proof: "1 $\Rightarrow$ 2" : u is a solution and $v\in W^{1,2}(\Omega), v-u_0\in V$ ($\Rightarrow v-u\in V$)

$0 \leq \frac{1}{2} B_{L\sigma}(\underset{\in V}{v-u}, \underset{\in V}{v-u})$

$= \frac{1}{2}B_{L\sigma}(v,v) - \frac{1}{2}B_{L\sigma}(u,u) + \frac{1}{2}B_{L\sigma}(u,u) - \frac{1}{2}B_{L\sigma}(u,v) - \frac{1}{2}B_{L\sigma}(v,u) + \frac{1}{2}B_{L\sigma}(u,u)$

$\overset{\text{symmetric}}{=} \frac{1}{2}\left(B_{L\sigma}(v,v) - B_{L\sigma}(u,u)\right) + B_{L\sigma}(u,u) - B_{L\sigma}(u,v)$

$\overset{\text{bilinear}}{=} \frac{1}{2}\left(B_{L\sigma}(v,v) - B_{L\sigma}(u,u)\right) + B_{L\sigma}(u,u-v)$

$\overset{WS,\ \varphi=u-v}{=} \frac{1}{2}\left(B_{L\sigma}(v,v) - B_{L\sigma}(u,u)\right) + \langle f,u-v\rangle + \int_{\Gamma_2\cup\Gamma_3} g(u-v) \qquad \Leftrightarrow \quad 2.$

"2. $\Rightarrow$ 1." set $v := u+\varepsilon\varphi$, $\varphi\in V$

$\frac{1}{2}B_{L\sigma}(u,u) - \langle f,u\rangle - \int_{\Gamma_2\cup\Gamma_3} gu \leq \frac{1}{2}B_{L\sigma}(u+\varepsilon\varphi, u+\varepsilon\varphi) - \langle f,u+\varepsilon\varphi\rangle - \int_{\Gamma_2\cup\Gamma_3} g(u+\varepsilon\varphi)$

$\overset{\text{bilinear \& symm.}}{=} \frac{1}{2}\left(B_{L\sigma}(u,u) + B_{L\sigma}(\varepsilon\varphi,\varepsilon\varphi) + 2\varepsilon B_{L\sigma}(u,\varphi)\right) - \langle f,u+\varepsilon\varphi\rangle - \int_{\Gamma_2\cup\Gamma_3} g(u+\varepsilon\varphi)$

$0 \leq \varepsilon B_{L\sigma}(u,\varphi) + \frac{\varepsilon^2}{2} B_{L\sigma}(\varphi,\varphi) - \varepsilon\langle f,\varphi\rangle - \varepsilon\int_{\Gamma_2\cup\Gamma_3} g\varphi$

$\varepsilon \to 0_+$: $0 \leq B_{L\sigma}(u,\varphi) - \langle f,\varphi\rangle - \int_{\Gamma_2\cup\Gamma_3} g\varphi$

true also for $(-\varphi)$ $\Rightarrow$ $0 = B_{L\sigma}(u,\varphi) - \langle f,\varphi\rangle - \int_{\Gamma_2\cup\Gamma_3} g\varphi \quad \Leftrightarrow \quad u$ is a WS

Remark: $B_{L\sigma}$ is elliptic and symmetric $\Rightarrow$ $L = ?$

$$Lu = -\operatorname{div} A\nabla u + bu + \vec{c}\cdot\nabla u + \operatorname{div}(\vec{d}u)$$

$$? \left(\left(\begin{aligned} B_{L\sigma}(u,v) &= \int_\Omega A\nabla u\cdot\nabla v + buv + \vec{c}\cdot\nabla u\, v - \vec{d}u\cdot\nabla v + \int_{\Gamma_3}\sigma u v \\ B_{L\sigma}(v,u) &= \int_\Omega A\nabla v\cdot\nabla u + bvu + \vec{c}\cdot\nabla v\, u - \vec{d}v\cdot\nabla u + \int_{\Gamma_3}\sigma v u \end{aligned}\right.\right.$$

$$\Rightarrow \quad A = A^T \quad \& \quad \vec{d} = -\vec{c}$$

Theorem (dual formulation). Let $Lu = -\operatorname{div}(A\nabla u)$ with A an elliptic and symmetric matrix. Assume that $\Gamma_1 \neq \emptyset$ and $\Gamma_3 = \emptyset$, $f \in V^*$, $g \in L^2(\Gamma_3)$ and $u_0 \in W^{1,2}(\Omega)$.

Then the following is equivalent:

1. u is a weak solution
2. $\nabla u = A^{-1}\vec{T}$, where $\vec{T}$ is a minimizer to $\int_\Omega \frac{1}{2}A^{-1}\vec{T}\cdot\vec{T} - \nabla u_0\cdot\vec{T}$

   over the set $\tilde{V} = \{\vec{T} \in L^2(\Omega,\mathbb{R}^d),\ \forall \varphi \in V \ \int_\Omega \vec{T}\cdot\nabla\varphi = \langle f,\varphi\rangle + \int_{\Gamma_2} g\varphi\}$

Remark: $\vec{T} \in \tilde{V}$ "$\Leftrightarrow$" $-\operatorname{div}\vec{T} = f$ in Ω, $\vec{T}\nu = g$ on Γ_2

Proof: "1. $\Rightarrow$ 2." u is a WS, define $\vec{T} := A\nabla u$.

Now $\vec{T} \in L^2(\Omega,\mathbb{R}^d)$ and also $\int_\Omega \vec{T}\cdot\nabla\varphi = \int_\Omega A\nabla u\cdot\nabla\varphi \overset{WS}{=} \langle f,\varphi\rangle + \int_{\Gamma_2} g\varphi \quad \Rightarrow \vec{T} \in \tilde{V}$

Take $\vec{V} \in \tilde{V}$: $0 \le \int_\Omega \frac{1}{2}A^{-1}(\vec{V}-\vec{T})(\vec{V}-\vec{T}) = \frac{1}{2}\int_\Omega A^{-1}\vec{V}\cdot\vec{V} - A^{-1}\vec{T}\cdot\vec{T} + \frac{1}{2}\int_\Omega A^{-1}\vec{T}\cdot\vec{T} - A^{-1}\vec{V}\cdot\vec{T} - A^{-1}\vec{T}\cdot\vec{V} + A^{-1}\vec{T}\cdot\vec{T}$

$$= \frac{1}{2}\int_\Omega A^{-1}\vec{V}\cdot\vec{V} - A^{-1}\vec{T}\cdot\vec{T} + \int_\Omega A^{-1}\vec{T}\cdot(\vec{T}-\vec{V})$$

$$= \left(\frac{1}{2}\int_\Omega A^{-1}\vec{V}\cdot\vec{V} - \int_\Omega \nabla u_0\cdot\vec{V}\right) - \left(\frac{1}{2}\int_\Omega A^{-1}\vec{T}\cdot\vec{T} - \int_\Omega \nabla u_0\cdot\vec{T}\right) + \int_\Omega \nabla u_0\cdot(\vec{V}-\vec{T}) + A^{-1}\vec{T}\cdot(\vec{T}-\vec{V})$$

$$\int_\Omega \nabla u_0\cdot(\vec{V}-\vec{T}) + A^{-1}\vec{T}\cdot(\vec{T}-\vec{V}) = \int_\Omega (A^{-1}\vec{T}-\nabla u_0)\cdot(\vec{T}-\vec{V}) \overset{\text{Def of } \vec{T}}{=} \int_\Omega (\nabla u - \nabla u_0)\cdot(\vec{T}-\vec{V})$$

$u - u_0 \in V \Rightarrow \int_\Omega \nabla(u-u_0)\cdot(\vec{T}-\vec{V}) = \int_\Omega \nabla(u-u_0)\cdot\vec{T} - \int_\Omega \nabla(u-u_0)\cdot\vec{V}$

$\vec{T},\vec{V} \in \tilde{V} \Rightarrow$
$$= \langle f, u-u_0\rangle + \int_{\Gamma_2} g(u-u_0) - \langle f, u-u_0\rangle - \int_{\Gamma_2} g(u-u_0) = 0$$

$$\Rightarrow \quad 0 \le \frac{1}{2}\int_\Omega A^{-1}\vec{V}\cdot\vec{V} - \int_\Omega \nabla u_0\cdot\vec{V} - \left(\frac{1}{2}\int_\Omega A^{-1}\vec{T}\cdot\vec{T} - \int_\Omega \nabla u_0\cdot\vec{T}\right)$$

$\Rightarrow$ $\vec{T}$ is a minimizer

"2. $\Rightarrow$ 1." we have $\vec{T} \in \tilde{V}$: $\forall \vec{V} \in \tilde{V}$ $\quad \frac{1}{2}\int_\Omega A^{-1}\vec{T}\cdot\vec{T} - \int_\Omega \nabla u_0\cdot\vec{T} \le \frac{1}{2}\int_\Omega A^{-1}\vec{V}\cdot\vec{V} - \int_\Omega \nabla u_0\cdot\vec{V}$

"Euler-Lagrange"

Take $\vec{W} \in L^2(\Omega,\mathbb{R}^d)$ $\forall \varphi \in V$ $\int_\Omega \vec{W}\,\nabla\varphi = 0$ $\quad$ ("$\Leftrightarrow$" $\operatorname{div}\vec{W} = 0$, $\quad \vec{W}\cdot\nu = 0$ on Γ_2)

Set $\vec{V} := \vec{T} + \varepsilon\vec{W} \in \tilde{V}$

$$\frac{1}{2}\int_\Omega A^{-1}\vec{T}\cdot\vec{T} - \int_\Omega \nabla u_0\cdot\vec{T} \le \frac{1}{2}\int_\Omega A^{-1}(\vec{T}+\varepsilon\vec{W})(\vec{T}+\varepsilon\vec{W}) - \int_\Omega \nabla u_0\cdot(\vec{T}+\varepsilon\vec{W})$$

$$\overset{\text{symm.}}{=} \frac{1}{2}\int_\Omega A^{-1}\vec{T}\cdot\vec{T} - \frac{\varepsilon^2}{2}\int_\Omega A^{-1}\vec{W}\cdot\vec{W} + \varepsilon\int_\Omega A^{-1}\vec{T}\cdot\vec{W} - \int_\Omega \nabla u_0\cdot(\vec{T}+\varepsilon\vec{W})$$

$0 \le \int_\Omega A^{-1}\vec{T}.\vec{w} - \nabla u_0.\vec{w} + \frac{\varepsilon}{2}\int_\Omega A\vec{w}.\vec{w}$, $\varepsilon \to 0+$

$0 \le \int_\Omega A^{-1}\vec{T}.\vec{w} - \nabla u_0.\vec{w}$, true also for $(-\vec{w})$

$0 = \int_\Omega A^{-1}\vec{T}.\vec{w} - \nabla u_0.\vec{w}$ Euler-Lagrange

Show that $A^{-1}\vec{T} = \nabla u$, where u is a WS:

Look for u, $u-u_0 \in V$ which solves $\int_\Omega \nabla u.\nabla\varphi = \underbrace{\int_\Omega A^{-1}\vec{T}.\nabla\varphi}_{= \langle F,\varphi\rangle_{V^*}}$ $\forall \varphi \in V$ (**) $(-\Delta u = -\mathrm{div}(A^{-1}\vec{T}))$

Lax-Milgram: $\exists!\, u$, $u-u_0 \in V$ and $\checkmark$ holds.

$\int_\Omega |\nabla u - A^{-1}\vec{T}|^2 = \int_\Omega (A^{-1}\vec{T} - \nabla u).(A^{-1}\vec{T} - \nabla u)$

$= \int_\Omega (A^{-1}\vec{T} - \nabla u_0).(A^{-1}\vec{T} - \nabla u) + \int_\Omega \nabla(u_0 - u)(A^{-1}\vec{T} - \nabla u) =: \mathrm{I} + \mathrm{II}$

(**) $\Rightarrow \int_\Omega (A^{-1}\vec{T} - \nabla u).\nabla\varphi = 0 \quad \forall \varphi \in V$

1. Euler-Lagrange, $\vec{w} := A^{-1}\vec{T} - \nabla u \Rightarrow \mathrm{I} = 0$

2. $u-u_0 \in V$ & u solves (**), $\varphi := u - u_0 \Rightarrow \mathrm{II} = 0$

$\Rightarrow \nabla u = A^{-1}\vec{T}$ a.e. $\Rightarrow A\nabla u = \vec{T}$

$\int_\Omega A\nabla u.\nabla\varphi = \int_\Omega \vec{T}.\nabla\varphi \overset{\vec{T}\in\tilde{V}}{=} \langle f,\varphi\rangle + \int_{\Gamma_2} g\varphi \iff u$ is a WS

Why is minimization "easier" than $\exists$ of WS?

Lemma (**reflexivity vs weak convergence**): Let X be a reflexive Banach space.

Let $\{u^n\}_{n=1}^\infty$: $\|u^n\|_X \le c$.

Then $\exists u \in X\ \exists \{u^{n_k}\}\ \forall F \in X^*$: $\lim_{n_k\to\infty} \langle F, u^{n_k}\rangle_X = \langle F, u\rangle$ (weak convergence)

How to minimize / is there a minimizer?

$\exists \min_{u-u_0\in V} \left(\frac{1}{2}B_{L,\sigma}(u,u) - \langle f,u\rangle - \int_{\Gamma_2\cup\Gamma_3} g u\right)$?

$I := \inf(- \,\text{"}\, -)$, take a sequence u^n : $I = \lim_{n\to\infty} \frac{1}{2}B_{L,\sigma}(u^n,u^n) - \langle f,u^n\rangle - \int_{\Gamma_2\cup\Gamma_3} g u^n$

$I \le \frac{1}{2}B_{L,\sigma}(u_0,u_0) - \langle f,u_0\rangle - \int_{\Gamma_2\cup\Gamma_3} g u_0$

$\exists n_0\ \forall n > n_0 : \frac{1}{2}B_{L,\sigma}(u^n,u^n) - \langle f,u^n\rangle - \int g u^n \le \frac{1}{2}B_{L,\sigma}(u_0,u_0) - \langle f,u_0\rangle - \int g u_0 + 1$

V-ellipticity: $\|u^n\|_{1,2} \le c(f,g,u_0)$

$\exists u^{n_k} \rightharpoonup u$ in $W^{1,2}(\Omega) \iff \forall F \in (W^{1,2})^*\ \langle F,u^n\rangle \xrightarrow{n\to\infty} \langle F,u\rangle$

$W^{1,2} \hookrightarrow\hookrightarrow L^2 \Rightarrow u^n \to u$ in $L^2(\Omega)$ $(\iff \|u^n - u\|_2 \to 0)$

$\|\nabla u^n\|_2 \le c$ $\exists \vec{V} \in L^2(\Omega,\mathbb{R}^d)$ $\nabla u^{n_k} \rightharpoonup \vec{V}$ in $L^2(\Omega,\mathbb{R}^d)$

$\iff \forall \vec{F} \in (L^2(\Omega,\mathbb{R}^d))^* : \langle \vec{F}, \nabla u^{n_k}\rangle \to \langle \vec{F},\vec{V}\rangle \overset{(L^2)^* = L^2}{\iff} \forall \vec{F} \in L^2(\Omega,\mathbb{R}^d) : \int_\Omega \vec{F}.\nabla u^{n_k} \to \int_\Omega \vec{F}.\vec{V}$

2. $\vec{V} = \nabla u$

$\varphi \in C_0^\infty(\Omega)$: $\displaystyle \int_\Omega u \frac{\partial \varphi}{\partial x_i} = \int_\Omega (u-u^{n_k}) \frac{\partial \varphi}{\partial x_i} + \int_\Omega u^{n_k} \frac{\partial \varphi}{\partial x_i} = \underbrace{\int_\Omega (u-u^{n_k}) \frac{\partial \varphi}{\partial x_i}}_{\to 0 \text{ in } L^2} - \underbrace{\int_\Omega \frac{\partial u^{n_k}}{\partial x_i} \varphi}_{= -\int_\Omega \nabla u^{n_k} \cdot \vec{F}}$

$\vec{F} := (0,\dots,0,\underset{i\text{-th place}}{\varphi},0,\dots,0)$ $\quad \to \quad -\int_\Omega \vec{V}\cdot\vec{F} = -\int_\Omega V_i \varphi \quad \Rightarrow \nabla u = \vec{V}$

$\Rightarrow \quad u^{n_k} \rightharpoonup u \quad \text{in } W^{1,2}(\Omega) \qquad \qquad \forall F \in (W^{1,2}(\Omega))^* : \langle F, u^{n_k}\rangle \to \langle F, u\rangle$

$\quad u^{n_k} \to u \quad \text{in } L^2(\Omega) \qquad \Longleftrightarrow \qquad \|u^{n_k} - u\|_2 \to 0$

$\quad \nabla u^{n_k} \rightharpoonup \nabla u \quad \text{in } L^2(\Omega, \mathbb{R}^d) \qquad \qquad \forall \vec{F} \in L^2(\Omega,\mathbb{R}^d): \int_\Omega (\nabla u^{n_k} - \nabla u)\cdot \vec{F} \to 0$

We will show that $\frac{1}{2} B_{L,\sigma}(u,u) - \langle f, u\rangle - \int_{\Gamma_2\cup\Gamma_3} g u \le I$ $\qquad$ ($\Rightarrow u$ is a minimizer)

$I = \lim_{n_k\to\infty} \frac{1}{2} B_{L,\sigma}(u^{n_k}, u^{n_k}) - \langle f, u^{n_k}\rangle - \int_{\Gamma_2\cup\Gamma_3} g u^{n_k}$

$0 \le \frac{1}{2} B_{L,\sigma}(u^{n_k} - u, u^{n_k} - u)$

$\quad = \frac{1}{2} B_{L,\sigma}(u^{n_k}, u^{n_k}) - \frac{1}{2} B_{L,\sigma}(u,u) - B_{L,\sigma}(u, u^{n_k} - u)$

take lim: $n_k\to\infty$

$\quad = \frac{1}{2} B_{L,\sigma}(u^{n_k}, u^{n_k}) - \langle f, u^{n_k}\rangle - \int g u^{n_k} - \left(\frac{1}{2} B_{L,\sigma}(u,u) - \langle f,u\rangle - \int g u\right) - B_{L,\sigma}(u, u^{n_k}-u) + \underbrace{\langle f, u^{n_k}-u\rangle}_{\to 0,\ \text{def. of WC}} + \underbrace{\int g (u^{n_k}-u)}_{\to 0,\ \text{compactness of the trace operator}}$

$0 \le I - \left(\frac{1}{2} B_{L,\sigma}(u,u) - \langle f,u\rangle - \int g u\right) - \lim_{n_k\to\infty} B_{L,\sigma}(u, u^{n_k}-u)$

CLAIM: $B_{L,\sigma}(u, u^{n_k}-u) \to 0$

$B_{L,\sigma}(u, u^{n_k}-u) = \int_\Omega \underbrace{A\nabla u}_{\in L^2} \cdot \nabla(u^{n_k}-u) + \underbrace{bu}_{\in L^2}(u^{n_k}-u) + \underbrace{\vec{c}\cdot\nabla u}_{\in L^2}(u^{n_k}-u) - \underbrace{\vec{d}u}_{\in L^2}\cdot\nabla(u^{n_k}-u) + \int_{\Gamma_3} \underbrace{\sigma u}_{\in L^2(\partial\Omega)} \underbrace{(u^{n_k}-u)}_{\to 0 \text{ in } L^2(\partial\Omega)}$

fixed object in L^2 & the weak convergence $\Rightarrow B_{L,\sigma}(u, u^{n_k}-u) \to 0$

$\Rightarrow u$ is a minimizer!

