VARIAČNÍ POČET

X - prostor s normou

Př. $X = C[a,b]$ $\quad \|u\|_X := \max_{x\in[a,b]} |u(x)|$

$X = C^1[a,b]$ $\quad \|u\|_X := \max_{x\in[a,b]} |u(x)| + \max_{x\in[a,b]} |u'(x)|$

$X = C[a,b] \times C[c,d]$ $\quad \|u\|_X = \|(u_1, u_2)\|_X$

$= \|u_1\|_{C[a,b]} + \|u_2\|_{C[c,d]}$

BUĎ $\Phi : X \to \mathbb{R}$, ŘEKNEME, ŽE

$D\Phi(y,h) = D_h\Phi(y) = \delta_h \Phi(y)$ (značení)

je <u>Gateauxův diferenciál</u> $\boxed{\Phi}$ v bodě $\boxed{y}$ ve směru $\boxed{h \in X}$

$$\Leftrightarrow \lim_{\varepsilon \to 0} \frac{\Phi(y+\varepsilon h) - \Phi(y)}{\varepsilon} = D\Phi(y,h)$$

obdobně nedefinuji derivace vyšších řádů

BUĎ $\Phi: X \to \mathbb{R}$. ŘEKNEME, ŽE <u>LINEÁRNÍ</u> OPERÁTOR

$F : X \to \mathbb{R}$ je Frechetův diferenciál Φ v bodě y

$$\Leftrightarrow \lim_{\|h\|_X \to 0} \frac{\Phi(y+h) - \Phi(y) - F(h)}{\|h\|_X} = 0$$

POKUD $\exists$ F-D pak $F(h) = D_h\Phi(y)$!

EXTRÉMY FUNKCIONÁLŮ

① Pokud má Φ v y_0 lokální extrém pak

$$D_h \Phi(y_0) = 0 \quad !?$$

nutná podmínka, y_0 splňující ← se nazývají **Extremály**

(Euler)

② Pokud Φ má v y_0 lokální minimum a $\exists\, \delta^2\Phi$

pak $\quad D^2_{hh}\Phi(y_0) \geq 0 \quad \forall h$

(nutná podmínka pro lokální minimum – Lagrange)

③ Postačující podmínka: Pokud y_0 je extremála a

$\exists\, \varepsilon > 0 \quad \forall\, y \in X \quad \|y - y_0\| \leq \varepsilon \quad D^2_{hh}\Phi(y) \geq 0 \quad \forall h$

④ Buď Φ konvexní $\Leftrightarrow$ $\forall\, \lambda \in [0,1]$ $\forall\, y_1, y_2$

platí $\quad \Phi(\lambda y_1 + (1-\lambda) y_2) \leq \lambda \Phi(y_1) + (1-\lambda)\Phi(y_2)$

Pak je každá extremála bodem globálního minima.

Funkcionál reprezentovaný integrálem

$$\Phi(y) = \int_a^b f(x, y, y', \ldots, y^{(n)})\, dx \qquad y \in C^n[a,b]$$

① NUTNÁ PODMÍNKA PRO EXISTENCI EXTRÉMU VEDE K EULER-LAGRANGEOVÝM ROVNICÍM

$$D_h \Phi(y) = \int_a^b \sum_{i=0}^{n} \frac{\partial f}{\partial y^{(i)}}(x, y, \ldots, y^{(n)})\, \underbrace{h^{(i)}}_{i\text{-tá derivace}}\, dx$$

$$0 = D_h \Phi(y) \ \forall h \iff \underbrace{\sum_{i=0}^{n} (-1)^i \left(\frac{\partial f}{\partial y^{(i)}}(x, y, \ldots, y^{(n)}) \right)^{(i)} = 0}_{\text{Euler-Lagrangeova rovnice}}$$

② Př: $\Phi(y) = \int_a^b f(y_1, y_2, y_1', y_2')$

vyšší: $X = C^1[a,b] \times C^1[a,b]$

Pro $h \in X$ $(h = (h_1, h_2))$

$$D_h \Phi(y) = \int_a^b \frac{\partial f}{\partial y_1} h_1 + \frac{\partial f}{\partial y_2} h_2 + \frac{\partial f}{\partial y_1'} h_1' + \frac{\partial f}{\partial y_2'} h_2'$$

$$= \int_a^b \left(\frac{\partial f}{\partial y_1} - \left(\frac{\partial f}{\partial y_1'} \right)' \right) h_1 + \left(\frac{\partial f}{\partial y_2} - \left(\frac{\partial f}{\partial y_2'} \right)' \right) h_2$$

E-L ROVNICE (SOUSTAVA)

$$\frac{\partial f}{\partial y_1} - \left(\frac{\partial f}{\partial y_1'} \right)' = 0$$

$$\frac{\partial f}{\partial y_2} - \left(\frac{\partial f}{\partial y_2'} \right)' = 0$$

PODMÍNKY PRO MINIMUM $f = f(x, y, z)$ z-jako y'

(3) Pokud y_0 je lokální minimum pak $f_{zz}(x, y_0(x), y_0'(x)) \geq 0$ na $[a,b]$

NENÍ POSTAČUJÍCÍ !!!

(4) Pokud $f(x, y, z)$ je konvexní vzhledem k (y, z), pak každé řešení E–L rovnic je <u>GLOBÁLNÍ MINIMUM</u>

(5) Buď y_0 extremála E–L a $f_{zz}(x, y_0(x), y_0'(x)) \geq 0$ na $[a,b]$

Jacobi: Definuj: $P(x) := f_{zz}(x, y_0(x), y_0'(x))$

$Q(x) := f_{yy}(\dots) - (f_{yz}(\dots))'$

v Jacobiho rovnici $-(P h')' + Q h = 0$

- POKUD EXISTUJE NETRIVIÁLNÍ ŘEŠENÍ SPLŇUJÍCÍ $h(a) = h(c) = 0$ pro nějaké $c \in (a,b]$ pak y_0 NENÍ EXTRÉM

- POKUD TAKOVÉ ŘEŠENÍ NEEXISTUJE, PAK y_0 je LOKÁLNÍ EXTRÉM

ÚLOHA S VAZBOU $G(y) = \int g(x, y, y') = k$

$\Rightarrow$ Lagrangeovy multiplikátory $\Rightarrow$

vázEXTRÉM $\Rightarrow$ $\left[\begin{array}{l} D_h(\Phi(y) + \lambda G(y)) = 0 \\ G(y) - k \end{array}\right]$

1) $\Phi(y) = \int_a^b \left(y^2 + (y')^2 \right) dx ; \quad y \in C^1[a,b]$

$$D_h\Phi(y) = D\Phi(y;h) = \lim_{t\to 0} \frac{\Phi(y+th) - \Phi(y)}{t}$$

$$= \lim_{t\to 0} \frac{1}{t} \int_a^b (y+th)^2 - y^2 + \left((y+th)'\right)^2 - (y')^2 \, dx$$

$$= \lim_{t\to 0} \frac{1}{t} \int_a^b 2thy + t^2h^2 + 2th'y' + t^2(h')^2 \, dx$$

$$= \lim_{t\to 0} \int_a^b 2hy + th^2 + 2h'y' + t(h')^2 \, dx$$

$$= 2\int_a^b hy + h'y'$$

$$D^2_{h,k}\Phi(y) = D^2\Phi(y,h,k) = \lim_{t\to 0} \frac{D_h\Phi(y+tk) - D_h\Phi(y)}{t}$$

$$= \lim_{t\to 0} \frac{2}{t} \int_a^b h(y+tk) - hy + h'(y+tk)' - h'y'$$

$$= \lim_{t\to 0} \frac{2}{t} \int_a^b thk + th'k' = 2\int_a^b hk + h'k'$$

$$D^3_{hkl}\Phi(y) = \lim_{t\to 0} \frac{2}{t} \int_a^b (hk + h'k') - (hk + h'k') = 0$$

2) $D_h\Phi(y) = \lim_{t\to 0} \frac{1}{t} \int_0^1 x^2 \left((y+th)^4 - \left((y+th)'\right)^2 - y^4 + (y')^2 \right)$

$$= \frac{d}{dt} \int_0^1 x^2 \left((y+th)^4 - (y'+th')^2 \right) dx \Big|_{t=0}$$

$$= \int_0^1 x^2 \left(4h(y+th)^3 - 2h'(y'+th') \right) dx \Big|_{t=0}$$

$$= \int_0^1 x^2 \left(4hy^3 - 2h'y' \right) dx$$

Jediný adept na Frechetův dif je

$$L(h) := \int_0^1 x^2 (4hg - 2h'g')\, dx$$

Ověříme, zda je to diferenciál (na $C^1[0,1]$ máme $\|h\| := \max_{x\in[0,1]} |h(x)| + \max_{x\in[0,1]} |h'(x)|$)

$$\lim_{\|h\|\to 0} \frac{\Phi(g+h) - \Phi(g) - L(h)}{\|h\|} = \lim_{h\to 0} \frac{1}{\|h\|} \int_0^1 x^2 \Big((g+h)^4 - (g'+h')^2 - g^4 + (g')^2 - 4g^3h + 2h'g' \Big)$$

$$= \lim_{h\to 0} \frac{1}{\|h\|} \int_0^1 x^2 \left(6g^2h^2 + 4gh^3 + h^4 - (h')^2 \right)$$

$$\leq \lim_{h\to 0} \frac{2\|h\|^2 + \|h\|^3 + \|h\|^4}{\|h\|} \int_0^1 x^2 \left(6g^2 + 4|g|\right) = 0$$

3) $\Phi(g) = \int_0^1 x^2 \sin(\pi g) + (g')^3 + g'' g''' + g e^{-(g'')^2}\, dx$ na $C^3[0,1]$

$$D_h\Phi(g) = \lim_{t\to 0} \frac{\Phi(g+th) - \Phi(g)}{t} = \frac{d}{dt} \Phi(g+th)\Big|_{t=0}$$

$$= \frac{d}{dt} \int_0^1 x^2 \sin(\pi(g+th)) + (g'+th')^3 + (g''+th'')(g'''+th''') + (g+th)\, e^{-(g''+th'')^2} \Big|_{t=0}$$

$$= \int_0^1 \pi h x^2 \cos(\pi(g+th)) + 3h'(g'+th')^2 + h''(g'''+th''') + h'''(g''+th'') + h e^{-(g''+th'')^2} - 2h''(g+th)\, e^{-(g''+th'')^2} \Big|_{t=0}$$

$$= \int_0^1 \pi x^2 h \cos \pi g + 3h'(g')^2 + h''g''' + h'''g'' + e^{-(g'')^2}(h - 2gh'')$$

$$D^2_{hk}\Phi(g) = \frac{d}{dt} D_h\Phi(g+tk)\Big|_{t=0} = \frac{d}{dt} \int_0^1 \pi x^2 \cos(\pi(g+tk)) + 3h'(g'+tk')^2 + h''(g'''+tk''') + h'''(g''+tk'') + e^{-(g''+tk'')^2}(h - 2h''(g+tk))$$

$$= \int_0^1 -\pi^2 x^2 hk \sin \pi g + 6h'k'g' + h''k''' + h'''k'' - 2k''(h - 2h'')\, e^{-(g'')^2} - 2h''k\, e^{-(g'')^2}$$

$$\phi(y_1,y_2) = \int_0^1 \left(x y_1^2 + (y_1')^2 (y_2')^2 + (y_2')^6 \right) \quad \text{na} \quad C[0,1] \times C[0,1]$$

$h \in C[0,1] \times C[0,1]$, $h = (h_1, h_2)$

$$D_h \phi(y_1,y_2) = \frac{d}{dt} \phi(y_1 + th_1, y_2 + th_2) \Big|_{t=0}$$

$$= \int_0^1 2x y_1 h_1 + 2 y_1' h_1' (y_2')^2 + 2 (y_1')^2 y_2' h_2' + 6 (y_2')^5 h_2'$$

(5) $\phi(y) := \int_{-1}^1 x^2 (y')^2$ nemá minimum na $M := \{ y \in C^1[-1,1] ;\ y(-1) = -1,\ y(1) = 1 \}$

a) $\phi(y) \geq 0 \quad \forall\, y \in M$

ale $\phi(y) > 0$, pokud $\phi(y) = 0$ pak $y' = 0$ na $(-1,1) \Rightarrow$

y je konstanta což je spor s $y(-1) \neq y(1)$!

b) ukážeme, že $\inf_{y \in M} \phi(y) = 0 \Rightarrow \phi$ nemá na M minimum

Volme $y_a := \dfrac{\operatorname{arctg}\left(\frac{x}{a}\right)}{\operatorname{arctg} \frac{1}{a}} \Rightarrow y_a \in M$

$$y_a' = \frac{1}{\operatorname{arctg} \frac{1}{a}} \, \frac{1}{a} \, \frac{1}{1 + \left(\frac{x}{a}\right)^2} = \frac{a}{\operatorname{arctg} \frac{1}{a}} \, \frac{1}{a^2 + x^2}$$

$$\phi(y_a) = \int_{-1}^1 \frac{a x^2}{a^2 + x^2} \, \frac{1}{\operatorname{arctg} \frac{1}{a}} \, dx \leq \int_{-1}^1 \frac{a}{\operatorname{arctg} \frac{1}{a}} = \frac{2a}{\operatorname{arctg} \frac{1}{a}}$$

$\to 0$ pro $a \to 0$

$\Rightarrow \inf \phi(y) = 0$

6) $\Phi(y) = \int_{-1}^{1} x^{\frac{2}{5}} (y')^2$ nemá extrém na M z 5)

Dk: POKUD BY Φ mělo extrém v nějakém $y \in M$

pak $D_h \Phi(y) = 0 \quad \forall h \in C^1[-1,1] \quad h(-1) = h(1) = 0$

$$0 = D_h \Phi(y) = 2 \int_{-1}^{1} x^{\frac{2}{5}} y' h'$$

$\Rightarrow$ (základní věta) $\quad x^{\frac{2}{5}} y' = \text{konst.}$

$= 0 \Rightarrow y' = 0$ - nelze

$\neq 0 \Rightarrow y' = \frac{\text{konst}}{x^{\frac{2}{5}}} \notin C^1$!

(lze i přes E-L rovnice $\left(x^{\frac{2}{5}} y' \right)' = 0$)

7) $\Phi(y) = \int_0^{2\pi} \underbrace{(y')^2 - y^2}_{f(y',y)} \qquad M = \{ y \in C^1[0, 2\pi] ; y(0) = y(2\pi) = 1 \}$

extremály = řešení E-L Φ.

E-L: $\quad 0 = \left(\frac{\partial f}{\partial y'} \right)' + \frac{\partial f}{\partial y} = -(2y')' - 2y$

$\Rightarrow \quad y'' = -y \Rightarrow \quad y = A \sin x + B \cos x$

A, B určíme z podmínky $y(0) = y(2\pi) = 1 \Rightarrow B = 1, \; A \in \mathbb{R}$

extremály $\quad y = A \sin x + \cos x$

8) $\Phi(y) = \int_0^1 y^2(x^n - y)$ $\quad n \in \mathbb{N}$ (dost velké)

$M = \{ y \in C^1[0,1],\ y(0) = y(1) = 0 \}$

a) E-L $f(x,y) = y^2(x^n - y)$

$$0 = \frac{\partial f}{\partial y} = 2y(x^n - y) - y^2 = y(2x^n - 3y)$$

pokud $y(x) \neq 0$ pro nějaké $x \in (0,1)$ ipak $\Downarrow$ $y(x) = \frac{2}{3}x^n$

pro každé $x \in (0,1) \Rightarrow y(1) = \frac{2}{3}$ a tedy $y \notin M$

$\Rightarrow$ JEDINÉ ŘEŠENÍ E-L rovnic patřící do M je $y_0 = 0$

b) SPOČTEME $D\Phi(y, h, k)$

$$D\Phi(y,h) = \int_0^1 \frac{\partial f}{\partial y} h = \int_0^1 h(2x^n y - 3y^2)$$

$$D\Phi(y,h,k) = \lim_{t \to 0} \frac{D\Phi(y+tk, h) - D\Phi(y,h)}{t} = \int_0^1 hk \frac{\partial^2 f}{\partial y}$$

$$= \int_0^1 hk(2x^n - 6y)$$

a tedy $D\Phi(y_0, h, h) = \int_0^1 2x^n h^2 > 0$ pro $h \neq 0$

c) najdeme ukážeme, že $\forall \varepsilon > 0$ $\exists y_1$; $\|y_1\| \leq \varepsilon$ $y_1 \in M$

a $\Phi(y_1) > 0$. ZŘEJMĚ $y_1 := \frac{\varepsilon}{2n} x^n(1-x)$ to splňuje

Chci najít $\forall \varepsilon\ \exists y_\varepsilon\ \|y_\varepsilon\| \leq \varepsilon \quad \Phi(y_\varepsilon) < 0$

Definuji $y_\varepsilon = x(x-\delta)^2$ pro $x \in (0,\delta)$ Pak $\|y_\varepsilon\| \leq 3\delta$

$= 0$ pro $x \geq 0$

$$\int_0^1 y_\varepsilon^2 (x^n - y_\varepsilon) = \int_0^\delta x^2 (x-\delta)^4 \left(x^n - x(x-\delta)^2\right)$$

$$= \int_0^\delta x^{n+2}(x-\delta)^4 - x^3(x-\delta)^6$$

Per partes
$$= \int_0^\delta -\frac{n+2}{5} x^{n+1}(x-\delta)^5 - x^3(x-\delta)^6$$

$$= \int_0^\delta \frac{(n+2)(n+1)}{5\cdot 6} x^n (x-\delta)^6 - x^3(x-\delta)^6$$

$$= \int_0^\delta \underset{\geq 0}{(x-\delta)^6}\, \underset{\geq 0}{x^3} \left(\underset{< 0}{\frac{(n+2)(n+1)x^{n-3}}{30} - 1} \right) < 0$$

pro $n = 3$

Extrémy funkcí – hledáním řešení E-L se zadanými okr. podmínkami poté ověřím, zda je to lokální.

a) $f(x,y,y') = x(y')^4 - 2y(y')^3$

$$\frac{\partial f}{\partial y} = -2(y')^3 \qquad \frac{\partial f}{\partial y'} = 4x(y')^3 - 6y(y')^2$$

$$\text{E-L}: \quad 0 = \frac{\partial f}{\partial y} - \left(\frac{\partial f}{\partial y'}\right)' = -2(y')^3 - \left(4x(y')^3 - 6y(y')^2\right)'$$

$$= -2(y')^3 - 4(y')^3 - 12x(y')^2y'' + 6(y')^3 + 12yy'y''$$

$$= -12y''y'(xy' - y) = -12y''y'\left(\frac{y}{x}\right)'x^2$$

MOŽNÁ ŘEŠENÍ $\Rightarrow$ $y = \text{konst.}$

$y = Ax + B$

$y = Ax$.

vzhledem k okrajovým podmínkám je jediné řešení $\boxed{y_0 = x - 1}$

Hessián f: $f_{yy} = 0$; $f_{y'y'} = \partial_{y'}\left(4x(y')^3 - 6y(y')^2\right)$

$$f_{y'y} = -6(y')^2 \qquad = 12x(y')^2 - 12yy' \Rightarrow f_{y'y'}(y_0) = 12$$

f není ani konvexní ani konkávní!

$$\Phi(y_0) = \int_1^2 x - 2(x-1) = \int_1^2 2 - x = 1 - \left[\frac{x^2}{2}\right]_1^2 = \frac{1}{2}$$

$$\Phi((x-1)^n) = \int_1^2 x\left(n(x-1)^{n-1}\right)^4 - 2(x-1)^n\left(n(x-1)^{n-1}\right)^3$$

$$= \int_1^2 x n^4 (x-1)^{4n-4} - 2n^3(x-1)^{4n-3}$$

$$= \int_1^2 (n^4 - 2n^3)(x-1)^{4n-3} + n^4(x-1)^{4n-4} = \frac{n^4 - 2n^3}{4n-2} + \frac{n^4}{4n-3}$$

$$\to \infty$$

nemůže mít globální maximum

$$D_h \Phi(g) = \int_1^2 4x(g')^3 h' - 2(g')^3 h - 6g(g')^2 h'$$

$$D_{h,k}\Phi(g) = \int_1^2 12x(g')^2 h'k' - 6(g')^2 h k' - 6(g')^2 h' k - 12 g g' h' k'$$

$$D_{h,h}\Phi(g_0) = \int_1^2 12x(h')^2 - 6hh' - 6h'h - 12(x-1)(h')^2$$

$$\underset{h(2)=h(1)}{=} 12\int_1^2 (h')^2 \quad \geq \quad \|h\|_{C^1}$$

NE?

$$\varphi(t) := \Phi(g_0 + th) = \int_1^2 x(1+th')^4 - (x-1+th)(1+th')^3$$

$$\varphi' = \int_1^2 4x(1+th')^3 h' - h(1+th')^3 - 3(x-1+th)(1+th')^2 h'$$

$$\varphi'' = \int_1^2 12x(1+th')^2(h')^2 - 3hh'(1+th')^2 - 3hh'(1+th')^2$$
$$- 6(x-1+th)(1+th')(h')^2$$

$$= \int_1^2 12x(1+th')^2(h')^2 - 6hh' - 12h(h')^2 t - 6h(h')^3 t^2$$
$$+ 6(1+th' - th' - x - th)(1+th')(h')^2$$

$$= \int_1^2 (12x+6)(1+th')^2(h')^2 - 12h(h')^2 t - 6h(h')^3 t^2 - 6(th' + x + th)(1+th')(h')^2$$

$$\geq \int_1^2 (h')^2 \Big(12x+6 - (\|h'\| + \|h'\|^2)\cdot 30 - 12\|h\| - 6\|h\|^2 - 6x - 6 + 18\|h\| \Big)$$

$$\geq \int_1^2 (h')^2 \big(6 - c(\|h\| + \|h'\|)\big),$$ kde $c > 3$ je nějaká konst.

$\Rightarrow$ pokud $\|h\| \leq \frac{3}{c}$ pak $\varphi'' > 0 \ \forall t \Rightarrow$ N

g_0 je lokální minimum

Zkusíme ještě druhou metodu

$P := f_{y'y'} = 12 > 0$ $\quad Q = f_{yy} - (f_{yz})' = 0$

KONJUGOVANÁ ÚLOHA $-(Ph')' + Qh = 0$

$\iff$

$h'' = 0 \Rightarrow h = Ax + B$

TATO FUNKCE JE NULOVÁ POUZE V JEDNOM BODĚ $\Rightarrow$ y_0 je lokální minimum

(10) $\Phi(y) = \int_2^3 \frac{x^3}{(y')^2}$ $\qquad y(2)=4,\ y(3)=9$

Globální max nemá – stačí vzít y tak, že $|y'| << \varepsilon$

$f(x,y') = \frac{x^3}{(y')^2}$ – konvexní v y'! → řešení E-L bude globální minimum

• E-L $\quad 0 = (f_{y'})' = \left(\frac{2x^3}{(y')^3}\right)' \quad = \quad \frac{x^3}{(y')^3} = \text{konst}$

$\Rightarrow \quad y' = 2Ax \quad \Rightarrow \quad \boxed{y = Ax^2 + B}$

• máme A, B $\quad \left.\begin{array}{l} 4 = y(2) = 4A + B \\ 9 = y(3) = 9A + B \end{array}\right\} \Rightarrow A = 1,\ B = 0$

• extremála $\boxed{y_0 = x^2}$ $\qquad f$ – konvexní → GLOB. MIN!

(11) $\Phi(y) = \int_0^1 (y')^2 + x^2 \quad ; \quad y(0) = -1 \ ; \ y(1) = 1$

$f(x,y') := (y')^2 + x^2$ – konvexní v y'!