Theorem (spectrum of symmetric operators – variational approach): Let $B_{L,\sigma}$ be symmetric V-elliptic and V-bounded. Then there exists $\{\lambda_k\}_{k=1}^\infty$, $0 < \lambda_1 \le \dots \le \lambda_k \le \dots$ and corresponding family of $\{u_k\}_{k=1}^\infty \subseteq V$, such that

- $B_{L,\sigma}(u^k, \varphi) = \lambda_k \int_\Omega u^k \varphi \qquad \forall \varphi \in V$
- $\lambda_k \to \infty$
- $\{u^k\}_{k=1}^\infty$ is a basis in V fulfilling $\int_\Omega u^i u^j = \delta^{ij}$ and $\forall i \ne j$ $B_{L,\sigma}(u^i, u^j) = 0$
- $P^N: V \to \text{linhull}\{u^1,\dots,u^N\}$, $\quad P^N u = \sum_{k=1}^N u^k \left(\int_\Omega u u^k\right)$. $\qquad$ 14.11. 2019

Then $\|P^N u\|_2 \le \|u\|_2$, $\quad B_{L,\sigma}(P^N u, P^N u) \le B_{L,\sigma}(u,u)$.

Proof: Step 1. construction of λ_k, u_k

1a) find $\lambda_1 := \inf_{\substack{u\in V\\ \|u\|_2=1}} B_{L,\sigma}(u,u)$ $\qquad$ (denote by u_1 the u in which the inf is attained)

1b) $V^N := \{u \in V,\ B_{L,\sigma}(u, u_k) = 0 \quad \forall k = 1,\dots,N\}$, $\quad \lambda_{N+1} := \inf_{\substack{u\in V^N\\ \|u\|_2=1}} B_{L,\sigma}(u,u)$ $\qquad$ (in u_{N+1} the inf is attained)

Remark: λ_1 is the best constant for the Poincaré inequality

$$\lambda_1 \|u\|_2^2 \leq B_{L,\sigma}(u,u) \qquad \forall u \in V$$

Step 2: the definition is meaningful

$$\exists u^n \in V, \quad \|u^n\|_2 = 1, \qquad \lambda = \lim_{n\to\infty} B_{L,\sigma}(u^n,u^n) \leq c$$

V-ellipticity: $\|u^n\|_V \leq c$, subsequence $u^{n_k} \rightharpoonup u$ in V

$V \hookrightarrow\hookrightarrow L^2 \qquad u^{n_k} \to u$ in $L^2(\Omega)$

$u^{n_k} \to u$ in L^2: $\quad \|u^n\|_2 = 1 \Rightarrow \|u\|_2 = 1$

already proven: $\quad B_{L,\sigma}(u,u) \underset{\uparrow}{\leq} \liminf_{n\to\infty} B_{L,\sigma}(u^n,u^n) = \lambda_1$

weak-lower-semicontinuity

$\Rightarrow B_{L,\sigma}(u,u) \leq \lambda_1$ & $\|u\|_2 = 1 \quad \Rightarrow \quad \lambda_1 = B_{L,\sigma}(u,u)$, denote $u_1 := u$

Step 3: λ_k and u_k are eigenvalues and eigenvectors

$$\forall v \in V \quad \|v\|_2 = 1 \qquad \lambda_1 \leq B_{L,\sigma}(v,v)$$

$$v := \frac{u_1+\varepsilon\psi}{\|u_1+\varepsilon\psi\|_2} \qquad \text{for } \psi \in V, \quad 0<\varepsilon \ll 1$$

$$\lambda_1 \leq B_{L,\sigma}\left(\frac{u_1+\varepsilon\psi}{\|u_1+\varepsilon\psi\|_2}, \frac{u_1+\varepsilon\psi}{\|u_1+\varepsilon\psi\|_2}\right)$$

$$\lambda_1 \|u_1+\varepsilon\psi\|_2^2 \leq B_{L,\sigma}(u_1+\varepsilon\psi, u_1+\varepsilon\psi)$$

$$\lambda_1\Big(\underbrace{\|u_1\|_2^2}_{=1} + \varepsilon^2\|\psi\|_2^2 + 2\int_\Omega \varepsilon u_1\psi\Big) \leq \underbrace{B_{L,\sigma}(u_1,u_1)}_{=\lambda_1} + \varepsilon^2 B_{L,\sigma}(\psi,\psi) + 2\varepsilon B_{L,\sigma}(u_1,\psi) \quad / \tfrac{1}{\varepsilon}$$

$$\lambda_1\Big(\varepsilon\|\psi\|_2^2 + 2\int_\Omega u_1\psi\Big) \leq \varepsilon B_{L,\sigma}(\psi,\psi) + 2B_{L,\sigma}(u_1,\psi) \qquad / \varepsilon \to 0+$$

$$\lambda_1 \int_\Omega u_1\psi \leq B_{L,\sigma}(u_1,\psi)$$

choose $v := \dfrac{u_1-\varepsilon\psi}{\|u_1-\varepsilon\psi\|_2} \quad \Rightarrow \quad \lambda_1\int_\Omega u_1\psi = B_{L,\sigma}(u_1,\psi) \qquad \forall \psi \in V$

For λ_{N+1} in the same way: $\quad \lambda_{N+1}\int_\Omega u_{N+1}\psi = B_{L,\sigma}(u_{N+1},\psi) \quad \forall \psi \in V^N \qquad (*)$

For $\psi \in \operatorname{linhull}\{u_1,\dots,u_N\}$:

$B_{L,\sigma}(u_{N+1},u_K) = 0 \quad \forall K \leq N$ because $u_{N+1} \in V^N$. In $(*)$, set $N+1=K$ and $\psi = u_{N+1}$.

[redacted] $\lambda_K \int_\Omega u_K u_{N+1} = B_{L,\sigma}(u_K,u_{N+1}) = 0 \quad \Rightarrow \quad \int_\Omega u_{N+1}u_K = 0 \quad \forall K \leq N$

$\Rightarrow \int_\Omega u_i u_j = \delta_{ij}, \quad B_{L,\sigma}(u_i,u_j) = 0 \quad i \neq j$

- $\lambda_K \to \infty$, it follows from the construction

Assume: $\lambda_1 \leq \lambda_2 \leq \dots$

$\lim_{k\to\infty} \lambda_K = \lambda < \infty$

$\lambda \geq \lambda_K = B_{L,\sigma}(u_K,u_K) \overset{\text{V-ellipticity}}{\geq} c\|u_K\|_V^2$, $V \hookrightarrow\hookrightarrow L^2$ $u_K \to u$ in $L^2(\Omega) \Rightarrow \{u_K\}_{K=1}^\infty$ is Cauchy on the other hand, $i \neq j$ $\|u_i - u_j\|_2^2 = \int_\Omega u_i^2 + u_j^2 - 2u_iu_j = 2$, contradiction!

Are $\{\lambda_k\}$ all eigenvalues?

Contradiction: assume that no.

Then $\exists \lambda \neq \lambda_k \ \forall k$, $u \in V$, $\|u\|_2 = 1$ $\quad \lambda \int_\Omega u\varphi = B_{L,\sigma}(u,\varphi) \quad \forall \varphi \in V$

$(\varphi := u \Rightarrow \lambda = B_{L,\sigma}(u,u)$ and I know that $\lambda_i = B_{L,\sigma}(u_i,u_i))$

$\exists i \in \mathbb{N}: \quad \lambda_i < \lambda < \lambda_{i+1} \quad \Rightarrow \quad B_{L,\sigma}(u_i,u_i) < \underbrace{B_{L,\sigma}(u,u)}_{u \in V^{i+1}} < B_{L,\sigma}(u_{i+1},u_{i+1})$

$B_{L,\sigma}(u,u) < B_{L,\sigma}(u_{i+1},u_{i+1}) = \inf\limits_{\substack{u \in V^{i+1} \\ \|u\|_2 = 1}} B_{L,\sigma}(u,u)$ $\quad ⚡$.

Exercise:
$$-\Delta u_1 + u_2 = f_1 \quad \text{in } \Omega$$
$$-\Delta u_2 + a u_1 = f_2 \quad \text{in } \Omega$$
$$u_1 = u_2 = 0 \quad \text{on } \partial\Omega$$

· find $\{w_k\}_{k=1}^\infty \subseteq W_0^{1,2}(\Omega)$ such that $-\Delta w_k = \lambda_k w_k$, $\{w_k\}_{k=1}^\infty$ forms a basis in $W_0^{1,2}(\Omega)$.

$u_1 = \sum_{i=1}^\infty a_1^i w_i \qquad f_1 = \sum_{i=1}^\infty f_1^i w_i$

$u_2 = \sum_{i=1}^\infty a_2^i w_i \qquad f_2 = \sum_{i=1}^\infty f_2^i w_i$

(u_1, u_2) is a weak solution $\Leftrightarrow \begin{cases} \int_\Omega \nabla u_1 \cdot \nabla\varphi + \int_\Omega u_2\varphi = \int_\Omega f_1\varphi & \forall \varphi \in W_0^{1,2}(\Omega) \\ \int_\Omega \nabla u_2 \cdot \nabla\varphi + \int_\Omega a u_1\varphi = \int_\Omega f_2\varphi & \forall \varphi \in W_0^{1,2}(\Omega) \end{cases}$

$$\Leftrightarrow \begin{cases} \int_\Omega \nabla u_1 \cdot \nabla w_k + \int_\Omega u_2 w_k = \int_\Omega f_1 w_k & \forall k = 1,\dots \\ \int_\Omega \nabla u_2 \cdot \nabla w_k + \int_\Omega a u_1 w_k = \int_\Omega f_2 w_k & \forall k = 1,\dots \end{cases}$$

$$\Leftrightarrow \begin{cases} \sum_{i=1}^\infty \int_\Omega a_1^i \nabla w_i \cdot \nabla w_k + \sum_{i=1}^\infty \int_\Omega a_2^i w_i w_k = \sum_{i=1}^\infty \int_\Omega f_1^i w_i w_k & \forall k = 1,\dots \\ \sum_{i=1}^\infty \int_\Omega a_2^i \nabla w_i \cdot \nabla w_k + \sum_{i=1}^\infty a \int_\Omega a_1^i w_i w_k = \sum_{i=1}^\infty \int_\Omega f_2^i w_i w_k & \forall k = 1,\dots \end{cases}$$

$$\Leftrightarrow \int_\Omega w_i w_j = \delta_{ij}: \begin{cases} \lambda_k a_1^k = \boxed{\int_\Omega a_1^k \nabla w_k \cdot \nabla w_k} + a_2^k = f_1^k & \forall k = 1,\dots \\ \lambda_k a_2^k = \boxed{\int_\Omega a_2^k \nabla w_k \cdot \nabla w_k} + a a_1^k = f_2^k & \forall k = 1,\dots \end{cases}$$

RESULT: $\begin{cases} \lambda_k a_1^k + a_2^k = f_1^k \\ \lambda_k a_2^k + a a_1^k = f_2^k \end{cases} \quad \forall k$

$A^k := \begin{pmatrix} \lambda_k & 1 \\ a & \lambda_k \end{pmatrix} \qquad A^k \vec{a}_k = f^k = (f_1^k, f_2^k)$

Homework: compute the eigenvalues and eigenvectors for the problem

$-\Delta u_k = \lambda_k u \quad$ in $\Omega \qquad \Omega = [0,1]^2$ or $\Omega = [0,1]^d$

$u = 0 \quad$ on $\partial\Omega \qquad$ hint: there is π..

3.7. Regularity of weak solution

Goal: when is u a WS better, i.e. in $W^{2,2}(\Omega)$ or in $W^{k,2}(\Omega)$?

Assume we have a weak solution to $Lu = f$ in Ω + bc

- better operator : $-\operatorname{div}(A\nabla u) + bu + \vec{c}\cdot\nabla u + \operatorname{div}(\vec{d}u) = f$

the regularity depends only on the leading operator:

$$-\operatorname{div}(A\nabla u) = -bu - \vec{c}\cdot\nabla u - \operatorname{div}(\vec{d}u) + f$$

Problem : $Lu = -\operatorname{div}(A\nabla u) = f$ in Ω

$u = 0$ on Γ_1

$(A\nabla u)\cdot\vec{\nu} = g$ on Γ_2

$(A\nabla u)\cdot\vec{\nu} + \sigma u = g$ on Γ_3

splitting :
- interior regularity : $u \in W^{2,2}_{loc}(\Omega)$
- boundary regularity : $u \in W^{2,2}(\Omega)$

Assumptions :
- IR : $A \in W^{k+1,\infty}(\Omega)$, $f \in W^{k,2}(\Omega) \Rightarrow u \in W^{k+2,2}_{loc}(\Omega)$
- BR :
  - smoothnes of $\partial\Omega$
  - compatibility on intersection of Γ_1 and Γ_2

Homework : boundary (ir)regularity

1. Take Ω : 

look for $u = |x|^\beta \sin(A\theta)$, find (β, A) : $\Delta u = 0$

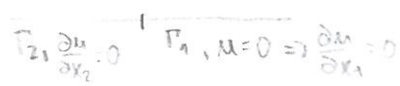
1. consider $u = 0$ on $\Gamma_1 \cup \Gamma_2$, check $u \in W^{1,2}$ a WS, check for which α $u \notin W^{2,2}$ (convex vs concave corner

2. $u = 0$ on Γ_1, $\nabla u \cdot \vec{\nu} = 0$ on Γ_2 ($\Rightarrow$ choice of "A")

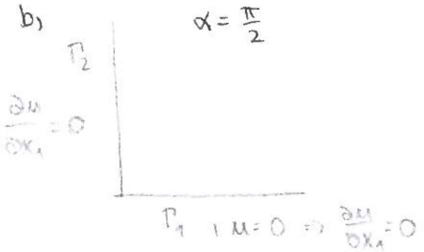
observe that for

a) $\alpha = \pi$

$\Rightarrow \nabla u = 0$ it is incompatible



b) $\alpha = \frac{\pi}{2}$



this is compatible

Theorem (interior regularity): Let A be an elliptic matrix and $u \in W^{1,2}(\Omega)$ solves

$$\int_\Omega A\nabla u \cdot \nabla \varphi = \int_\Omega f\varphi \qquad \forall \varphi \in W^{1,2}_0(\Omega) \qquad (\Rightarrow \ -\operatorname{div}(A\nabla u) = f)$$

Then if $A \in W^{k+1,\infty}(\Omega, \mathbb{R}^{d\times d})$ and $f \in W^{k,2}(\Omega)$, then $u \in W^{k+2,2}_{loc}(\Omega)$, and $\forall \tilde\Omega \subseteq \overline{\tilde\Omega} \subseteq \Omega \ \exists C(\tilde\Omega, A)$ such that $\|u\|_{W^{k+2,2}(\tilde\Omega)} \le C(\tilde\Omega, A)\left(\|f\|_{W^{k,2}(\Omega)} + \|u\|_{W^{1,2}(\Omega)}\right)$.