E-L $\qquad y'' = 0 \quad \Rightarrow \quad y = Ax^2 + B$

$\left.\begin{array}{l} -1 = y(0) = B \\ 1 = y(1) = A + B = A - 1 \end{array}\right\} \begin{array}{l} B = -1 \\ A = 2 \end{array}$

$\Rightarrow \quad y_0 = 2x^2 - 1$ – globální minimum

(12) $\Phi(y) = \int_0^a \left(1 - e^{-(y')^2}\right)$ $\qquad y(0)=0 \quad y(a)=b \quad a>0$

$f(y') = \left(1 - e^{-(y')^2}\right)$

$f_{zz}(z) = \partial_z\left(2z e^{-z^2}\right) = 2e^{-z^2} - 4z^2 e^{-z^2}$

$= 2e^{-z^2}\left(1 - 2z^2\right)$

konvexní pro $|z| \le \frac{1}{\sqrt{2}}$

E-L $\left(y' e^{-(y')^2}\right)' = 0 \Rightarrow \underbrace{y' e^{-(y')^2} = \text{konst.}}$

protože $y \in C^1 \Rightarrow$ $\quad y' = \text{konst} \Rightarrow y = Ax + B$

$0 = y(0) = B \qquad b = y(a) = Aa \Rightarrow A = \frac{b}{a}$

$\Rightarrow y_0 = \frac{b}{a}x$ lokální minim pro $|b| < \frac{a}{\sqrt{2}}$

lok max pro $|b| > \frac{a}{2}$

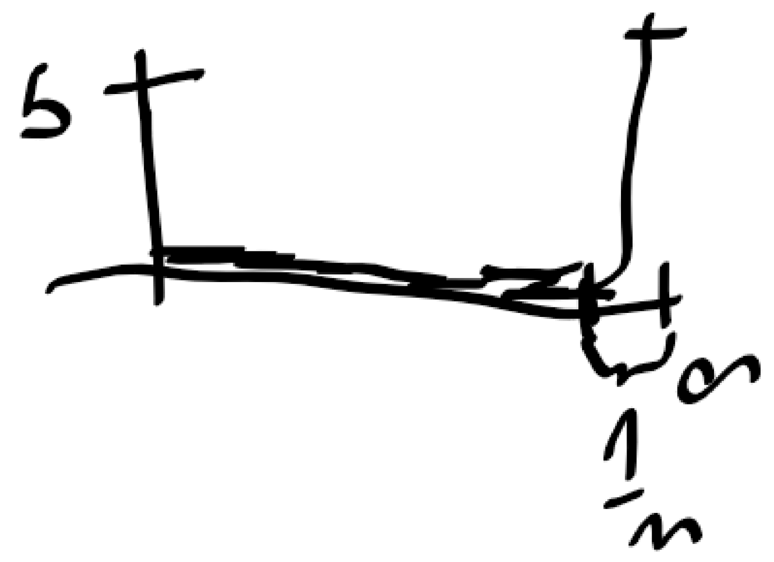
- nemá globální extrém

$\sup \Phi(y) = a$ $\qquad$ volím $y_n =$

$|y_n| \sim n$

$|y_n'| \to \infty$

$\inf \Phi(y) = 0$ $\qquad$ volím $y_n =$



$y_n' \to 0$

(13) $\Phi(y) = \int_0^1 y(y')^2$ $\qquad y(0)=p>0 \qquad y(1)=q>0$

$$f(t,y,y') = y(y')^2 \qquad f_y = (y')^2, \quad f_{y'} = 2y'y$$

Nemá globální extrémy $\sup = +\infty$, $\inf = -\infty$

E-L $\quad (y')^2 - (2y'y)' = 0$

$$0 = (y')^2 - 2y''y - 2(y')^2 = -(y')^2 - 2y''y$$

$$= y'y\left(\frac{y'}{y} - \frac{2y''}{y'}\right) = y'y\left(\ln|y| - \ln(y')^2\right)'$$

$$= \frac{1}{2}(y^2)'\left(\ln\frac{|y|}{(y')^2}\right)'$$

Buď $\underbrace{y = \text{konst}}_{\text{jen pokud } p=q}$ nebo $\frac{|y|}{(y')^2} = \text{konst}$

pak $y_0 = p$ a $\Phi(y_0)=0$

lokální minimum

$$\sqrt{|y|} = At+B$$

$$y = \pm(At+B)^2$$ + dodatečná podmínka $y=0$

$$\left.\begin{aligned} 0<p&=y(0) = \pm B^2 \\ 0<q&=y(1) = (A+\sqrt{p})^2 \end{aligned}\right\} \quad y_0 = \left((\sqrt{q}-\sqrt{p})t + \sqrt{p}\right)^2$$

$$D_h\Phi(y) = \int_0^1 (y')^2 h + 2yy'h'$$

$$D_{hh}\Phi(y) = \int_0^1 2y'h'h + 2y(h')^2 + 2y'hh' = 2\int_0^1 y(h')^2 + (h^2)'y'$$

$$D_{hh}\Phi(y_0) = 2\int_0^1 \left[(\sqrt{q}-\sqrt{p})t+\sqrt{p}\right](h')^2 - 2h^2(\sqrt{q}-\sqrt{p})$$ – nezáporné pro $p \geq q$

$\Rightarrow$ lok. minimum

vázané extrémy:

(14) $\Phi(y) = \int_0^\pi \underbrace{(y')^2}_{=f(y')}$; $y(0) = y(\pi) = 0$; $g(y) = \int_0^\pi y^2 = 1$

METODA 1) Lagrangeovy multiplikátory

$$F(y) = \Phi(y) - \lambda\, g(y) = \int_0^\pi (y')^2 - \lambda y^2$$

E-L pro F $\Rightarrow$ $-y'' - \lambda y = 0$

$y'' = -\lambda y$ $\quad \lambda = 0 \quad \to y = Ax + B$

$\lambda > 0 \quad y = A \sin\sqrt{\lambda}x + B\cos\sqrt{\lambda}x$

$\lambda < 0 \quad y = A\sinh(\sqrt{-\lambda}x) + B\cosh\ldots$

okrajové podmínky $\lambda < 0 \Rightarrow y = 0$ – nesplňuje $\int y^2 = 1$!

$\lambda > 0 \Rightarrow y = A\sin\sqrt{\lambda}x \Rightarrow \lambda = n^2$

a A volíme tak, aby $\int y^2 = 1$

$\to \int A^2 \sin^2 nx = \frac{A^2}{2}\pi \Rightarrow A = \sqrt{\frac{2}{\pi}}$

$\lambda < 0 \Rightarrow y = 0$

nemá maximum, má jen minimum pro $n = 1$

B) METODA $\quad \varphi(t) = \Phi\left(\underbrace{\frac{y+th}{\int (y+th)^2}}_{z} \right)$

hledám y, takové aby $\quad z$ splňuje $\int z^2 = 1$

$$\varphi'(t) = 0 \quad \text{pro } t=0$$

E-L

$$\varphi(t) = \int_0^\pi \frac{(y'+th')^2}{\int_0^\pi (y+th)^2} \quad ; \quad \varphi'(t) = \frac{\int_0^\pi 2(y'+th')h'}{\int_0^\pi (y+th)^2} - \frac{\int_0^\pi (y'+th')^2}{\left(\int_0^\pi (y+th)^2\right)^2} \int_0^\pi (y+th)h$$

$$\varphi'(0) = \frac{\int_0^\pi 2y'h'}{\int_0^\pi y^2} - \frac{\int_0^\pi (y')^2}{\left(\int_0^\pi y^2\right)^2} \int_0^\pi 2yh \qquad \left(\int y^2 = 1 \right.$$

$$\Rightarrow \varphi'(0) = 2\int_0^\pi y'h' - yh \int_0^\pi (y')^2 \Rightarrow \text{E-L} \quad \boxed{-y'' = y \int_0^\pi (y')^2}$$

(VÝHODA - VIDÍME ROVNOU STRUKTURU $-y'' = y\lambda$

ale $\lambda > 0$?)

(15) $\Phi(y) = \int_0^1 (y')^2$ $\quad y(0) = 1,\ y(1) = 6$ $\quad \int_0^1 y = 3$

$$F(y) = \int_0^1 (y')^2 - \lambda y$$

E-L $\quad -2y'' - \lambda = 0 \Rightarrow y'' = \text{konst.}$

$$y = Ax^2 + Bx + C$$
$$y(0) = 1 \Rightarrow C = 1$$
$$y(1) = 6 \Rightarrow A + B = 5$$

$$y = Ax^2 + (5-A)x + 1$$
$$\int_0^1 y = \frac{A}{3} + \frac{5-A}{2} + 1 = \frac{21-A}{6} \Rightarrow A = 15$$

$y = 15x^2 - 10x + 1$: globální min

globální max není!

(16) $\Phi(y) = \int_0^1 x^2 + (y')^2$ $\quad y(0) = y(1) = 0$ $\quad \int_0^1 y^2 = 2$

$F(y) = \int_0^1 x^2 + (y')^2 - \lambda y^2$ $\quad$ E-L:

stejné jako (14)

(17) $J(y) = \int_0^1 (y')^2 \qquad 0 = y(0) \qquad \frac{\pi}{4} = y(1) \qquad \int_0^1 y - (y')^2 = \frac{1}{12}$

$$F(y) = \int_0^1 (y')^2 - \lambda y + \lambda (y')^2$$

E–L $\qquad -2(1+\lambda) y'' - \lambda = 0 \Rightarrow y'' = \text{const.}$

$$y = Ax^2 + Bx + C \qquad y = Ax^2 + \left(\frac{\pi}{4} - A\right)x$$

$$\frac{1}{12} = \int_0^1 y - y'y' = \int_0^1 y - y''y - \left[y y'\right]_0^1$$

$$= \int_0^1 (1-2A) y - \frac{\pi}{4}\left(A + \frac{\pi}{4}\right) = \frac{(1-2A)A}{3} + \frac{(1-2A)\left(\frac{\pi}{4} - A\right)}{2} - \frac{\pi}{4}\left(A + \frac{\pi}{4}\right)$$

Dopočítu A

Aplikace ve fyzice

(18) $L = L(x, \dot{x})$ $\quad$ ... $x_0(t)$ extremála tj. řeší E-L rovnice $\int_a^b L\, dt$

$$\Rightarrow \quad -\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i}\right) + \frac{\partial L}{\partial x_i} = 0$$

Pak: $$\frac{d}{dt} E = \frac{d}{dt}\left(\sum\left(\dot{x}_i \frac{\partial L}{\partial \dot{x}_i} - L\right)\right) =$$

$$= \sum \ddot{x}_i \frac{\partial L}{\partial \dot{x}_i} + \dot{x}_i \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i}\dot{x}_i - \frac{\partial L}{\partial \dot{x}_i}\ddot{x}_i$$

$$= \sum \dot{x}\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i}\right) = 0$$

(19) Buď $L(t, x_i, \dot{x}_i)$ $\quad i = 1, \dots, n$ $\quad$ nezávisí na x_m

$$\frac{d}{dt} p_m = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_m}\right) = \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_m}\right) - \underbrace{\frac{\partial L}{\partial x_m}}_{=0} =$$

$=$ to jsou E-L rovnice pro extremálu

(20) E-L $0 = \left(\frac{\partial L}{\partial \dot q_i}\right)^{\cdot} - \frac{\partial L}{\partial q}$

$$= \left(\frac{\partial T}{\partial \dot q_i}\right)^{\cdot} - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} \qquad i = 1, \dots, n$$

(21) E-L $L = \frac{1}{2} \sum_{ij} g_{ij} \dot x_i \dot x_j$ (pozn.! uvažovat $g_{ij} = g_{ji}$)

$$\frac{\partial L}{\partial \dot x_i} = \sum_j g_{ij} \dot x_j \qquad \forall i$$

E-L $0 = \left(\frac{\partial L}{\partial \dot x_i}\right)^{\cdot} = \sum_j g_{ij} \ddot x_j$

Nyní $\frac{d}{dt}\left(\sum g_{ij} \dot x_i \dot x_j\right) = 2 \sum_{j,i} g_{ij} \dot x_i \ddot x_j =$

$$= \sum_i \dot x_i \left(\sum_j g_{ij} \ddot x_j \right) = 0$$

(22) – (24)

Máme L – Lagrangián, zadefinujme zobecněnou $L(t, q_i, \dot q_i)$ hybnost $\boxed{p_i := \frac{\partial L}{\partial \dot q_i}}$

Definujme Hamiltonián (jako při hybnosti a poli)

$$H(p_i, q_i) := \sum_i p_i \dot q_i - L$$

Hamiltonovy rovnice (pro elhindy)

$$\boxed{\begin{aligned} \dot{p}_i &= -\frac{\partial H}{\partial q_i} \\ \dot{q}_i &= \frac{\partial H}{\partial p_i} \end{aligned}}$$

22 $L = 2y_1 y_2 - 2y_1^2 + (\dot{y}_1)^2 - (\dot{y}_2)^2$

$$p_1 = \frac{\partial L}{\partial \dot{y}_1} = 2\dot{y}_1$$

$$p_2 = \frac{\partial L}{\partial \dot{y}_2} = -2\dot{y}_2$$

$$H(p_1, p_2, y_1, y_2) = p_1\dot{y}_1 + p_2\dot{y}_2 - 2y_1y_2 + 2y_1^2 - (\dot{y}_1)^2 + (\dot{y}_2)^2$$

(nahradím $\dot{y}_1$ a $\dot{y}_2$)

$$= \frac{p_1^2}{2} - \frac{p_2^2}{2} - 2y_1y_2 + 2y_1^2 - \frac{p_1^2}{4} + \frac{p_2^2}{4}$$

$$= \frac{p_1^2 - p_2^2}{4} + 2y_1(y_1 - y_2)$$

Rovnice

$$\dot{p}_1 = -\frac{\partial H}{\partial y_1} = +2y_2 - 4y_1$$

$$\dot{p}_2 = -\frac{\partial H}{\partial y_2} = +2y_1$$

$$\dot{y}_1 = \frac{\partial H}{\partial p_1} = \frac{p_1}{2}$$

$$\dot{y}_2 = -\frac{p_2}{2}$$

(23) $$L = \sqrt{t^2+y^2}\,\sqrt{1+(\dot{y})^2}$$

$$p = \frac{\partial L}{\partial \dot{y}} = \frac{\sqrt{t^2+y^2}}{\sqrt{1+(\dot{y})^2}}\,\dot{y}$$

$$H = p\dot{y} - L = \frac{\sqrt{t^2+y^2}}{\sqrt{1+(\dot{y})^2}}(\dot{y})^2 - \sqrt{t^2+y^2}\,\sqrt{1+(\dot{y})^2}$$

$$p^2(1+(\dot{y})^2) = (t^2+y^2)(\dot{y})^2 \Rightarrow \frac{p^2}{t^2+y^2-p^2} = (\dot{y})^2$$

$$\dot{y} = \frac{p}{\sqrt{t^2+y^2-p^2}}$$

$$= \frac{p^2}{\sqrt{t^2+y^2-p^2}} - \sqrt{t^2+y^2}\,\sqrt{1+\frac{p^2}{t^2+y^2-p^2}}$$

$$= \frac{p^2 - (t^2+y^2)}{\sqrt{t^2+y^2-p^2}} = -\sqrt{t^2+y^2-p^2}$$

atd

(24) $L = t^2 + g_1(\dot{g}_1)^2 + g_2(\dot{g}_2)^2$

$$p_1 = \frac{\partial L}{\partial \dot{g}_1} = 2 g_1 \dot{g}_1$$

$$p_2 = 2 g_2 \dot{g}_2$$

$$H = p_1 \dot{g}_1 + p_2 \dot{g}_2 - t^2 - g_1(\dot{g}_1)^2 - g_2(\dot{g}_2)^2$$

$$= g_1(\dot{g}_1)^2 + g_2(\dot{g}_2)^2 - t^2 = \frac{p_1^2}{4g_1} + \frac{p_2^2}{4g_2} - t^2$$

$$\dot{p}_1 = + \frac{4p_1^2}{g_1^2} \qquad \dot{p}_2 = \frac{4p_2^2}{g_2}$$

$$\dot{g}_1 = \frac{p_1}{2g_1} \qquad \dot{g}_2 = \frac{p_2}{2g_2}$$

POSLOUPNOSTI FUNKCÍ

$f^n : J \to \mathbb{R}$, kde J je interval
(otevřený, uzavřený, polouzavřený ...)

Def: $f^n \to f$ <u>bodově</u> v J
$\overset{\text{def}}{\iff}$ $\boxed{\forall x \in J}\ \forall \varepsilon > 0\ \exists n_0\ \forall n \geq n_0\ |f^n(x) - f(x)| < \varepsilon$

Def: $f^n \rightrightarrows f$ <u>stejnoměrně</u> v J
$\overset{\text{def}}{\iff}$ $\forall \varepsilon > 0\ \exists n_0\ \forall n \geq n_0\ \boxed{\forall x \in J}\ |f^n(x) - f(x)| < \varepsilon$

NA POŘADÍ KVANTIFIKÁTORŮ ZÁLEŽÍ!!

Def: $f^n \overset{\text{loc}}{\rightrightarrows} f$ $\forall$ uzavřený interval $J' \subseteq J$ platí $f^n \rightrightarrows f$ na J'

Př: $f^n(x) = x^n$ na $(0,1)$

určitě $f^n \to 0$ na $(0,1)$

ale platí
$f^n \overset{\text{loc}}{\rightrightarrows} 0$ na $(0,1)$

ale $f^n \not\rightrightarrows 0$ na $(0,1)$ protože
volme $x_n := \left(\frac{1}{2}\right)^{\frac{1}{n}} \in (0,1)$. Potom $|f^n(x_n) - 0|$
$= \frac{1}{2} \not\to 0$!

① $$\frac{e^x \sin x \sin 2x \dots \sin(nx)}{\sqrt{n}} = f^n(x)$$

$$|f^n(x)| \leq \frac{e^x}{\sqrt{n}} \to 0 \quad \text{bodově na } (-\infty, \infty)$$

$$\text{a tedy } f^n \to 0 \qquad \text{-//-}$$

② $$f^n(x) = \frac{1 + x^{2n+1}}{1 + x^{2n}}$$

$$f^n(0) = f^n(1) = 1 \to 1$$

$$f^n(-1) = 0 \to 0$$

$|x| > 1$ $$f^n(x) = \frac{\frac{1}{x^{2n}} + x}{\frac{1}{x^{2n}} + 1} \to x$$

$|x| < 1$ $$f^n(x) \to 1$$

Tedy $$f^n \to \max\{1, x\} \quad \text{na } (-1, \infty)$$

$$f^n \to x \quad \text{na } (-\infty, -1)$$

$$f^n(-1) \to 0$$

③ $$f^n(x) = \sin(\pi x n)$$

Pokud pro $x \in \mathbb{R}$ limita existuje, pak $\lim_{n\to\infty} f^n(x) = \lim_{n\to\infty} f^{kn}(x) = L$

$$\Rightarrow L^2 = \lim_{n\to\infty} \sin^2(2\pi x n) = 4\lim (\sin^2 x n \pi) \cos^2(\pi x n)$$

$$= 4 \lim (\sin^2 x n \pi (1 - \sin^2 x n \pi))$$

$$= 4 L^2 (1 - L^2) \Rightarrow L = 0 \text{ nebo } L = \frac{\sqrt{3}}{2}$$

$\Rightarrow$ limita existuje pro $x \in \mathbb{N}$

Nutná podmínka pro stejnoměrnou konvergenci je konvergence bodová!
Proto zjišťujeme většinou bodovou a poté stejnoměrnou

(4) $f^n(x) := x^n - x^{n+1}$ na $[0,1]$

$= x^n(1-x) \to 0$ bodově na $[0,1]$

$(f^n)' = n x^{n-1} - (n+1) x^n = x^{n-1}(n-(n+1)x)$

je f^n má maximum v $x = \frac{n}{n+1}$; $f^n\left(\frac{n}{n+1}\right) = \left(\frac{n}{n+1}\right)^n \left(1 - \frac{n}{n+1}\right)$

$= \frac{1}{\left(1+\frac{1}{n}\right)^n} \cdot \frac{1}{n+1}$

$\downarrow \frac{1}{e} \quad \downarrow 0$

a tedy $\sup_{x\in[0,1]} |f^n(x)| \leq \frac{1}{n+1} \to 0$

$f^n \rightrightarrows 0$ na $[0,1]$

(5) $f^n = x^n - x^{2n} \to 0$ bodově $f(0) = f(1) = 0$

$(f^n)' = n x^{n-1} - 2n x^{2n-1} = n x^{n-1}(1 - 2x^n)$

f^n má maximum v bodě $x = 2^{-\frac{1}{n}}$ $\quad f^n(2^{-\frac{1}{n}}) =$

$= \left(2^{-\frac{1}{n}}\right)^n - \left(2^{-\frac{1}{n}}\right)^{2n} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

$\Rightarrow \sup_{x\in[0,1]} |f^n(x)| = \frac{1}{4} \Rightarrow f^n \not\rightrightarrows 0$ na $[0,1]$

ale $f^n \rightrightarrows 0$ na $[0, 1-\varepsilon)$ $\forall \varepsilon > 0$

a $f^n \overset{loc}{\rightrightarrows} 0$ na $[0,1)$

(6) $f^n = \operatorname{arctg} nx$ na $(0,\infty)$

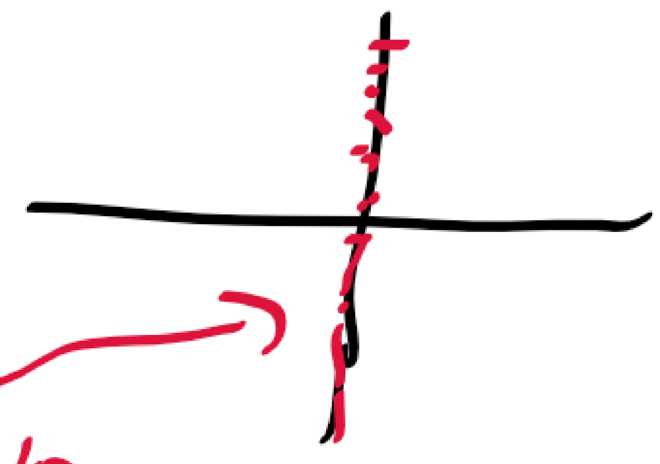
ALE $f^n \overset{\boxed{loc}}{\rightrightarrows} \frac{\pi}{2}$ na $(0,\infty)$

$f^n \to \frac{\pi}{2}$ bodově na $(0,\infty)$

ALE $f^n\left(\frac{1}{n}\right) = \frac{\pi}{4} \Rightarrow f^n \not\rightrightarrows \frac{\pi}{2}$ na $(0,\infty)$

(7) $f^n(x) = \dfrac{nx}{1+n^2x^2}$ $\qquad x \in \mathbb{C}$

není definována v bodech $\quad x = \pm \frac{i}{n}$!



PROBLÉM NA SVISLÉ OSE

$$f^n(x) = \frac{x}{\frac{1}{n} + nx^2} = \boxed{\frac{1}{n}\,\frac{x}{\frac{1}{n^2}+x^2}}$$

pokud $|x| \geq \varepsilon \Rightarrow f^n \rightrightarrows 0$

pro $|x| < \varepsilon \Rightarrow f^n \to 0$ bodově kromě bodů $\pm \frac{i}{n}$

musí být $x = \frac{z}{n}$, kde $|z| = 1 \to f(x) = \frac{z}{1+z^2}$

tedy nemůže konvergovat v "nule"

$$f^n \overset{lok}{\rightrightarrows} 0 \quad \mathbb{C}\setminus\{0\}$$

(8) $f^n = \sin \pi x^n$ $\qquad$ na $[0,1]$

$f^n \to 0$ na $[0,1]$

ALE!!! $f^n\left(\left(\frac{1}{2}\right)^{1/n}\right) = \sin\frac{\pi}{2} = 1 \not\to 0$

$\Rightarrow f^n \not\rightrightarrows 0$ na $[0,1]$

ale $f^n \overset{lok}{\rightrightarrows} 0$ na $[0,1)$ $\qquad \left(|\sin \pi x^n| \leq \pi x^n \to 0 \right)$

(9) $f^n(x) = \frac{x}{n} \ln \frac{x}{n}$ $\qquad x > 0$

$f^n(x) \to 0$ bodově

$$|f^n| = \left| \frac{x \ln x}{n} - \frac{x \ln n}{n} \right| \leq \frac{1}{n}|x \ln x| + |x| \frac{\ln n}{n}$$

$\Rightarrow f^n \rightrightarrows 0$ na $(0, \varepsilon)$

ALE $f^n(n) - 2\ln 2 \not\to 0$ a tedy

$f^n \not\rightrightarrows 0$ na (ε, ∞)

POUZE $f^n \overset{loc}{\rightrightarrows} f$ na $(0, \infty)$

(10) $f^n = (1+x^n)^{1/n}$ na $[0, \infty)$

$f^n = e^{\frac{1}{n}\ln(1+x^n)} \to \begin{cases} 1 & x \le 1 \\ x & x > 1 \end{cases}$

$x \in [0,1]$ $1 \le f^n(x) \le 2^{\frac{1}{n}} \to 1 \Rightarrow f^n \rightrightarrows 1$ na $[0,1]$

$x \ge 1$ $0 \le f^n(x) - x = \dfrac{(1+x^n) - x^n}{\sum_{i=0}^{n-1} \left[(1+x^n)^{\frac{1}{n}}\right]^{n-1-i} x^i} = \dfrac{1}{\sum_{i=0}^{n-1} \left[(1+x^n)^{\frac{1}{n}}\right]^{n-1-i} x^i}$

$\le \dfrac{1}{n-1} \to 0$

$f^n \rightrightarrows x$ na $[1, \infty]$

(11) $f^n = \dfrac{1+x^{n+1}}{1+x^n}$ $x \in [0, \infty)$

$f^n \to 1$ na $[0,1]$

$f^n = \dfrac{\frac{1}{x^n} + x}{\frac{1}{x^n} + 1} \to x$ pro $x > 1$

$f^n(x) - \max(1, x) = \begin{cases} \dfrac{1+x^{n+1}}{1+x^n} - 1 & x < 1 \\ \dfrac{1+x^{n+1}}{1+x^n} - x & x \ge 1 \end{cases}$

$$|f^n(x) - f(x)| = \begin{cases} \dfrac{x^{n+1} - x^n}{1 + x^n} \leq x^n - x^{n+1} \rightrightarrows 0 & \text{viz } 2 \text{ pro } x \in [0,1] \\ \left| \dfrac{1 + x^{n+1} - x - x^{n+1}}{1+x^n} \right| = \dfrac{x-1}{1+x^n} \rightrightarrows 0 \end{cases}$$

(12) $$\int_0^1 \frac{nx}{1+n^2x^2}\,dx = \left[\frac{\ln(1+n^2x^2)}{2n} \right]_0^1 = \frac{\ln(1+n^2)}{2n} \to 0$$

$$\int_0^1 \lim_{n\to\infty} \frac{nx}{1+n^2x} \,dx = \int_0^1 0 = 0$$

ROVNOST PLATÍ I KDYŽ $\frac{nx}{1+n^2x^2} \not\rightrightarrows 0$ (STAČÍ $\overset{lu}{\rightrightarrows}$)

(13) $$\int_0^1 \frac{nx}{1+n^2x^4} = \sqrt{n} \int_0^1 \frac{\sqrt{n}x}{1+(\sqrt{n}x)^4} \overset{\sqrt{n}x = y}{=} \int_0^{\sqrt{n}} \frac{y}{1+y^4}\,dy$$

$$\to \int_0^\infty \frac{y}{1+y^4}$$

ale $$\int_0^1 \lim_{n\to\infty} \frac{nx}{1+n^2x^4}\,dx = 0$$ LIMITY SE NEKOMUTUJÍ !?