Proof: for $k=0$

$$v \in W^{1,2}(\tilde\Omega) \iff (v \in L^2(\tilde\Omega) \ \wedge \ \Delta_k^h v \in L^2(\tilde\Omega) \ \forall h)$$

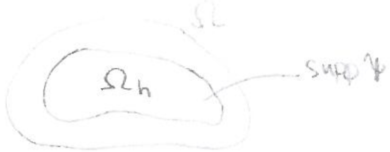
(remember: $\Delta_k^h v \in L^2(\tilde\Omega) \iff \forall h>0, \int_{\tilde\Omega_h} \frac{|v(x+he_k)-v(x)|^2}{h^2} \le C$, $\tilde\Omega_h := \{x \in \tilde\Omega, \operatorname{dist}(x,\partial\tilde\Omega) > h\}$)

we want $u \in W^{2,2} \iff \frac{\partial u}{\partial x_i} \in W^{1,2} \iff$ we need to check $\int \left|\frac{\partial u}{\partial x_i}(x+he_k) - \frac{\partial u(x)}{\partial x_i}\right|$

Equation for differences:

$$\int A(x)\nabla u(x) \cdot \nabla \varphi(x) = \int_\Omega f(x)\varphi(x) \qquad \forall \varphi \in W^{1,2}_0(\Omega)$$

$h>0$ $\quad \psi \in W^{1,2}_0(\Omega)$, $\psi(x) = 0 \ \forall x \ \operatorname{dist}(x,\partial\Omega) < h$



set $\varphi(x) := \psi(x-he_k) \in W^{1,2}_0(\Omega)$

$$\int_\Omega A(x)\nabla u(x) \cdot \nabla\psi(x-he_k) = \int_\Omega f(x)\psi(x-he_k)\,dx \qquad \text{define } f(x)=0 \ \forall x \notin \Omega$$

$$\int_\Omega A(x+he_k)\nabla u(x+he_k) \cdot \nabla\psi(x) = \int_\Omega f(x+he_k)\psi(x)\,dx \qquad (\text{eq } 1)$$

set $\varphi(x) = \psi(x)$: $\quad \int_\Omega A(x)\nabla u(x) \cdot \nabla\psi = \int_\Omega f(x)\psi(x)\,dx \qquad (\text{eq } 2)$

$$(\text{eq}1)-(\text{eq}2) = \int_\Omega (A(x+he_k)\nabla u(x+he_k) - A(x)\nabla u(x)) \cdot \nabla\psi(x) = \int_\Omega (f(x+he_k)-f(x))\psi(x)\,dx \qquad \begin{array}{l}\forall \psi \in W^{1,2}\\ \psi(x)=0\\ \operatorname{dist}(x,\partial\Omega)<h\end{array}$$

set $\psi(x) := (u(x+he_k)-u(x))\tau^2(x)$, $\quad \tau(x) = 0$ if $\operatorname{dist}(x,\partial\Omega) < h$

$\tau(x) = 1$ if $\operatorname{dist}(x,\partial\Omega) > 2h_0$

$\tau \in C^\infty(\Omega)$ $\quad h < h_0$, $\quad |\nabla\tau| \le \frac{C}{h_0}$

$$\int_\Omega A(x+he_k)(\nabla u(x+he_k) - \nabla u(x)) \cdot \nabla\psi(x) = \int_\Omega (f(x+he_k)-f(x))\psi(x) + \int_\Omega (A(x) - A(x+he_k))\nabla u(x) \cdot \nabla\psi(x)$$

$$\int_\Omega A(x+he_k)\nabla(u(x+he_k)-u(x)) \cdot \nabla(u(x+he_k)-u(x))\tau^2(x) =: \boxed{\text{GOOD}}$$

$$= -\int_\Omega A(x+he_k)\nabla(u(x+he_k-u(x)) \cdot \nabla\tau^2(x)(u(x+he_k)-u(x)) \qquad =: \mathrm{I}$$

$$+\int_\Omega (A(x)-A(x+he_k))\nabla u(x) \cdot \left(\nabla(u(x+he_k)-u(x))\tau^2(x) + \nabla\tau^2(x)(u(x+he_k)-u(x))\right) \qquad =: \mathrm{II}$$

$$+\int_\Omega (f(x+he_k - f(x))(u(x+he_k)-u(x))\tau^2(x) \qquad =: \mathrm{III}$$

ellipticity of $A \Rightarrow \int_\Omega |\nabla u(x+he_k) - \nabla u(x)|^2\tau^2(x) \le C\,\boxed{\text{GOOD}}$

$$|\mathrm{II}| \le \|A(x) - A(x+he_k)\|_\infty \|\nabla u\|_2 \left(\|\nabla(u(x+he_k)-u(x))\tau\|_2 + \|\nabla\tau^2\|_\infty \|u(x+he_k)-u(x)\|_2\right)$$

$$\le h\|A\|_{1,\infty}\|\nabla u\|_2 \left(C\sqrt{\boxed{\text{GOOD}}} + \frac{C}{h_0} h\|u\|_{1,2}\right)$$

Young

$$\le h^2\|u\|_{1,2}^2 \left(\frac{\|A\|_{1,\infty}C}{h_0}\right) + \varepsilon\,\boxed{\text{GOOD}} + \frac{C}{\varepsilon}h^2\|A\|_{1,\infty}^2\|u\|_{1,2}^2$$

28.11.2018 $\quad |\mathrm{II}| \leq 2 \int_\Omega \underbrace{|A(x+he_k)|}_{L^\infty} \underbrace{\tau\, |\nabla u(x+he_k) - \nabla u(x)|}_{L^2} \underbrace{|u(x+he_k) - u(x)|}_{L^2} \underbrace{|\nabla \tau|}_{L^\infty}$

$$\leq 2\, \|A\|_\infty \|\nabla\tau\|_\infty\, c \sqrt{\boxed{\text{GOOD}}}\; h\, \underbrace{\frac{\|\, |u(x+he_k) - u(x)|\, \chi_{\operatorname{supp}\tau} \|_2}{h}}$$

characterization of $W^{1,2}$

$$\leq c\, \|A\|_\infty \|\nabla\tau\|_2 \sqrt{\boxed{\text{GOOD}}}\; h\, \|\nabla u\|_2$$

$$\overset{\text{Young}}{\leq} \varepsilon \boxed{\text{GOOD}} + \frac{c}{\varepsilon} \|A\|_\infty^2 \|\nabla\tau\|_\infty^2 \|u\|_{1,2}^2 h^2$$

$$|\mathrm{III}| = \left| \int_\Omega (f(x+he_k) - f(x))\,(u(x+he_k) - u(x))\,\tau^2(x) \right|$$

$$= \left| \int_\Omega f(x+he_k)\,(u(x+he_k) - u(x))\,\tau^2(x) - \int_\Omega f(x)\,(u(x+he_k) - u(x))\,\tau^2(x) \right|$$

$$= \left| \int_\Omega f(x)\,(u(x) - u(x-he_k))\,\tau^2(x-he_k) - \int_\Omega f(x)\,(u(x+he_k) - u(x))\,\tau^2(x) \right|$$

$$= \left| \int_\Omega \underbrace{f(x)}_{L^2} \left[(u(x) - u(x-he_k))\,\tau^2(x-he_k) - (u(x+he_k) - u(x))\,\tau^2(x) \right] \right|$$

$$\leq \|f\|_2\, \| [\ldots] \|_2 \qquad\qquad [\ldots] =: \omega$$

auxiliary function $g(x) := (u(x+he_k) - u(x))\,\tau^2(x)$

$$g(x-he_k) - g(x) = (u(x) - u(x-he_k))\,\tau^2(x-he_k) - (u(x+he_k) - u(x))\,\tau^2(x) = \omega$$

$$\|\omega\|_2^2 = h^2 \int_\Omega \frac{|g(x-he_k) - g(x)|^2}{h^2} \leq c h^2 \int_\Omega |\nabla g|^2 = c h^2 \int_\Omega \left| (\nabla u(x+he_k) - \nabla u(x))\,\tau^2(x) + (u(x+he_k) - u(x))\,\nabla\tau^2 \right|$$

$$\leq 2 c h^2 \Big(\underbrace{\int_\Omega |\nabla u(x+he_k) - \nabla u(x)|^2 \tau^2(x)}_{\leq \boxed{\text{GOOD}}} + \int_\Omega |u(x+he_k) - u(x)|^2 \tau^2(x)\, \|\tau\|_\infty^2 \Big)$$

$$\Rightarrow |\mathrm{III}| \leq \tilde{c}\, \|f\|_2\, h \left(\sqrt{\boxed{\text{GOOD}}} + \|u\|_2 \|\nabla\tau\|_\infty \right) \overset{\text{Young}}{\leq} \varepsilon \boxed{\text{GOOD}} + \frac{\tilde{c}}{\varepsilon} h^2 \left(\|f\|_2^2 + \|u\|_{1,2}^2 \|\nabla\tau\|_\infty^2 \right)$$

ALTOGETHER

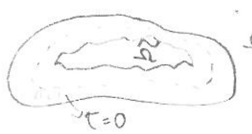
$$\boxed{\text{GOOD}} = \mathrm{I} + \mathrm{II} + \mathrm{III} \leq 3\varepsilon \boxed{\text{GOOD}} + \frac{c}{\varepsilon} h^2 \left(\|f\|_2^2 + \|u\|_{1,2}^2 \|\nabla\tau\|_\infty^2 \right)$$

choose $\varepsilon = \frac{1}{6}$: $\quad \boxed{\text{GOOD}} \leq \frac{2c}{\frac{1}{6}} h^2 (\ldots) = 12 c h^2 \left(\|f\|_2^2 + \|u\|_{1,2}^2 \|\nabla\tau\|_\infty^2 \right)$

divide by h^2 $\quad \int_\Omega \frac{|\nabla u(x+he_k) - \nabla u(x)|^2}{h^2} \tau^2(x) \leq c \left(\|f\|_2^2 + \|u\|_{1,2}^2 \right) \|\nabla\tau\|_\infty^2$ $\quad$ I assume it is always ≥ 1

$$\sup_{h \leq h_0} \int_\Omega \frac{|\nabla u(x+he_k) - \nabla u(x)|^2 \tau^2}{h^2} \leq c \left(\|f\|_2^2 + \|u\|_{1,2}^2 \right) \|\nabla\tau\|_\infty^2$$

Take open $\tilde{\Omega} \subseteq \overline{\tilde{\Omega}} \subseteq \Omega$



Find $\tau = 1$ in $\tilde{\Omega}$,

$\tau \in C_0^\infty(\Omega)$, $\tau \in [0,1]$; $\|\nabla\tau\|_\infty \leq [\operatorname{dist}(\tilde{\Omega}, \partial\Omega)]^{-1} c$

$$\sup_{h \leq \frac{\operatorname{dist}(\tilde{\Omega}, \partial\Omega)}{2}} \int_{\tilde{\Omega}} \frac{|\nabla u(x+he_k) - \nabla u(x)|^2}{h^2} \leq \frac{c}{\operatorname{dist}(\tilde{\Omega}, \partial\Omega)^2} \left(\|f\|_2^2 + \|u\|_{1,2}^2 \right)$$

$$\Rightarrow \int_{\tilde{\Omega}} \underset{\substack{\| \\ \sum_{i,j} \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \right)^2}}{|\nabla^2 u|^2} \leq \frac{c}{\operatorname{dist}(\tilde{\Omega}, \partial\Omega)^2} \left(\|f\|_2^2 + \|u\|_{1,2}^2 \right)$$

finish the proof for $k = 0$

BOOTSTRAP – $K = 1, 2, \dots$

WF: $\int_\Omega A(x)\nabla u(x)\cdot\nabla\varphi(x) = \int_\Omega f(x)\varphi(x) \qquad \forall \varphi\in W_0^{1,2}(\Omega)$

if $A\in W^{1,\infty}(\Omega)$ and $f\in L^2(\Omega)$ $\Rightarrow$ $u\in W^{2,2}_{loc}(\Omega)$ $\qquad (\Leftrightarrow u\in W^{2,2}(\tilde\Omega)\ \forall \tilde\Omega\subseteq\overline{\tilde\Omega}\subseteq\Omega)$

Denote $u_m = \frac{\partial u}{\partial x_m}\in W^{1,2}_{loc}(\Omega)$, take $\varphi(x) := \frac{\partial\psi}{\partial x_m}$ with $\psi\in C_0^\infty(\Omega)$

$\int_\Omega a_{ij}(x)\,\partial_j u(x)\,\partial_i(\partial_m\psi(x)) = \int_\Omega f(x)\,\partial_m\psi(x)$

$K=1$ ($\Leftrightarrow A\in W^{2,\infty}$, $f\in W^{1,2}$)

IBP: $\int_\Omega \partial_m(a_{ij}(x)\,\partial_j u(x))\,\partial_i\psi(x) = \int_\Omega \partial_m f(x)\,\psi(x)$

$\int_\Omega a_{ij}(x)\,\partial_j u_m(x)\,\partial_i\psi(x) = \int_\Omega \partial_m f(x)\,\psi(x) - \partial_m a_{ij}(x)\,\partial_j u(x)\,\partial_i\psi(x)$

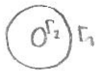
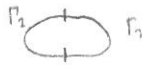
$= \int_\Omega \psi(x)\left[\partial_m f(x) + \partial_i(\partial_m a_{ij}(x)\,\partial_j u(x))\right]$

and $u_m\in W^{1,2}_{loc}(\Omega)$, we have just got the weak formulation of

$$-\operatorname{div}(A\nabla u_m) = \underset{L^2}{\underset{\cap}{\partial_m f}} + \underbrace{\operatorname{div}(\partial_m(A\nabla u))}_{\approx \nabla^2 A\nabla u + \nabla A\nabla^2 u\in L^2_{loc}(\Omega)},$$

$\Rightarrow u_m\in W^{2,2}_{loc} \Rightarrow \frac{\partial u}{\partial m}\in W^{2,2}_{loc}\ \forall m \Rightarrow u\in W^{3,2}_{loc}$

Theorem (regularity up to the boundary): Let u be a WS to $-\operatorname{div} A\nabla u = f$ in Ω, $u = u_0$ on Γ_1, $A\nabla u\cdot\vec\nu = g$ on Γ_2, $A\nabla u\cdot\vec\nu + \sigma u = g$ on Γ_3.
Let $\Omega\in C^{k+1}$, $A\in W^{k,\infty}$, $f\in W^{k-1,2}$, $g\in W^{-\frac12+k,2}(\partial\Omega)$, $\sigma\in W^{k,\infty}(\partial\Omega)$ and $\Gamma_1,\Gamma_2,\Gamma_3$ smooth open (in topology of $\partial\Omega$) and $\overline{\Gamma_i}\cap\overline{\Gamma_j}=\emptyset$ if $i\neq j$. Then $u\in W^{k+1,2}(\Omega)$.

(GOOD DOMAIN:  , bad domain )

Proof. Step 1. Regularity near flat boundary

Step 2. Transfer the result of Step 1 to small parts of $\partial\Omega$

Step 3. Combine Step 2 and local (interior) regularity result

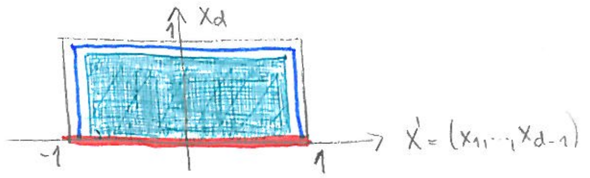
Step 1. $\Omega = (-1,1)^{d-1}\times(0,1)$

Consider $u\in W^{1,2}(\Omega)$, $u = 0$ on $(x',0)$, $|x'|<1$ is a solution to $-\operatorname{div} A\nabla u = f$ in Ω

We want to get $u\in W^{2,2}\,(-1+\delta, 1-\delta)\times(0,1-\delta)$

Step 1a. Tangential derivatives ∂x_i $\quad i = 1,\dots,d-1$ $\qquad$ ("same" as local regularity)

Step 1b. Normal derivatives ∂x_d $\qquad$ (we read it from eq)



Step 1a. WF $\quad \int_\Omega A(x)\nabla u(x)\nabla\varphi(x) = \int_\Omega f(x)\varphi(x) \qquad \forall \varphi \in W_0^{1,2}(\Omega)$

Fix $m \in \{1,\dots,d-1\}$, $\tau \in C^\infty(\Omega)$ $\quad \tau = 1$ in $\blacksquare$ $\qquad \tau \in [0,1]$

$\tau = 0$ in $\Omega \setminus \blacksquare$ $\qquad |\nabla \tau| \le \frac{c}{\delta}$

$g(x) := (u(x+he_m) - u(x))\tau^2(x) \quad , \; h < \frac{\delta}{2}$

$\varphi(x) := g(x-he_m) - g(x)$

Q: $\varphi \in W_0^{1,2}(\Omega)$???

$\varphi \in W^{1,2}(\Omega)$ ✓ $\quad , \; \varphi(x',0) = g(x'-he_m,0) - g(x',0)$

$= (u(x',0) - u(x'-he_m,0))\,\tau^2(x'-he_m,0) - (u(x'+he_m,0) - u(x',0))\,\tau^2(x',0)$

$= 0$ if $|x'| < 1$ $\quad$ ($u = 0$ if $|x'|<1$ and τ is cut off fn)

on the remaining part of the bdary it follows from $\tau = 0$ on $\Omega\setminus\blacksquare$ and $h < \frac{\delta}{2}$

$\int_\Omega A\nabla u \cdot \nabla\varphi = \int_\Omega f\varphi$

$\int_\Omega A(x)\nabla u(x) \cdot (\nabla g(x-he_m) - \nabla g(x)) = \int_\Omega f(x)\,(g(x-he_m) - g(x))$

$\int_\Omega (A(x+he_m)\nabla u(x+he_m) - A(x)\nabla u(x)) \cdot \nabla g(x) = \int_\Omega f(x)\,(g(x-he_m) - g(x))$

$\int_\Omega A(x+he_m)(\nabla u(x+he_m) - \nabla u(x)) \cdot \nabla((u(x+he_m) - u(x))\tau^2(x))$

$= \int_\Omega (A(x) - A(x+he_m))\nabla u(x) \cdot \nabla((u(x+he_m) - u(x))\tau^2(x)) + \int_\Omega f(x)(g(x-he_m) - g(x))$

$\Rightarrow$ (local regularity) $\quad \int_\Omega \frac{|\nabla u(x+he_m) - \nabla u(x)|^2 \tau^2(x)}{h^2} \le C\|\nabla\tau\|_\infty^2 (\|f\|_2^2 + \|u\|_{1,2}^2) \quad \forall m \in \{1,\dots,d-1\}$, $h < \frac{\delta}{2}$

$\forall m \in \{1,\dots,d-1\}$: $\frac{\partial}{\partial x_m}\nabla u \in L^2(\blacksquare) \iff \frac{\partial^2 u}{\partial x_i \partial x_j} \in L^2(\blacksquare)$ except $\frac{\partial^2 u}{\partial x_d \partial x_d}$

Step 1b. local regularity $\Rightarrow u \in W^{2,2}_{loc}(\Omega)$

($\frac{\partial^2 u}{\partial x_d \partial x_d}$ is well defined, we just don't know whether it is L^2 up to the bdary)

$\forall \varphi \in C_0^\infty(\Omega) \qquad \int_\Omega A\nabla u \cdot \nabla\varphi = \int_\Omega f\varphi$

$-\int_\Omega \operatorname{div}(A\nabla u)\varphi = \int_\Omega f\varphi \quad \Rightarrow \quad -\operatorname{div}(A\nabla u) = f$ a.e. in Ω

$f(x) = -\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}(x)\right) = -\frac{\partial}{\partial x_i} a_{ij}(x)\frac{\partial u}{\partial x_j}(x) - a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}u(x)$

$a_{dd}(x)\frac{\partial^2 u}{\partial x_d \partial x_d}(x) = \underbrace{-f(x) + \sum_{i,j} \partial_i a_{ij}(x)\frac{\partial u}{\partial x_j}(x) - \sum_{\substack{i,j=1,\dots,d \\ \text{except } i=j=d}} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}(x)}$

$\left\| a_{dd}\frac{\partial^2 u}{\partial x_d \partial x_d}\right\|_{L^2(\blacksquare)} = \| \qquad \|_{L^2(\blacksquare)} \overset{1a}{\le} \|f\|_{L^2(\Omega)} + C\|a\|_{1,\infty}\left(\|u\|_{1,2} + \sum_{m=1}^{d-1}\|\partial_m \nabla u\|_{L^2(\blacksquare)}\right)$

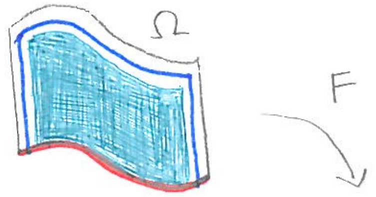
$\le \frac{c}{\delta}\left(\|f\|_{L^2(\Omega)} + \|u\|_{1,2}\right)$

elipticity of A: $\forall \lambda \in \mathbb{R}^d \quad a_{ij}\lambda_i\lambda_j \ge c_1|\lambda|^2$, set $\lambda = (0,\dots,0,1) \Rightarrow a_{dd} \ge c_1|\lambda|^2 = c_1$

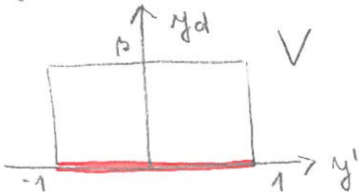
$\Rightarrow c_1\left\|\frac{\partial^2 u}{\partial x_d \partial x_d}\right\|_{L^2(\blacksquare)} \le \left\|a_{dd}\frac{\partial^2 u}{\partial x_d \partial x_d}\right\|_{L^2(\blacksquare)} \le \frac{c}{\delta}\left(\|f\|_{L^2(\Omega)} + \|u\|_{1,2}\right)$

Step 2. $\quad -\mathrm{div}(A\nabla u) = f \quad$ in Ω

$u = 0 \quad$ on Γ



? $u \in W^{2,2}($  $)$



$\Omega = \{(x', x_d),\ a(x') < x_d < a(x') + \beta\}$

$\Gamma = \{(x', x_d)\ ,\ a(x') = x_d\}$

$\Omega \in C^{1,1}\ ,\quad a \in C^{1,1}(-1,1)^{d-1}\ ,\quad V := (-1,1)^{d-1} \times (0,\beta)$

$F : \Omega \to V\ ,\quad (x', x_d) \mapsto (y', y_d)\ ,\quad y' := x'\ ,\quad y_d := x_d - a(x')$

$\tilde{u}(y) := u(F^{-1}(y))$

Find eq in $V \quad -\mathrm{div}(\tilde{A}\nabla\tilde{u}) = \tilde{f}$

WF for $u \quad \forall \varphi \in W_0^{1,2}(\Omega) \quad \int_\Omega A(x)\nabla u(x) \cdot \nabla\varphi(x) = \int_\Omega f(x)\varphi(x)$

$\psi \in C_0^\infty(V) \qquad$ set $\varphi(x) := \psi(F(x))$

$\varphi \overset{?}{\in} W_0^{1,2}(\Omega)\ ,\quad \varphi = 0$ on $\partial\Omega$

$$\frac{\partial\varphi(x)}{\partial x_i} = \sum_{j=1}^{d} \frac{\partial\psi}{\partial y_j}(F(x))\,\frac{\partial F_j}{\partial x_i}(x) \quad , \quad \frac{\partial F}{\partial x} = \begin{pmatrix} 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ & & \ddots & 0 \\ -\partial_1 a & -\partial_2 a & \dots & 1 \end{pmatrix}$$

$$\int_\Omega \sum_{i,j,k} a_{ij}(x)\,\frac{\partial u(x)}{\partial x_j}\,\frac{\partial\psi(F(x))}{\partial y_k}\,\frac{\partial F_k(x)}{\partial x_i}\,dx = \int_\Omega f(x)\,\psi(F(x))$$

$$\frac{\partial u(x)}{\partial x_j} = \frac{\partial}{\partial x_j}\tilde{u}(F(x)) = \sum_{m=1}^{d} \frac{\partial\tilde{u}(F(x))}{\partial y_m}\,\frac{\partial F_m(x)}{\partial x_j}$$

$$\int_\Omega \sum_{i,j,k,m} a_{ij}(x)\,\frac{\partial\tilde{u}(F(x))}{\partial y_m}\,\frac{\partial\psi(F(x))}{\partial y_k}\,\frac{\partial F_k(x)}{\partial x_i}\,\frac{\partial F_m(x)}{\partial x_j} = \int_\Omega f(x)\,\psi(F(x))$$

$$\int_V \sum_{i,j,k,m} a_{ij}(F^{-1}(y))\,\frac{\partial F_k(F^{-1}(y))}{\partial x_i}\,\frac{\partial F_m(F^{-1}(y))}{\partial x_j}\,\frac{\partial\psi(y)}{\partial y_k}\,\frac{\partial\tilde{u}(y)}{\partial y_m}\,dy = \int_V f(F^{-1}(y))\psi(y)\,dy$$

set $\tilde{f}(y) := f(F^{-1}(y))\ ,\quad \tilde{a}_{km}(y) = \sum_{i,j} a_{ij}(F^{-1}(y))\,\frac{\partial F_k}{\partial x_i}(F^{-1}(y))\,\frac{\partial F_m}{\partial x_j}(F^{-1}(y)) \quad \left(\tilde{A} = \frac{\partial F}{\partial x} A \left(\frac{\partial F}{\partial x}\right)^T\right)$

$$\int_V \sum_{k,m} \tilde{a}_{km}(y)\,\frac{\partial\tilde{u}(y)}{\partial y_m}\,\frac{\partial\psi(y)}{\partial y_k}\,dy = \int_V \tilde{f}(y)\psi(y)\,dy$$

$\Rightarrow \quad -\mathrm{div}(\tilde{A}\nabla\tilde{u}) = \tilde{f}$ in V,

we need $\quad \tilde{f} \in L^2(V)$ ✓ $\quad , \ \tilde{A} \in W^{1,\infty}(V) \quad$ (check whether derivatives exist and are bdd)

$\nabla\tilde{A} \sim \nabla^2 F\, A\, \nabla F^T + \nabla F\, \nabla A\, \nabla F^T + \nabla F\, A\, \nabla^2 F^T$

$\nabla^2 F \sim \nabla^2 a \in L^\infty \quad$ because $\Omega \in C^{1,1}$

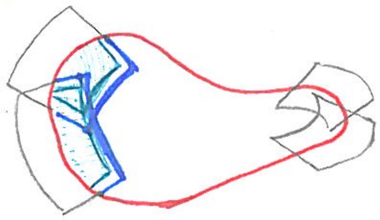
$\Rightarrow \quad \tilde{u} \in W^{2,2}($  $)\ ,\quad$ I want to go back to u

$$\frac{\partial^2 u}{\partial x_i \partial x_j}(x) = \frac{\partial^2\tilde{u}(F(x))}{\partial x_i \partial x_j} = \text{many terms } (\sim \nabla^2\tilde{u}\,\nabla F + \nabla\tilde{u}\,\nabla^2 F)$$

$\Rightarrow \quad u \in W^{2,2}($  $)$

Step 3. $\quad -\operatorname{div}(A\nabla u) = f$ in Ω, $\quad u = 0$ on $\partial\Omega$

$\Omega \in C^{1,1}$



$u \in W^{2,2}(\square) \quad \forall \square$

$\Rightarrow u \in W^{2,2}(\Omega)$, $\quad \|u\|_{2,2} \leq c(\Omega, A)(\|f\|_2 + \|u\|_{1,2})$

I can again bootstrap for $k = 1, 2, \dots$

this was only for homogeneous data

Final notes: 1. homogeneous Neumann - the same proof

2. $u \neq 0$ on $\partial\Omega$, $\quad u_0 \in W^{k,2}(\Omega)$

$$-\operatorname{div}(A\nabla u) = f$$

$$-\operatorname{div}(A\nabla \underbrace{(u-u_0)}_{\in W^{1,2}_0}) = \underbrace{f}_{\in L^2} - \underbrace{\operatorname{div}(A\nabla u_0)}_{\in L^2} \Rightarrow u - u_0 \in W^{k,2} \Rightarrow u \in W^{k,2}$$

3. if nonhomogeneous Neumann : $g \neq 0$

then $g \in W^{-\frac{1}{2}+k,2}(\partial\Omega) \quad \Rightarrow \tilde{g} \in W^{k,2}(\Omega)$, $\tilde{g} = g$ on $\partial\Omega$

WF: $\quad \int_\Omega A\nabla u \cdot \nabla\varphi = \int_\Omega f\varphi + \int_{\partial\Omega} g\varphi$

$\Omega \in C^{1,1}$, extend $\vec{\nu}$ inside Ω, $\vec{\nu} \in W^{1,\infty}$ (if fn is $W^{1,\infty}$ on closed set $(\partial\Omega)$, I can extend it to $\mathbb{R}^d$ s.t. it remains $W^{1,\infty}$)

$$\int_{\partial\Omega} g\varphi = \int_{\partial\Omega} g\varphi\vec{\nu}\cdot\vec{\nu} = \int_\Omega \operatorname{div}(g\varphi\vec{\nu}) = \int_\Omega \nabla g\cdot\vec{\nu}\varphi + \nabla\varphi\cdot\vec{\nu}g + g\varphi \operatorname{div}\vec{\nu}$$

$$\int_\Omega (A\nabla u - g\vec{\nu})\cdot\nabla\varphi = \int_\Omega f\varphi + \nabla g\cdot\vec{\nu}\varphi + g\varphi\operatorname{div}\vec{\nu}$$

$$\int_\Omega A(\nabla u - A^{-1}g\vec{\nu})\cdot\nabla\varphi$$

to prove that $\nabla u - A^{-1}g\vec{\nu} \in W^{1,2}(\Omega) \quad \rightarrow u \in W^{2,2}(\Omega)$

5.12.201

Chapter 4. Bochner integral

$u: I \to X$, I - time interval "$(0,T)$", X - ~~Bochner~~ Banach space

Definition: We say that s is a **simple function** ($s: I \to X$) if there exists a sequence $\{I_j\}_{j=1}^n$ of disjoint intervals and $\{x_j\}_{j=1}^n \subseteq X$ such that $s = \sum_{j=1}^n x_j \chi_{I_j}$

Definition (**measurability**): We say that $f: I \to X$ is measurable (strongly measurable, Bochner measurable) if $\exists \{s^n\}_{n=1}^\infty$ a sequence of simple functions such that

$\|f(t) - s^n(t)\|_X \to 0$, $n \to \infty$ a.e. in $(0,T)$

Theorem (**measurability**): Function $f: I \to X$ is measurable iff

1. f is almost separably valued, and
2. f is weakly measurable.

Ad 1. $\exists E \subseteq I$ $|E| = 0$, $f(I \setminus E)$ is separable

(if X is separable, this is for free)

Ad 2. $\forall F \in X^*$ $\langle F, f(t) \rangle_X$ is Lebesgue measurable wrt $t \in (0,T)$

Definition (**Bochner integral for simple function**): Let $s: I \to X$ be a simple function. We define

(Bochner integral) $\int_I s(t)\,dt = \sum_{j=1}^n x_j |I_j|$ $\left(\int_I s(t)\,dt \in X\right)$

Definition (**Bochner integral for measurable functions**): Let $f: I \to X$ be Bochner measurable.

We say that f is Bochner integrable if $\exists \{s^n\}_{n=1}^\infty$ simple functions such that

- $s^n(t) \to f(t)$ for a.a. $t \in (0,T)$,
- $\int_I \|s^n(t) - f(t)\|_X \to 0$ $n \to \infty$.

If f is Bochner integrable we set $\int_I f(t)\,dt := \lim_{n\to\infty} \int_I s^n(t)\,dt$.

Remark: The definition is meaningful and independent of the choice of $\{s^n\}$.

We will show that $\int_I s^n(t)\,dt$ is a Cauchy sequence in X.

$$\left\|\int_I s^n(t)\,dt - \int_I s^m(t)\,dt\right\|_X = \left\|\sum_{j=1}^k x_j |I_j| - \sum_{j=1}^l y_j |Y_j|\right\|_X = \left\|\sum (x_p - y_p)|I_p|\right\|_X$$

$$\leq \int_I \|s^n(t) - s^m(t)\|_X \leq \int_I \|s^n - f\|_X + \|s^m - f\|_X$$

Definition ($L^p(0,T;X)$ **spaces**): Let X be a Banach space.

$$L^p(0,T;X) := \{ f : I \to X \text{ Bochner integrable and } \int_I \|f(t)\|_X^p < \infty \} \qquad p \in \langle 1,\infty)$$

$$L^\infty(0,T;X) := \{ \quad - \text{"} - \quad \text{and } \| \|f(t)\|_X \|_\infty < \infty \}$$

$$\|f\|_{L^p(0,T;X)} := \left(\int_I \|f(t)\|_X^p \, dt \right)^{1/p} \qquad p \in \langle 1,\infty)$$

$$:= \| \|f(t)\|_X \|_{L^\infty(I)} \qquad p = \infty$$

Typically, X is a Lebesgue or Sobolev space.

Theorem (**Dual spaces**): Let X be a Banach space, separable and $p \in \langle 1,\infty)$.

Then $(L^p(0,T;X))^* = L^{p'}(0,T;X^*)$ (isometrically isomorfic)

Difficult HW: 1. Show that $L^\infty(0,1;L^\infty(0,1)) \neq L^\infty(0,1)^2$ (without the use of the density of C^∞ functions)

2. Show that $L^2(0,1;L^2(0,1)) = L^2(0,1)^2$

Theorem (**approximation**): Let X be a separable Banach space and $p \in \langle 1,\infty)$.

Then $C_0^\infty(0,T;X)$ is dense in $L^p(0,T;X)$.

4.1. Sobolev - Bochner spaces

$f : I \to X$, we want to define $\frac{\partial}{\partial t} f$ (weak derivative)

Definition (**weak time derivative**): Let $f : I \to X$ be Bochner integrable. We say that $g : I \to X$ is a weak derivative of f iff g is Bochner integrable and $\forall \tau \in C_0^\infty(I)$: $\int_I f(t)\tau'(t)\,dt = -\int_I g(t)\tau(t)\,dt$.

$W^{1,1}(0,1) \hookrightarrow C(\overline{0,1})$

Homework: $f \in L^1(0,T;X)$, $\partial_t f \in L^1(0,T;X) \Rightarrow f \in C([0,T],X)$

Definition (**Sobolev-Bochner space**):

$$W^{1,p}(0,T;X) := \{ f \in L^p(0,T;X), \partial_t f \in L^p(0,T;X) \}$$

$$\|f\|_{W^{1,p}(0,T;X)} := \left(\int_I \|f\|_X^p + \|\partial_t f\|_X^p \right)^{1/p} \qquad p \in \langle 1,\infty)$$

$$:= \operatorname*{esssup}_{t \in I} \left(\|f(t)\|_X + \|\partial_t f(t)\|_X \right) \qquad p = \infty$$

Theorem: $W^{1,p}(0,T;X)$ is Banach space for $p \in \langle 1,\infty\rangle$

- separable if $p < \infty$ and X is separable
- reflexive if $p \in (1,\infty)$ and X is reflexive

4.2 Time derivative and Gelfand triple

Motivation: heat equation
$$\frac{\partial}{\partial t} u - \Delta u = 0 \quad \text{in } \Omega \times (0,T)$$
$$u = 0 \quad \text{on } \partial\Omega \times (0,T)$$
$$u(0) = u_0 \quad \text{in } \Omega \qquad (u(0,x) = u_0(x) \text{ a.e. in } \Omega)$$

A priori estimates: test by "u"

$$\int_\Omega \partial_t u \, u \, dx - \int_\Omega \Delta u \, u \, dx = 0$$
$$\frac{1}{2} \int_\Omega \frac{d}{dt} \|u\|^2 dx + \int_\Omega |\nabla u|^2 dx = 0 \qquad \Big| \int_0^\tau dt$$
$$\frac{1}{2} \|u(\tau)\|_2^2 + \int_0^\tau \int_\Omega |\nabla u|^2 dx = \frac{1}{2} \|u_0\|_2^2$$
$$\Rightarrow \quad \|u\|_{L^\infty(0,T;L^2(\Omega))} \le \|u_0\|_{L^2(\Omega)}$$
$$c_1 \int_0^T \|u\|_{1,2}^2 dt \le \int_0^T \int_\Omega |\nabla u|^2 \le 2 \|u_0\|^2_{L^2(\Omega)}$$

Natural space: $u \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; W_0^{1,2}(\Omega))$ (from AE)

What about the time derivative?

$$\frac{\partial}{\partial t} u = \Delta u \quad , \quad \varphi \in C_0^\infty(0,T; C_0^\infty(\Omega))$$
$$\text{"} \int_0^T \int_\Omega \frac{\partial}{\partial t} u \varphi = \int_0^T \int_\Omega \Delta u \varphi \text{"} = - \int_0^T \int_\Omega \nabla u \cdot \nabla \varphi \quad \rightarrow \text{this exists}$$

I don't know whether these guys are well-defined (integrable)

give it a meaning: introduce duality

$$\langle \partial_t u, \varphi \rangle_{W_0^{1,2}(\Omega)} = - \int_\Omega \nabla u \cdot \nabla \varphi \qquad \forall \varphi \in W_0^{1,2}(\Omega) \text{ and a.a. } t \in (0,T)$$
$$\int_0^T \|\partial_t u\|^2_{(W_0^{1,2}(\Omega))^*} = \int_0^T \sup_{\substack{\varphi \in W_0^{1,2}(\Omega) \\ \|\varphi\|_{1,2} \le 1}} (\langle \partial_t u, \varphi \rangle)^2 dt = \int_0^T \sup \Big| \int_\Omega \nabla u \cdot \nabla \varphi \Big|^2 \le \int_0^T \|\nabla u\|_2^2 \le 2 \|u_0\|_2^2$$

Therefore also $u \in W^{1,2}(0,T; (W_0^{1,2}(\Omega))^*)$ (from eq)

Also, $u \in C([0,T], L^2(\Omega))$ specifies, in which sense the ~~[illegible]~~ initial values are attained

Altogether: $u \in L^\infty(0,T, L^2(\Omega)) \cap L^2(0,T, W_0^{1,2}(\Omega)) \cap W^{1,2}(0,T; (W_0^{1,2}(\Omega))^*) \cap C([0,T]; L^2(\Omega))$.

Definition (**Gelfand triple**): We say that X, H, X^* is a Gelfand triple iff X is a Banach space, H is a Hilbert space and $X \overset{\text{densely}}{\hookrightarrow} H \cong H^* \overset{\text{densely}}{\hookrightarrow} X^*$.

Example: $X = W_0^{1,2}(\Omega)$, $H = L^2(\Omega)$, $X^* = (W_0^{1,2}(\Omega))^*$.

$X \overset{\text{densely}}{\hookrightarrow} H$ is easy, also $H \cong H^*$.

We want $H^* \overset{\text{densely}}{\hookrightarrow} X^*$.

$f \in (W_0^{1,2}(\Omega))^*$, $\exists! \, u \in W_0^{1,2}(\Omega)$: $-\Delta u = f$ in Ω, $u = 0$ on $\partial\Omega$ (Lax-Milgram),

$$\int_\Omega \nabla u \cdot \nabla \varphi = \langle f, \varphi \rangle \qquad \forall \varphi \in W_0^{1,2}(\Omega)$$

Find $\{u^n\}_{n=1}^{\infty}$, $u^n \in C_0^\infty(\Omega)$ and $u^n \to u$ in $W_0^{1,2}(\Omega)$

$\langle f, \varphi \rangle_{W_0^{1,2}(\Omega)} = \int_\Omega \nabla u \cdot \nabla \varphi = \lim_{n\to\infty} \int_\Omega \nabla u^n \cdot \nabla \varphi = -\lim_{n\to\infty} \int_\Omega \Delta u^n \varphi$

I know that $-\Delta u^n \in L^2(\Omega)$, define $f^n := -\Delta u^n \in L^2(\Omega)$

then $\langle f, \varphi \rangle_{W_0^{1,2}(\Omega)} = \lim_{n\to\infty} \int_\Omega f^n \varphi$

Definition: Let X, H, X^* be a Gelfand triple, $\Phi: H \to H^*$ Riesz representation.