(14) $$\lim_{x\to 1-} \lim_{n\to\infty} x^n = \lim_{x\to 1} 0 = 0$$

$$\lim_{n\to\infty} \lim_{x\to 1_-} x^n = \lim_{n\to\infty} 1 = 1$$

DŮVOD :- $x^n \not\rightrightarrows$ na $[0,1]$

PROTOŽE POKUD BY PLATILO $x^n \rightrightarrows 0$ na $[0,1]$
pak $\lim_{x\to 1_-} \lim_{n\to\infty} = \lim_{n\to\infty} \lim_{x\to 1_-}$

ŘADY FUNKCÍ

Bodová absolutní / neabsolutní konvergence

(1) $\sum_{n=1}^{\infty} \ln^n x$ $\qquad D_f: x > 0$

označ $y := \ln x \quad \Rightarrow \quad \sum y^n$ geometrická řada

konverguje absolutně $\Leftrightarrow |y| < 1 \Leftrightarrow x \in (e^{-1}, e)$

$\left(\sum_{n=1}^{\infty} \ln^n x = \frac{\ln x}{1 - \ln x} \right)$

(2) $\sum_{n=1}^{\infty} \frac{x^n}{1 + x^{2n}}$

$x = 1: \quad \sum_{n=1}^{\infty} \frac{1}{1+1}$ DIVERGENCE

$x = -1: \quad \sum \frac{(-1)^n}{1+1}$ DIVERGENCE

$|x| \neq 1$

$$\left| \frac{x^n}{1 + x^{2n}} \right| \leq \frac{|x|^n}{1 + |x|^{2n}} \leq \begin{cases} |x|^n \\ \frac{1}{|x|^n} \quad x \neq 0 \end{cases}$$

$$\Rightarrow \left| \frac{x^n}{1 + x^{2n}} \right| \leq \left[\min \left(|x|, \frac{1}{|x|} \right) \right]^n$$

a protože $|x| \neq 1$ máme odhad pomocí geometrické řady

a tedy absolutní konvergenci!

③ $\sum (-1)^n \left(\frac{1-x}{1+x} \right)^n$

označím $y := \frac{1-x}{1+x}$

$\Rightarrow \sum (-1)^n y^n$ konverguje (absolutně) $\Leftrightarrow |y| < 1$

$\Leftrightarrow \left| \frac{1-x}{1+x} \right| < 1 \Leftrightarrow x > 0$

④ $\sum\limits_{n=1} x^n \operatorname{tg} \frac{x}{2^n}$ $\qquad x \neq k\pi$!?

nutná podmínka $\lim\limits_n \left(x^n \operatorname{tg} \frac{x}{2^n}\right) = \lim\limits_n \frac{1}{\cos \frac{x}{2^n}} \cdot \lim \frac{\sin \frac{x}{2^n}}{\frac{x}{2^n}} \cdot \lim \frac{x^{n+1}}{2^n}$

$= 0 \Leftrightarrow |x| < 2$

Zároveň je konvergence absolutní

⑤ $\sum e^{-nx} \cos x$

a) $x = (2k+1)\frac{\pi}{2} \Rightarrow \sum e^{-nx} \cdot 0 < 0$

b) $x \leq 0 \quad \lim e^{-nx} \cos x \boxed{\neq} 0$

není splněna nutná podmínka — DIVERGUJE

c) $x > 0 \quad |e^{-nx} \cos x| \leq (e^{-x})^n$ — KONVERGUJE ABSOLUTNĚ

(6) $\sum_{n=1} \frac{(-1)^n}{(x+n)^p}$ $\quad p \in \mathbb{R}$

$p \in \mathbb{R} \Rightarrow$ potřebujeme $x+n \geq 0$ a tedy $x > -1$

a) $p \leq 0$ není splněna nutná podmínka - DIVERGUJE

b) $p \in (0,1]$ $\quad \left|\frac{(-1)^n}{(x+n)^p}\right| = \frac{1}{(x+n)^p}$ - NEMŮŽE KONVERGOVAT ABS

ale $\frac{1}{(x+n)^p} \searrow 0$ monotónně a $(-1)^n$ má omezený posloupnost částečných součtů $\Rightarrow$ KONVERGUJE

c) $p > 1$ - KONVERGUJE ABSOLUTNĚ

(7) $\sum_{n=1}^{\infty} \frac{x^n}{n+y^n}$ $\quad y \in \mathbb{R}_0^+$

jde o mocninou řadu, poloměr konvergence je

$$R = \lim_{n \to \infty} (n+y^n)^{1/n} = \begin{cases} y & \text{pro } y > 1 \\ 1 & \text{pro } y \leq 1 \end{cases}$$

$$= \max(1, y)$$

pro $|x| < \max(1,y)$ konverguje absolutně

a) $\max(1,y) = y > 1$ a $|x| = y$ pak není splněná nutná podmínka - DIVERGUJE

b) $\max(1,y) = 1$, $x = 1$ - DIVERGUJE $\sim \frac{1}{n}$

$x = -1$ - KONVERGUJE $\sim \frac{(-1)^n}{n}$

STEJNOMĚRNÁ KONVERGENCE

$$\sum_{n=1}^{\infty} f_n(x) \quad \text{na } \Omega$$

1) NUTNÁ PODMÍNKA $f_n \rightrightarrows 0$ na Ω !

2) B-C podmínka $\sum f_n \rightrightarrows$ na Ω

$\Leftrightarrow \forall \varepsilon\ \exists n_0\ \forall n \geq n_0\ \forall p \geq 1\ \forall x \in \Omega$

$$\left| \sum_{k=n}^{n+p} f_k(x) \right| < \varepsilon$$

3) POSTAČUJÍCÍ (WEIERSTRASS)

$|f_n(x)| \leq g_n(x)$ a $\sum g_n \rightrightarrows$ na Ω

$\Rightarrow \sum f_n \rightrightarrows$ na Ω

4) DIRICHLET • $a_n(x)$ má stejně omezenou posloupnost částečných součtů

$\Leftrightarrow \exists C\ \forall k \in \mathbb{N}\ \forall x \in \Omega$

$$\left| \sum_{n=1}^{k} a_n(x) \right| < C$$

• $b_n \rightrightarrows 0$ a $b_n(x) \geq b_{n+1}(x)$

pak $\sum a_n b_n \rightrightarrows$ na Ω

5) Abel: $\left.\begin{array}{l} a_n \text{ – STEJNĚ OMEZENÁ} \\ \sum b_n \rightrightarrows \text{ na } \Omega \end{array}\right\} \sum a_n b_n \rightrightarrows$

f) $\sum_{n=1}^{\infty} (1-x) x^n$

$$\sum_{n=1}^{N} (1-x) x^n = x(1-x) \frac{1-x^N}{1-x} = x(1-x^N)$$

Vidíme, že pro $x \in [0,1]$

$$\sum_{n=1}^{\infty} (1-x) x^n = \begin{cases} x & x<1 \\ 0 & x=1 \end{cases}$$

Zvolme $\varepsilon > 0$, pak $\forall x \in [0, 1-\varepsilon]$

$$\left|x - \sum_{n=1}^{N} (1-x)x^n\right| = \left|x - x(1-x^N)\right| = x^{N+1} \le \underset{\to 0}{(1-\varepsilon)^{N+1}}$$

a tedy na $[0, 1-\varepsilon]$ konverguje stejnoměrně

b) Pomocí B-C podmínky ukážeme, že na $[0,1]$ nekonverguje stejnoměrně, tj

$$\underbrace{\exists \varepsilon > 0 \ \forall n_0 \ \exists n, p, x \ \left|\sum_{k=n}^{n+p} \varphi_k(x)\right| > \varepsilon}_{\text{negace B-C}}$$

$$\sum_{k=n}^{n+p} (1-x) x^k = (1-x) x^n \frac{1-x^p}{1-x} = x^n (1-x^p)$$

Volím $p = 2n$, $x = \left(\frac{1}{2}\right)^{\frac{1}{n}}$ $\Big\}$ $\overset{\shortparallel}{\frac{1}{2}} \left(1 - \frac{1}{4}\right) = \frac{3}{8} \not\to 0!!!$

$$\sum_{n=1}^{\infty} \frac{\sin nx}{(n^2+x^2)^{1/3}}$$

DŮLEŽITÝ !

$\forall x \in \mathbb{R}$ řada konverguje neabsolutně.

NA STEJNOMĚRNOU KONVERGENCI POUŽIJEME DIRICHLETOVO KRITÉRIUM

PRO $x \neq k\pi$

$$\sum_{n=1}^{N} i\sin nx + \cos nx = \sum_{n=1}^{N} e^{inx} = \frac{e^{iNx}-1}{e^{ix}-1}\, e^{ix}$$

$$= \frac{(\cos x + i\sin x)(\cos Nx - 1 + i\sin x)(\cos x - 1 - i\sin x)}{(\cos x - 1)^2 + \sin^2 x}$$

$$= \frac{-\,\|\,-}{2(1-\cos x)}$$

$$\Rightarrow \left|\sum_{n=1}^{N} \sin nx\right| \leq \frac{100}{1-\cos x}$$

a tedy $\left| \quad \text{"} \quad \right| \leq \frac{100}{1-\cos\varepsilon}$ $\quad \forall x \in (\varepsilon, 2\pi-\varepsilon)$

STEJNĚ OMEZENÉ!

$$\frac{1}{(n^2+x^2)^{\frac{1}{3}}} \rightrightarrows 0 \qquad \text{a monotónní!}$$

$\Rightarrow \sum \frac{\sin nx}{(n^2+x^2)^{1/3}}$ konverguje stejnoměrně na $[\varepsilon, 2\pi-\varepsilon]$

• NEKONVERGUJE STEJNOMĚRNĚ NA $(0, 2\pi)$.

Dk: SPOR S B-C $\left(x = \frac{1}{2n}\right)$

$$\sum_{k=n}^{2n} \frac{\sin(kx)}{(x^2+k^2)^{1/3}} = \sum_{k=n}^{2n} \frac{\sin \frac{k}{2n}}{\left(\left(\frac{1}{2n}\right)^2+k^2\right)^{1/3}}$$

$$\geq \sum_{k=n}^{2n} \frac{\sin\left(\frac{n}{2n}\right)}{8 n^{\frac{2}{3}}} = n^{1/3} \sin \frac{1}{2} \to \infty$$

(10) $\displaystyle\sum_{n=1}^{\infty} \frac{x}{1+n^4x^2}$ a) $[-k,k]$ b) $(-\infty, \infty)$

Víme, že $\forall x \in \mathbb{R}$ řada konverguje bodově

Dále $f_n(x) = \dfrac{x}{1+n^4x^2}$ $\quad f_n' = \dfrac{1+n^4x^2 - 2n^4x^2}{(\ )^2}$

$$= \frac{1-n^4x^2}{(\ )^2}$$

extrémy $x = \pm \frac{1}{n^2}$

$$\Rightarrow \left|\frac{x}{1+x^2n^4}\right| \leq \frac{\frac{1}{n^2}}{1+n^4 \cdot \frac{1}{n^4}} = \frac{1}{2n^2}$$

a $\sum \frac{1}{2n^2} \Rightarrow\!\!\!\!\rightarrow$ na $\mathbb{R}$ (nezávisí na x!)

(11) $\sum \ln\left(1+\frac{x^2}{n\ln^2 n}\right)$

nemůže konvergovat stejnoměrně na $\mathbb{R}$

protože $\sup_x \ln\left(1+\frac{x^2}{n\ln^2 n}\right) = \infty$

ale pro $x \in [-K, K]$ máme

$$\left|\ln\left(1+\frac{x^2}{n\ln^2 n}\right)\right| \leq \frac{x^2}{n\ln^2 n} \leq \frac{K^2}{n\ln^2 n}$$

a řada $\sum_{n=2}^{\infty} \frac{K^2}{n\ln^2 n}$ konverguje

tedy $\sum \ln\left(1+\frac{x^2}{n\ln^2 n}\right) \rightrightarrows$ na $[-K, K]$

(12) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} x^{\frac{2}{n}}$

b) $\frac{x^{\frac{2}{n}}}{\sqrt{n}} \not\rightrightarrows 0$ na $[0, \infty)$

tedy na $[0, \infty)$ nekonverguje stejnoměrně

a) $x^{\frac{2}{n}}$ monotónní a stejně omezená na $[0, K]$

$\frac{1}{\sqrt{n}} \searrow 0$: Dirichlet $\Rightarrow \sum \frac{(-1)^n}{\sqrt{n}}$ K

ABEL $\Rightarrow \sum \frac{(-1)^n}{\sqrt{n}} x^{\frac{2}{n}}$ K STEJNOMĚRNĚ

(1) $\sum_{n=1}^{\infty} x^{\alpha} e^{nx}$ $\qquad \alpha \in \mathbb{N}_0$

$$\sum_{n=1}^{N} x^{\alpha} e^{nx} = \underbrace{x^{\alpha} \frac{e^{Nx}-1}{e^{x}-1} e^{x}}_{\text{konverguje stejnoměrně ??}} \quad \text{pro } x \neq 0 \Big\} = S_N$$

$$= 0 \qquad \text{pro } x=0 \text{ a } \alpha>0$$

$$= N \qquad \text{pro } x=0 \text{ a } \alpha=0$$

c) $x \in [0,1]$ nemůže konvergovat $\to \emptyset$

b) pro $\alpha=0$ nemůže konvergovat stejnoměrně na $[-1,0]$

ale konverguje lokálně stejnoměrně $e^{Nx} \rightrightarrows 0$ na $[-1,-\varepsilon]$

pro $\alpha \geq 1$ – celá konverguje lokálně stejnoměrně.

ALE na $[-1,0]$ nekonverguje, protože pro $\alpha=1$

$$S_N\left(-\tfrac{1}{N}\right) = \underbrace{-\frac{1}{N} \frac{e^{-1}-1}{e^{-\frac{1}{N}}-1} e^{-\frac{1}{N}} \xrightarrow{N\to\infty} e^{-1}-1}_{\text{nemůže konvergovat}}$$

ale $S_N(0) \to 0$

ALE pro $\boxed{\alpha>1}$ konverguje stejnoměrně na $[-1,0]$

$$S_N(x) \rightrightarrows e^{x} \frac{x^{\alpha}}{1-e^{x}}$$

a) konverguje $\rightrightarrows$ protože $S_N \rightrightarrows$ na $[-\delta,-1]$

(14) $\sum \sin\left(\pi (x^2+k^2)^{1/2}\right)\left(\frac{x^2}{1+x^2}\right)^{1/n}$ $(-\infty, \infty)$??

není splněna nutná podmínka – nekonverguje

(15) $\sum\limits_{n=2}^{\infty} \frac{(-1)^n}{n+\sin x}$

i) $\frac{1}{n+\sin x} \geq \frac{1}{n+1+\sin x}$

ii) $\left| \frac{1}{n+\sin x} \right| \leq \frac{1}{n-1} \rightrightarrows 0$

Dirichlet $\Rightarrow$ $\sum \frac{(-1)^n}{n+\sin x} \rightrightarrows$ na $(-\infty, \infty)$!

(16) $\sum \frac{(-1)^n}{n} \, x \ln\left(\frac{x}{n}\right)$

$$= \sum \frac{(-1)^n}{n} x(\ln x - \ln n)$$

$$= x \ln x \sum \frac{(-1)^n}{n} - x \sum \frac{(-1)^n}{n} \ln n$$

(17) $\sum\limits_{n=1}^{\infty} (-1)^n \frac{n-1}{n+1} \frac{1}{n^{1/100}} e^{-nx}$

$$= \sum (-1)^n \underbrace{\frac{e^{-nx}}{n^{\frac{1}{100}}}}_{\searrow\searrow 0} - \underbrace{2 \frac{(-1)^n}{(n+1) n^{\frac{1}{100}}} e^{-nx}}_{|\cdot| \leq \frac{1}{n^{1+\frac{1}{100}}} \text{ konverg.}}$$

Dirichlet K $\Rightarrow$ konvergence stej. na $[0, \infty)$

Lebesgueův integrál – Fubini – substituce

- λ_m: m-rozměrná Lebesgueova míra

  často $d\lambda_m = d\lambda = dx_1 \dots dx_m$ atd..

- Vždy uvažujeme pouze MĚŘITELNÉ funkce.
- s.v. = skoro všude = až na množině míry nula

$\boxed{\text{Fubini}}$ $m = n + l$, $n, l \in \mathbb{N}$. Buď $\Omega \subseteq \mathbb{R}^m$ měřitelná.

Označ. $\forall x \in \mathbb{R}^n$ $\Omega(x) := \{ y \in \mathbb{R}^l : (x,y) \in \Omega \}$

$\forall y \in \mathbb{R}^l$ $\Omega(y) := \{ x \in \mathbb{R}^n : (x,y) \in \Omega \}$

Buď f integrovatelná na Ω pak

$$\int_\Omega f(x_1,\dots,x_n,y_1,\dots,y_l)\, d\lambda_m = \int_{\mathbb{R}^n} \left(\int_{\Omega(x)} f(x_1,\dots,x_n,y_1,\dots,y_l)\, dy_1 \dots dy_l \right) dx_1 \dots dx_n$$

$$= \int_{\mathbb{R}^l} \left(\int_{\Omega(y)} f(x_1,\dots,x_n,y_1,\dots,y_l)\, \underbrace{dx_1 \dots dx_n}_{d\lambda_n} \right) \underbrace{dy_1 \dots dy_l}_{d\lambda_l}$$

SUBSTITUCE: Buď G měřitelná, $\varphi: G \to \mathbb{R}^n$ prostá a C^1

Pak $$\int_{\varphi(G)} f(x)\, dx = \int_G f(\varphi(y)) \cdot J_\varphi \, dy$$

kde $\varphi(y) = \left| \det \nabla \varphi(y) \right|$

VÝZNAČNÉ SUBSTITUCE

a) polární

$x = r\cos\varphi$ $\qquad \varphi \in (0, 2\pi)$ $\qquad J = \det\begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix} = r$

$y = r\sin\varphi$ $\qquad r > 0$

b) "eliptické"

$x = \frac{r}{a}\cos\varphi$ $\qquad \varphi \in (0, 2\pi)$ $\qquad J = \begin{pmatrix} \frac{\cos\varphi}{a} & -\frac{r}{a}\sin\varphi \\ \frac{\sin\varphi}{b} & \frac{r}{b}\cos\varphi \end{pmatrix}$

$y = \frac{r}{b}\sin\varphi$ $\qquad r > 0$

$$= \boxed{\frac{r}{ab}}$$

c) sférické

$r > 0 \quad \varphi \in (0, 2\pi) \quad \psi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$x = r\cos\varphi\cos\psi$

$y = r\sin\varphi\cos\psi$

$z = r\sin\psi$

$$J = \det\begin{pmatrix} \cos\varphi\cos\psi & -r\sin\varphi\cos\psi & -r\cos\varphi\sin\psi \\ \sin\varphi\cos\psi & r\cos\varphi\cos\psi & -r\sin\varphi\sin\psi \\ \sin\psi & 0 & r\cos\psi \end{pmatrix}$$

$$= r^2\cos^2\varphi\cos^3\psi + r^2\sin^2\varphi\cos\psi\sin^2\psi + r^2\cos^2\varphi\sin^2\psi\cos\psi$$

$$+ r^2\sin^2\varphi\cos^3\psi = r^2\cos^2\psi\cos\psi + r\sin^2\psi\cos\psi =$$

$$= \underline{\underline{r\cos\psi}}$$

1) $\int_\Omega f(x+y)\,dx\,dy \qquad \Omega = \{(x,y);\ |x|+|y| \le 1\}$

$$x = \frac{u+v}{2}, \quad y = \frac{u-v}{2} \Rightarrow x+y = u,\quad x-y = v \rightarrow u \in (-1,1),\ v \in (-1,1)$$

$$\frac{\partial x}{\partial u} = \frac{1}{2}, \quad \frac{\partial x}{\partial v} = \frac{1}{2} \qquad J = \left| \det \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \right| = \frac{1}{2}$$

$$\frac{\partial y}{\partial u} = \frac{1}{2} \qquad \frac{\partial y}{\partial v} = -\frac{1}{2}$$

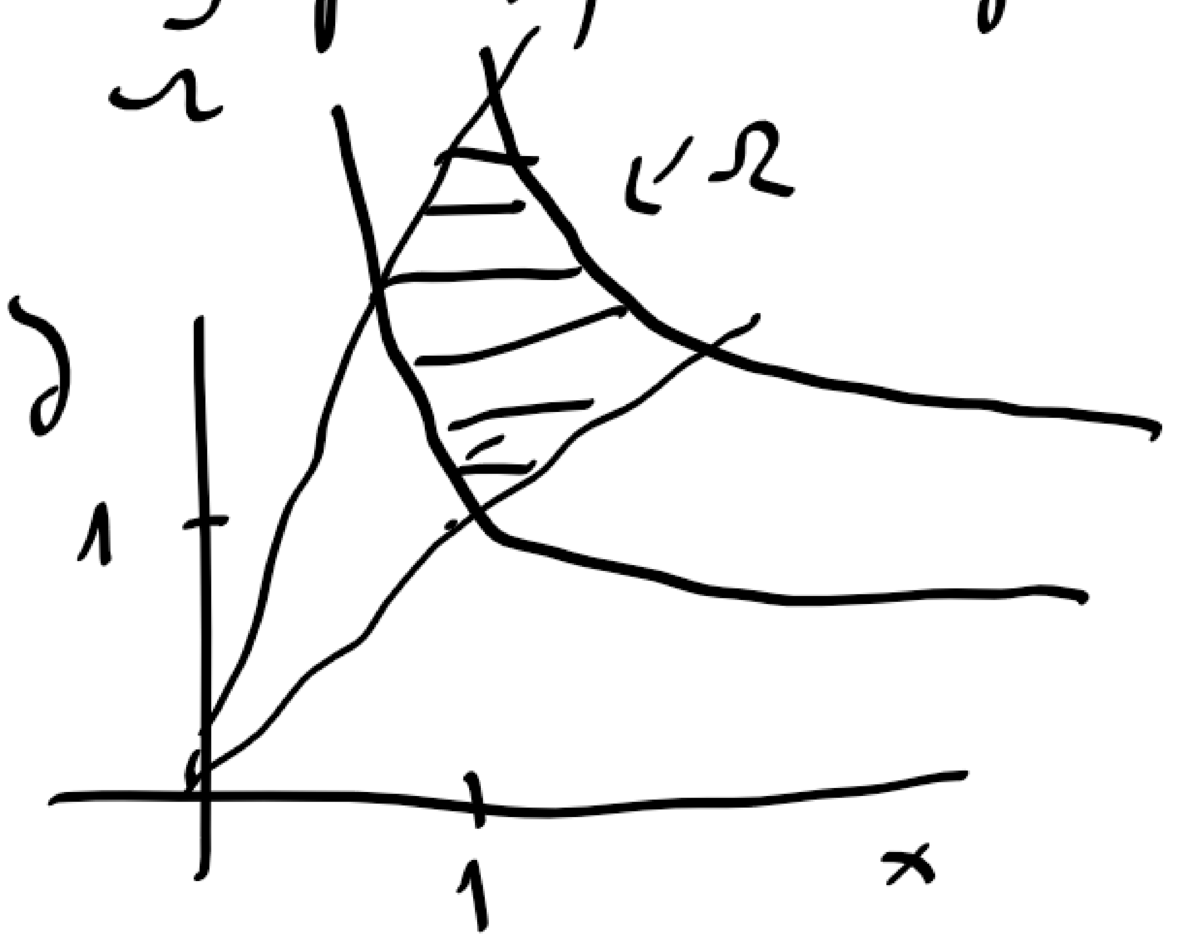
$$\int_\Omega f(x+y)\,dx\,dz = \iint_{(-1,1)^2} f(u)\cdot\frac{1}{2}\,du\,dv = \int_{-1}^{1} f(u)\,du.$$

2) $\int_\Omega f(x,y)\,dx\,dy$ Ω ohraničené

$xy = 1,\ xy = 2$

$y = x,\ y = 4x$

$x, y > 0$



$$\left.\begin{aligned} xy &= s \\ \frac{y}{x} &= t \end{aligned}\right\} \quad \begin{aligned} x^2 t &= s \Rightarrow x = \sqrt{\frac{s}{t}} \\ y^2 &= st \Rightarrow y = \sqrt{st} \end{aligned}$$

$$\partial_s x = \frac{1}{2\sqrt{st}}, \quad \partial_t x = -\frac{1}{2}\frac{\sqrt{s}}{t^{3/2}}$$

$$\partial_s y = \frac{1}{2}\sqrt{\frac{t}{s}} \qquad \partial_t y = \frac{1}{2}\sqrt{\frac{s}{t}}$$