We define $i: X \to X^*$, such that for $x_0, x \in X$ and I identity mapping

$$\langle i x_0, x \rangle_X := (I x_0, I x)_H = \langle \Phi I x_0, I x \rangle_H$$

duality in X — scalar product in H — duality in H

$i: X \to X^*$ is dense! (Because $X \overset{dense}{\hookrightarrow} H$ and $H^* \overset{dense}{\hookrightarrow} X^*$)

Take $y \in X^*$, $\langle y, x \rangle_X := \lim_{n\to\infty} \langle i x_0^n, x \rangle_X$, where $i x_0^n \to y$ in X^*

If $y \in H^*$, by definition this is scalar product in H.

If X is reflexive the second density ($H^* \overset{dense}{\hookrightarrow} X^*$) is for free by Hahn-Banach.

Theorem (integration by parts for Sobolev Bochner function): Let $p \in (1, \infty)$, X, H, X^* be a Gelfand triple, $u, v \in L^p(0,T; X)$, $\partial_t u, \partial_t v \in L^{p'}(0,T; X^*)$.

Then $u, v \in C([0,T]; H)$ and $\forall\, t_1, t_2 \in (0,T)$

$$\int_{t_1}^{t_2} \langle \partial_t u, v \rangle_X = -\int_{t_1}^{t_2} \langle \partial_t v, u \rangle + (u(t_2), v(t_2))_H - (u(t_1), v(t_1))_H$$

12.12.2018 Proof: Scheme. 1. mollification in terms of Steklov averages $u_h(t) := \frac{1}{h} \int_t^{t+h} u(\tau) d\tau$ (Bochner integral)

2. integration by parts for the average function

$$\int_{t_1}^{t_2} \langle \partial_t u_{h_1}, v_{h_2} \rangle + \int_{t_1}^{t_2} \langle \partial_t v_{h_2}, u_{h_1} \rangle = (u_{h_1}(t_2), v_{h_2}(t_2))_H - (u_{h_1}(t_1), v_{h_2}(t_1))_H$$

3. let $h_1, h_2 \to 0+$.

Step 1. $\forall h\ \forall t \le T-h$ $u_h(t) := \frac{1}{h} \int_t^{t+h} u(\tau) d\tau$

if $u \in L^p(0,T; X) \Rightarrow u_h \to u$ in $L^p(0,T; X)$

$u_h(t) \to u(t)$ in X for a.a. $t \in (0,T)$ } based on Lebesgue integration

$$(\partial_t u)_h(t) = \frac{1}{h} \int_t^{t+h} \partial_t u(\tau) d\tau$$

$(\partial_t u)_h \to \partial_t u$ in $L^{p'}(0,T; X^*)$

$(\partial_t u)_h(t) \to \partial_t u(t)$ in X^* for a.a. t

Now, $\partial_t u_h = (\partial_t u)_h = \frac{u(t+h) - u(t)}{h}$ (*)

$\varphi \in C_0^\infty(0, T-h)$: $\int_0^{T-h} u_h(t) \varphi'(t) dt = \frac{1}{h} \int_0^{T-h} \varphi'(t) \int_t^{t+h} u(\tau) d\tau = \frac{1}{h} \int_0^{T-h} \varphi'(t) \left(\int_0^{t+h} u(\tau) - \int_0^t u(t) \right)$

$$= -\frac{1}{h} \int_0^{T-h} \varphi(t) (u(t+h) - u(t))$$

$$\int_0^{T-h} \varphi'(t) \int_0^t u(\tau)\,d\tau\,dt = \int_0^{T-h}\int_0^{T-h} \varphi'(t)\, u(\tau)\, \chi_{\tau\le t}\, d\tau\, dt$$

$$= \int_0^{T-h} u(\tau) \int_\tau^{T-h} \varphi'(t)\,dt\,d\tau \underset{\uparrow}{=} -\int_0^{T-h} u(\tau)\,\varphi(\tau)\,d\tau$$

φ is compactly supported $\Rightarrow \varphi(T-h)=0$

$$\Rightarrow \partial_t u_h(t) = \frac{u(t+h)-u(t)}{h}.$$

Proof of $(\partial_t u)_h = \partial_t u_h = \frac{u(t+h)-u(t)}{h}$

$\overset{\|}{\frac{1}{h}\int_0^{t+h} \partial_t u(\tau)\,d\tau}$

Take $\varphi \in C_0^\infty(0,T-h)$

$$\int_0^{T-h} \varphi(t) \int_t^{t+h} \partial_t u(\tau)\,d\tau = \int_0^{T-h} \varphi(t) \left(\int_0^{t+h} u(\tau)\,d\tau - \int_0^t u(\tau)\,d\tau \right)$$

$$\int_0^{T-h} \varphi(t) \int_0^t u(\tau)\,d\tau\,dt = \int_0^{T-h} \partial_t u(\tau) \int_\tau^{T-h} \varphi(t)\,dt\,d\tau$$

$$\Rightarrow \int_0^{T-h} \varphi(t) \int_t^{t+h} \partial_t u(\tau)\,d\tau = \int_0^{T-h} u(\tau)\,\varphi(\tau)\,d\tau - u(0)\int_0^{T-h}\varphi(t)\,dt \quad \Big(+ u(T-h) \overset{\overset{0}{\|}}{\int_{T-h}^{T-h} \varphi(t)\,dt} \Big)$$

Summary: $u_h(t) = \frac{1}{h}\int_t^{t+h} u(\tau)\,d\tau \qquad \partial_t u_h(t) = \frac{u(t+h)-u(t)}{h}$

$u_h \to u$ in $L^p(0,T;X) \qquad \partial_t u_h \to \partial_t u$ in $L^{p'}(0,T;X^*)$

Step 2. We take u_{h_1}, v_{h_2} $\quad \forall\ 0<t_1<t_2<T-\max(h_1,h_2)$

We want $\int_{t_1}^{t_2} \langle \partial_t u_{h_1}, v_{h_2}\rangle_X + \langle \partial_t v_{h_2}, u_{h_1}\rangle_X = (u_{h_1}(t_2), v_{h_2}(t_2))_H - (u_{h_1}(t_1), v_{h_2}(t_1))_H$

$$\Leftrightarrow \int_{t_1}^{t_2} \Big\langle \underbrace{\frac{u(t+h_1)-u(t)}{h_1}}_{\in X \text{ for a.a. } t}, \underbrace{-\!\!\!\!\!\!\int_t^{t+h_2} v}_{\in X \text{ for a.a. } t} \Big\rangle_X + \Big\langle \frac{v(t+h_2)-v(t)}{h_2}, -\!\!\!\!\!\!\int_t^{t+h_1} u(\tau)\,d\tau \Big\rangle_X = \Big(-\!\!\!\!\!\!\int_{t_2}^{t_2+h_1} u, -\!\!\!\!\!\!\int_{t_2}^{t_2+h_2} v \Big)_H - \Big(-\!\!\!\!\!\!\int_{t_1}^{t_1+h_1} u, -\!\!\!\!\!\!\int_{t_1}^{t_1+h_2} v \Big)_H$$

using Gelfand

$$\Leftrightarrow \int_{t_1}^{t_2} \big(\quad -''- \quad \big)_H + \underbrace{\int_{t_1}^{t_2} \big(\ -''-\ \big)_H}_{=:(\sigma 1)} = \underbrace{\quad -''- \quad}_{=:(\sigma 2)} \qquad (\sigma)$$

Proof of (σ).

$$\int_{t_1}^{t_2} \Big(\frac{u(t+h_1)-u(t)}{h_1}, -\!\!\!\!\!\!\int_t^{t+h_2} v(\tau)\,d\tau \Big)_H dt = \frac{1}{h_1h_2} \int_{t_1}^{t_2} \Big(u(t+h_1)-u(t), \int_{t_1}^{t+h_2} v(\tau)\,d\tau - \int_{t_1}^t v(\tau)\,d\tau \Big)_H dt$$

$$= \frac{1}{h_1h_2} \int_{t_1}^{t_2} \Big(u(t+h_1)-u(t), \int_{t_1-h_2}^{t} v(\tau+h_2) - \int_{t_1}^t v(\tau)\,d\tau \Big)_H dt$$

$$= \frac{1}{h_1h_2} \int_{t_1}^{t_2} \Big(u(t+h_1)-u(t), \int_{t_1}^{t} v(\tau+h_2) - v(\tau)\,d\tau \Big)_H dt + \frac{1}{h_1h_2} \int_{t_1}^{t_2} \Big(u(t+h_1)-u(t), \int_{t_1-h_2}^{t_1} v(\tau+h_2) \Big)_H dt$$

$$= \frac{1}{h_1h_2} \int_{t_1}^{t_2}\int_{t_1}^{t_2} \big(u(t+h_1)-u(t), v(\tau+h_2)-v(\tau) \big)_H \chi_{\tau\le t}\,d\tau\,dt + \frac{1}{h_1h_2} \int_{t_1}^{t_2} \Big(u(t+h_1)-u(t), \int_{t_1}^{t_1+h} v(\tau)\,d\tau \Big)_H dt$$

$$= \frac{1}{h_1h_2} \int_{t_1}^{t_2} \Big(v(\tau+h_2)-v(\tau), \int_\tau^{t_2} (u(t+h_1)-u(t))\,dt \Big)_H d\tau + \quad -''-$$

$$= \frac{1}{h_1h_2} \int_{t_1}^{t_2} \Big(v(\tau+h_2)-v(\tau), \int_{\tau+h_1}^{t_2+h_1} u(t)\,dt - \int_\tau^{t_2} u(t)\,dt \Big)_H d\tau + \quad -''-$$

$$= \frac{1}{h_1h_2} \int_{t_1}^{t_2} \Big(v(\tau+h_2)-v(\tau), \int_{t_2}^{t_2+h_1} u(t)\,dt - \int_\tau^{\tau+h_1} u(t)\,dt \Big)_H d\tau + \quad -''-$$

$$= \underbrace{-\int_{t_1}^{t_2} \Big(\frac{v(\tau+h_2)-v(\tau)}{h_2}, \frac{1}{h_1}\int_\tau^{\tau+h_1} u(t)\,dt \Big)_H d\tau}_{=-(\sigma 1)} + \underbrace{\frac{1}{h_1h_2}\int_{t_1}^{t_2} \Big(v(\tau+h_2)-v(\tau), \int_{t_2}^{t_2+h_1} u(t)\,dt \Big)_H d\tau}_{=:I_1} + \underbrace{\frac{1}{h_1h_2}\int_{t_1}^{t_2} \Big(u(t+h_1)-u(t), \int_{t_1}^{t_1+h_2} v \Big)_H d}_{=:I_2}$$

$$I_1 = \frac{1}{h_1 h_2}\left(\int_{t_2}^{t_2+h_2} v(t)\,dt - \int_{t_1}^{t_1+h_2} v(t)\,dt \,,\, \int_{t_2}^{t_2+h_1} u(t)\,dt\right)_H$$

$$I_2 = \frac{1}{h_1 h_2}\left(\int_{t_2}^{t_2+h_1} u(t)\,dt - \int_{t_1}^{t_1+h_1} u(t)\,dt \,,\, \int_{t_1}^{t_1+h_2} v(\tau)\,d\tau\right)_H$$

$$\Rightarrow \quad I_1 + I_2 = (\circlearrowleft 2)$$

Step 3. $h_1, h_2 \to 0+$

Take u, v, u_n, v_n

$$\int_{t_1}^{t_2} \langle \partial_t u_n, v_n\rangle_X + \langle \partial_t v_n, u_n\rangle_X = (u_n(t_2), v_n(t_2))_H - (u_n(t_1), v_n(t_1))_H$$

$\partial_t u_n, \partial_t v_n \to \partial_t u, \partial_t v \quad$ in $L^{p'}(0,T; X^*)$

$u_n, v_n \to u, v \quad$ in $L^p(0,T; X)$

also, $\quad u_n(t), v_n(t) \to u(t), v(t) \quad$ in X for a.a.t

but $X \hookrightarrow H \quad u_n(t), v_n(t) \to u(t), v(t)$ in H for a.a.t

$h \to 0+$: $\quad$ LHS $\to \int_{t_1}^{t_2} \langle \partial_t u, v\rangle_X + \langle \partial_t v, u\rangle_X$

$\quad$ for a.a. t, RHS $\to \quad (u(t_2), v(t_2))_H - (u(t_1), v(t_1))_H$

$u \in C([0,T], H)$: $\quad$ we show that u_n is Cauchy in $C([0,T]; H)$

$u_{n_1}, u_{n_2} \quad$ use integration by parts for $u_{n_1} - u_{n_2}$

$$\int_{t_1}^{t_2} \langle \partial_t(u_{n_1}-u_{n_2}), u_{n_1}-u_{n_2}\rangle_X + \langle u_{n_1}-u_{n_2}, \partial_t(u_{n_1}-u_{n_2})\rangle_X = \|(u_{n_1}-u_{n_2})(t_2)\|_H^2 - \|(u_{n_1}-u_{n_2})(t_1)\|_H^2$$

take some $t_1 \in (0, \frac{T}{4})$ for which $u_n(t_1) \to u(t_1)$ in H

$\Rightarrow u_n$ is Cauchy in $C(\frac{T}{4}, T; H) \quad \Rightarrow \quad u \in C(\frac{T}{4}, T; H)$

take some $t_2 \in (\frac{3T}{4}, T)$, $\quad \ldots\ldots \quad \Rightarrow \quad u \in C(0, \frac{3T}{4}; H)$

$\Rightarrow u \in C([0,T]; H)$

5. Parabolic equations

Ω - open set in $\mathbb{R}^d$, $\quad T > 0$, $\quad L$ - elliptic operator

$$\partial_t u + Lu = f \quad \text{in } Q := (0,T) \times \Omega$$

$$u(0) = u_0 \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } (0,T) \times \partial\Omega$$

$$Lu(t,x) = -\operatorname{div}(A(t,x)\nabla u(t,x)) + b(t,x)u(t,x) + \vec{c}(t,x)\nabla u(t,x) + \operatorname{div}(\vec{d}(t,x)u(t,x))$$

$\vec{d}, \vec{c} \in L^\infty(Q, \mathbb{R}^d)$, $\quad b \in L^\infty(Q)$, $\quad A \in L^\infty(Q, \mathbb{R}^{d\times d})$

$$\sum_{i,j} A_{ij}(t,x)\lambda_i \lambda_j \geq c_1 |\lambda|^2, \quad \lambda \in \mathbb{R}^d \qquad \text{for a.a. } (t,x) \in Q$$

5.1 Formal apriori estimates

Multiply by u and $\int_\Omega$ + use IBP

$$\int_\Omega \partial_t u\, u + \int_\Omega A\nabla u\cdot\nabla u + bu^2 + \vec{c}\cdot\nabla u\, u - \vec{d}\cdot\nabla u\, u = \int_\Omega f u = \langle f, u\rangle_V \quad , V = W_0^{1,2}(\Omega)$$

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2 + c_1\|\nabla u(t)\|_2^2 \le \|f\|_{V^*}\|u\|_V + \|\vec{d}\|_{L^\infty(Q)}\|\nabla u(t)\|_2\|u(t)\|_2$$

$$+ \|b\|_{L^\infty(Q)}\|u\|_2^2 + \|\vec{c}\|_{L^\infty(Q)}\|\nabla u(t)\|_2\|u(t)\|_2$$

$\|\nabla u(t)\|_2 \le \|u\|_V$

$$\le C(b,\vec{c},\vec{d})\left(\|u\|_V\left(\|f\|_{V^*} + \|u(t)\|_2\right) + \|u(t)\|_2^2\right)$$

$$\overset{\text{Young}}{\le} \varepsilon\|u\|_V + C(b,\vec{c},\vec{d},\varepsilon)\left(\|u(t)\|_2^2 + \|f\|_{V^*}^2\right)$$

Choose $\varepsilon \ll 1$ small enough

$$\frac{d}{dt}\|u(t)\|_2^2 + \tilde{c}\|u(t)\|_V^2 \le C(b,\vec{c},\vec{d},c_1,\Omega)\left(\|u(t)\|_2^2 + \|f\|_{V^*}^2\right)$$

Gronwall lemma: Assume that $u' \le gu + f$ in $(0,T)$, where $g, f \in L^1(0,T)$, and $u \ge 0$.

Then $u(t) \le C(\|g\|_{L^1})\left(\|f\|_{L^1} + u(0)\right)$

Proof: Multiply by $e^{-\int_0^t g(\tau)d\tau}$

$$u'(t)e^{-\int_0^t g(\tau)d\tau} - e^{-\int_0^t g(\tau)d\tau}g(t)u(t) \le e^{-\int_0^t g(\tau)d\tau}f(t)$$

$$\left(u(t)e^{-\int_0^t g(\tau)d\tau}\right)' \le e^{-\int_0^t g(\tau)d\tau}f(t) \quad / \int_0^t$$

$$u(t)e^{-\int_0^t g(\tau)d\tau} \le u(0) + \int_0^t f(y)e^{-\int_0^y g(\tau)d\tau}dy$$

$$u(t) \le e^{\int_0^t g(\tau)d\tau}\left(u(0) + \int_0^t f(y)e^{-\int_0^y g(\tau)d\tau}\right) \overset{t\le T}{\le} e^{\|g\|_{L^1}}\left(u(0) + \|f\|_{L^1}\right)$$

Apply GL to $\frac{d}{dt}\|u(t)\|_2^2 \le C\left(\|u(t)\|_2^2 + \|f(t)\|_{V^*}^2\right)$

$$\|u(t)\|_2^2 \le e^{CT}\left(\|u_0\|_2^2 + \int_0^T \|f\|_{V^*}^2\,dt\right) \Rightarrow \|u\|_{L^\infty(0,T;L^2)} \le C\left(\|u_0\|_2^2 + \|f\|_{L^2(V^*)}^2\right)$$

$$\int_0^T \frac{d}{dt}\|u(t)\|_2^2 + \tilde{c}\|u\|_V^2 \le \int_0^T C(\dots)\left(\|u(t)\|_2^2 + \|f\|_{V^*}^2\right)$$

$$\|u(T)\|_2^2 + \tilde{c}\int_0^T \|u\|_V^2 \le \|u(0)\|_2^2 + C(T,\dots)\left(\|u(0)\|_2^2 + \|f\|_{L^2(V^*)}^2\right)$$

$$\le C(T,A,b,\vec{c},\vec{d},\Omega)\left(\|u_0\|_2^2 + \|f\|_{L^2(V^*)}^2\right)$$

Estimate for $\partial_t u$ (Gelfand triple: $\underset{V}{W_0^{1,2}} \hookrightarrow \underset{H}{L^2} \hookrightarrow \underset{V^*}{(W_0^{1,2})^*}$)

$$\|\partial_t u(t)\|_{V^*} = \sup_{\|\varphi\|_V\le 1}\langle \partial_t u(t), \varphi\rangle_V = \sup_{\|\varphi\|\le 1}\int_\Omega \partial_t u\,\varphi = \sup_{\|\varphi\|\le 1}\int_\Omega(-Lu + f)\varphi$$

$$= \sup_{\|\varphi\|\le 1}\left(\int_\Omega -A\nabla u\cdot\nabla\varphi - bu\varphi - \vec{c}\cdot\nabla u\,\varphi + (\vec{d}u)\cdot\nabla\varphi + \langle f, \varphi\rangle\right)$$

$$\le C\sup_{\|\varphi\|\le 1}\left(\|u(t)\|_V\|\varphi\|_V + \|f(t)\|_{V^*}\|\varphi\|_V\right)$$

$$\le C\left(\|u(t)\|_V + \|f(t)\|_{V^*}\right)$$

$$\int_0^T \|\partial_t u(t)\|_{V^*}^2 \le 2C\int_0^T\left(\|u(t)\|_V^2 + \|f(t)\|_{V^*}^2\right) \overset{\text{Apriori estimates}}{\le} \tilde{\tilde{C}}\left(\int_0^T \|f\|_{V^*}^2 + \|u_0\|_2^2\right)$$

5.2 Definition of weak solution $\quad (V = W_0^{1,2}(\Omega))$

<u>Def</u>: Let $\Omega \subseteq \mathbb{R}^d$ open bounded, L be an elliptic operator, $u_0 \in L^2(\Omega)$, $f \in L^2(0,T;V^*)$.

We say that $u: Q \to \mathbb{R}$ is a **weak solution** to $\begin{cases} \partial_t u + Lu = f & \text{in } Q \\ u = 0 & \text{on } (0,T)\times\partial\Omega \\ u(0) = u_0 & \text{in } \Omega \end{cases}$

iff $u \in L^2(0,T;V)$, $\partial_t u \in L^2(0,T;V^*)$

$u(0) = u_0$ in Ω

and for a.a. $t \in (0,T)$ and $\forall \varphi \in V$ there holds

$$\langle \partial_t u(t), \varphi \rangle_V + \int_\Omega A\nabla u \cdot \nabla\varphi + bu\varphi + \vec{c}\cdot\nabla u\varphi - \vec{d}\cdot\nabla\varphi\, u = \langle f(t), \varphi \rangle_{V^*}$$

<u>Remark.</u> We know $u \in C([0,T]; L^2(\Omega))$ so it makes sense to say $u(0) = u_0$ (a.e in Ω) in topology of L^2

<u>Alternative definition of WS:</u> We say that u is a WS if $u \in L^2(0,T;V)$ and $\forall \varphi \in C_0^\infty((-\infty,T)\times\Omega)$ there holds:

$$\int_Q -u\,\partial_t\varphi + A\nabla u\nabla\varphi + bu\varphi + \vec{c}\cdot\nabla u\varphi - \vec{d}\cdot\nabla\varphi\, u = \int_\Omega u_0(x)\varphi(0,x)\,dx + \int_0^T \langle f, \varphi \rangle$$

Easy exercise (definition 1 $\Rightarrow$ definition 2)

5.3 Existence and uniqueness of WS

<u>Theorem</u>: Let $\Omega \subseteq \mathbb{R}^d$ be open bounded, $f \in L^2(0,T;V^*)$, $u_0 \in L^2(\Omega)$ and L be an elliptic operator. Then **$\exists!$ u a weak solution.**

Proof: Uniqueness. Let u_1, u_2 be two solutions, then $w := u_1 - u_2$ is a weak solution to $\quad \partial_t w + Lw = 0$ in Q

$w = 0$ on $(0,T)\times\partial\Omega$

$w(0) = 0$ in Ω

for a.t $\quad \langle \partial_t w, \varphi \rangle + \int_\Omega A\nabla w\cdot\nabla\varphi + bw\varphi + \vec{c}\cdot\nabla w\varphi - \vec{d}\cdot\nabla\varphi\, w = 0 \quad \forall \varphi \in V$

$\varphi := w(t) \quad \langle \partial_t w, w \rangle + \int_\Omega A\nabla w\cdot\nabla w + bww + \vec{c}\cdot\nabla w w - \vec{d}\cdot\nabla w\, w = 0$

Repeat formal apriori estimates (5.1)

$$\langle \partial_t w, w \rangle + \tilde{c}\,\|w(t)\|_V^2 \le c\,(\|w(t)\|_2^2)$$

$$\tfrac{1}{2}\|w(t)\|_2^2 = \int_0^t \langle \partial_t w, w \rangle_V \le c\int_0^t \|w(t)\|_2^2$$

$$\frac{d}{dt}\underbrace{\int_0^t \|w(\tau)\|_2^2\, d\tau}_{=: g(t)} = \|w(t)\|_2^2 \le \tilde{c}\int_0^t \|w(t)\|_2^2$$

$$\frac{d}{dt} g(t) \le \tilde{c}\, g(t) \overset{\text{Gronwall}}{\Rightarrow} g(t) \le c\, g(0)$$

$$\Rightarrow \int_0^t \|w(\tau)\|_2^2\, d\tau \le \int_0^0 \|w(\tau)\|_2^2\, d\tau = 0$$

$\Rightarrow \quad w = 0$ for a.a $t \in (0,T)$!

Existence. There $\exists$ basis $\{w_j\}_{j=1}^{\infty}$ of V, w_j is orthogonal in V and orthonormal in $L^2(\Omega)$, $\|P^N u\|_V \le C\|u\|_V$, $P^N u \to u$ as $N \to \infty$

P^N is the projection to $\{w_j\}_{j=1}^{N}$, $P^N u = \sum_{j=1}^{N} a_j w_j$ with $a_j = \int_\Omega u\, w_j$

I look for $u^n(t,x) := \sum_{j=1}^{n} a_j^n(t)\, w_j(x)$ (Galerkin approximation)

$$u^n(0,x) = (P^n u_0)(x) = \sum_{j=1}^{n} \left(\int_\Omega u_0 w_j\right) w_j(x)$$

My unknown is now a_j^n, I construct n equations: system

$$\int_\Omega \partial_t u^n w_j + \int_\Omega A \nabla u^n \cdot \nabla w_j + b u^n w_j + \vec{c}\,\nabla u^n w_j - \vec{d}\cdot\nabla w_j\, u^n = \langle f, w_j\rangle, \quad j = 1,\dots,n \quad (W\text{-}F)^n$$

$$\int_\Omega \sum_{i=1}^{n} \partial_t a_i^n(t)\, w_i w_j + \sum_{i=1}^{n} a_i^n(t) \int_\Omega A\nabla w_i \nabla w_j + b w_i w_j + \vec{c}\cdot\nabla w_i w_j - \vec{d}\cdot\nabla w_j\, w_i = \langle f, w_j\rangle$$

orthonormality of $\{w_j\}$

$$\partial_t a_j^n(t) = -\sum_{i=1}^{n} a_i^n(t)\, B_{ij}(t) + F_j(t)$$

$$B_{ij}(t) := \int_\Omega A(t,x)\nabla w_i(x)\cdot\nabla w_j(x) + b(t,x) w_i(x) w_j(x) + \vec{c}(t,x)\cdot\nabla w_i(x)\, w_j(x) - \vec{d}(t,x)\cdot\nabla w_j(x)\, w_i(x)$$

$$F_j(t) = \langle f(t), w_j\rangle_V$$

$$a_j(0) = \int_\Omega u_0 w_j$$

$B_{ij} \in L^\infty(0,T)$, $F_j \in L^2(0,T)$

$\exists\, \tilde{T} > 0$ such that there exists a^n a solution, $a^n \in AC$
Moreover, either $\tilde{T} = T$ or $|a^n(t)| \to \infty$ as $t \to \tilde{T}_-$
} Carathéodory theory

Apriori estimate. multiply the j-th equation of $(W\text{-}F)^n$ by $a_j^n(t)$ and take $\sum_{j=1}^{n}$

$$\sum_{j=1}^{n} \int_\Omega \partial_t u^n w_j(t)\, a_j^n(t) = \int_\Omega \partial_t u^n u^n$$

$$\frac{d}{dt}\|u^n\|_2^2 + \|u^n\|_V^2 \le \tilde{c}\left(\|u^n\|_2^2 + \|f\|_{V^*}^2\right) \quad \text{in } (0,\tilde{T})$$

$$\|u^n(t)\|_2^2 + \int_0^{\tilde{T}} \|u^n\|_V^2 \le \tilde{c}\Big(\underbrace{\|u^n(0)\|_2^2}_{P^n u_0} + \int_0^{\tilde{T}} \|f\|_{V^*}^2\Big) \quad \forall t \in (0,\tilde{T})$$

$$\sum_{i=1}^{n} a_i^2(t) \qquad \le \tilde{c}\left(\|u_0\|_2^2 + \int_0^{\tilde{T}} \|f\|_{V^*}^2\right)$$

$$\Rightarrow \tilde{T} = T, \quad \|u^n\|_{L^2(0,T,V)} + \|u^n\|_{L^\infty(0,T,L^2(\Omega))} \le \tilde{c}\left(\|u_0\|_2 + \|f\|_{L^2(0,T,V^*)}\right)$$

RHS is independent of $n \Rightarrow$ the estimate is uniform

19.12.2018 Galerkin approximation:

$$\int_\Omega \partial_t u^n w_j + \int_\Omega A\nabla u^n \nabla w_j + b u^n w_j + \vec{c}\nabla u^n w_j - \vec{d}\nabla w_j u^n = \langle f, w_j\rangle, \quad j=1,\dots,n \qquad (W\text{-}F)^n$$

$$u^n(0) = P^n u_0$$

$$\|u^n(t)\|_2^2 + \int_0^T \|u^n\|_{1,2}^2 \le \tilde{c}\left(\|u_0\|_2^2 + \int_0^T \|f\|_{V^*}^2\, dt\right)$$

uniform estimate.

Estimate on $\partial_t u^n$:

$$\|\partial_t u^n(t)\|_{V^*} \overset{def}{=} \sup_{\varphi\in V, \|\varphi\|\le 1} \langle \partial_t u^n, \varphi\rangle_V \overset{Gelfand}{=} \sup_{\|\varphi\|\le 1} \int_\Omega \partial_t u^n \varphi \qquad \int_\Omega w_j w_i = 0 \quad i\ne j$$

$$\partial_t u^n = \sum_{j=1}^n \partial_t a_j^n(t) w_j, \quad \varphi = P^n\varphi + (I-P^n)\varphi, \quad P^n\varphi = \sum_{i=1}^n \varphi_i w_i, \quad (I-P^n)\varphi = \sum_{i=n+1}^\infty \varphi_i w_i$$

$$\overset{orthogonality}{=} \sup_{\|\varphi\|\le 1} \int_\Omega \partial_t u^n P^n\varphi \overset{(W-F)^n}{=} \sup_{\|\varphi\|\le 1} -\int_\Omega A\nabla u^n \nabla P^n(\varphi) + b u^n P^n\varphi + \vec{c}\cdot\nabla u^n P^n(\varphi) - \vec{d}\cdot\nabla P^n\varphi\, u^n + \langle f, P^n\varphi\rangle$$

$$\overset{Hölder}{\le} \sup_{\|\varphi\|\le 1} \tilde{c}\left(\|u^n\|_V \|P^n\varphi\|_V + \|f(t)\|_{V^*}\|P^n\varphi\|_V\right)$$

$\|P^n\varphi\|_V \le c\|\varphi\|_V$ (c is independent of n !)

$$\le \sup_{\|\varphi\|\le 1} \tilde{c}\|\varphi\|_V \left(\|u^n(t)\|_V + \|f(t)\|_{V^*}\right) \le \tilde{\tilde{c}}\left(\|u^n(t)\|_V^2 + \|f(t)\|_{V^*}^2\right)^{1/2}$$

$$\Rightarrow \int_0^T \|\partial_t u^n\|_{V^*}^2 \le \tilde{\tilde{c}} \int_0^T \|u^n\|_V^2 + \|f\|_{V^*}^2 \overset{apriori\ estimate}{\le} \tilde{\tilde{\tilde{c}}}\left(\|u_0\|_2^2 + \int_0^T \|f\|_{V^*}^2\right)$$

Limit $n\to\infty$.

$$\|u^n\|_{L^2(0,T;V)} + \|\partial_t u^n\|_{L^2(0,T;V^*)} \le c(f,u_0)$$

$\exists$ subsequence u^{n_k}

$u^{n_k} \rightharpoonup u$ weakly in $L^2(0,T;V)$

$\partial_t u^{n_k} \rightharpoonup \partial_t u$ weakly in $L^2(0,T;V^*)$ check at home that this limit is ∂_t of this limit

in particular,

$u^{n_k} \rightharpoonup u$ weakly in $L^2(0,T;L^2(\Omega))$

$\nabla u^{n_k} \rightharpoonup \nabla u$ weakly in $L^2(0,T;L^2(\Omega,\mathbb{R}^d))$

Take $\tau\in C_0^\infty(0,T)$, multiply $(W\text{-}F)^n$ by τ and $\int_0^T$:

$$\int_0^T \langle \partial_t u^{n_k}, \tau w_j\rangle + \int_0^T\int_\Omega A\nabla u^{n_k}\cdot\nabla w_j \tau + b u^{n_k} w_j\tau + \vec{c}\cdot\nabla u^{n_k}\tau w_j - \vec{d}\tau\cdot\nabla w_j u^{n_k} = \int_0^T \langle f, \tau w_j\rangle \quad j=1,\dots,n_k$$

Fix j and let $n_k\to\infty$:

$$\lim_{n_k\to\infty} \int_0^T \langle \partial_t u^{n_k}, \tau w_j\rangle_V \underset{\uparrow}{=} \int_0^T \langle \partial_t u, \tau w_j\rangle$$

τw_j is fixed object in $L^2(0,T;V) = (L^2(0,T;V^*))^*$ & $\partial_t u^{n_k}$ converges weakly

$$\lim_{n_k\to\infty} \int_0^T\int_\Omega A\nabla u^{n_k}\cdot\nabla w_j\tau \underset{\uparrow}{=} \int_0^T\int_\Omega A\nabla u\cdot\nabla w_j\tau$$

$A\nabla w_j\tau \in L^2(0,T;L^2(\Omega,\mathbb{R}^d))$ fixed & ∇u^{n_k} converges weakly; the same with other terms

After taking the limit $n_k \to \infty$ we get

$$\int_0^T \langle \partial_t u, \tau w_j \rangle + \int_0^T \left(\int_\Omega A\nabla u \cdot \nabla w_j + b u w_j + \vec{c}\cdot\nabla u\, w_j - \vec{d}\cdot\nabla w_j\, u \right)\tau = \int_0^T \langle f, w_j \rangle \tau \quad \forall j \in \mathbb{N} \quad (W\text{-}F)$$

$\{w_j\}$ is a basis of $V \Rightarrow$ (W-F) holds $\forall w \in V \quad \forall \tau \in C_0^\infty(0,T)$:

$$\int_0^T \langle \partial_t u, w \rangle \tau + \int_0^T \tau \int_\Omega A\nabla u\cdot\nabla w + b u w + \vec{c}\cdot\nabla u\, w - \vec{d}\,\nabla w\, u = \int_0^T \tau \langle f, w \rangle$$

τ is arbitrary $\Rightarrow$ for a.a. $t \in (0,T)$ $\forall w \in V$:

$$\langle \partial_t u, w \rangle + \int_\Omega A\nabla u\cdot\nabla w + b u w + \vec{c}\cdot\nabla u\, w - \vec{d}\,\nabla w\, u = \langle f, w \rangle \qquad (W\text{-}F)u$$

Attaining u_0: $\tau \in C_0^\infty(-\infty, T)$

$$\int_0^T \langle \partial_t u^{n_k}, \tau w_j \rangle + \int_0^T \int_\Omega A\nabla u^{n_k}\cdot\nabla w_j\, \tau + b u^{n_k} w_j \tau + \vec{c}\cdot\nabla u^{n_k} \tau w_j - \vec{d}\cdot\nabla w_j\, u^{n_k}\tau = \int_0^T \langle f, \tau w_j \rangle$$

$$\int_0^T \langle \partial_t u^{n_k}, \tau w_j \rangle = \int_0^T \int_\Omega \partial_t u^{n_k} w_j \tau = -\int_0^T \int_\Omega u^{n_k} w_j \partial_t \tau - \int_\Omega u^{n_k}(0) w_j \tau(0) \qquad j = 1,\dots,n_k$$

$$= -\int_0^T \int_\Omega u^{n_k} w_j \partial_t \tau - \int_\Omega P^{n_k} u_0\, w_j \tau(0)$$

Let $n_k \to \infty$.

$$-\int_0^T \int_\Omega u\, w_j \partial_t \tau = \int_\Omega u_0 w_j \tau(0) + \int_0^T \langle f, \tau w_j \rangle - \int_0^T \int_\Omega A\nabla u\cdot\nabla w_j \tau + b u w_j \tau + \vec{c}\,\nabla u\, \tau w_j - \vec{d}\,\nabla w_j\, u \tau$$

$$\overset{(W\text{-}F)u}{=} \int_\Omega u_0 w_j \tau(0) + \int_0^T \langle \partial_t u, w_j \tau \rangle$$

$$= \int_\Omega u_0 w_j \tau(0) \underbrace{- \int_0^T \langle u, w_j \partial_t \tau \rangle - \int_\Omega u(0) w_j \tau(0)}_{\text{by Gelfand triple } = -\int_0^T \int_\Omega u\, w_j \partial_t \tau}$$

$\left.\begin{array}{l} u \in L^2(V) \\ \partial_t u \in L^2(V^*) \end{array}\right\} \Rightarrow u \in C([0,T], L^2(\Omega))$

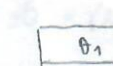
it makes sense to talk about $u(0)$

$\Rightarrow \int_\Omega u_0 w_j = \int_\Omega u(0) w_j \quad \forall j \quad$ (choose $\tau(0) = 1$)

w_j is basis of $V \to$ ~~$\int_\Omega u_0 w_j = \int_\Omega u(0)$~~ $\quad u_0 = u(0)$