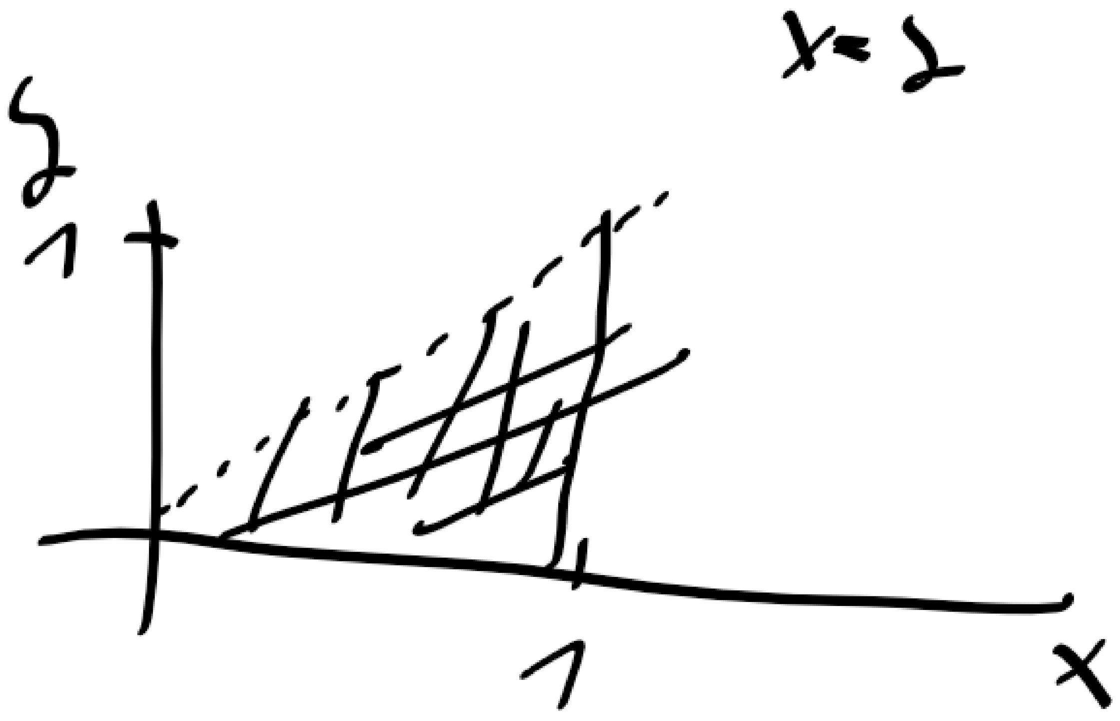
$$J = \left| \det \begin{pmatrix} \frac{1}{2}\frac{1}{\sqrt{st}} & -\frac{1}{2}\frac{\sqrt{s}}{t^{3/2}} \\ \frac{1}{2}\sqrt{\frac{t}{s}} & \frac{1}{2}\sqrt{\frac{s}{t}} \end{pmatrix} \right| = \frac{1}{4}\left| \frac{1}{t} + \frac{1}{t} \right| = \frac{2}{t}$$

$s \in (1,2)$

$t \in (1,4)$

$$\int_\Omega f(x,y)\,dx\,dy = 2\int_1^2 \left(\int_1^4 \frac{f(s)}{t}\,dt \right) ds = 2\ln 4 \int_1^2 f(s)\,ds$$

(3) $\int_0^1 \left(\int_y^1 f(x)\,dx \right) dy = \int_\Omega f(x)\,dx\,dy$, kde

$\Omega =$ 

$\Omega = (x,y : x,y \geq 0 \quad y \leq x \leq 1)$

(4) a) $(x^2+y^2)^2 = 2a^2(x^2-y^2)$

tedy najít míru množiny $(x^2+y^2)^2 \leq 2a^2(x^2-y^2)$

$x = r\cos\varphi$
$y = r\sin\varphi$ $\quad J = r$

$r^4 \leq 2a^2 r^2 (\cos^2\varphi - \sin^2\varphi)$

$r^2 \leq 2a^2 \cos 2\varphi$

$\int_\Omega 1\,dx\,dy = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[\int_0^{\sqrt{2a^2\cos 2\varphi}} r\,dr \right] d\varphi$ $\quad \Rightarrow \varphi \in (-\frac{\pi}{4}, \frac{\pi}{4})$

$r \in (0, \sqrt{2a^2\cos 2\varphi})$

$= \frac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2a^2 \cos 2\varphi = \left[\frac{a^2 \sin 2\varphi}{2} \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$

$= \underline{\underline{a^2}}$

b) $(x^3+y^3)^2 = xy$ $\quad \Omega : (x^3+y^3)^2 < xy \Rightarrow x,y > 0$

$\Updownarrow$

$x = r^{\frac{2}{3}} \cos^{\frac{2}{3}}\varphi$ $\quad \varphi \in (0, \frac{\pi}{2})$ (polovina Ω) $\quad r^4 < r^{\frac{4}{3}} \cos^{\frac{2}{3}} \sin^{\frac{2}{3}}$

$y = r^{\frac{2}{3}} \sin^{\frac{2}{3}}\varphi$

$\Updownarrow$

$r^{\frac{8}{3}} < (2\cos\varphi\sin\varphi)^{2/3} \frac{1}{2^{2/3}}$

$\partial_r x = \frac{2}{3} r^{-\frac{1}{3}} \cos^{\frac{2}{3}}$

$\partial_\varphi x = -\frac{2}{3} r^{\frac{2}{3}} \cos^{-\frac{1}{3}} \sin$ $\quad = (\sin 2\varphi)^{2/3} \frac{1}{2^{\frac{2}{3}}}$

$\partial_r y = \frac{2}{3} r^{-\frac{1}{3}} \sin^{\frac{2}{3}}$ $\quad \partial_\varphi y = \frac{2}{3} r^{\frac{2}{3}} \sin^{-\frac{1}{3}} \cos$

$$J = \det \begin{pmatrix} \frac{2}{3} r^{-\frac{1}{3}} \cos^{\frac{2}{3}} & -\frac{2}{3} r^{\frac{2}{3}} \cos^{-\frac{1}{3}} \sin \\ \frac{2}{3} r^{-\frac{1}{3}} \sin^{\frac{2}{3}} & \frac{2}{3} r^{\frac{2}{3}} \sin^{-\frac{1}{3}} \cos \end{pmatrix}$$

$$= \frac{4}{9} r^{1/3} \left(\cos^{\frac{5}{3}} \sin^{-\frac{1}{3}} + \sin^{\frac{5}{3}} \cos^{-\frac{1}{3}} \right)$$

$$= \frac{4}{9} r^{1/3} \cos^{-\frac{1}{3}} \sin^{-\frac{1}{3}} \left(\cos^2 + \sin^2 \right) = \frac{4}{9} r^{\frac{1}{3}} \cos^{-\frac{1}{3}} \sin^{-\frac{1}{3}}$$

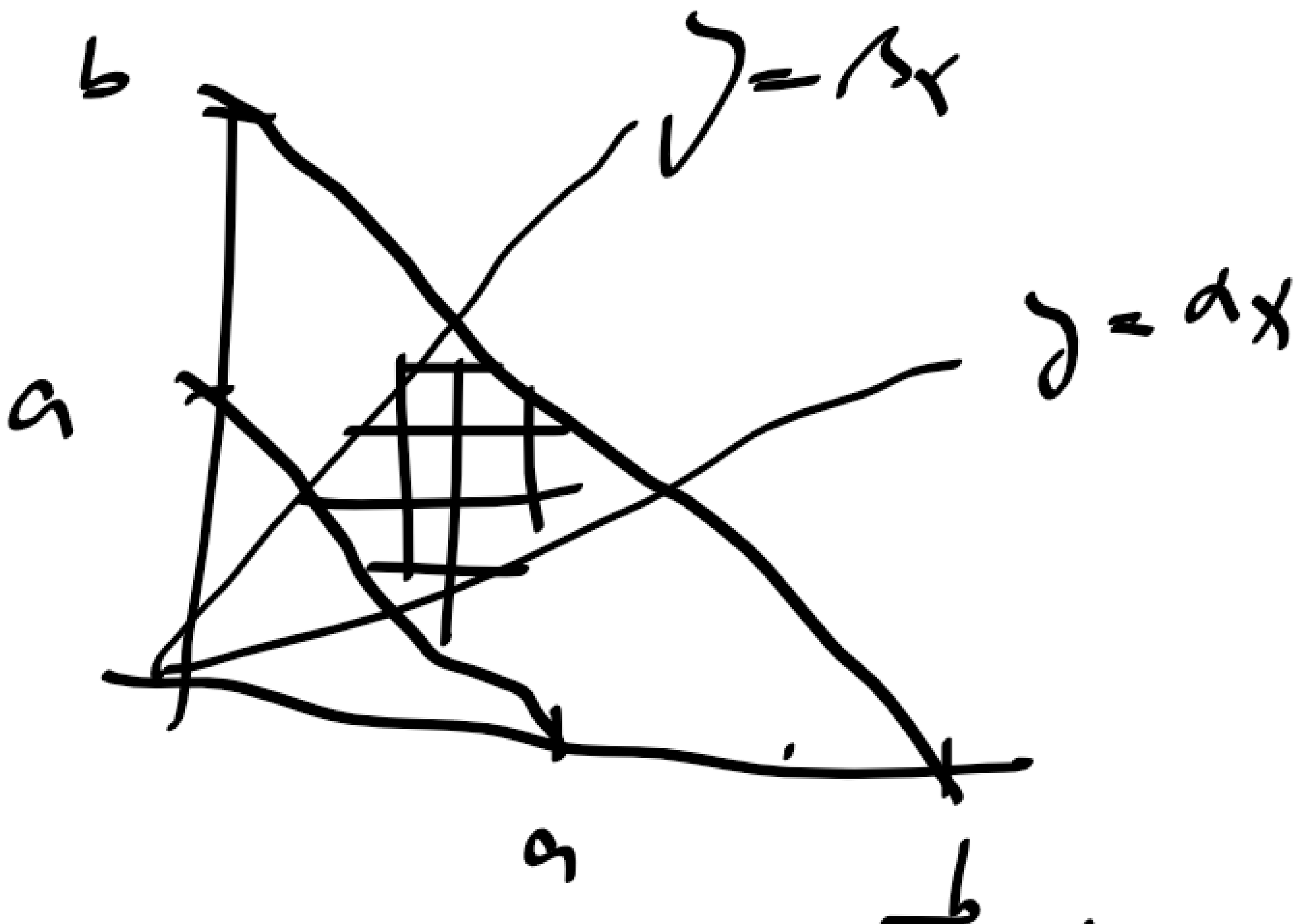
$$\int_\Omega 1 \, dx\, dy = 2 \int_0^{\frac{\pi}{2}} \int_0^{\left((\sin 2\varphi)^{2/3} \frac{1}{2^{\frac{2}{3}}} \right)^{3/8}} 1 \cdot \frac{4}{9} r^{\frac{1}{3}} \cos^{-\frac{1}{3}} \sin^{-\frac{1}{3}} \, dr\, d\varphi$$

$$= \frac{2}{3} \int_0^{\frac{\pi}{2}} \cos^{-\frac{1}{3}} \sin^{-\frac{1}{3}} \left[r^{\frac{4}{3}} \right]_0^{\left((\sin 2\varphi)^{2/3} \frac{1}{2^{\frac{2}{3}}} \right)^{3/8}} =$$

$$= \frac{2}{3} \int_0^{\frac{\pi}{2}} \cos^{-\frac{1}{3}} \sin^{-\frac{1}{3}} \left(\sin 2\varphi \right)^{1/3} \frac{1}{2^{1/3}} = \frac{2}{3} \int_0^{\frac{\pi}{2}} = \frac{\pi}{3}$$

c) $x+y=a$, $x+y=b$, $y=\alpha x$, $y=\beta x$

$0<a<b$, $0<\alpha<\beta$



$$x+y=u, \quad \frac{y}{x}=v \Rightarrow x = \frac{u}{v+1}, \quad y = u - \frac{u}{v+1}$$

$$\nabla\varphi = \begin{pmatrix} \frac{1}{v+1} & -\frac{u}{(v+1)^2} \\ 1-\frac{1}{v+1} & +\frac{u}{(v+1)^2} \end{pmatrix} \Rightarrow J = \frac{u}{(v+1)^3} + \frac{u}{(v+1)^2} - \frac{u}{(v+1)^3}$$

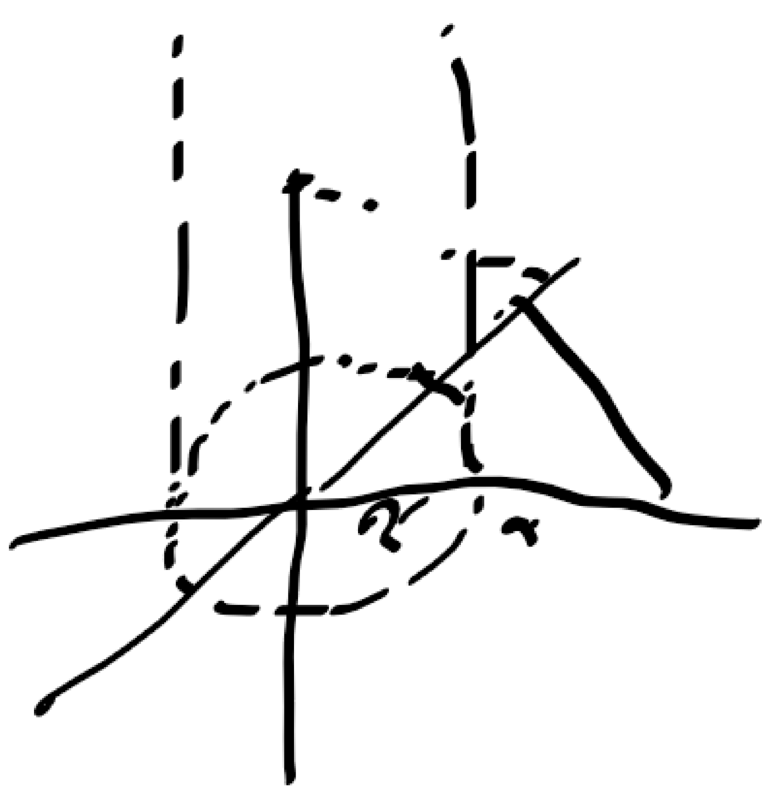
$$= \frac{u}{(v+1)^2}$$

$$\int_\Omega 1 = \int_a^b \int_\alpha^\beta \frac{u}{(v+1)^2} \, dv\, du$$

$$= \frac{b^2-a^2}{2} \left(\frac{1}{\alpha+1} - \frac{1}{\beta+1} \right)$$

5 objem mezi plochami

a) $x+y+z=a \qquad x^2+y^2=R^2, \quad x=0, y=0, z=0, a>R\sqrt{2}$



$x^2+y^2 < R^2 \quad x,y>0$

$0<z<a-x-y$

$x=r\cos\varphi$
$y=r\sin\varphi$
$z=z$

$\mathcal{J}=r$

$r\in(0,R)$
$\varphi\in(0,\frac{\pi}{2})$
$z\in(0, a-r\cos\varphi-r\sin\varphi)$

$$|\Omega|=\int 1\,dx\,dy\,dz=$$
$$=\int_0^{\frac{\pi}{2}}\int_0^R\int_0^{a-r\cos\varphi-r\sin\varphi} r\,dz\,dr\,d\varphi$$
$$=\int_0^{\frac{\pi}{2}}\int_0^R ar-r^2\cos\varphi-r^2\sin\varphi\,dr\,d\varphi$$
$$=\int_0^{\frac{\pi}{2}}\frac{aR^2}{2}-\frac{R^3}{3}\cos\varphi-\frac{R^3}{3}\sin\varphi\,d\varphi$$
$$=\frac{a\pi R^2}{4}-\frac{2R^3}{3}$$

b) $z=xy, \quad z=0, \quad x+y+z=1$

$\Omega \qquad x,y\geq 0 \qquad x+y<1 \qquad 0<z<\min(1-x-y, xy)$

$$\int_\Omega 1=\int_0^1\left(\int_0^{1-x}\left(\int_0^{\min(1-x-y,xy)}1\,dz\right)dy\right)dx$$

$$=\int_0^1\left(\int_0^{1-x}\min(1-x-y, xy)\,dy\,dx\right)=$$

$$= \int_0^1 \left(\int_0^{1-x} \min(1-x-y,\ xy)\, dy\, dx \right) = \left(1-x-y < xy \iff \frac{1-x}{1+x} = y \right)$$

$$= \int_0^1 \left(\int_0^{\frac{1-x}{1+x}} xy\, dy + \int_{\frac{1-x}{1+x}}^{1-x} (1-x-y)\,dy \right) dx$$

$$= \int_0^1 \frac{x(1-x)^2}{(1+x)^2 2} + (1-x)\left((1-x) - \frac{1-x}{1+x}\right) - \frac{(1-x)^2}{2} + \frac{(1-x)^2}{(1+x)^2 2}$$

$$= \int_0^1 \frac{x(1-x)^2}{2(1+x)^2} + \frac{(1-x)^2 x}{(1+x)} - \frac{(1-x)^2}{2} + \frac{(1-x)^2}{2(1+x)^2}$$

$$= \int_0^1 \frac{(1-x)^2}{2(1+x)} + \frac{(1-x)^2 x}{(1+x)} - \frac{(1-x)^2}{2} = \int \frac{(1-x)^2 + 2x(1-x)^2 - (x+1)(1-x)^2}{2(1+x)}$$

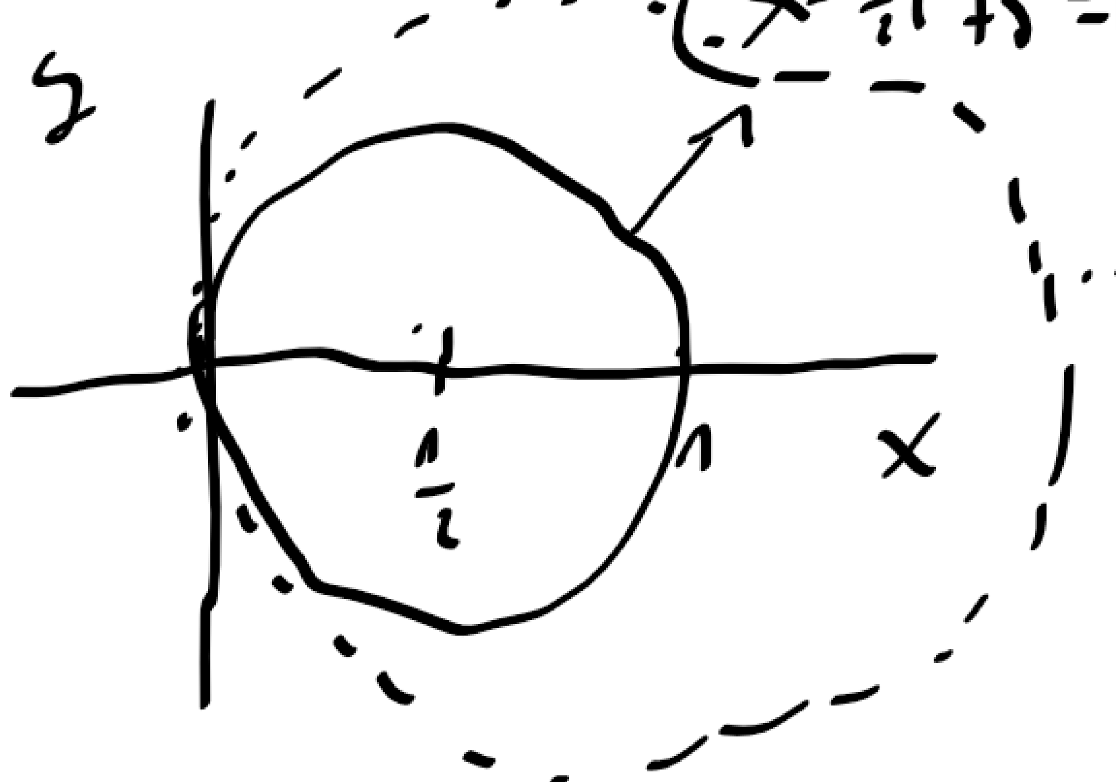
$$= \int_0^1 \frac{(1-x)^2}{2(1+x)}(1+2x-x-1) = \int_0^1 \frac{x(1-x)^2}{2(1+x)} = \int_0^1 \frac{(1-x)^2}{2} - \frac{(1-x)^2}{(1+x)2}$$

$$= \frac{1}{6} - \frac{1}{2} + \int_0^1 \frac{2}{(1+x)} - 2\int_0^1 \frac{1}{(x+1)^2} = \frac{1}{6} - \frac{1}{2} + 2\ln 2 - 1$$

c) $z = x^2+y^2$ $\quad x^2+y^2 = x$, $\quad x^2+y^2 = 2x$ $\quad z = 0$

$(x-\frac{1}{2})^2+y^2=(\frac{1}{2})^2$

$\to (x-1)^2+y^2=1$



$x < x^2+y^2 < 2x$

$\Updownarrow$

$r\cos\varphi < r^2 < 2r\cos\varphi$

$x = r\cos\varphi$

$y = r\sin\varphi$

$z = z$

$J = r, \quad 0 \le z < r^2$

$\cos\varphi < r < 2\cos\varphi$

$\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$$\int 1 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{\cos\varphi}^{2\cos\varphi}\left(\int_0^{r^2} r\,dz\right) dr\,d\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{\cos\varphi}^{2\cos\varphi} r^3\,dr\,d\varphi = \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(2^4-1)\cos^4\varphi\,d\varphi$$

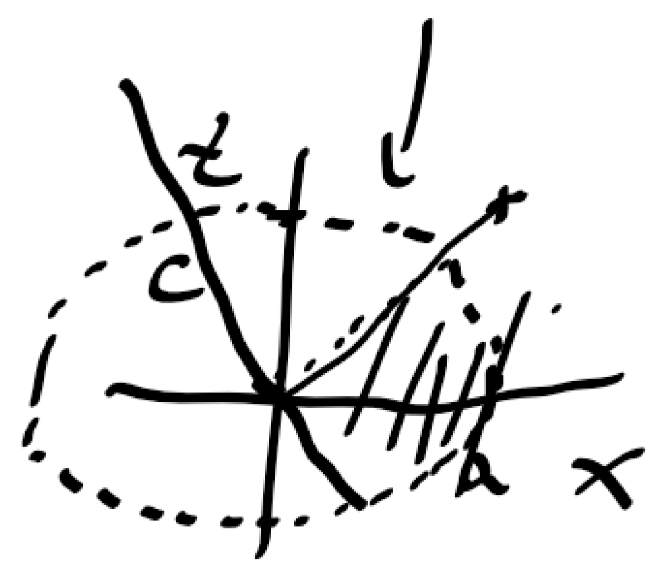
$$= \frac{15}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 - \cos^2\sin^2 \quad \to \quad = \frac{15}{4}\left(\frac{\pi}{2} - \frac{\pi}{8}\right)$$

d) $z = e^{-(x^2+y^2)}$, $z=0$, $x^2+y^2=R^2$

$x = r\cos\varphi$
$y = r\sin\varphi$ $\qquad J = r \qquad r\in(0,R)$, $\varphi\in(0,2\pi)$
$z = z$ $\qquad\qquad z\in(0, e^{-r^2})$

$$\int_\Omega 1 = \int_0^{2\pi}\left(\int_0^R\left(\int_0^{e^{-r^2}} r\,dz\right)dr\right)d\varphi = \int_0^{2\pi}\int_0^R re^{-r^2}\,dr\,d\varphi = 2\pi\int_0^R re^{-r^2}\,dr$$
$$= \pi\left[-e^{-r^2}\right]_0^R = \pi\left(1-e^{-R^2}\right)$$

e) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1 \qquad \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=\dfrac{z^2}{c^2} \qquad a,b,c>0$



$$\left|\begin{aligned} x &= ar\cos\varphi\cos\psi \\ y &= br\sin\varphi\cos\psi \\ z &= cr\sin\psi \end{aligned}\right|$$

$r\in(0,1)$ $\qquad \cos^2\psi > \sin^2\psi$

$\varphi\in(0,2\pi)$ $\qquad \psi\in\left(-\frac{\pi}{4},\frac{\pi}{4}\right)$

$$J = \det\begin{pmatrix} a\cos\varphi\cos\psi & -ar\sin\varphi\cos\psi & -ar\cos\varphi\sin\psi \\ b\sin\varphi\cos\psi & br\cos\varphi\cos\psi & -br\sin\varphi\sin\psi \\ c\sin\psi & 0 & cr\cos\psi \end{pmatrix}$$

$$J = abcr^2\left(\cos^2\varphi\cos^3\psi + \sin^2\varphi\sin^2\psi\cos\psi + \cos^2\varphi\cos\psi\sin^2\psi + \sin^2\varphi\cos^3\psi\right) = abcr^2\cos\psi$$

$$\int_\Omega = \int_0^{2\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_0^1 abcr^2\cos\psi\,dr\,d\psi\,d\varphi = \frac{2\pi}{3}abc\sqrt{2}$$

j) $\left(\frac{x}{a}+\frac{y}{b}\right)^2 + \frac{z^2}{c^2} = 1, \quad x=0,\ y=0,\ z=0 \quad a,b,c>0$

$x, y, z \geq 0$

$x = a r (\cos\varphi \cos\psi)^{1/2}$

$y = b r (\cos\varphi \sin\psi)^{1/2}$

$z = c r \sin\varphi$

$1 > \left(\frac{x}{a}+\frac{y}{b}\right)^2 + \frac{z^2}{c^2} = r^2$

$r \in (0,1), \quad \varphi \in (0, \frac{\pi}{2})$

$\psi \in (0, \frac{\pi}{2})$

$$J = \det \begin{pmatrix} a(\cos\varphi\cos\psi)^{1/2} & -\frac{ar}{2}\left(\frac{\cos\psi}{\cos\varphi}\right)^{1/2}\sin\varphi & -\frac{ar}{2}\left(\frac{\cos\varphi}{\cos\psi}\right)^{1/2}\sin\psi \\ b(\cos\varphi\sin\psi)^{1/2} & -\frac{br}{2}\left(\frac{\sin\psi}{\cos\varphi}\right)^{1/2}\sin\varphi & +\frac{br}{2}\left(\frac{\cos\varphi}{\sin\psi}\right)^{1/2}\cos\psi \\ c\sin\varphi & cr\cos\varphi & 0 \end{pmatrix}$$

$$= \frac{abcr^2}{2}\left(\left(\frac{\cos\psi}{\sin\psi}\right)^{1/2}\sin^2\varphi\cos\psi + \cos^2\varphi\left(\frac{\sin\psi}{\cos\psi}\right)^{1/2}\sin\psi + \left(\frac{\sin\psi}{\cos\psi}\right)^{1/2}\sin^2\varphi\sin\psi + \cos^2\varphi\left(\frac{\cos\psi}{\sin\psi}\right)^{1/2}\cos\psi\right)$$

$$= \frac{abcr^2}{2}\left(\left(\frac{\cos\psi}{\sin\psi}\right)^{1/2}\cos\psi + \left(\frac{\sin\psi}{\cos\psi}\right)^{1/2}\sin\psi\right)$$

$$= \frac{abcr^2}{2}\left(\frac{1}{\sin\psi\cos\psi}\right)^{\cancel{1/2}} = \ldots \text{ad}$$

$$\int_0^a \int_0^{b(1-\frac{x}{a})} \int_0^{c\sqrt{1-(\frac{x}{a}+\frac{y}{b})^2}} 1 \cdot dz\, dy\, dx = c \int_0^a \int_0^{b(1-\frac{x}{a})} \sqrt{1-\left(\frac{x}{a}+\frac{y}{b}\right)^2}\, dy\, dx$$

$= \text{dd.}$

⑥ a) $\int_{\mathbb{R}^2} e^{-x^2-y^2}\,dx\,dy \overset{\substack{x=r\cos\varphi\\ y=r\sin\varphi}}{=} \int_0^{\infty}\int_0^{2\pi} r e^{-r^2}\,dr\,d\varphi = -\left[\pi e^{-r^2}\right]_0^{\infty}$

$= +\pi$

KÜRZEST $I = \int_0^{\infty} e^{-x^2}\,dx$

$I^2 = \left(\int_0^{\infty} e^{-x^2}dx\right)\cdot\left(\int_0^{\infty} e^{-y^2}dy\right) \overset{\text{FUBINI}}{=} \frac{1}{4}\int_{\mathbb{R}^2} e^{-x^2-y^2}\,dx\,dy = \frac{\pi}{4}$

$\Rightarrow \quad I = \frac{\sqrt{\pi}}{2}$

$p, q > 1$

b) $\int_{\Omega} x^{-p} y^{-q}\,dx\,dy \qquad \Omega = \quad xy \geq 1, \; x \geq 1$

$= \lim_{R\to\infty} \int_1^R \int_{\frac{1}{x}}^{R} x^{-p} y^{-q}\,dy\,dx = \lim \int_1^R x^{-p}\left[\frac{+y^{-q+1}}{-q+1}\right]_{\frac{1}{x}}^{R}$

$= \lim \int_1^R \frac{+x^{-p} R^{-q+1}}{-q+1} - \frac{x^{-p+q-1}}{-q+1} = \lim\left[\frac{x^{-p+1} R^{-q+1}}{(-q+1)(-p+1)}\right]_1^R$

$- \left[\frac{x^{-p+q}}{(-q+1)(-p+q)}\right]_1^R = -\frac{1}{(-q+1)(-p+q)}$

pokud $q < p$!?

c) $\int\limits_{\Omega} (x+y)^{-p}\, dx\, dy$ $\qquad x+y \geq 1 \qquad 0 \leq x \leq 1$

$x = u$

$y = -u + t$ $\qquad t > 1$

$u \in (0,1)$

$J = \det \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = 1$

$= \int\limits_0^1 \int\limits_1^{\infty} t^{-p}\, dt\, du = \frac{1}{p-1}$ punkt $\underline{\underline{p > 1}}$!