5.4 Regularity "theory" for parabolic equation

- smoothing effect (= even if u_0 is bad (irregular), $u(t)$ is better $\forall t > 0$)



θ-temperature

- first improve information about $\partial_t u$ and then apply the elliptic theory

$\Rightarrow$ first show $\partial_t u \in L^2(0,T; L^2) \Rightarrow Lu(t) = f - \partial_t u(t)$

elliptic theory $\Rightarrow \quad \|u(t)\|_{2,2}^2 \le C\left(\|f(t)\|_2^2 + \|\partial_t u(t)\|_2^2\right) \quad \Rightarrow \quad u \in L^2(0,T; W^{2,2}(\Omega))$

- improve $\partial_t u^n$ (eg. $\int_0^T \|\partial_t u^n\|_2^2 \le C \Rightarrow \int_0^T \|\partial_t u\|_2^2 \le C$)

$$\int_\Omega \partial_t u^n w_j + \int_\Omega A\nabla u^n\cdot\nabla w_j + b u^n w_j + \vec{c}\cdot\nabla u^n w_j - \vec{d}\cdot\nabla w_j\, u^n = \langle f, w_j \rangle$$

test by $\partial_t u^n$ ($u^n = \sum_{i=1}^n a_i^n w_i$, multiply the j-th by $\partial_t a_j^n$ and $\sum_{j=1}^n$)

$$\Rightarrow \|\partial_t u^n(t)\|_2^2 + \int A\nabla u^n \nabla \partial_t u^n = -\int_\Omega b u^n \partial_t u^n + \vec{c}\,\nabla u^n \partial_t u^n - \vec{d}\cdot\nabla\partial_t u^n\, u^n + \langle f, \partial_t u^n \rangle$$

$$\int_\Omega A\nabla u^n\cdot\nabla\partial_t u^n = \int_\Omega \frac{A+A^T}{2}\nabla u^n\cdot\nabla\partial_t u^n + \int_\Omega \frac{A-A^T}{2}\nabla u^n\cdot\nabla\partial_t u^n$$

$$= \frac{1}{2}\int_\Omega \partial_t\left(\frac{A+A^T}{2}\nabla u^n\cdot\nabla u^n\right) - \frac{1}{4}\int \partial_t(A+A^T)\nabla u^n\cdot\nabla u^n - \frac{1}{2}\sum_{ij}\int_\Omega \partial_j(A_{ij} - A_{ji})\partial_i u^n)\partial_t u^n$$

u has zero Trace $\Rightarrow \partial_t u$ has zero Trace this produces symmetricity, I need not take $\frac{A+A^T}{2}$

$$= \frac{1}{2}\frac{d}{dt}\int_\Omega A\,\overbrace{\nabla u^n \cdot \nabla u^n} - \frac{1}{4}\int_\Omega \partial_t (A+A^T)\nabla u^n\cdot\nabla u^n - \frac{1}{2}\sum_{i,j}\int_\Omega \partial_j (A_{ij}-A_{ji})\partial_i u^n\,\partial_t u^n$$

Summary.

$$\|\partial_t u^n\|_2^2 + \frac{1}{2}\frac{d}{dt}\int_\Omega A\nabla u^n\cdot\nabla u^n \le \|\partial_t u^n\|_2 \left(\|b\|_\infty \|u^n\|_2 + \|\vec{c}\|_\infty \|\nabla u^n\|_2 + \|\vec{d}\|_{1,\infty}\|u\|_{1,2} + \|f\|_2 + \|A\|_{1,\infty}\|\nabla u^n\|_2\right) + \|\nabla u^n\|_2^2 \|\partial_t A\|_\infty$$

$$\|\partial_t u^n\|_2^2 + \frac{d}{dt}\int_\Omega A\nabla u^n\cdot\nabla u^n \overset{\text{Young}}{\le} C\left(\|b\|_\infty^2 + \|\vec{c}\|_\infty^2 + \|\vec{d}\|_{1,\infty}^2 + \|A\|_{1,\infty}^2 + \|\partial_t A\|_\infty\right)\cdot \|u^n\|_{1,2}^2 + \|f\|_2^2$$

$$\int_{t_1}^{t_2} \|\partial_t u^n\|_2^2 + \int_\Omega A\nabla u^n(t_2)\nabla u^n(t_2) \le \int_\Omega A\nabla u^n(t_1)\cdot\nabla u^n(t_1) + \int_0^T (RHS)$$

if $\nabla u_0 \in L^2$ $\Rightarrow$ take $t_1 = 0$ $\Rightarrow \int_0^T \|\partial_t u^n\|_2^2 + \sup_{t\in(0,T)} \|\nabla u^n(t)\|_2^2 \le C(DATA, \|u_0\|_2)$

if $\nabla u_0 \notin L^2$: $\delta > 0$, $t_2 \ge \delta > t_1$ and $\int_0^\delta dt_1$

$$1_i: \le \int_0^\delta \left(\int_{t_1}^{t_2} \|\partial_t u^n\|_2^2 + \int_\Omega A\nabla u^n(t_2)\nabla u^n(t_2)\right)dt_1 \le \int_0^\delta\int_\Omega A\nabla u^n\cdot\nabla u^n + \int_0^\delta\int_0^T (RHS) \le C(\text{apriori estimate})$$

$$2_i: \ge \int_0^\delta \left(\int_\delta^{t_2} \|\partial_t u^n\|_2^2\,dt + \int_\Omega A\nabla u^n(t_2)\cdot\nabla u^n(t_2)\right)$$

$$\Rightarrow \quad = \delta\left(\int_\delta^{t_2} \|\partial_t u^n\|_2^2 + \int_\Omega A\nabla u^n(t_2)\cdot\nabla u^n(t_2)\right)$$

Theorem: Let $\vec{b}, \vec{c}, \vec{d} \in L^\infty$, $\operatorname{div}\vec{d} \in L^\infty$, $A, \nabla A, \partial_t A \in L^\infty$, $f \in L^2(0,T;L^2)$.

Then $\forall \delta > 0$ $\int_\delta^T \|\partial_t u\|_2^2 + \sup_{t\ge\delta} \|\nabla u(t)\|_2^2 \le \frac{c}{\delta}$

Moreover, if $u_0 \in W_0^{1,2}(\Omega)$ then $\partial_t u \in L^2(0,T;L^2)$, $\nabla u \in L^\infty(0,T;L^2)$.

Boot strap.

$$\partial_t u + Lu = f \quad / \partial_t$$

$$\partial_{tt} u + \partial_t (Lu) = \partial_t f$$

$$\partial_t (Lu) = \partial_t(-\operatorname{div} A\nabla u + bu + \vec{c}\cdot\nabla u + \operatorname{div}(\vec{d}u)) = L\partial_t u + (-\operatorname{div}\partial_t A\nabla u + \partial_t b u + \partial_t \vec{c}\cdot\nabla u + \operatorname{div}(\partial_t \vec{d} u))$$

$$\partial_{tt} u + L\partial_t u = \partial_t f - (\quad)$$

$$v = \partial_t u: \quad \partial_t v + Lv = \underbrace{\partial_t f - (-\operatorname{div}\partial_t A\nabla u + \partial_t b u + \partial_t \vec{c}\cdot\nabla u + \operatorname{div}(\partial_t \vec{d} u))}_{\text{must belong to } L^2(0,T;V^*)}$$

initial condition: $v(0) = \partial_t u(0) \overset{\text{from eq.}}{=} -Lu_0 + f(0) \overset{\text{must}}{\in} L^2(\Omega)$

if $RHS \in L^2(0,T;V^*)$ and $-Lu_0 + f(0) \in L^2(\Omega)$ then

$$\partial_t v \in L^2(0,T;V^*),\ v \in L^2(0,T;V) \iff \partial_{tt} u \in L^2(0,T;V^*),\ \partial_t u \in L^2(0,T;W_0^{1,2}(\Omega))$$

Theorem: Let $\partial_t f \in L^2(0,T;L^2)$, $\partial_t A, \partial_t b, \partial_t \vec{c}, \partial_t \vec{d} \in L^\infty$.

Then $\forall \delta > 0$ $\partial_{tt} u \in L^2(\delta,T;V^*)$ and $\partial_t u \in L^2(\delta,T;W_0^{1,2}(\Omega))$.

Moreover, if $-Lu_0 + f(0) \in L^2$ then $\partial_{tt} u \in L^2(0,T;V^*)$ and $\partial_t u \in L^2(0,T;W_0^{1,2}(\Omega))$.

Homework: $\partial_t u - \Delta u + \vec{c}\cdot\nabla u = f$, $u(0) = u_0 = 0$, $f \in C^\infty$.

Find minimal p such that if $\vec{c} \in L^p(0,T;L^\infty(\Omega,\mathbb{R}^d))$ then $u \in L^\infty(W_0^{1,2})$ and $\partial_t u \in L^2(0,T;L^2)$

6. Linear hyperbolic equations of second order

$$\partial_{tt} u + Lu = f \quad \text{in } (0,T)\times\Omega$$

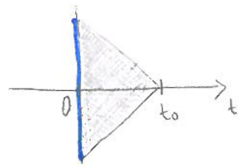
$$u(0) = u_0 \quad \text{in } \Omega$$

$$\partial_t u(0) = v_0 \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } (0,T)\times\partial\Omega \quad (\text{for simplicity})$$

- estimates are got by testing the eq by $\partial_t u$
- uniqueness is got by testing the eq by $\int_0^\tau u$ (antiderivative)
- there is no smoothing effect $\Leftrightarrow$ there is no time direction
- Finite speed of propagation

HYPERBOLIC:

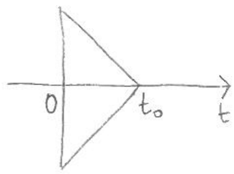


$u_0^1, v_0^1, u_0^2, v_0^2$

$u_0^1 = u_0^2$ and $v_0^1 = v_0^2$ on |

then $u^1 = u^2$ on ▶

PARABOLIC:



the same picture but $t_0 = 0$

6.1 Formal apriori estimates

Multiply by $\partial_t u$ and $\int_\Omega$ and integrate by parts

$$\int_\Omega \partial_{tt} u\, \partial_t u + \int_\Omega \left(\frac{A+A^T}{2}\right) \nabla u\, \nabla \partial_t u = -\int_\Omega \left(\frac{A-A^T}{2}\right)\nabla u\, \nabla\partial_t u + f\partial_t u - bu\partial_t u - \vec{c}\cdot\nabla u\, \partial_t u + \vec{d}\cdot\nabla\partial_t u\, u$$

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega |\partial_t u|^2 + A\nabla u\cdot\nabla u\right) = \int_\Omega \partial_t\left(\frac{A+A^T}{4}\right)\nabla u\cdot\nabla u + \int_\Omega \partial_i (A_{ij} - A_{ji})\,\partial_j u\, \partial_t u$$

$$+\int_\Omega f\,\partial_t u - bu\,\partial_t u - \vec{c}\cdot\nabla u\,\partial_t u - \operatorname{div}\vec{d}\,\partial_t u\, u - \vec{d}\cdot\nabla u\,\partial_t u$$

$$\leq \tilde{c}\Big(\|\partial_t A\|_\infty \|\nabla u\|_2^2 + \|\nabla A\|_\infty \|\partial_t u\|_2 \|\nabla u\|_2 \cancel{\vec{d}\cdot\nabla u\,\partial_t u} + \|f\|_2\|\partial_t u\|_2 + \|\vec{c}\|_\infty \|\nabla u\|_2 \|\partial_t u\|_2$$

$$+ \|\partial_t u\|_2 \|u\|_2 \left(\|\operatorname{div}\vec{d}\|_\infty + \|b\|_\infty\right)\Big)$$

$$\|\nabla u\|_2^2 \leq \tilde{c}\int_\Omega A\nabla u\cdot\nabla u, \qquad \|u\|_2^2 \leq \tilde{c}\,\|\nabla u\|_2^2 \qquad \text{Poincaré}$$

$$\underset{\text{Young ineq}}{\leq} \tilde{c}\left(\|\partial_t u\|_2^2 + \int_\Omega A\nabla u\cdot\nabla u\right)\left[1 + \|\partial_t A\|_\infty + \|\nabla A\|_\infty + \|\vec{c}\|_\infty + \|\operatorname{div}\vec{d}\|_\infty + \|b\|_\infty\right] + \|f\|_2^2$$

Gronwall lemma:

$$\|\partial_t u(t)\|_2^2 + \int_\Omega A\nabla u(t)\cdot\nabla u(t) \leq c\left(\int_0^T \|f\|_2^2 + \|\partial_t u(0)\|_2^2 + \int_\Omega A\nabla u(0)\,\nabla u(0)\right)$$

with $c = e^{\int_0^T [\dots]}$

We need that $u_0 \in W_0^{1,2}$, $v_0 \in L^2$

Estimate for $\partial_{tt} u$

$$\|\partial_{tt} u\|_{V^*} = \sup_{\|\varphi\|\leq 1} \langle \partial_{tt} u, \varphi\rangle \overset{\text{eq}}{\underset{\text{usc}}{=}} \sup_{\|\varphi\|\leq 1} \left(\int_\Omega -A\nabla u\cdot\nabla\varphi + bu\varphi + \vec{c}\cdot\nabla u\varphi - \vec{d}\cdot\nabla\varphi\, u + \int_\Omega f u\right)$$

$$\le \sup_{\|\varphi\|\le 1} \left(\|A\|_\infty \|\nabla u\|_2 + \|b\|_\infty \|u\|_2 + \|\vec{c}\|_\infty \|\nabla u\|_2 + \|\vec{d}\|_\infty \|u\|_2 + \|f\|_2 \right) \|\varphi\|_V$$

$$\int_0^T \|\partial_{tt} u\|_{V^*}^2 \le \int_0^T (\quad)^2 \le c([\,]) \left(\|u_0\|_2^2 + \|\nabla u_0\|_2^2 \right)$$

Definition: Let L be an elliptic operator such that $\int_0^T \left(\|\partial_t A\|_\infty + \|A\|_{1,\infty} + \|b\|_\infty + \|\vec{c}\|_\infty + \|\vec{d}\|_{1,\infty} \right) < \infty$ and $f \in L^2(0,T; L^2(\Omega))$. Let $u_0 \in W_0^{1,2}(\Omega)$ and $v_0 \in L^2(\Omega)$. We say that u is a **weak sol.** iff $u \in L^2(0,T;V)$, $\partial_t u \in L^2(0,T, L^2(\Omega))$, $\partial_{tt} u \in L^2(0,T, V^*)$ and for a.a. $t \in (0,T)$ and $\forall \varphi \in V$

$$\langle \partial_{tt} u, \varphi \rangle + \int_\Omega A \nabla u \cdot \nabla \varphi + b u \varphi + \vec{c} \cdot \nabla u \varphi - \vec{d} \cdot \nabla \varphi u = \int_\Omega f \varphi \qquad \text{and } u(0) = u_0, \ \partial_t u(0) = v_0 :$$

$u(0) = u_0$: we know $\partial_t u \in L^2(0,T; L^2) \Rightarrow u \in C(0,T; L^2) \Rightarrow u(0) = u_0$ in the sense of L^2.

$\partial_t u(0) = v_0$: $\partial_{tt} u \in L^2(0,T; V^*) \Rightarrow \partial_t u \in C(0,T; V^*) \Rightarrow \partial_t u(0) = v_0$ in sense of V^*.

Theorem: **$\exists!$ weak solution.**

Proof. Existence: 1. Galerkin approximation

2. uniform estimates

3. limit

1. Take $\{w_j\}_{j \in \mathbb{N}}$ an orthogonal basis in V, orthonormal in $L^2(\Omega)$.

Look for $u^n := \sum_{i=1}^n a_i^n(t) w_i(x)$, solution to

$$\int_\Omega \partial_{tt} u^n w_i + \int_\Omega A \nabla u^n \cdot \nabla w_i + b u^n w_i + \vec{c} \cdot \nabla u^n w_i - \vec{d} \cdot \nabla w_i u^n = \int_\Omega f w_i \quad \forall i = 1, \dots, n$$

$$u^n(0) = P^n u_0, \quad \partial_t u^n(0) = P^n v_0$$

Carathéodory theory $\Rightarrow$ $\exists u^n$ a solution

2. Estimates. Multiply the i-th eq by $\partial_t a_i^n$ and $\sum_{i=1}^n$ (= testing by $\partial_t u^n$)

$$\|\partial_t u^n(t)\|_2^2 + \|\nabla u^n(t)\|_2^2 \le c(A, b, \vec{c}, \vec{d}) \left(\|u_0\|_{1,2}^2 + \|v_0\|_2^2 + \int_0^T \|f\|_2^2 \right)$$

9.1.2019 $\|\partial_{tt} u^n(t)\|_{(W_0^{1,2})^*} = \sup_{\substack{w \in W_0^{1,2} \\ \|w\|=1}} \langle \partial_{tt} u^n(t), w \rangle = \sup_w \int_\Omega \partial_{tt} u^n w$

$\overset{\text{orthon. basis}}{=} \sup_w \int_\Omega \partial_{tt} u^n P^n w \overset{\text{eq.}}{=} \sup_w - \int_\Omega A \nabla u^n \nabla P^n w + b u^n P^n w + \vec{c} \cdot \nabla u^n P^n w - \vec{d} u^n \cdot \nabla P^n w - f P^n w$

$$\le \sup_w \|P^n w\|_{1,2} \, c(A, b, \vec{c}, \vec{d}) \left(\|u^n\|_{1,2} + \|f\|_2 \right)$$

$$\le \sup_w c \|w\|_{1,2} \text{ -''- } \le c(A, b, \vec{c}, \vec{d}) \left(\|u^n\|_{1,2}^2 + \|f\|_2^2 \right)^{1/2}$$

$$\int_0^T \|\partial_{tt} u^n\|_{(W_0^{1,2})^*}^2 \le \int_0^T c(\,) \left(\|u^n\|_{1,2}^2 + \|f\|_2^2 \right) \le C(A, b, \vec{c}, \vec{d}, \|u_0\|_2, \|v_0\|_2, \|f\|_{L^2 L^2})$$

$\Rightarrow \exists u : u^{n_k} \rightharpoonup^* u$ in $L^\infty(0,T; W_0^{1,2}(\Omega))$

(that means: $u^{n_k} \rightharpoonup^* u$ in $L^\infty(0,T; L^2(\Omega))$ $\Leftrightarrow$ $\forall v \in L^1(L^2): \int u^{n_k} v \to \int u v$ $n \to \infty$

$\nabla u^{n_k} \rightharpoonup^* \nabla u$ in $L^\infty(0,T; L^2(\Omega))$)