7

$$x_T = \frac{\int_\Omega x \, dx\,dy}{\int_\Omega 1 \, dx\,dy}$$

$x, y, a > 0$

$x^{\frac{2}{3}} + y^{\frac{2}{3}} \le a^{\frac{2}{3}}$

$$\overset{\text{Fubini}}{=} \frac{\int_0^a \left(\int_0^{\left(a^{\frac{2}{3}} - y^{\frac{2}{3}}\right)^{3/2}} x \, dx \right) dy}{\int_0^a \int_0^{\left(a^{\frac{2}{3}} - y^{\frac{2}{3}}\right)^{3/2}} 1 \, dx\, dy} = \frac{\frac{1}{2} \int_0^a \left(a^{\frac{2}{3}} - y^{\frac{2}{3}}\right) dy}{\int_0^a \left(a^{\frac{2}{3}} - y^{\frac{2}{3}}\right)^{3/2} dy}$$

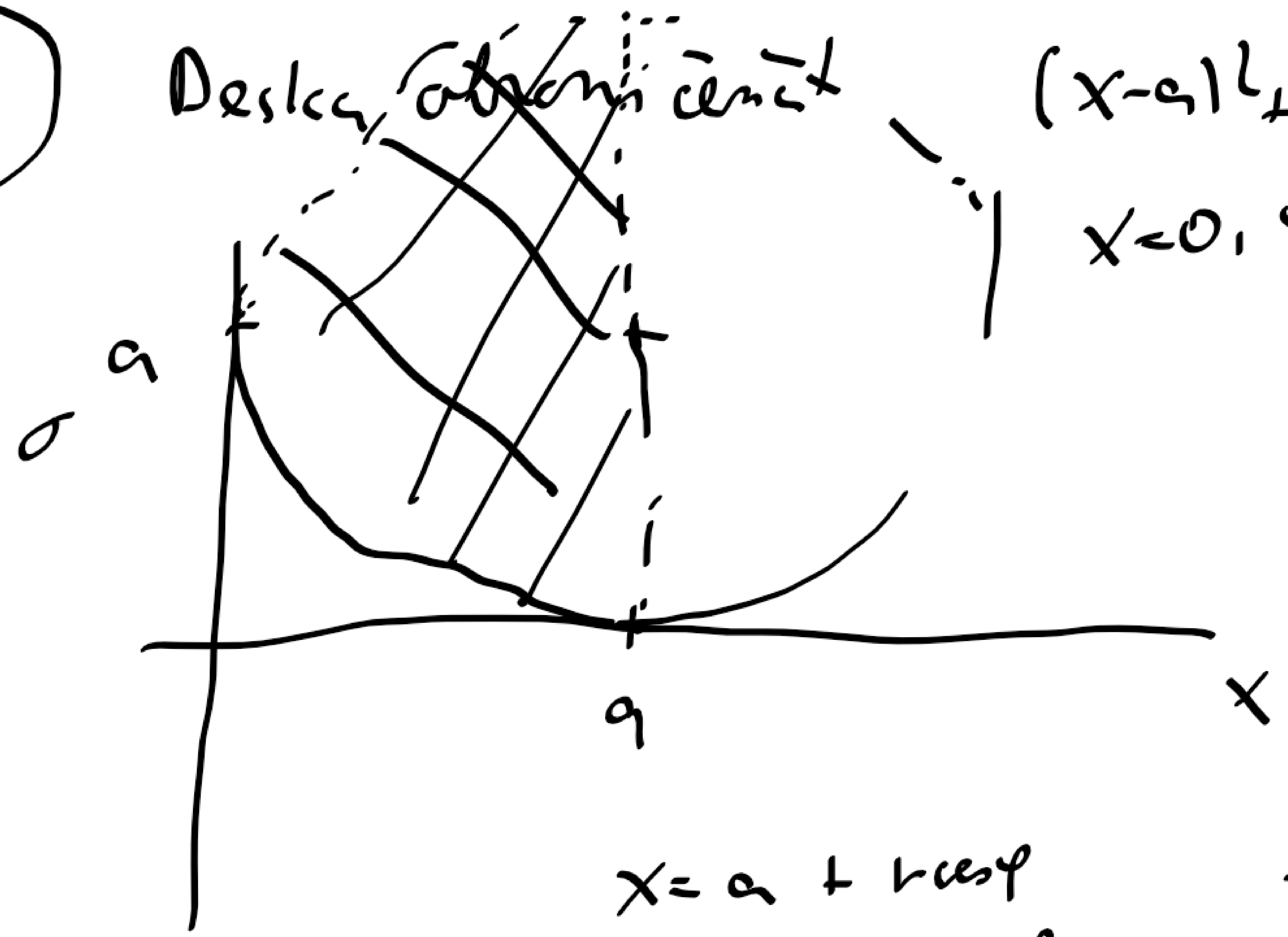
$y = a \sin^3 t$

$dy = 3a \sin^2 t \cos t$

$$= \frac{\frac{1}{2} \, a^{\frac{5}{3}} \left(1 - \frac{3}{5}\right)}{\int_0^{\frac{\pi}{2}} \left(a^{\frac{2}{3}} - a^{\frac{2}{3}} \sin^2 t\right)^{3/2} 3a \sin^2 t \cos t} =$$

$$= \frac{\frac{a^{-\frac{1}{3}}}{15}}{\int_0^{\frac{\pi}{2}} \cos^4 t \sin^2 t} = \ldots$$

8 Deska, obszar cienki



$(x-a)^2+(y-a)^2=a^2$

$x=0,\ y=0 \qquad 0\le x\le a$

$x = a + r\cos\varphi$

$y = a + r\sin\varphi \qquad J = r \qquad r\in(0,a) \qquad \varphi\in\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$

$$J_x=\iint_\Omega y^2\,dx\,dy = \int_0^a\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (a^2+r^2\sin^2\varphi+2ar\sin\varphi)\,r$$

$$=\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{a^4}{2}+\frac{a^4}{4}\sin^2\varphi+\frac{2a^4}{3}\sin\varphi\,d\varphi = \frac{a^4\pi}{2}+\frac{a^4\pi}{8}$$

$$=\frac{5a^4\pi}{8}$$

$$J_y=\iint_\Omega x^2\,dx\,dy = \int_0^a\int_0^{\frac{3\pi}{2}} (a^2+r^2\cos^2\varphi+2ar\cos\varphi)\,r$$

$$=\int \frac{a^4}{2}+\frac{a^4}{4}\cos^2\varphi+\frac{2a^4}{3}\cos\varphi$$

$$=\frac{5a^4\pi}{8}-\frac{4a^4}{3}$$

INTEGRÁLY ZÁVISLÉ NA PARAMETRU

- Obecně $\Omega \subseteq \mathbb{R}^d$ měřitelná, $I \subseteq \mathbb{R}$, $f : \Omega \times I \to \mathbb{R}$

  $f(x, \alpha)$ $x \in \Omega \subset \mathbb{R}^d$, $\alpha \in I \subset \mathbb{R}$

  $\forall \alpha$ je f měřitelná !

- Můžeme uvažovat i posloupnost $\tilde{f}_n(x) = f(x, n)$ a studovat chování pro $n \to \infty$

ZÁMĚNA LIMITY A INTEGRÁLU

Buď $f^n(x) \to f(x)$ skoro všude v $\Omega \subseteq \mathbb{R}$

Pak

$$\lim_{n \to \infty} \int_\Omega f^n(x)\,dx = \int f(x)\,dx$$

pokud je splněn jeden z následujících předpokladů:

1) $f^n \rightrightarrows f$ v Ω (stejnoměrná konvergence)

2) $f^n \nearrow f$ v Ω (monotónní konvergence - Levi)

3) $\exists g : \Omega \to \mathbb{R}$ $\int_\Omega |g| < \infty$ a $\forall n$ $|f^n(x)| \le g(x)$

   (integrovatelná majoranta - Lebesgue)

4) $\exists g^n : \Omega \to \mathbb{R}$ a $g : \Omega \to \mathbb{R}$, tak že $|f^n(x)| \le g^n(x)$ a

   $$\lim_{n \to \infty} \int_\Omega |g^n(x) - g(x)|\,dx = 0$$ (konvergující majoranta)

5) $\forall \varepsilon > 0$ $\exists \delta > 0$ $\forall$ měřitelnou $\tilde{\Omega} \subseteq \Omega$ $\forall n$ platí

   $|\tilde{\Omega}| \le \delta \Rightarrow \int_{\tilde{\Omega}} |f^n(x)| \le \varepsilon$ (VITALI - stejně stejnoměrná integrovatelnost)

ZÁVISLOST na parametru

Označ $F(\alpha) := \int_\Omega f(x,\alpha)\,dx$

1) pokud existuje $g:\Omega\to\mathbb{R}$ (majoranta), tak že $\int_\Omega |g| < \infty$ $\forall \alpha\in I$ platí $|f(x,\alpha)| \le g(x)$ a pokud je pro s.v. $x\in\Omega$ $f(x,\cdot)$ spojitá na I pak $F(\alpha)$ je spojitá na I

2) pokud $\exists \alpha_0\in I$ tak že $|F(\alpha_0)| < \infty$ a pokud existuje $g:\Omega\to\mathbb{R}$ $\int|g|<\infty$ $\left|\frac{\partial f(x,\alpha)}{\partial\alpha}\right| < |g(x)|$ pak $F'(\alpha) = \int_\Omega \frac{\partial f(x,\alpha)}{\partial\alpha}\,dx$.

(1) $\lim\limits_{n\to\infty} \int_0^1 \frac{x^n}{n}\, dx$

$$\frac{x^n}{n} \to 0 \quad \forall x\in[0,1) \text{ a tedy i skoro všude v } [0,1].$$

Máme 2 možnosti jak použít větu o záměně limity a integrálu

a) $\frac{x^n}{n} \searrow 0$ MONOTÓNNĚ?

b) $\left|\frac{x^n}{n}\right| \le 1 =: g(x)$ integrovatelná majoranta nezávislá na n.

$$\Rightarrow \lim_{n\to\infty} \int_0^1 \frac{x^n}{n}\, dx = 0$$

(2) $\lim\limits_{n\to\infty} \int_0^\infty \underbrace{\frac{\ln(x+n)}{n} e^{-x} \cos x}_{f(x,n)}\, dx$

$\forall x\in(0,\infty)$ $f(x,n) \to 0$

$$a \quad |f(x,n)| \le \frac{e^{-x}\ln(x+n)}{n} \le e^{-x}\ln(2x) + \frac{e^{-x}\ln(2n)}{n}$$
$$\le \underline{\underline{e^{-x}\ln 2x + 2e^{-x}}}$$

nezávisí na 'n' + je integrovatelná!!

$$\Rightarrow \lim_{n\to\infty} \int_0^\infty f(n,x) = 0$$

(3) $\lim\limits_{n\to\infty} \int_0^\infty \underbrace{\frac{x^n}{1+x^{2n}}}_{f(x,n)}\, dx$ – víme $f(x,n) \to \begin{cases} 0 & \text{pro } x\neq 1 \\ \frac{1}{2} & \text{pro } x=1 \end{cases}$

$$\Rightarrow f(x,n) \to 0 \text{ s.v. v } (0,\infty)$$

$\frac{x^n}{1+x^{2n}} \le \min\left\{1, \frac{1}{x^2}\right\}$ $\forall n\ge 2$ a $\int_0^\infty \min\left(1, \frac{1}{x^2}\right) < \infty$.

$$\Rightarrow \lim_{n\to\infty} \int_0^\infty f(x,n) = 0$$

(4) $\lim_{n\to\infty} \int_0^\infty e^{-nx} x^2 = 0$

protože $e^{-nx}x^2 \to 0 \quad \forall x \in (0,\infty)$

a $|e^{-nx}x^2| \le e^{-x}x^2$ integrovatelná majoranta.

(5) $\int_0^\infty \frac{x}{e^x-1}\, dx$ (víme, že $\sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$)

máme nalézt hodnotu !?

$$\frac{x}{e^x-1} = \frac{x}{e^x} \underbrace{\frac{1}{1-\underset{<1}{e^{-x}}}}_{\text{součet geometrické řady}} = xe^{-x}\sum_{n=0}^\infty e^{-nx} = \sum_{n=1}^\infty xe^{-nx}$$

navíc platí $\sum_{k=1}^n xe^{-kx} \nearrow \frac{x}{e^x-1}$ a můžeme použít

větu o monotónní konvergenci

$$\int_0^\infty \frac{x}{e^x-1} = \int_0^\infty \sum_{k=1}^\infty xe^{-nx} = \sum_{n=1}^\infty \int_0^\infty xe^{-nx}\,dx = \sum_{n=1}^\infty \left[-\frac{xe^{-nx}}{n} - \frac{e^{-nx}}{n^2}\right]_0^\infty$$

$$= \sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$$

(6) $\frac{\ln\frac{1}{x}}{1-x^2} = -\ln x \sum_{k=0} x^{2k}$

$$\Rightarrow \int_0^1 \frac{\ln\frac{1}{x}}{1-x^2} = \sum \int_0^1 -\ln x\, x^{2k} = \sum \int_0^1 \frac{x^{2k}}{2k+1} = \sum_{k=0}^\infty \frac{1}{(2k+1)^2}$$

víme $\sum \frac{1}{k^2} = \frac{\pi^2}{6} \Rightarrow \sum \frac{1}{(2k)^2} = \frac{\pi^2}{24}$

$$\Rightarrow \sum_{k=1} \frac{1}{k^2} - \frac{1}{(2k)^2} = \frac{\pi^2}{8} = \underline{\underline{\sum_{k=0} \frac{1}{(2k+1)^2}}}$$

⑦ $$\int_0^\infty \frac{x^3}{e^x-1} = \int_0^\infty x^3 e^{-x} \frac{1}{1-e^{-x}}$$

$$= \int_0^\infty x^3 e^{-x} \sum_{k=0}^\infty e^{-kx}\, dx = \sum_{k=1}^\infty \int_0^\infty x^3 e^{-kx} =$$

$$\overset{P.P.}{=} \sum_{k=1} \int_0^\infty \frac{3x^2 e^{-kx}}{k} \overset{P.P.}{=} \sum_{k=1} \int_0^\infty \frac{6x e^{-kx}}{k^2} = \sum \int_0^\infty \frac{6}{k^3} e^{-kx}$$

$$= \sum 6 \left[\frac{e^{-kx}}{-k^4}\right]_0^\infty = \underbrace{\sum_{k=1} 6 \frac{1}{k^4} = \frac{\pi^4}{15}}_{\text{Tabulka mezipočtů}}$$

⑧ Ukažte, že integrál je spojitou funkcí parametru

a) $$\int_0^\pi \frac{\sin x}{x^\alpha (\pi-x)^\alpha} dx \qquad f(x,\alpha) := \frac{\sin x}{x^\alpha(\pi-x)^\alpha}$$

spojitě skrz "α" musím najít majorantu

$\varepsilon > 0$ libovolné a uvažuji

$$\alpha \in \left(-\frac{1}{\varepsilon},\ 2-\varepsilon\right)$$

$$\left|\frac{\sin x}{x^\alpha(\pi-x)^\alpha}\right| \le \chi_{[0,\frac{\pi}{2}]}\left(\frac{2}{\pi}\right)^\alpha \frac{\sin x}{x}\frac{1}{x^{\alpha-1}} + \chi_{[\frac{\pi}{2},\pi]}\left(\frac{2}{\pi}\right)^\alpha \frac{\sin(\pi-x)}{(\pi-x)}\frac{1}{(\pi-x)^{\alpha-1}}$$

$$! \le \underbrace{\left(\frac{2}{\pi}\right)^{-\frac{1}{\varepsilon}}\left(\chi_{[0,\frac{\pi}{2}]}\frac{1}{x^{1-\varepsilon}} + \chi_{[\frac{\pi}{2},\pi]}\frac{1}{(\pi-x)^{1-\varepsilon}}\right)}_{\text{INTEGROVATELNÁ MAJORANTA NEZÁVISLÁ NA "}\alpha\text{"!}}$$

DŮLEŽITÉ!! JE POTŘEBA UVAŽOVAT INTERVAL $\left(-\frac{1}{\varepsilon}, 2-\varepsilon\right)$

JINAK NENALEZNU MAJORANTU!

b) $\int_0^{?} \underbrace{x^2 \cos ax}_{\text{spojitá v } \dot{c}} \, dx \qquad a \in (-\infty, \infty)$

$| x^2 \cos a x | \leq x^2$ – integrálna majoranta nezávislá na „a"

c) $\int_1^{\infty} \underbrace{\frac{\cos x}{x^a}}_{\text{spojité}} dx \qquad 1 < a < \infty$

$a \in (1+\varepsilon, \infty) \Rightarrow \left| \frac{\cos x}{x^a} \right| \leq \frac{1}{x^{1+\varepsilon}}$ = MAJORANTA

$\Rightarrow \int_1^{\infty} \frac{\cos x}{x^a} dx$ je spojitá fcia „a" na intervale $(1+\varepsilon, \infty)$.

$\varepsilon > 0$ bolo ľubovoľné a teda je to pravda na $(1, \infty)$

(a) $\int_0^{\infty} \underbrace{\frac{\operatorname{arctg} ax}{x(1+x^2)}}_{g(x,\alpha)} = F(\alpha) \qquad D_f = \mathbb{R}$

1) $F(0) = 0$

2) $\frac{\partial g(x,\alpha)}{\partial \alpha} = \frac{1}{1+(a x)^2} \cdot \frac{1}{1+x^2} \qquad \leq \frac{1}{1+x^2}$ majoranta

$\Rightarrow F'(\alpha) = \int_0^{\infty} \frac{1}{1+a^2x^2} \cdot \frac{1}{1+x^2} dx = \frac{1}{1-a^2} \int_0^{\infty} \frac{1}{1+x^2} - \frac{a^2}{1+a^2x^2} dx$

$\alpha \neq 1$

$= \frac{1}{1-a^2} \left[\operatorname{arctg} x - a \operatorname{arctg} a x \right]_0^{\infty} = \frac{\pi}{2(1+|a|)}$

$\Rightarrow F(\alpha) = \frac{\pi}{2} \ln(1+\alpha) \qquad$ Pre $\alpha \geq 0$

alternatívne: $\int_0^{\infty} \frac{\operatorname{arctg} ax}{x(1+x^2)} dx = \int_0^{\infty} \int_0^{a} \frac{\partial}{\partial \alpha} \left(\frac{\operatorname{arctg} \alpha x}{x(1+x^2)} \right) dx = \int_0^{\infty} \int_0^{a} \frac{1}{(1+x^2)(1+a^2x^2)} d\alpha dx$

$\overset{\text{Fubini}}{=} \int_0^{a} \int_0^{\infty} \frac{1}{(1+x^2)(1+a^2x^2)} = \frac{\pi}{2} \ln(1+a)$

(10) $\int_0^\infty \frac{e^{-ax^2} - e^{-bx^2}}{x}\,dx$ Definován pro $a=b$ a pro $a,b>0$

Metoda pomocí derivace : $f(x,a) := \frac{e^{-ax^2} - e^{-bx^2}}{x}$

1) $\int_0^\infty f(x,b) = 0$

2) $\frac{\partial f(x,a)}{\partial a} = -xe^{-ax^2}$

$$\Rightarrow \frac{\partial}{\partial a} \int_0^\infty \frac{e^{-ax^2} - e^{-bx^2}}{x} = -\int_0^\infty xe^{-ax^2} = \left[\frac{+e^{-ax^2}}{2a}\right]_0^\infty = -\frac{1}{2a}$$

$$\Rightarrow \int_0^\infty \frac{e^{-ax^2} - e^{-bx^2}}{x} = -\frac{\ln 2a}{2} + \frac{\ln 2b}{2} = \frac{1}{2}\ln\frac{b}{a}$$

$\uparrow$ INTEGRAČNÍ KONSTANTA

METOD „ALA" FUBINI

$$\int_0^\infty \frac{e^{-ax^2} - e^{-bx^2}}{x} = \int_0^\infty \left(\int_b^a \frac{d}{dc} \frac{e^{-cx^2}}{x}\,dc\right) dx$$

$$= -\int_0^\infty \int_b^a x e^{-cx^2}\,dc\,dx = -\int_b^a \left(\int_0^\infty xe^{-cx^2}dx\right) dc$$

$$= -\int_b^a \left[-\frac{e^{-cx^2}}{2c}\right]_0^\infty dc = -\frac{1}{2}\int_b^a \frac{1}{c}\,dc = \frac{1}{2}\ln\frac{b}{a}$$

(11) $\int_0^\infty \frac{e^{-ax^2} - e^{-bx^2}}{x^2}\,dx$ (Víme, že $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$)

$$\parallel$$

$$\int_0^\infty \int_b^a \frac{\partial}{\partial c} \frac{e^{-cx^2}}{x^2}\,dc\,dx = -\int_0^\infty \int_b^a e^{-cx^2}dc\,dx$$

$$= \int_b^a \left(\int_0^\infty e^{-cx^2}dx\right)dc \overset{\sqrt{c}x=y}{\underset{dx=\frac{dy}{\sqrt{c}}}{=}} -\int_b^a \frac{1}{\sqrt{c}}\int_0^\infty e^{-y^2}dy\,dc = -\int_b^a \frac{\sqrt{\pi}}{2}\frac{1}{\sqrt{c}}\,dc$$

$$= \sqrt{\pi}\left(\sqrt{b} - \sqrt{a}\right)$$

(12) $$\int_0^1 \frac{x^b - x^a}{\ln x}\,dx = \int_0^1 \left(\int_a^b \frac{\partial}{\partial c} \frac{x^c}{\ln x}\,dc \right) dx$$

$$= \int_0^1 \int_a^b x^c\,dc\,dx = \int_a^b \left(\int_0^1 x^c\,dx \right) dc$$

$$= \int_a^b \frac{1}{c+1}\,dc = \ln\left(\frac{b+1}{a+1}\right)$$

$\underline{\underline{a, b > -1}}$

NEBO $F(b) = \int_0^1 \overbrace{\frac{x^b - x^a}{\ln x}}^{f(b,x)}$ $\qquad F(a) = 0$ ✓

$\frac{\partial f}{\partial b} = x^b \leq \underbrace{x^{-1+\varepsilon}}_{\text{majoranta}}$ pro $b > -1+\varepsilon$

$$F'(b) = \int_0^1 x^b\,dx = \frac{1}{b+1}$$

$F(b) = \ln(b+1) + C \rightarrow$ volím C tak aby $F(a) = 0$

$$\Rightarrow F(b) = \ln\left(\frac{b+1}{a+1}\right)$$

(13) $\int_0^\infty \underbrace{\frac{\operatorname{arctg} ax - \operatorname{arctg} bx}{x^2}}_{f(a,b,x)} = F(a,b)$ má smysl $\forall a, b$

Uvažuji $0 < \varepsilon < a, b < \frac{1}{\varepsilon}$

$$\frac{\partial f}{\partial a} = \frac{1}{1+(ax)^2} \cdot \frac{1}{x} \cdot \operatorname{arctg} bx$$

$$\left|\frac{\partial f}{\partial a}\right| \leq \frac{1}{\varepsilon} \cdot \frac{1}{1+\varepsilon^2 x^2} \quad \text{majoranta}$$

$$\frac{\partial F}{\partial a}(a,b) = \int_0^\infty \frac{1}{1+(ax)^2} \cdot \frac{1}{x} \operatorname{arctg} bx$$

stejně $\frac{\partial F}{\partial a \partial b} = \int_0^\infty \frac{1}{1+(ax)^2} \cdot \frac{1}{(1+bx)^2} = \frac{1}{a^2-b^2} \int_0^\infty \frac{a^2}{1+(ax)^2} - \frac{b^2}{1+(bx)^2}$

$$= \frac{1}{a^2-b^2} \Big[a \operatorname{arctg} ax - b \operatorname{arctg} bx \Big]_0^\infty = \frac{1}{a+b} \frac{\pi}{2}$$

$\Rightarrow \frac{\partial F}{\partial a} = \frac{\pi}{2} \ln(a+b) \underbrace{- \frac{\pi}{2} \ln a}$ protože $\frac{\partial F}{\partial a}(a,0) = 0$

$$\frac{\partial F}{\partial a} = \frac{\pi}{2} \ln(a+b) \underbrace{- \frac{\pi}{2}\ln a}_{} \quad \text{protože } \frac{\partial F}{\partial a}(a,0)=0$$

$$\Rightarrow \quad F(a,b) = \frac{\pi}{2} \int \ln(a+b) - \ln a \, da$$

$$= (a+b)\ln(a+b) \qquad - a\ln a \quad + c$$

ale $F(b,b)=0 \Rightarrow \quad F(a,b) = \frac{\pi}{2}\left((a+b)\ln(a+b) - a\ln a - b\ln b\right)$

(14) $F(b) = \int_0^\infty \underbrace{e^{-ax^2}\cos bx}_{f}\,dx$ $\qquad$ ČVAŠUJI $\qquad a > \varepsilon > 0$

$$\frac{\partial f}{\partial b} = -x\sin bx\, e^{-ax^2} \qquad \left|\frac{\partial f}{\partial b}\right| \le x e^{-\varepsilon x^2} \text{ majoranta}$$

$$F'(b) = -\int_0^\infty e^{-ax^2} x \sin bx \overset{P.P}{=} \left[\frac{e^{-ax^2}}{2a}\sin bx\right]_0^\infty - \int_0^\infty \frac{e^{-ax^2}}{2a} b\cos bx\,dx$$

$$= -\frac{b}{2a}F(b)$$

$$\Rightarrow \quad F'(b) = -\frac{b}{2a}F(b) \quad \Rightarrow \quad (\ln F(b))' = \left(-\frac{b^2}{4a}\right)'$$

$$\Rightarrow \quad F(b) = e^{-\frac{b^2}{4a}} \cdot C(a)$$

$$F(0) = \int e^{-ax^2} = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

$$\Rightarrow \quad F(b) = \frac{1}{2}\sqrt{\frac{\pi}{a}}\, e^{-\frac{b^2}{4a}}$$

(15) $F(b) = \int_0^\infty e^{-ax^2}\cosh bx$ $\qquad a > \varepsilon > 0$

$$F'(b) = \int_0^\infty e^{-ax^2} x \sinh bx\,dx \qquad \left(\text{majoranta je } \; b < \frac{1}{\varepsilon} \quad x e^{-\varepsilon x^2 + \frac{x}{\varepsilon}}\right)$$

$$\overset{P.P}{=} \left[-\frac{e^{-ax^2}}{2a}\sinh bx\right]_0^\infty + \int_0^\infty \frac{b}{2a} e^{-ax^2}\cosh bx = \frac{b}{2a}F(b)$$

$$F'(b) = \frac{b}{2a}F(b) \Rightarrow$$

$$F(b) = e^{\frac{b^2}{4a}} \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

(16) $F(a)=\int_0^{\frac{\pi}{2}} \underbrace{\frac{\ln(1+a\sin^2 x)}{\sin^2 x}}_{f}\, dx$ $\qquad F(0)=0$

$\frac{\partial f}{\partial a} = \frac{1}{1+a\sin^2 x}$ $\qquad a > -1+\varepsilon$ $\Rightarrow$ majoranta $\frac{1}{1-(1-\varepsilon)\sin^2 x}$

$$\int_0^{\frac{\pi}{2}} \frac{dx}{1+a\sin^2 x} \overset{\operatorname{tg} x = t}{=} \int_0^\infty \frac{1}{(1+t^2)\left(1+a\frac{t^2}{1+t^2}\right)}\,dt = \int_0^\infty \frac{1}{1+t^2(a+1)}\,dt$$

$$= \left[\frac{\operatorname{arctg} \sqrt{a+1}\, t}{\sqrt{a+1}}\right]_0^\infty = \frac{\pi}{2\sqrt{a+1}}$$

$\Rightarrow F'(a) = \frac{\pi}{2}\frac{1}{\sqrt{a+1}}$

$\Rightarrow F(a) = F(a) - F(0) = \int_0^a \frac{\pi}{2}\frac{1}{\sqrt{s+1}}\,ds = \left[\pi\sqrt{s+1}\right]_0^a = \pi\left(\sqrt{a+1}-1\right)$

(17) $F_\pm(b) = \int_0^\pi \ln(a \pm b\cos x)\,dx$ $\qquad a>|b|$ $\qquad F(0) = \pi \ln a$

$F_\pm'(b) = \int_0^\pi \frac{\pm\cos x}{a\pm b\cos x}\,dx$ $\qquad |b| < a-\varepsilon$ $\Rightarrow$ majoranta $\frac{1}{\varepsilon}$!

$$\Rightarrow F_+'(b) + F_-'(b) = \int_0^\pi \cos x\left(\frac{1}{a+b\cos x} - \frac{1}{a-b\cos x}\right)dx$$

$$= -\int_0^\pi \frac{2b\cos^2 x}{a^2-b^2\cos^2 x}\,dx = -4b\int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{a^2-b^2\cos^2 x}\,dx \overset{\operatorname{tg} x = t}{=} -4b\int_0^\infty \frac{\frac{1}{1+t^2}}{(1+t^2)\left(a^2-\frac{b^2}{1+t^2}\right)}$$

$$= -4b\int_0^\infty \frac{1}{(1+t^2)\left(a^2(1+t^2)-b^2\right)}\,dt$$

$$= +\frac{4}{b}\int_0^\infty \frac{1}{1+t^2} - \frac{a^2}{a^2(1+t^2)-b^2}\,dt = \frac{4}{b}\left[\operatorname{arctg} t - \frac{a}{\sqrt{a^2-b^2}}\operatorname{arctg}\frac{at}{\sqrt{a^2-b^2}}\right]_0^\infty$$

$$= \frac{2\pi}{b}\left(1 - \frac{1}{\sqrt{1-\left(\frac{b}{a}\right)^2}}\right)$$

Protože $F_+ = F_-$ máme $F_+'(b) = \frac{2\pi}{b} - \frac{2\pi}{b\sqrt{1-\left(\frac{b}{a}\right)^2}}$

$$F_t'(b) = \frac{2\pi}{b} - \frac{2\pi}{b\sqrt{1-(\frac{b}{a})^2}}$$

$$\Rightarrow F_t = \int \ldots = 2\pi \ln b - \int \frac{2\pi}{b\sqrt{1-(\frac{b}{a})^2}}\,db$$

pro $b \geq 0$ F jednodel

$b = a\sin t$; $db = a\cos t\,dt$

$$= 2\pi\ln b - \int \frac{2\pi a\cos t\,dt}{a\sin t\cos t} = 2\pi\ln b - \int \pi\sin t\left(\frac{1}{1-\cos t}+\frac{1}{1+\cos t}\right)dt$$

$$= 2\pi\ln b - \pi\ln\left(\frac{1-\cos t}{1+\cos t}\right) = 2\pi\ln b - \pi\ln\left(\frac{1-\sqrt{1-\frac{b^2}{a^2}}}{1+\sqrt{1-\frac{b^2}{a^2}}}\right)+C$$

$$= \pi\ln\frac{b^2\left(1+\sqrt{1+\frac{b^2}{a^2}}\right)}{1-\sqrt{1+\frac{b^2}{a^2}}}+C = \pi\ln\left(a^2\left(1+\sqrt{1+\frac{b^2}{a^2}}\right)^2\right)+C$$

$F(0)=\pi\ln a \Rightarrow$ $F = \pi\ln a + 2\pi\ln\left(1+\sqrt{1+\frac{b^2}{a^2}}\right) - 2\pi\ln 2$

(1f) $F(a,b)=\int_0^\infty xe^{-ax}\cos bx\,dx$ $\qquad a>0$, $b\in\mathbb{R}$

Nejdřív spočteme $G(a,b)=\int_0^\infty e^{-ax}\cos bx\,dx$

$$=\left[-\frac{e^{-ax}}{a}\cos bx\right]_0^\infty - \int_0^\infty \frac{be^{-ax}}{a}\sin bx\,dx$$

$$=\frac1a + \left[\frac{be^{-ax}}{a^2}\sin bx\right]_0^\infty - \int_0^\infty \frac{b^2e^{-ax}}{a^2}\cos bx\,dx$$

$$=\frac1a - \frac{b^2}{a^2}G(a,b) \Rightarrow G(a,b)=\frac{a}{a^2+b^2}$$

Na druhou stranu $-\frac{\partial G}{\partial a} = \int_0^\infty xe^{-ax}\cos bx\,dx = F(a,b)$

$$\Rightarrow F(a,b) = \frac{\partial}{\partial a}\frac{a}{a^2+b^2} = \frac{1}{a^2+b^2} - \frac{2a^2}{(a^2+b^2)^2} = \frac{b^2-a^2}{(a^2+b^2)^2}$$

(1g) $F(a)=\int_0^\infty e^{-ax}\frac{1-\cos x}{x}\,dx$ $\qquad a>0$

pro $a\in(\delta,\infty)$

$$\frac{\partial F}{\partial a} = -\int_0^\infty e^{-ax}(1-\cos x)\,dx$$ (majoranta $e^{-\delta x}(2)$)

$$=\left[\frac{e^{-ax}}{a}\right]_0^\infty + \frac{a}{a^2+1} = -\frac1a+\frac{a}{a^2+1} = -\frac{1}{a(a^2+1)}$$

$\Rightarrow F(a) = -\ln a + \frac12\ln(a^2+1)+C = \ln\frac{\sqrt{a^2+1}}{a}+C$ $\Big|$ $F(\infty)=0 \Rightarrow F=\ln\left(\frac{\sqrt{a^2+1}}{a}\right)$

20 $F(a) = \int_0^\infty e^{-ax} \frac{\sinh x}{x}\, dx \qquad a>1 \quad !$

$F'(a) = -\int_0^\infty e^{-ax} \sinh x\, dx$ majorante $a\in(1+\delta, \infty)$

$= +\int_0^\infty e^{-ax}\left(\frac{e^{-x}-e^{x}}{2}\right) dx = \frac12 \int_0^\infty e^{-x(a+1)} - e^{-x(a-1)}\, dx$

$= \frac12 \left[\frac{e^{-x(a+1)}}{-(a+1)} - \frac{e^{-x(a-1)}}{-(a-1)} \right]_0^\infty = \frac12 \left(\frac{1}{a+1} - \frac{1}{a-1} \right)$

$\Rightarrow F(a) = \frac12 \left(\ln(a+1) - \ln(a-1)\right) + C = \frac12 \ln\left(\frac{a+1}{a-1}\right) + C$

$F(\infty) = 0 \Rightarrow \quad F(a) = \frac12 \ln\left(\frac{a+1}{a-1}\right)$

21 $F(\alpha) = \int_0^1 x^{-\alpha} \ln^n x \qquad \alpha<1 \quad n\in\mathbb{N}$

$F'(\alpha) = -\int_0^1 x^{-\alpha} \ln^{n+1} x\, dx \qquad \alpha\in(-\infty, 1-\varepsilon)$

majorant $\boxed{\ln^{n+1} x \quad x^{-1+\varepsilon}}$

$\overset{PP}{=} -\left[\frac{x^{-\alpha+1}}{-\alpha+1} \ln^{n+1} x \right]_0^1 + \int_0^1 \frac{x^{-\alpha+1}}{-\alpha+1} \frac{(n+1)\ln^n x}{x} = \frac{n+1}{1-\alpha} F(\alpha)$

$\Rightarrow F'(\alpha) = \frac{n+1}{1-\alpha} F(\alpha) \quad \Leftrightarrow \quad \ln F(\alpha) = -(n+1)\ln(1-\alpha) + C$

$\Rightarrow F(\alpha) = \frac{1}{(1-\alpha)^{n+1}} + C$

$F(-\infty) = 0 \Rightarrow \quad F(\alpha) = \frac{1}{(1-\alpha)^{n+1}}$

22 $F(a,b) = \int_0^{\frac{\pi}{2}} \ln(a^2 + b^2 \operatorname{tg}^2 x)\, dx \qquad a,b\in\mathbb{R}, \quad F(a,0) = \frac{\pi}{2}\ln a^2$

$\frac{\partial F}{\partial b} = \int_0^{\frac{\pi}{2}} \frac{2b \operatorname{tg}^2 x}{a^2 + b^2 \operatorname{tg}^2 x}\, dx \overset{t=\operatorname{tg} x,\ dx = \frac{1}{1+t^2} dt}{=} \int_0^\infty \frac{2b\, t^2}{a^2+b^2t^2} \frac{1}{1+t^2}\, dt$

$= \frac{2b}{b^2-a^2} \int_0^\infty \frac{b^2}{a^2+b^2t^2} - \frac{1}{1+t^2}\, dt = \frac{2b}{b^2-a^2}\left[\frac{b}{a} \operatorname{arctg} \frac{b}{a} t - \operatorname{arctg} t \right]_0^\infty$

$= \frac{\pi}{2} \frac{2b(b-a)}{a(b^2-a^2)} = \pi \frac{b}{a} \frac{1}{a+b}$

$a, b > 0$

$$\frac{\partial F}{\partial b} = \pi \frac{b}{a} \frac{1}{a+b} = \frac{\pi}{a} - \frac{\pi}{a+b}$$

$$F = \frac{\pi}{a} - \pi \ln(a+b) + C \qquad F(0) = \pi \ln a$$

$$F(a) = 2\pi \ln a - \pi \ln(a+b) = \pi \ln\left(\frac{a^2}{a+b}\right)$$

23) $$F(a) = \int_{x,y>0} e^{-\left(x+y+\frac{a^3}{xy}\right)} x^{-1/3} y^{-2/3}\, dx\, dy \qquad a>0$$

$$F'(a) = -3\int_{x,y>0} e^{-\left(x+y+\frac{a^3}{xy}\right)} a^2 x^{-4/3} y^{-5/3}\, dx\, dy \qquad a > \varepsilon$$

(24) Define $F(a) = \int_0^{\frac{\pi}{2}} \ln(\sin ax)$

Put $F'(a) = \int_0^{\frac{\pi}{2}} \frac{x}{\operatorname{tg} ax}\,dx$ — zajímá nás $F'(1)$

$$F'(a) = \left[x \frac{\ln \sin ax}{a}\right]_0^{\frac{\pi}{2}} - \frac{1}{a}\int_0^{\frac{\pi}{2}} x \ln \sin ax$$

$$\Rightarrow aF'(a) = \frac{\pi}{2} \ln \sin a\frac{\pi}{2} - F(a)$$

$$= (aF)' = \frac{\pi}{2} \ln \sin \frac{\pi a}{2}$$

$$aF = \int \frac{\pi}{2} \ln \sin \frac{\pi a}{2}$$

(25) $\int_0^{\infty} \frac{\sin^4 x}{x^2}\,dx$, nejdříve trik na pojednání

$$\sin^4 x = \sin^2 x(1-\cos^2 x) = \sin^2 x - \frac{1}{4}\sin^2 2x$$

$$\Rightarrow \int_0^{\infty} \frac{\sin^4 x}{x^2}\,dx = \int_0^{\infty} \frac{\sin^2 x}{x^2} - \frac{2}{4}\int_0^{\infty} \frac{\sin^2 2x}{(2x)^2}\,2dx = \frac{1}{2}\int_0^{\infty} \frac{\sin^2 x}{x^2}\,dx$$

$2x = y$

Def $F(a) = \frac{1}{2}\int_0^{\infty} \frac{e^{-xa}\sin^2 x}{x^2}\,dx$ — F je spojitá na $[0,\infty)$, $F(\infty) = 0$, $F'(a) \to 0$ pro $a \to \infty$

$$F'(a) = -\frac{1}{2}\int_0^{\infty} \frac{e^{-ax}\sin^2 x}{x}$$

$$F''(a) = \frac{1}{2}\int_0^{\infty} e^{-ax}\sin^2 x\,dx = \frac{1}{4}\int_0^{\infty} e^{-ax}(1-\cos 2x)\,dx$$

$$= \left[-\frac{e^{-ax}}{4a}\right]_0^{\infty} - \frac{1}{4}\int_0^{\infty} e^{-ax}\cos 2x = \frac{1}{4a} - \frac{1}{4}\int_0^{\infty} e^{-ax}\cos 2x$$

$$I = \int_0^{\infty} e^{-ax}\cos 2x = \left[-\frac{e^{-ax}}{a}\cos 2x\right]_0^{\infty} + 2\int_0^{\infty} -\frac{e^{-ax}}{a}\sin 2x$$

$$= \frac{1}{a} + \left[\frac{2e^{-ax}}{a^2}\sin 2x\right]_0^{\infty} - \int_0^{\infty} \frac{4e^{-ax}}{a^2}\cos 2x$$

$$= \frac{1}{a} - \frac{4}{a^2} I \Rightarrow I\left(\frac{4+a^2}{a^2}\right) = \frac{1}{a} \Rightarrow I = \frac{a}{4+a^2}$$

$$\Rightarrow F''(a) = \frac{1}{4a} - \frac{2a}{8(4+a^2)} \Rightarrow F'(a) = \frac{1}{4}\ln a - \frac{1}{8}\ln(4+a^2) + C$$

$$= \frac{1}{8}\ln\left(\frac{a^2}{a^2+4}\right) + C \Rightarrow C = 0$$

$$F'(a) = \frac{1}{4}\ln a - \frac{1}{8}\ln(4+a^2) + C$$

$$= \frac{1}{8}\ln\left(\frac{a^2}{a^2+4}\right) + C \Rightarrow C=0$$

$$F(a) = \int \frac{1}{4}\ln a - \frac{1}{8}\ln(4+a^2) = \frac{1}{4}\, a\ln a - \frac{a}{4} - \frac{1}{8}\, a\ln(4+a^2)$$

$$+\frac{1}{4}\int \frac{a^2}{4+a^2} = \frac{a}{4}(\ln a - 1) - \frac{1}{8}a\ln(4+a^2)$$

$$+\frac{1}{4}a - \frac{1}{2}\arctan\left(\frac{a}{2}\right) + C$$

$$F(\infty) = 0 \rightarrow C = \frac{\pi}{4}$$

$$= \frac{a}{8}\ln\left(\frac{a^2}{a^2+4}\right) - \frac{1}{2}\arctan\frac{a}{2} + C$$

$$F(a) = \frac{a}{8}\ln\left(\frac{a^2}{a^2+4}\right) - \frac{1}{2}\arctan\frac{a}{2} + \frac{\pi}{4}$$

$$\Rightarrow \int_0^\infty \frac{\sin^4 x}{x^2}\,dx = F(0) = \underline{\underline{\frac{\pi}{4}}}$$

KŘIVKOVÝ INTEGRÁL

S - křivka v $\mathbb{R}^n$ $\quad S \subseteq \mathbb{R}^n$ $\quad \exists\, \varphi_i,\ i=1,\dots,n$ a interval (a,b), $\vec{\varphi}(t)$ je prosté zobrazení do $\mathbb{R}^n$

$$S = \{ (x_1,\dots,x_n) \in \mathbb{R}^n ;\ x_i(t) = \varphi_i(t),\ t \in (a,b) \}$$

orientace - směr vektorů po křivce „jdeme"

INTEGRÁL 1. druhu

$$\int_C f(x)\,ds := \int_a^b f(x(t)) \sqrt{\sum (\varphi_i'(t))^2}\, dt$$

(nezávisí na orientaci)

INTEGRÁL II. druhu

$$\int_C \vec{f} \cdot ds = \int_C \sum_i f_i(x)\, dx_i := \int_a^b \vec{f}(t) \cdot \vec{\varphi}'(t)\, dt$$

POZOR!! závisí na orientaci

PROSTOTA křivky, když $\sum (\varphi'(t))^2 > 0$ v (a,b) !!

① Nakreslete si sami :)

② rovina $x+y+z=a$ koule $x^2+y^2+z^2=R^2$

$\Downarrow$

$z=a-x-y \implies x^2+y^2+(a-x-y)^2=R^2$

$2x^2+2y^2-2ax-2ay=R^2-a^2$

$\left(x-\frac{a}{2}\right)^2+\left(y-\frac{a}{2}\right)^2 = \frac{R^2-a^2-\frac{a^2}{2}}{2} = \frac{R^2-\frac{3}{2}a^2}{2}$ $\vee$ 0

$x=\frac{a}{2}+\sqrt{\frac{R^2-\frac{3}{2}a^2}{2}}\cos t$

$y=\frac{a}{2}+$ —,,— $\sin t$

$z=-\sqrt{\frac{R^2-\frac32 a^2}{2}}\,(\cos t+\sin t)$

$t\in[0,2\pi)$

③ Kužel $z^2=(x^2+y^2)$

rovina: $m_1x+m_2y+m_3z=A$ $\quad \vec{m}=(m_1,m_2,m_3)$ - normála

a) $\vec{m}=(0,0,1) \implies x^2+y^2=A^2$ (kružnice)

$x=A\cos t$
$y=A\sin t$
$z=A$

b) $\vec{m}=(\sin\varphi,0,\cos\varphi) \implies \varphi\neq 0$

$\sin\varphi\, x+\cos\varphi\, z=A$

$\implies \sin^2\varphi\, x^2=A^2+\cos^2\varphi\, z^2-2A\cos\varphi\, z$

$y^2=z^2-x^2=z^2(1-\cot^2\varphi)+\frac{2A\cos\varphi}{\sin^2\varphi}z-A^2$

$\varphi=\frac{\pi}{4} \implies y^2=\sqrt{2}Az-A^2 \to$ PARABOLA

$\varphi>\frac{\pi}{4}$ $\quad y^2=\left(\frac{z}{\sin\varphi}\sqrt{\sin^2-\cos^2\varphi}+\frac{A\cos\varphi}{\sin\varphi\sqrt{\ }}\right)^2-A^2\left(\frac{\sin^2|\sin^2-\cos^2|+\cos}{\sin^2(\sin^2-\cos^2)}\right)$

hyperbola

$\varphi<\frac{\pi}{4} \implies y^2=-\left(\frac{z}{\sin\varphi}\sqrt{\cos^2-\sin^2}-\frac{A\cos\varphi}{\sin\varphi\sqrt{\ }}\right)^2+A^2\left(\frac{\cos^2\varphi}{\sin^2(\cos^2-\sin^2)}-1\right)$

elipsa

(4) sami

(5) $x = t$ $\quad \vec{\varphi}(t) = (t, \ln t)$ $\quad t \in (1,2)$ (nezáleží na okraji.)