$\partial_t u^{n_k} \rightharpoonup^* \partial_t u$ in $L^\infty(0,T; L^2)$

$\partial_{tt} u^{n_k} \rightharpoonup \partial_{tt} u$ in $L^2(0,T; (W_0^{1,2})^*)$

3. Let $n \to \infty$. $\varphi \in C^\infty([0,T])$

$$\underbrace{\int_0^T\int_\Omega \partial_{tt} u^n w_i \varphi(t)}_{\int_0^T \langle \partial_{tt} u^n, w_i \varphi(t)\rangle} + \underbrace{A\nabla u^n . \nabla w_i \varphi(t)}_{= \int_0^T\int_\Omega A\nabla u^n \nabla w_i \varphi(t)} + b u^n w_i \varphi(t) + \vec{c}.\nabla u^n w_i \varphi(t) - \vec{d}.\nabla w_i u^n \varphi(t) = \int_0^T\int_\Omega f w_i \varphi(t)$$

$\in L^2(0,T; W_0^{1,2}(\Omega))$ $\quad \nabla u \searrow L^2 \quad \in L^2(0,T; L^2)$

$\to$: $n_k \to \infty$: (use weak convergence results)

$$\int_0^T \langle \partial_{tt} u, w_i \varphi(t)\rangle + \int_0^T\int_\Omega A\nabla u.\nabla w_i \varphi(t) + b u w_i \varphi(t) + \vec{c}.\nabla u w_i \varphi(t) - \vec{d}.\nabla w_i u \varphi(t) = \int_0^T\int_\Omega f w_i \varphi$$

$\Rightarrow$ for a.a. $t \in (0,T)$: $\langle \partial_{tt} u, w_i\rangle + \int_\Omega A\nabla u.\nabla w_i + b u w_i + \vec{c}\nabla u w_i - \vec{d}.\nabla w_i u = \int_\Omega f w_i$

basis

$\Rightarrow$ for a.a. $t \in (0,T)$ and $\forall w \in W_0^{1,2}(\Omega)$: $\langle \partial_{tt} u, w\rangle + \int_\Omega A\nabla u.\nabla w + \dots = \int_\Omega f w$

4. Initial conditions $\quad \varphi \in C_0^\infty(-\infty, T)$

$$\int_0^T \left(\int_\Omega A\nabla u.\nabla w + b u w + \vec{c}.\nabla u w - \vec{d}.\nabla w u - f w\right)\varphi(t) = -\int_0^T \langle \partial_{tt} u, w \varphi(t)\rangle$$

$$\overset{IBP}{=} \int_0^T\int_\Omega \partial_t u \, w \varphi' + \langle \partial_t u(0), w\varphi(0)\rangle \qquad \left(\partial_{tt} u \in L^2(0,T; (W_0^{1,2})^*) \Rightarrow\right.$$

$$= -\int_0^T\int_\Omega u w \varphi'' - \int_\Omega u(0) w \varphi'(0) + \langle \partial_t u(0), w\varphi(0)\rangle \qquad \left.\partial_t u \in C([0,T], (W_0^{1,2}(\Omega))^*)\right)$$

4^n. the same on the level of Galerkin approximation

$$\int_0^T \left(\int_\Omega A\nabla u^n \nabla w_i + b u^n w_i + \vec{c}.\nabla u^n w_i - \vec{d}.\nabla w_i u^n - f w\right)\varphi(t)$$

$$= -\int_0^T\int_\Omega u^n w_i \varphi'' - \int_\Omega u^n(0) w_i \varphi'(0) + \langle \partial_t u^n(0), w_i \varphi(0)\rangle$$

$$= -\int_0^T\int_\Omega u^n w_i \varphi'' - \int_\Omega P^n u_0 w_i \varphi'(0) + \langle P^n v_0, w_i \varphi(0)\rangle, \quad P^n v_0 \to v_0 \text{ in } L^2, \ P^n u_0 \to u_0 \text{ in } W^{1,2}$$

4^∞: $\int_0^T \left(\int_\Omega A\nabla u.\nabla w_i + b u w_i + \vec{c}.\nabla u w_i - \vec{d}.\nabla w_i u - f w_i\right)\varphi = -\int_0^T\int_\Omega u w_i \varphi'' - \int_\Omega u_0 w_i \varphi'(0) + \langle v_0, w_i \varphi(0)\rangle$

Compare 4. and 4^∞:

$$\langle \partial_t u(0), w_i \varphi(0)\rangle - \int_\Omega u(0) w_i \varphi'(0) = \langle v_0, w_i \varphi(0)\rangle - \int_\Omega u_0 w_i \varphi'(0)$$

choose φ: $\varphi(0) = 1$, $\varphi'(0) = 0 \Rightarrow \partial_t u(0) = v_0$; $\varphi(0) = 0$, $\varphi'(0) = 1 \Rightarrow u(0) = u_0$.

Uniqueness * if $u_0 = v_0 = 0$ and $f = 0$ then every weak solution is zero. (*due to linearity)

$$\langle \partial_{tt} u, w\rangle + \int_\Omega A\nabla u.\nabla w + b u w + \vec{c}.\nabla u w - \vec{d}.\nabla w u = 0 \qquad \forall w \in W_0^{1,2}(\Omega)$$

NEVER SET $w = \partial_t u$! (to get apriori estimates OK, on the level of GA OK, not here)

take $s \in (0,T)$, set $w(t) = \begin{cases} \int_t^s u(\tau)\,d\tau & t \le s \\ 0 & s \le t \le T \end{cases}$ $\qquad \partial_t w = -u$ on $(0,s)$

by parts, $w(s) = u(0) = 0$

$$-\int_0^s\int_\Omega \partial_t u \, \partial_t w + \int_0^s\int_\Omega A\nabla u.\nabla w = -\int_0^s\int_\Omega b u w + \vec{c}.\nabla u w - \vec{d}.\nabla w u$$

LHS $\quad -\int_0^s\int_\Omega \partial_t u \, \partial_t w = \int_0^s\int_\Omega \partial_t u \, u = \frac{1}{2}\int_0^s \frac{d}{dt}\|u\|_2^2 = \frac{\|u(s)\|_2^2}{2}$

$$\int_0^s\int_\Omega A\nabla u.\nabla w = -\int_0^s\int_\Omega A\,\partial_t \nabla w.\nabla w = -\int_0^s\int_\Omega \frac{A+A^T}{2}\partial_t \nabla w.\nabla w - \int_0^s\int_\Omega \frac{A-A^T}{2}\partial_t \nabla w.\nabla w$$

$$= -\frac{1}{2}\int_0^s\int_\Omega \partial_t\left(\frac{A+A^T}{2}\nabla w.\nabla w\right) + \frac{1}{2}\int_0^s \partial_t\left(\frac{A+A^T}{2}\right)\nabla w.\nabla w - \int_0^s\int_\Omega \frac{A-A^T}{2}\partial_t \nabla w.\nabla w$$

$$= \frac{1}{2}\int_\Omega A\nabla w(0).\nabla w(0) + \frac{1}{2}\int_0^s\int_\Omega \partial_t A \nabla w.\nabla w - \int_0^s\int_\Omega \frac{A-A^T}{2}\partial_t \nabla w.\nabla w$$

Summary: $\frac{1}{2}\int_\Omega A\nabla w(0)\cdot\nabla w(0) \geq \frac{1}{2}C_1\|\nabla w(0)\|_2^2 \geq \frac{1}{2}\tilde{C}\|w(0)\|_{1,2}^2 = \frac{1}{2}\tilde{C}\|\int_0^s u\|_{1,2}^2$

$$\frac{1}{2}\big(\|u(s)\|_2^2 + \tilde{C}\|\underbrace{\int_0^s u}_{V(s)}\|_{1,2}^2\big) \leq -\frac{1}{2}\int_0^s\int_\Omega \partial_t A\nabla w\cdot\nabla w + \int_0^s\int_\Omega \frac{A-A^T}{2}\partial_t\nabla w\cdot\nabla w - \int_0^s\int_\Omega buw + \vec{c}\nabla u w - \vec{d}\nabla w u$$

$$\Big|\int_0^s\int_\Omega \partial_t A\nabla w\cdot\nabla w\Big| \leq \|\partial_t A\|_\infty \int_0^s \|w\|_{1,2}^2 \qquad w(t) = \int_t^s u = V(s) - V(t)$$

$$\leq 4\|\partial_t A\|_\infty \int_0^s \|V(s)\|_{1,2}^2 + \|V(t)\|_{1,2}^2\,dt$$

$$= 4\|\partial_t A\|_\infty\, s\,\|V(s)\|_{1,2}^2 + c(A)\int_0^s\|V(t)\|_{1,2}^2\,dt$$ well prepared for Gronwall

$$\Big|\int_0^s\int_\Omega \frac{A-A^T}{2}\partial_t\nabla w\cdot\nabla w\Big| = \Big|\int_0^s\int_\Omega \frac{\nabla(A-A^T)}{2}\partial_t w\cdot\nabla w\Big| \leq \|\nabla A\|_\infty\int_0^s\|\partial_t w\|_2\|\nabla w\|_2$$

$$\leq \|\nabla A\|_\infty\int_0^s\|u(t)\|_2\|V(s)-V(t)\|_{1,2} \leq c(A)\int_0^s\|u(t)\|_2^2\,dt + c(A)\int_0^s\|V(t)\|_{1,2}^2\,dt + c(A)s\|V(s)\|_{1,2}^2$$

$$\Big|\int_0^s\int_\Omega buw\Big| \leq \|b\|_\infty\int_0^s\|u(t)\|_2\|w(t)\|_{1,2} \leq c(b)\Big(\int_0^s\|u(t)\|_2^2 + \|V(t)\|_{1,2}^2\,dt + s\|V(s)\|_{1,2}^2\Big)$$

$$\Big|\int_0^s\int_\Omega \vec{d}\nabla w u\Big| \leq c(\vec{d})\ \ (-\,\text{"}\,-)$$

$$\Big|\int_0^s\int_\Omega \vec{c}\cdot\nabla u w\Big| \leq \Big|\int_0^s\int_\Omega \vec{c}\,\partial_t\nabla w\, w\Big| \leq 1.\ \Big|\int_0^s\int_\Omega \partial_t\vec{c}\cdot\nabla w\, w + \vec{c}\cdot\nabla w\, w\Big|$$

$$\text{or } 2.\ \Big|\int_0^s\int_\Omega -\operatorname{div}\vec{c}\,\underbrace{\partial_t w}_{-u}\, w - \vec{c}\cdot\nabla w\,\underbrace{\partial_t w}_{-u}\Big|$$

we use 2: $\leq 2\|\vec{c}\|_{1,\infty}\int_0^s\|u\|_2\|w\|_{1,2} \leq c(\vec{c})\Big(\int_0^s\|u(t)\|_2^2 + \|V(t)\|_{1,2}^2\,dt + s\|V(s)\|_{1,2}^2\Big)$

Summary 2:

$$\cancel{\tfrac{1}{2}}\big(\|u(s)\|_2^2 + \tilde{C}\|\underbrace{\int_0^s u}_{=V(s)}\|_{1,2}^2\big) = c(A,b,\vec{c},\vec{d})\Big(\int_0^s\|u(t)\|_2^2 + \|V(t)\|_{1,2}^2\,dt + s\|V(s)\|_{1,2}^2\Big)$$

consider $s\in(0,\tilde{C}/2c(A,b,\vec{c},\vec{d}))$, then $\ldots\leq \frac{\tilde{C}}{2}\|V(s)\|_{1,2}^2 + c(A,b,\vec{c},\vec{d})\int_0^s\|u(t)\|_2^2 + \|V(t)\|_{1,2}^2$

$$\underbrace{\|u(s)\|_2^2 + \frac{\tilde{C}}{2}\|V(s)\|_{1,2}^2}_{\frac{d}{ds}\int_0^s\|u(t)\|_2^2 + \frac{\tilde{C}}{2}\|V(t)\|_{1,2}^2} \leq c(A,b,\vec{c},\vec{d})\int_0^s\|u(t)\|_2^2 + \|V(t)\|_{1,2}^2\,dt$$

$$\leq c(A,b,\vec{c},\vec{d})\Big(1+\frac{2}{\tilde{C}}\Big)\int_0^s\|u(t)\|_2^2 + \frac{\tilde{C}}{2}\|V(t)\|_{1,2}^2\,dt$$

$$\overset{\text{Gronwall}}{\Longrightarrow}\ \|u(s)\|_2^2 + \frac{\tilde{C}}{2}\|V(s)\|_{1,2}^2 = 0 \qquad \forall s\in\Big(0,\frac{\tilde{C}}{2c(A,b,\vec{c},\vec{d})}\Big)$$

Higher regularity:

$$\partial_{tt}u - \Delta u = f \quad \text{in } (0,T)\times\Omega$$
$$u = 0 \quad \text{on } \partial\Omega\times(0,T)$$
$$u(0) = u_0,\ \partial_t u(0) = v_0 \quad \text{in } \Omega$$

Formally apply ∂_t and test by $\partial_{tt}u$

$$\int_\Omega \partial_{ttt}u\,\partial_{tt}u + \int_\Omega \partial_t\nabla u\,\partial_{tt}\nabla u = \int_\Omega \partial_t f\,\partial_{tt}u$$

$$\frac{d}{dt}\Big(\frac{1}{2}\|\partial_{tt}u\|_2^2 + \frac{1}{2}\|\partial_t\nabla u\|_2^2\Big) \leq \|\partial_t f\|_2\|\partial_{tt}u\|_2 \leq \|\partial_t f\|_2^2 + \|\partial_{tt}u\|_2^2$$

$$\|\partial_{tt}u(t)\|_2^2 + \|\partial_t\nabla u(t)\|_2^2 \leq c\Big(\int_0^T\|\partial_t f\|_2^2 + \|\partial_{tt}u(0)\|_2^2 + \|\partial_t\nabla u(0)\|_2^2\Big)$$

$$\leq c\Big(\int_0^T\|\partial_t f\|_2^2 + \|v_0\|_{1,2}^2 + \|f(0)+\Delta u_0\|_2^2\Big)$$

$$\leq c\Big(\|\partial_t f\|_{L^2L^2} + \|f\|_{L^2L^2} + \|u_0\|_{2,2} + \|v_0\|_{1,2}\Big)^2$$

Theorem: If $f\in W^{1,2}(0,T;L^2)$, $u_0\in W^{2,2}$ and $v_0\in W^{1,2}$. Then $\partial_t u\in L^\infty(0,T;W_0^{1,2}(\Omega))$ and $\partial_{tt}u\in L^\infty(0,T;L^2)$

higher time ~~der~~ regularity

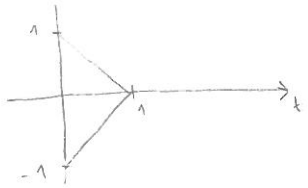
to improve space regularity: $-\Delta u = -\partial_{tt} u + f$ and use elliptic theory

This was done only formally, for rigorous proof, do the same on the level of GA.

Finite speed of propagation.

Only for wave equation, formally.

Assume that u_1, u_2 are weak solutions in $\Omega \times (0,T)$ and moreover $u_1(0) = u_2(0)$ and $\partial_t u_1(0) = \partial_t u_2(0)$ in $B_1(0)$. Then $u_1(x,t) = u_2(x,t)$ $\forall (x,t) \in$ $|x| + t \leq 1$



if I have general operator, the constant 1 would change

$\Leftrightarrow$ if w is a solution with $f=0$ and $w(0) = \partial_t w(0) = 0$ in $B_1(0)$, then $w(x,t) = 0$, $|x|+t \leq 1$.

$\partial_{tt} w - \Delta w = 0 \qquad \partial_t w\,(1-|x|-t)_+ \qquad B_1(0) \subseteq \Omega$

$$\int_\Omega \partial_t \frac{|\partial_t w|^2}{2}(1-|x|-t)_+ + \int_\Omega \nabla w\, \partial_t \nabla w\,(1-|x|-t)_+ = -\int_\Omega \nabla w\, \partial_t w \cdot \nabla(1-|x|-t)_+$$

$$\int_0^1\int_\Omega \partial_t \left(\frac{|\partial_t w|^2}{2} + \frac{|\nabla w|^2}{2}\right)(1-|x|-t)_+ = -\int_0^1\int_\Omega \nabla w\, \partial_t w\, \nabla(1-|x|-t)_+$$

$$= (1-\varepsilon)\int_0^1\int_\Omega \nabla w \frac{x}{|x|}\partial_t w\, \chi_{(1-|x|-t)>0}$$

NO. Change $(1-|x|-t)_+$ to $(1-\varepsilon-(1-\varepsilon)|x|-t)_+$

Then $\partial_t w(x,0)\,(1-\varepsilon-(1-\varepsilon)|x|)_+ = 0$

and $\partial_t w(x,1)\,(1-\varepsilon-(1-\varepsilon)|x|-1)_+ = 0$

$$\int_0^1\int_\Omega \partial_t \left(\frac{|\partial_t w|^2}{2} + \frac{|\nabla w|^2}{2}\right)(1-\varepsilon-(1-\varepsilon)|x|-t)_+ = (1-\varepsilon)\int_0^1\int_\Omega \nabla w \cdot \frac{x}{|x|}\partial_t w\, \chi_{(1-\varepsilon-(1-\varepsilon)|x|-t)>0}$$

integrate by parts (value at $t=0$ and $t=1$ is zero!)

$$\int_0^1\int_\Omega \frac{|\partial_t w|^2 + |\nabla w|^2}{2}\chi_{(1-\varepsilon-(1-\varepsilon)|x|-t)_+} \leq (1-\varepsilon)\int_0^1\int_\Omega |\nabla w|\,|\partial_t w|\,\chi_{(1-\varepsilon-(1-\varepsilon)|x|-t)>0}$$

$$\overset{\text{Young}}{\leq} (1-\varepsilon)\int_0^1\int_\Omega \frac{|\partial_t w|^2 + |\nabla w|^2}{2}\chi_{(1-\varepsilon-(1-\varepsilon)|x|-t)_+}$$

$$\Rightarrow \int_0^1\int_\Omega \frac{|\partial_t w|^2 + |\nabla w|^2}{2}\chi_{(1-\varepsilon-(1-\varepsilon)|x|-t)_+} = 0$$

$\Rightarrow \partial_t w(x,t) = \nabla w(x,t) = 0 \quad \forall (x,t):\quad 1-\varepsilon > (1-\varepsilon)|x| + t$

$\Leftrightarrow \forall (x,t): \quad 1 > |x| + t \quad !$

$\Rightarrow w = \text{const}$ in (x,t); $1 > |x| + t$

but for $w(x,0) = 0$. $|x| < 1$

$\Rightarrow w = 0$ for $1 > |x| + t$