$y = \ln t$ $\quad \vec{\varphi}'(t) = (1, \frac{1}{t})$ $\Rightarrow \|\varphi'(t)\| = \sqrt{\frac{t^2+1}{t^2}}$

$$\int_C x^2 ds = \int_1^2 t^2 \sqrt{\frac{t^2+1}{t^2}}\, dt = \int_1^2 t\sqrt{t^2+1}\, dt = \left[\frac{1}{3}(t^2+1)^{3/2}\right]_1^2$$

$$= \frac{1}{3}\left(5^{3/2} - 2^{3/2}\right)$$

(6) $C: \; x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ $\quad x = a(\cos t)^3$ $\quad t \in (0, 2\pi)$

$y = a(\sin t)^3$

$dx = -3a \cos^2 t \sin t$

$dy = 3a \sin^2 t \cos t$

$ds = 3a \left(\cos^4 t \sin^2 t + \sin^4 t \cos^2 t \right)^{1/2}$

Díky symetrii C a integrandu $\quad = 3a \; |\sin t \cos t|$

stačí integrovat pouze $t \in (0, \frac{\pi}{2})$

$$\int_C \left(x^{\frac{4}{3}} + y^{\frac{4}{3}}\right) ds = 4 \int_0^{\frac{\pi}{2}} a^{\frac{4}{3}} (\cos^4 t + \sin^4 t) \cdot 3a \sin t \cos t \, dt$$

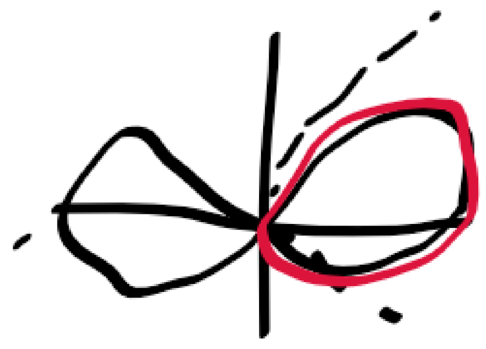
$$= 12\, a^{\frac{4}{3}} \int_0^{\frac{\pi}{2}} \cos^5 t \sin t + \sin^5 t \cos t \; dt$$

$$2a^{\frac{4}{3}} \left[-\cos^6 + \sin^6 t \right]_0^{\frac{\pi}{2}} = 4a^{\frac{4}{3}}$$

(7) $\int_C |y| ds$ $\quad C: \; (x^2+y^2)^2 = a^2(x^2 - y^2)$

$\left.\begin{array}{l} x = ar\cos t \\ y = ar\sin t \end{array}\right\}$ $a^4 r^4 = a^4 r^2 (\cos^2 t - \sin^2 t)$

$r^2 = \cos 2t \Rightarrow t \in (-\frac{\pi}{4}, \frac{\pi}{4})$ $\cup (\frac{3\pi}{4}, \frac{5\pi}{4})$



stačí integrovat přes 

tj $t \in (-\frac{\pi}{4}, \frac{\pi}{4})$ $\Rightarrow$ $x = a\sqrt{\cos 2t}\, \cos t$

$y = a\sqrt{\cos 2t}\, \sin t$

$dx = -\frac{a\cos t}{\sqrt{\cos 2t}} \sin 2t - a\sqrt{\cos 2t}\, \sin t$

$dy = -\frac{a \sin t}{\sqrt{\cos 2t}} \sin 2t + a\sqrt{\cos 2t}\, \cos t$

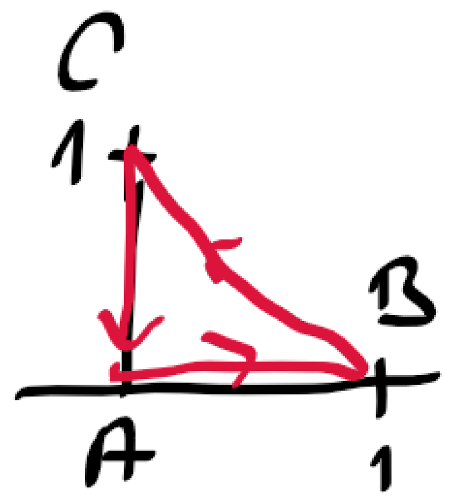
$$dx = \left(-\frac{a\cos t}{\sqrt{\cos 2t}}\sin 2t - a\sqrt{\cos 2t}\,\sin t\right)dt$$
$$dy = \left(-\frac{a\sin t}{\sqrt{\cos 2t}}\sin 2t + a\sqrt{\cos 2t}\,\cos t\right)dt$$

$$ds = a\left(\frac{\cos^2 t\,\sin^2 2t}{\cos 2t} + \sin^2 t\cos 2t + \sin^2 2t + \frac{\sin^2 t\,\sin^2 2t}{\cos 2t} + \cos 2t\cos^2 t - \sin^2 2t\right)^{1/2} dt$$

$$= a\left(\frac{1}{\cos 2t}\right)^{1/2} dt$$

$$\int_C |y|\,ds = 2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} a^2\sqrt{\cos 2t}\,|\sin t|\,\frac{1}{\sqrt{\cos 2t}}\,dt = 4a^2\int_0^{\frac{\pi}{4}}\sin t\,dt = 4a^2\left(1-\frac{\sqrt{2}}{2}\right)$$

8



C – krivka

$$\oint_C \vec{j}\cdot\vec{ds} = \int_A^B \vec{j}\cdot\vec{ds} + \int_B^C \vec{j}\cdot\vec{ds} + \int_C^A \vec{j}\cdot\vec{ds}$$

kde $\vec{j} = \left[(x^2+y^2),\ (x^2-y^2)\right]$

krivka: $x = t$, $y = 0$, $t\in(0,1)$

$dx = dt$

$dy = 0\,dt$

$$\int_A^B \vec{j}\cdot\vec{ds} = \int_0^1 t^2\,dt = \frac{1}{3}$$

krivka $x = 1-t$, $y = t$, $t\in(0,1)$

$dx = -dt$

$dy = dt$

$$\int_B^C \vec{j}\cdot\vec{ds} = \int_0^1 -\left((1-t)^2+t^2\right)dt + \left((1-t)^2-t^2\right)dt = -2\int_0^1 t^2\,dt = -\frac{2}{3}$$

krivka: $x = 0$, $y = (1-t)$, $t\in(0,1)$

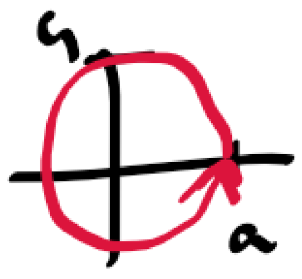
$dx = 0\,dt$

$dy = -dt$

$$\int_C^A \vec{j}\cdot\vec{ds} = \int_0^1 (1-t)^2\,dt = \left[\frac{-(1-t)^3}{3}\right]_0^1 = \frac{1}{3}$$

$$\Rightarrow \oint_C \vec{j}\cdot\vec{ds} = 0$$

(9) $\int_C \frac{(x+y)\,dx - (x-y)\,dy}{x^2+y^2} = \left(\int_C \frac{(x+y,\ x-y)}{x^2+y^2} \cdot d\vec{\varphi} \right)$



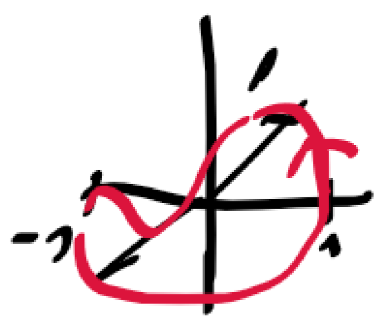
$x = a\cos t$ $\quad t \in \langle 0, 2\pi)$ $\quad dx = -a\sin t\, dt$

$y = a\sin t$ $\quad dy = a\cos t\, dt$

$$\int_C \dots = \int_0^{2\pi} \left[-(\cos t + \sin t)\sin t - (\cos t - \sin t)\cos t \right] dt$$

$$= \int_0^{2\pi} -\cos t \sin t - \sin^2 t - \cos^2 t + \sin t \cos t \; dt = -2\pi$$

(10) $\int_C y\,dx + z\,dy + x\,dz$



$x = \cos t$ $\quad t \in \langle 0, 2\pi)$ $\quad dx = -\sin t\, dt$

$y = \sin t$ $\quad dy = \cos t\, dt$

$z = \cos t \sin t$ $\quad dz = \cos 2t\, dt$

Popis $\vec{\varphi}(t) = (\cos t, \sin t, \cos t \sin t) \Rightarrow \vec{\varphi}'(t) = (-\sin t, \cos t, \cos 2t)$

$$\int_C = \int_0^{2\pi} -\sin^2 t\, dt + \cos^2 t \sin t\, dt + \cos t \cos 2t\, dt$$

$$= -\pi + \int_0^{2\pi} \cos t (1 - 2\sin^2 t)\, dt = -\pi$$

Další příklady jsou založeny na pozorování, že v případě, kdy $\vec{f}$ má potenciál, se křivkový integrál spočítá "lehce"

C – křivka, A – počáteční bod, B – koncový bod $\quad \vec{\varphi}(0) = \vec{A}$, $\vec{\varphi}(b) = \vec{B}$ $\quad \vec{\varphi} = C$

$$\int_C \nabla U(x_1, \dots, x_n) \cdot d\vec{s} = \int_0^b \nabla U(\vec{\varphi}(t)) \cdot \vec{\varphi}'(t)\, dt = \int_0^b \frac{d}{dt}\left(U(\vec{\varphi}(t))\right) dt$$

$$= U(\vec{\varphi}(b)) - U(\vec{\varphi}(a)) = U(B) - U(A)$$

musí platit $U \in C^1$ možná "C^1"?

(11) $$\int_C f(x^2+y^2+z^2)(x\,dx+y\,dy+z\,dz) =$$

$$= \frac{1}{2}\int_C \nabla f(x^2+z^2+y^2)\cdot \vec{ds}$$

$\vec{ds} = (dx, dy, dz)$

pokud C je uzavřená a $\vec{A}=(x_0,y_0,z_0)\in C$

pak $\int_C = \frac{1}{2} f(x_0^2+y_0^2+z_0^2) - \frac{1}{2} f(x_0^2+y_0^2+z_0^2) = \underline{\underline{0}}$

(12) $U := \frac{x^5}{5} - y^5 + 2x^2y^3 \Rightarrow \nabla U = (x^4+4xy^3,\ 6x^2y^2-5y^4)$

$\Rightarrow \int_A^B (x^4+4xy^3)\,dx + (6x^2y^2-5y^4)\,dy = \int_A^B \nabla U\cdot\vec{ds}$ (INTEGRÁL NEZÁVISLÝ NA CESTĚ!)

$= U(B)-U(A) = \frac{3^5}{5}+\frac{2^5}{5}-1+8$

(13) $U := \left(x^3 - \frac{1}{x} + y^3 - \frac{1}{y} + x^2y^2 + \frac{x^2}{y^2}\right)$

pak $\int_A^B \left(2xy^2+3x^2+\frac{1}{x^2}+\frac{2x}{y}\right)dx + \left(2x^2y+3y^2+\frac{1}{y^2}-\frac{2x^2}{y^3}\right)dy =$

$= \int_A^B \nabla U\cdot\vec{ds} = U(B)-U(A) = 2^3-\frac{1}{2}+4+\frac{1}{4}-2^3+\frac{1}{2}-4-4$

$= -4+\frac{1}{4}$

ALE JEN DÍKY TOMU ŽE C se nachází v I. kvadrantu, kde je $U\in C^1$

(14) $$\int_A^B \frac{x\,dx+y\,dy+z\,dz}{\sqrt{x^2+y^2+z^2}} = \int_A^B \nabla\left(\sqrt{x^2+y^2+z^2}\right)\cdot\vec{ds}$$

$B=(0,b,0)$, $A=(0,0,a)$

$= \sqrt{0^2+b^2+0^2} - \sqrt{0^2+0^2+a^2} = |b|-|a|$

ale C musí projít mimo počátek!

(15)

drát C $x = a\cos t$, $y = a\sin t$, $z = bt$, $t \in \langle 0, 2\pi)$, hustota $\mu(x,y,z) = k(x^2+y^2+z^2)$

$\vec{\varphi}(t) = (a\cos t, a\sin t, bt)$

$M = \int_C \mu \, ds$

$\vec{\varphi}'(t) = (-a\sin t, a\cos t, b)$

$|\vec{\varphi}'(t)| = \sqrt{a^2+b^2}$

$$= \int_0^{2\pi} k(a^2\cos^2 t + a^2\sin^2 t + b^2t^2)\sqrt{a^2+b^2}\, dt = \int_0^{2\pi} ka^2\sqrt{a^2+b^2} + kb^2t^2\sqrt{a^2+b^2}$$

$$= 2\pi k\sqrt{a^2+b^2}\, a^2 + \frac{8\pi^3}{3} k b^2 \sqrt{a^2+b^2}$$

(16) C: $x = a\cos t$, $y = a\sin t$, $t \in \langle 0, 2\alpha)$ — konstantní - homogenní drát

Hledám $T \in \mathbb{R}^2$ aby $\int_C [(x,y) - \vec{T}]\, \rho \, ds = 0$

$dx = -a\sin t\, dt$, $dy = a\cos t$, $ds = a\,dt$

$$\Rightarrow 0 = \int_0^{2\alpha} [(a\cos t - T_x), (a\sin t - T_y)]\, \rho a\, dt =$$

$$= a\rho \left([a\sin t - tT_x]_0^{2\alpha}, [-a\cos t - tT_y]_0^{2\alpha} \right)$$

$$= a\rho \left(a\sin 2\alpha - 2\alpha T_x, a(1-\cos 2\alpha) - 2\alpha T_y \right)$$

$$\Rightarrow \vec{T} = \left(\frac{a\sin 2\alpha}{2\alpha}, \frac{a(1-\cos 2\alpha)}{2\alpha} \right)$$

(17) Půlkružnice C: $x = R\cos t$, $y = R\sin t$, $t \in \langle 0, \pi)$ délka $C = \pi R$

hmotnost M

hustota $\frac{M}{\pi R} = \rho$

Grav. konst.

$$\text{síla } \vec{F} = \varkappa \int_C \frac{(x,y)}{\|(x,y)\|^3} m \rho \, ds = \int_0^{\pi} \frac{(R\cos t, R\sin t)}{R^3} \rho m \varkappa \underbrace{R\,dt}_{ds}$$

$$= \frac{M\varkappa m}{\pi R^2} \int_0^{\pi} (\cos t, \sin t)\, dt = \left(0, \frac{2mM\varkappa}{\pi R^2}\right)$$

PLOŠNÝ INTEGRÁL I. druhu

(A) 2-plochy ve 3d (náš svět)

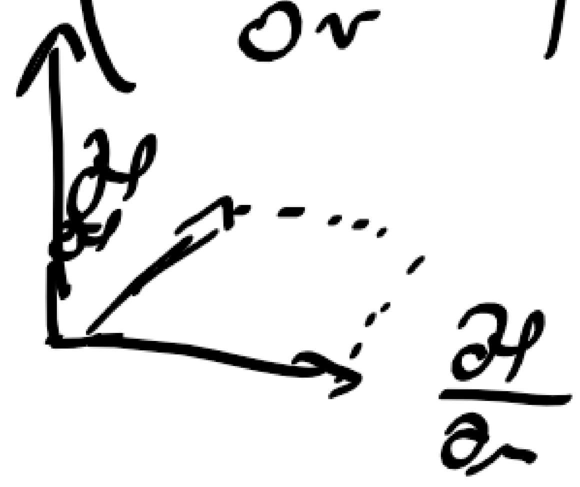
plocha S popsána $\vec{\varphi}(u,v)$ $u,v \in M \subset \mathbb{R}^2$

$\vec{\varphi} \in C^1$ – prostá

$\frac{\partial \vec{\varphi}}{\partial u}$ – tečný vektor

$\frac{\partial \vec{\varphi}}{\partial v}$ – tečný vektor

$J = \begin{pmatrix} \frac{\partial \vec{\varphi}}{\partial u} \\ \frac{\partial \vec{\varphi}}{\partial v} \end{pmatrix}$ musí mít hodnost $\boxed{2}$



$$dS = \left| \frac{\partial \vec{\varphi}}{\partial u} \times \frac{\partial \vec{\varphi}}{\partial v} \right| du\,dv$$

$$\Rightarrow \int_S f(x,y,z)\,dS = \int_M f(\vec{\varphi}(u,v)) \left| \frac{\partial \vec{\varphi}}{\partial u} \times \frac{\partial \vec{\varphi}}{\partial v} \right| du\,dv$$

$$\frac{\partial \vec{\varphi}}{\partial u} \times \frac{\partial \vec{\varphi}}{\partial v} := \left(\frac{\partial \varphi_2}{\partial u}\frac{\partial \varphi_3}{\partial v} - \frac{\partial \varphi_3}{\partial u}\frac{\partial \varphi_2}{\partial v},\ \frac{\partial \varphi_3}{\partial u}\frac{\partial \varphi_1}{\partial v} - \frac{\partial \varphi_1}{\partial u}\frac{\partial \varphi_3}{\partial v},\ \frac{\partial \varphi_1}{\partial u}\frac{\partial \varphi_2}{\partial v} - \frac{\partial \varphi_2}{\partial u}\frac{\partial \varphi_1}{\partial v}\right.$$

OBECNĚ k-plochy v N dimenzích $k < N$

$M \subset \mathbb{R}^k$ $\vec{\varphi}: M \to \mathbb{R}^n$ (parametrizace plochy)

$\vec{y} \mapsto \vec{\varphi}(\vec{y})$

Definuj Grammovu matici $G_{ij} := \frac{\partial \vec{\varphi}}{\partial y_i} \cdot \frac{\partial \vec{\varphi}}{\partial y_j}$ (symetrická $k \times k$ matice)

$$dS = \sqrt{\det G}\, dy_1 \dots dy_k$$

$$\int_S f(x)\,dS = \int_M f(\vec{\varphi}(\vec{y})) \sqrt{\det G}\, dy$$

Př: $k=2,\ n=3$ $G = \begin{pmatrix} \left|\frac{\partial \varphi}{\partial u}\right|^2 & \frac{\partial \vec{\varphi}}{\partial u}\cdot\frac{\partial \vec{\varphi}}{\partial v} \\ & \left|\frac{\partial \varphi}{\partial v}\right|^2 \end{pmatrix}$

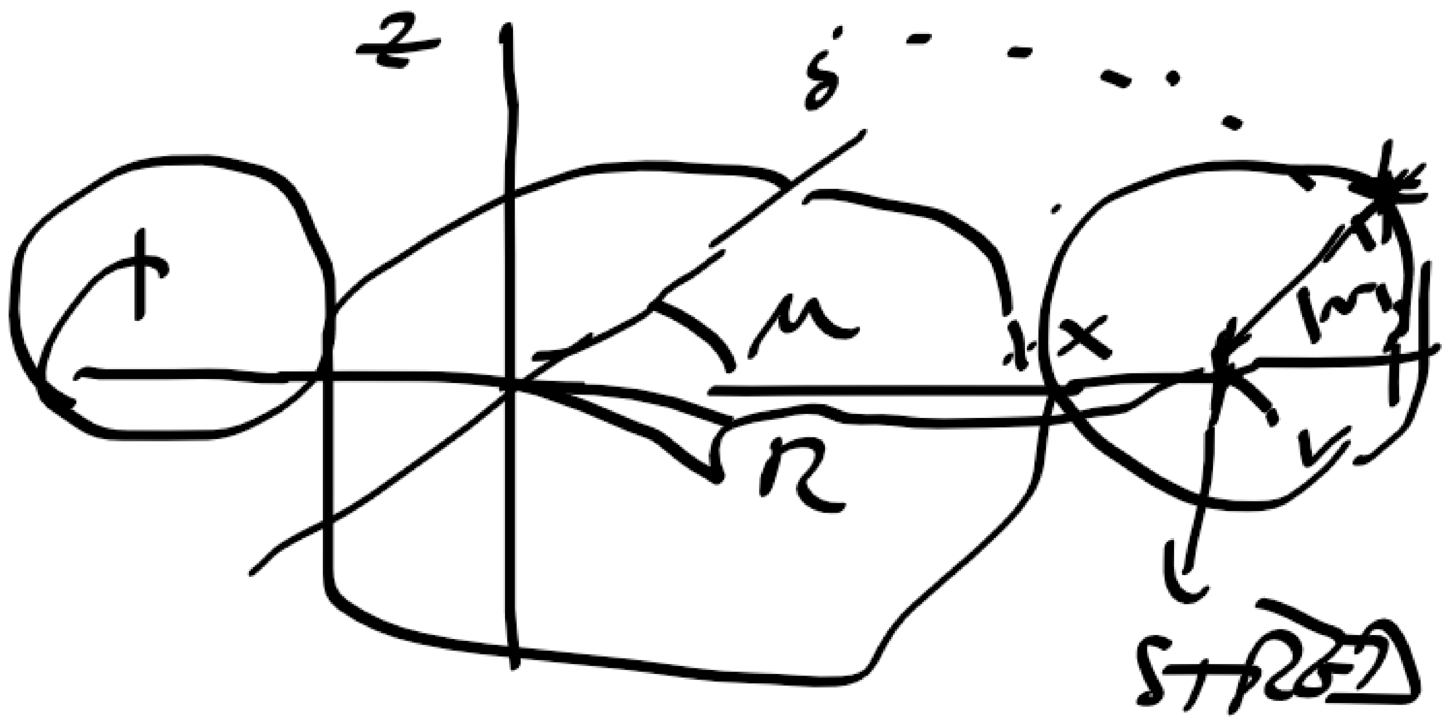
$$dS = \sqrt{\left|\frac{\partial \varphi}{\partial u}\right|^2 \left|\frac{\partial \varphi}{\partial v}\right|^2 - \left(\frac{\partial \varphi}{\partial u}\cdot\frac{\partial \varphi}{\partial v}\right)^2}$$

Př: $\vec{\varphi}(u,v) = (u, v, h(u,v))$; $\frac{\partial \vec{\varphi}}{\partial u} = \left(1, 0, \frac{\partial h}{\partial u}\right)$, $\frac{\partial \vec{\varphi}}{\partial v} = \left(0, 1, \frac{\partial h}{\partial v}\right)$

$$\frac{\partial \vec{\varphi}}{\partial u} \times \frac{\partial \vec{\varphi}}{\partial v} = \left(-\frac{\partial h}{\partial u}, -\frac{\partial h}{\partial v}, 1\right)$$

$$|\cdot| = \sqrt{1 + |\nabla h|^2}$$

(1) TORUS



$R > r$

STŘED $S_x = (R+r)\cos u$
$S_y = (R+r)\sin u$
$S_z = 0$

$\vec{\varphi}$: $\varphi_1 = (R + r\cos v)\cos u$
$\varphi_2 = (R + r\cos v)\sin u$
$\varphi_3 = r\sin v$

(2) PODOBNĚ JAKO TORUS; JEN MUSÍME "PŘEKROUTIT" STRANY

$x = (R + r\cos\frac{u}{2})\cos u$ $\quad r \in [-a, a] \quad a < R$
$y = (R + r\cos\frac{u}{2})\cos u$ $\quad u \in [0, 2\pi)$
$z = r\sin\frac{u}{2}$

Pozor! normál orientaci!

(3) POVRCH JE POTŘEBA ROZDĚLIT NA 6 PLOCH!
KAŽDÁ NA "HRANÁ" A TAK MŮŽEME TĚŽKO POPSAT, ČI ZOBRAZIT

(4) $x = a|\cos\varphi\cos\psi|^{\frac{2}{a}}\ \mathrm{sign}(\cos\varphi)\ \mathrm{sign}(\cos\psi)$
$y = a|\cos\varphi\sin\psi|^{\frac{2}{a}}\ \mathrm{sign}(\cos\varphi)\ \mathrm{sign}(\sin\psi)$
$z = a|\sin\varphi|^{\frac{2}{a}}\ \mathrm{sign}(\sin\varphi)$

(5) $S = S_1 \cup S_2 \cup S_3 + E$ $\quad$ E-hrany, musíme zvolení! Mám rád

$S_1 =$ $x = r\cos t$ $\quad t \in (0, 2\pi)$
$y = r\sin t$ $\quad u \in (-a, a)$
$z = u$

$S_2 =$ $x = u\cos t$ $\quad u \in (0, r)$
$y = u\sin t$ $\quad t \in (0, 2\pi)$
$z = a$

S_3 $x = u\cos t$
$y = u\sin t$
$z = -a$

(6) $S:$ $x = u$ $\quad u \in (0, 1)$
$y = v$ $\quad v \in (0, 1-u)$
$z = \sqrt{2uv}$

$$dS = \sqrt{1 + \left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2}\, du\, dv = \sqrt{1 + \frac{v}{2u} + \frac{u}{2v}}\; du\, dv$$

$$|S| = \int_0^1 \int_0^{1-u} \sqrt{1 + \frac{v}{2u} + \frac{u}{2v}}\; dv\, du = \int_0^1 \int_0^{1-u} \frac{u+v}{\sqrt{2uv}}\, du\, dv = \int_0^1 \int_0^{1-u} \left(\frac{\sqrt{u}}{\sqrt{2v}} + \frac{\sqrt{v}}{\sqrt{2u}}\right) dv\, du$$

$$= 2\int_0^1 \left[\frac{2}{\sqrt{2}}\sqrt{uv} \quad \dots\right]_0^{1-u} du = 2\int_0^1 \frac{2}{\sqrt{2}}\sqrt{u(1-u)}$$

$$= 2\int_0^1 (1-u)\left(\frac{2}{\sqrt{2}}\sqrt{\frac{u}{1-u}}\right) du$$

$$\frac{u}{1-u} = t^2 \qquad t \in (0, \infty) \qquad u = \frac{t^2}{1+t^2} \qquad du = \frac{2t}{(1+t^2)^2}\, dt$$

$$= 2\int_0^1 \frac{1}{1+t^2}\left(\frac{2}{\sqrt{2}}\, t \quad \cdot\right) \frac{2t}{(1+t^2)^2}\, dt$$

$$= \int_0^1 \frac{8}{\sqrt{2}} \frac{t^2}{(1+t^2)^3}\, dt = -\frac{2}{\sqrt{2}} \int_0^1 t\left((1+t^2)^{-2}\right)' dt = -\frac{2}{\sqrt{2}\cdot 4}$$

$$+ \frac{2}{\sqrt{2}} \int_0^1 \frac{1}{(1+t^2)^2}\, dt = -\frac{\sqrt{2}}{4} + \sqrt{2}\int_0^1 \frac{1}{(1+t^2)} - \sqrt{2}\int_0^1 \frac{t^2}{(1+t^2)^2}$$

$$= -\frac{\sqrt{2}}{4} + \sqrt{2} + \frac{\sqrt{2}}{2}\int_0^1 t\left(\frac{1}{(1+t^2)}\right)' = -\frac{\sqrt{2}}{4} + \sqrt{2} + \frac{\sqrt{2}}{8} - \frac{\sqrt{2}}{2} = \sqrt{2}\left(\frac{1}{2} - \frac{1}{8}\right)$$

7 a) $x = 1 + r\cos t$

$y = r\sin t$

$z = \sqrt{r^2\cos^2 t + r^2\sin^2 t + 1 + 2r\cos t}$

$r \in (0,1)$

$t \in (0, 2\pi)$

$$\frac{\partial \varphi}{\partial r} = \frac{\partial}{\partial r}\left(1 + r\cos t, r\sin t, \sqrt{}\right)$$

$$= \left(\cos t, \sin t, \frac{1}{2}\frac{2r + 2\cos t}{\sqrt{}}\right)$$

$$\frac{\partial \varphi}{\partial t} = \left(-r\sin t, r\cos t, \frac{-2r\sin t}{2\sqrt{}}\right)$$

$$\frac{\partial \varphi}{\partial r} \times \frac{\partial \varphi}{\partial t} = \left(\frac{-r\sin^2 t}{\sqrt{}} - \frac{r^2\cos t + r\cos^2 t}{\sqrt{}}, \frac{-r^2\sin t - r\sin t\cos t}{\sqrt{}} + \frac{r\sin t\cos t}{\sqrt{}}, r\right)$$

$$\left|\frac{\partial \varphi}{\partial r} \times \frac{\partial \varphi}{\partial t}\right| = \sqrt{\frac{(r + r^2\cos t)^2}{r^2 + 1 + 2r\cos t} + \frac{r^4\sin^2}{r^2 + 1 + 2r\cos t} + r^2} = \sqrt{2r^2} = r\sqrt{2}$$

$$S = \int_0^1\int_0^{2\pi} r\sqrt{2}\, dt\, d\varphi = \pi\sqrt{2}$$

b) Gramova matice $G = \begin{pmatrix} 1 + \frac{(r + \cos t)^2}{r^2 + 1 + 2r\cos t} & + \frac{(r\cos t)\sin t}{r^2 + 1 + 2r\cos t} \\ & r^2 + \frac{r^2\sin^2 t}{r^2 + 1 + 2r\cos t} \end{pmatrix}$

$\sqrt{\det G}$ = atd..

c) $M \subset \mathbb{R}$ $\quad M = x^2 + y^2 \leq 2x$

$$\vec{\varphi} = \left(x, y, \sqrt{x^2 + y^2}\right) \qquad ds = \sqrt{1 + |\nabla\sqrt{x^2+y^2}|^2}\, dx\, dy$$

$$= \sqrt{2}\, dx\, dy$$

$$S = \int_M \sqrt{2}\, dx\, dy = \pi\sqrt{2}$$

$\uparrow$ kruh o poloměru 1.

(8)

$x = \rho\cos\varphi$ $\qquad \rho \in (0, a)$

$y = \rho\sin\varphi$ $\qquad \varphi \in (0, 2\pi)$

$z = h\varphi$

$$\frac{\partial\varphi}{\partial\rho} = (\cos\varphi, \sin\varphi, 0)$$

$$\frac{\partial\varphi}{\partial\varphi} = (-\rho\sin\varphi, \rho\cos\varphi, h)$$

$$\frac{\partial\varphi}{\partial\rho} \times \frac{\partial\varphi}{\partial\varphi} = (h\sin\varphi, -h\cos\varphi, \rho)$$

$$|\partial \ldots| = \sqrt{h^2+\rho^2}$$

$$S = \int_0^a\int_0^{2\pi} \sqrt{h^2+\rho^2}\, d\rho\, d\varphi$$

$$= 2\pi h \int_0^a \sqrt{1+\left(\frac{\rho}{h}\right)^2}\, d\rho$$

$\rho = h\sinh t \qquad \operatorname{argsinh}\frac{a}{h}$

$$= 2\pi h^2 \int_0 \cosh^2 t\, dt$$

$$\pi h^2 \left[\frac{e^{2t} - e^{-2t} + 4t}{4}\right]$$

$$= \pi h^2 \left[\frac{e^{2t} - 2 + e^{-2t}}{4} + \frac{4t-2}{4}\right]$$

$$= \pi h^2 \left[(\sinh t)^2 + t\right]$$

$$\pi a^2 + \pi h^2 + \pi \operatorname{argsinh}\frac{a}{h}$$

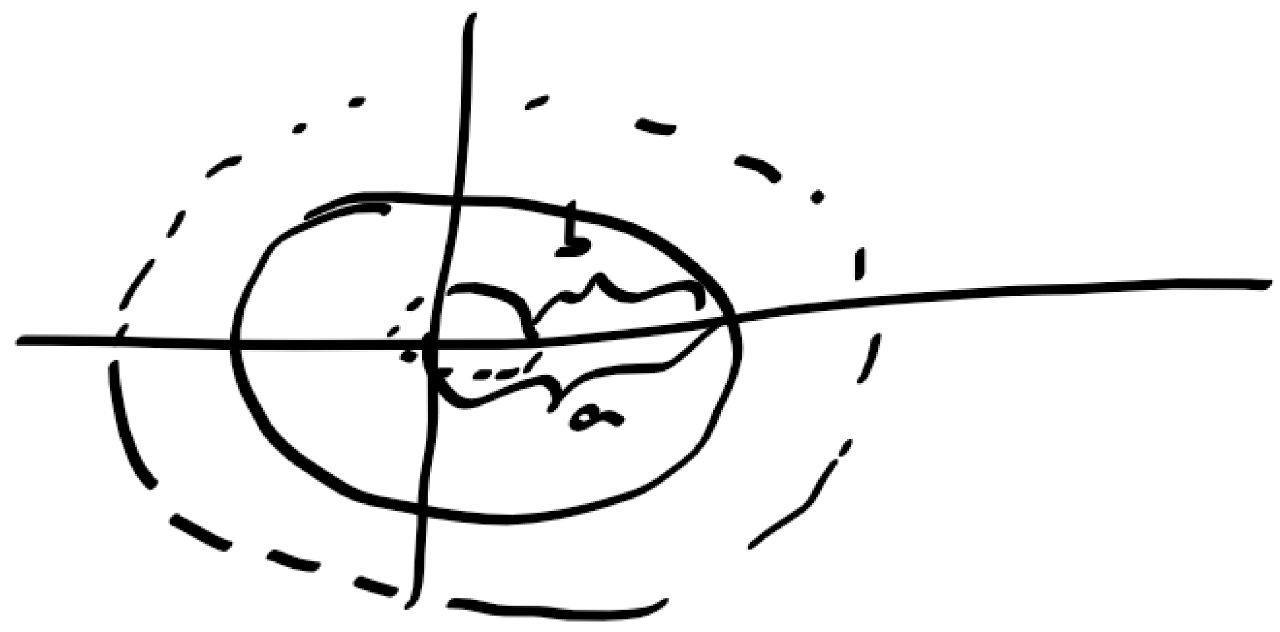
(9) analytický Torus – ve směru $\qquad a > b$

1) parametricky $\quad x = (a + b\cos u)\cos v$

$y = (a + b\cos u)\sin v$

$z = b\sin u$

$u, v \in (0, 2\pi)$



2) rovnicí $\quad M = B_{a+b} \setminus B_{a-b}$

$x = u \qquad (u, v) \in M$

$y = v$

$z = \sqrt{b^2 - (\sqrt{x^2+y^2} - a)^2}$

půlka toru

ad 1) $\dfrac{\partial\varphi}{\partial u} = (-b\sin u\cos v, -b\sin u\sin v, b\cos u)$

$\dfrac{\partial\varphi}{\partial v} = (-(a+b\cos u)\sin v, (a+b\cos u)\cos v, 0$

$$\frac{\partial f}{\partial u} = \left(-b\sin u \cos v, -b\sin u \sin v, b\cos u \right)$$

$$\frac{\partial f}{\partial v} = \left(-(a+b\cos u)\sin v, (a+b\cos u)\cos v, 0 \right)$$

$$\frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v} = \Big(-b(a+b\cos u)\cos u \cos v, -b\cos u \sin v (a+b\cos u), b(a+b\cos u)\sin u \Big)$$

$$\left| \; \cdot \; \right| = b(a+b\cos u)$$

$$\boxed{S = \int_0^{2\pi}\int_0^{2\pi} b(a+b\cos u)\, du\, dv = 4\pi^2 ab}$$

ad 2)

$x = u$ $\quad (u,v) \in M$ $\qquad ds = \sqrt{1+|\nabla z|^2}\, du\, dv$

$y = v$

$$z = \sqrt{b^2 - \left(\sqrt{x^2+y^2} - a\right)^2} \qquad = \sqrt{1 + \frac{\left(\sqrt{x^2+y^2} - a\right)^2}{b^2 - \left(\sqrt{\;} - a\right)^2}}$$

$$S = 2b \int_M \frac{1}{\sqrt{b^2 - (\sqrt{\;} - a)^2}}\, du\, dv$$

M: $u = r\cos t$ $\quad t \in (0, 2\pi)$

$v = r\sin t$ $\quad r \in (a-b, a+b)$

ad

(10) $x = \cos t$ $\qquad t \in (0, 2\pi)$

$y = \sin t$ $\qquad z \in \left(-\sqrt{(1-\sin^2 t)}, \sqrt{(1-\sin^2 t)}\right)$

$z = z$

$$\frac{\partial f}{\partial t} \left(-\sin t, \cos t, 0\right) \qquad \frac{\partial f}{\partial t} \times \frac{\partial f}{\partial z} = \left(\cos t, -\sin t, 0\right)$$

$$\frac{\partial f}{\partial z} = (0, 0, 1) \qquad |\;\cdot\;| = 1$$

$$S = \int_0^{2\pi} \int_{-|\cos t|}^{|\cos t|} 1\, dz\, dt = 2\int_0^{2\pi} |\cos t|\, dt = 8\int_0^{\frac{\pi}{2}} \cos t\, dt = 8$$

(12) $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

$$x = a\cos\varphi\cos\psi \qquad \psi\in(0,2\pi)$$
$$y = b\cos\varphi\sin\psi \qquad \varphi\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$$
$$z = c\sin\varphi$$

$$\vec{dS} = \left(-a\sin\varphi\cos\psi,\ -b\sin\varphi\sin\psi,\ c\cos\varphi\right)\times\left(-a\cos\varphi\sin\psi,\ b\cos\varphi\cos\psi, 0\right)$$
$$=\left(-bc\cos^2\varphi\cos\psi,\ -ac\cos^2\varphi\sin\psi,\ -ab\sin\varphi\cos\varphi\right)$$
$$=(dS_1, dS_2, dS_3)$$

ROVNICE TEČNÉ ROVINY $\quad dS_1 x + dS_2 y + dS_3 z + A = 0$

URČÍM $A := -a\cos\varphi\cos\psi\, dS_1 + b\cos\varphi\sin\psi\, dS_2 - c\sin\varphi\, dS_3$

a máme $h(\varphi,\psi) = \frac{|A|}{|\vec{dS}|}$

$$\int_S \frac{dS}{h} = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \frac{|\vec{dS}|^2}{|A|}\, d\varphi\, d\psi = \iint_0^{2\pi\ \frac{\pi}{2}} \frac{b^2c^2\cos^4\varphi\cos^2\psi + a^2c^2\cos^4\varphi\sin^2\psi + a^2b^2\sin^2\varphi\cos^2\varphi}{abc\quad\cos\varphi}\, d\varphi\, d\psi$$
$$= \frac{1}{abc}\iint b^2c^2\cos^3\varphi\cos^2\psi + a^2c^2\cos^3\varphi\sin^2\psi + a^2b^2\sin^2\varphi\cos\varphi\, d\varphi\, d\psi$$
$$= \frac{\pi}{abc}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} c^2(a^2+b^2)\underbrace{\cos^3\varphi}_{(1-\sin^2)\cos\varphi} + 2a^2b^2\sin^2\varphi\cos\varphi\, d\varphi =$$

$$\frac{\pi}{abc}\left[c^2(a^2+b^2)\left(\sin\varphi - \frac{\sin^3\varphi}{3}\right) + 2a^2b^2\frac{\sin^3\varphi}{3}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$
$$= \frac{\pi}{abc}\left(c^2(a^2+b^2)\frac{4}{3} + \frac{4a^2b^2}{3}\right)$$
$$= \frac{4}{3}\pi abc\left(\frac{a}{b}+\frac{b}{a}+\frac{ab}{c}\right)$$

(13) $M = x^2+y^2 \le R^2$ (x, y, xy) $\partial_x = (1,0,y)$

$\partial_y = (0,1,x)$

$$\vec{dS} = (1,0,y) \times (0,1,x) = (-y,-x,1)$$

$$\int_S \frac{1}{\sqrt{x^2+y^2}}\, dS = \int_M \frac{\sqrt{x^2+y^2+1}}{\sqrt{x^2+y^2}}\, dx\,dy = \int_M \sqrt{1+\frac{1}{x^2+y^2}}\, dx\,dy$$

$x = r\cos t$
$y = r\sin t$

$$\to \int_0^R \int_0^{2\pi} \sqrt{1+\frac{1}{r^2}}\, r\,dr\,d\varphi = 2\pi \int_0^R \sqrt{1+r^2}\, dr$$

$$= \pi \left[r\sqrt{1+r^2} \right]_0^R + \pi \int_0^R \frac{1}{\sqrt{r^2+1}}\, dr = \pi \left(R\sqrt{1+R^2} + \operatorname{arsinh} R \right)$$

(14) dsl. $(x, y, (1-x-y))$ $x \in (0,1)$, $y \in (0,1-x)$

$$dS = \sqrt{3}\; dx\,dy \qquad |(1,0,-1) \times (0,1,-1)| = |(1,1,1)| = \sqrt{3}$$

$$|S| = \int_0^1 \int_0^{1-x} \sqrt{3}\, dx\,dy = \sqrt{3} \int_0^1 (1-x)\,dx = \sqrt{3}\left[x - \frac{x^2}{2}\right]_0^1 = \frac{\sqrt{3}}{2}$$

hustota. $\rho = \dfrac{M}{\frac{\sqrt{3}}{2}}$

$$J_x = \int_S \rho x^2\, dS \qquad J_y = \int_S \rho y^2\, dS \quad \ldots$$

$$J_x = \frac{2M}{\sqrt{3}} \int_0^1 \int_0^{1-x} x^2\, dy\,dx = \frac{2M}{\sqrt{3}} \int_0^1 x^2 - x^3\, dx = \frac{2M}{\sqrt{3}} \left(\frac{1}{2} - \frac{1}{3} \right) = \frac{M}{3\sqrt{3}}$$

$J_y = \ldots = J_x$

$$J_z = \frac{2M}{\sqrt{3}} \int_0^1 \int_0^{1-x} (1-x-y)^2\, dy\,dx = -\frac{2M}{3\sqrt{3}} \int_0^1 \left[(1-x-y)^3 \right]_0^{(1-x)} dx$$

$$= \frac{2M}{3\sqrt{3}} \int_0^1 (1-x)^3\, dx = -\frac{2M}{3\sqrt{3}\cdot 4} \left[(1-x)^4 \right]_0^1 = \frac{M}{6\sqrt{3}}$$

(15) rejdání m_1

hustota $\frac{m_1}{4\pi r^2} = \rho$

$x = r\cos\varphi\cos\psi$
$y = r\cos\varphi\sin\psi$
$z = r\sin\varphi$

$$F = \int_S \frac{(x, y, z-d)}{\|(x, y, z-d)\|^3} \varkappa \rho M \, dS =$$

$$\vec{dS} = (-r\sin\varphi\cos\psi, -r\sin\varphi\sin\psi, r\cos\varphi) \times (-r\cos\varphi\sin\psi, r\cos\varphi\cos\psi, 0)$$

$$= r^2(-\cos^2\varphi\cos\psi, -\cos^2\varphi\sin\psi, -\cos\varphi\sin\varphi)$$

$F_x = F_y = 0$

$$F_z = \frac{G m_1 m_2}{-4\pi r^2} \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(d - r\sin\varphi) r^2 \cos\varphi}{(r^2 + d^2 - 2dr\sin\varphi)^{\frac{3}{2}}} d\varphi \, d\psi = \frac{-G 2\pi m_1}{4\pi r^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(d - r\sin\varphi)\cos\varphi}{(r^2 + d^2 - 2dr\sin\varphi)^{3/2}}$$

$$= -\frac{G}{2} m_1 m_2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (d - r\sin\varphi) \left((r^2 + d^2 - 2dr\sin\varphi)^{-\frac{1}{2}}\right)' \frac{1}{dr} \, d\varphi$$

$$= -\frac{G}{2} m_1 m_2 \left[\frac{(d - r\sin\varphi)(r^2 + d^2 - 2dr\sin\varphi)^{-1/2}}{dr} \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$- \frac{G}{2} m_1 m_2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos\varphi}{d} (r^2 + d^2 - 2dr\sin\varphi)^{-1/2} \, d\varphi$$

$$= + \frac{G}{2} \frac{m_1 m_2}{d^2 r} \left[(r^2 + d^2 - 2dr\sin\varphi)^{1/2} \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \boxed{\frac{G m_1 m_2}{d^2}}$$

SÍLA MEZI DVĚMA STĚNAMI = SÍLA MEZI DVĚMA HMOTNÝMI BODY VE STŘEDU!

(16) kužel $z=\sqrt{x^2+y^2}$ useknutý válcem $x^2+y^2=ax$

hledám (T_x, T_y, T_z) tak

$$(x-\tfrac{a}{2})^2+y^2=\frac{a^2}{4}$$

$$\int_S (x-T_x,\ y-T_y,\ z-T_z)\, ds = 0$$

$$t\in(0,2\pi)$$

$$r\in(0,1)$$

$$x=\frac{a}{2}+\frac{a}{2}r\cos t$$

$$y=\frac{a}{2}r\sin t$$

$$z=\sqrt{\frac{a^2}{4}+\frac{a^2}{4}+\frac{a^2}{2}r\cos t}=\frac{a}{2}\sqrt{r^2+1+2r\cos t}$$

$$\vec{ds}=\left(+\frac{a}{2}\cos t,\ \frac{a}{2}\sin t,\ \frac{a}{2}\frac{r+\cos t}{\sqrt{\ }}\right)\times\left(-\frac{a}{2}r\sin t,\ \frac{a}{2}r\cos t,\ -\frac{a}{2}\frac{r\sin t}{\sqrt{\ }}\right)$$

$$=\frac{a^2r}{4}\left(-\frac{\sin^2 t}{\sqrt{\ }}-\frac{r\cos t+\cos^2 t}{\sqrt{\ }},\ \frac{-r\sin t-\sin t\cos t}{\sqrt{\ }}+\frac{\sin t\cos t}{\sqrt{\ }},\ 1\right)$$

$$|\vec{ds}|=\frac{a^2r}{4}\sqrt{\frac{(1+r\cos t)^2+r^2\sin^2 t}{r^2+1+2r\cos t}+1}=\frac{\sqrt{2}}{4}a^2r$$

$$\int_0^1\int_0^{2\pi}\left(\frac{a}{2}+\frac{a}{2}r\cos t-T_x,\ \frac{a}{2}r\sin t-T_y,\ \frac{a}{2}\sqrt{r^2+1+2r\cos t}-T_z\right)\frac{\sqrt{2}}{4}a^2r\, dr\, dt$$

$$=\dots$$

(17) $x = a\cos\varphi\cos\psi$ $\qquad \varphi\in(0,\frac{\pi}{2})$

$y = a\cos\varphi\sin\psi$ $\qquad \psi\in(0,\frac{\pi}{2})$

$z = a\sin\varphi$

$$\vec{ds} = (-a\sin\varphi\cos\psi, -a\sin\varphi\sin\psi, a\cos\varphi)\times(-a\cos\varphi\sin\psi, a\cos\varphi\cos\psi, 0)$$

$$= a^2(-\cos^2\varphi\cos\psi, -\cos^2\varphi\sin\psi, -\sin\varphi\cos\varphi)$$

$$|ds| = a^2\cos\varphi$$

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} (a\cos\varphi\cos\psi - T_x, a\cos\varphi\sin\psi - T_y, a\sin\varphi - T_z)\, a^2\cos\varphi\, d\varphi\, d\psi$$

$$= \int_0^{\frac{\pi}{2}} \left(a\cos\varphi - \frac{\pi}{2}T_x, a\cos\varphi - \frac{\pi}{2}T_y, \frac{\pi}{2}(a\sin\varphi - T_z)\right) a^2\cos\varphi\, d\varphi$$

$$= \frac{2}{a}\left(\frac{a\pi}{4} - \frac{\pi^2}{4}T_x, \frac{a\pi}{4} - \frac{\pi^2}{4}T_y, \frac{\pi}{2}\left(\frac{a}{2} - \frac{\pi}{2}T_z\right)\right)$$

$$\Rightarrow T = \left(\frac{a}{\pi}, \frac{a}{\pi}, \frac{a}{\pi}\right)$$

(18) $x = u\cos v$ $\qquad u\in(0,a)$

$y = u\sin v$ $\qquad v\in(0,\pi)$

$z = hv$

$$\vec{ds} = (\cos v, \sin v, 0)\times(-u\sin v, u\cos v, h)$$

$$= (h\sin v, -h\cos v, u) \Rightarrow |\vec{ds}| = \sqrt{h^2+u^2}$$

$$0 = \int_0^{\pi}\int_0^{a} (T_x - u\cos v, T_y - u\sin v, T_z - hv)\sqrt{h^2+u^2}\, du\, dv$$

$$= \int_0^{a} \left(\pi T_x, \pi T_y - 2u, \pi T_z - \frac{h\pi^2}{2}\right)\sqrt{h^2+u^2}\, du$$

$\Rightarrow T_x = 0, T_z = \frac{\pi h}{2}$; and ....

1a) Díky symetrii stačí uvažovat $P = (0,0,d)$

kde $0 < d \neq a$

$$x = a\cos\varphi\cos\psi \qquad \psi \in (0, 2\pi)$$
$$y = a\cos\varphi\sin\psi \qquad \varphi \in \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right)$$
$$z = a\sin\varphi$$

$$\vec{ds} = (-a\sin\varphi\cos\psi, -a\sin\varphi\sin\psi, a\cos\varphi) \times (-a\cos\varphi\sin\psi, a\cos\varphi\cos\psi, 0)$$
$$= a^2(\cos^2\varphi\cos\psi, -\cos^2\varphi\sin\psi, -\sin\varphi\cos\varphi)$$

$$|ds| = a^2\cos\varphi$$

$$\int_S \frac{1}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}}\, ds = \int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a^2\cos\varphi}{\sqrt{a^2\cos^2\varphi + (a\sin\varphi - d)^2}}\, d\varphi\, d\psi$$

$$= 2\pi a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos\varphi\, d\varphi}{\sqrt{a^2+d^2 - 2ad\sin\varphi}}\, d\varphi = \left[-\frac{2\pi a}{d}\sqrt{a^2+d^2-2ad\sin\varphi}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= -\frac{2\pi a}{d}\left(\sqrt{a^2+d^2-2ad} - \sqrt{a^2+d^2+2ad}\right) = \frac{2\pi a}{d}(a+d - |d-a|)$$

$$= \begin{cases} \dfrac{4\pi a^2}{d} & d > a \\ 4\pi a & d < a \end{cases}$$

→ uvnitř sféry nepůsobí žádná síla!

2a) $F = \int_S p\, ds$, kde p je tlak $\quad p = -z\rho g$

$$S: \quad x = 0 \quad y = y \quad z = z \qquad y \in (-a, a) \qquad z \in \left[\tfrac{h}{a^2}(y^2 - a^2), 0\right]$$

$$\vec{ds} = (0,1,0) \times (0,0,1) = (1,0,0) \quad ; \quad |\vec{ds}| = 1$$

$$F = \int_{-a}^{a}\int_{\frac{h}{a^2}(y^2-a^2)}^{0} -z\rho g\, dz\, dy = \frac{\rho g}{2}\int_{-a}^{a} \frac{h^2}{a^4}(y^2-a^2)^2\, dy$$

$$= \frac{h^2\rho g}{2a^4}\int_{-a}^{a} y^4 - 2y^2a^2 + a^4\, dy = \frac{h^2\rho g}{2a^4}\left(\frac{2a^5}{5} - \frac{2a^5}{3} \cdot 2 + 2a^5\right)$$

$$= \frac{h^2\rho g a \cdot 8}{15}$$

PLOŠNÝ INTEGRÁL II. DRUHU

$$\int_S \vec{j}\cdot d\vec{S} := \underbrace{\int_S \vec{j}\cdot\vec{n}\, dS}_{\text{integrál I. druhu}}$$

, kde $\vec{n}$ je velikost vnější normály

- S – musí mít orientaci – musíme vědět, co je směr „ven" – vektor $\vec{n}$

Pokud máme parametrizaci $\vec{\varphi}: M \to \mathbb{R}^3$ $\quad M \subseteq \mathbb{R}^2$

pak $\quad d\vec{S} = \frac{\partial\vec{\varphi}}{\partial u}\times\frac{\partial\vec{\varphi}}{\partial v}$ $\quad$ (záleží na znaménku!)

a můžeme psát $\displaystyle\int_S \vec{j}\cdot d\vec{S} = \int_S \vec{j}\cdot\vec{n}\, dS = \boxed{\int_M \vec{j}(\vec{\varphi}(u,v))\left(\frac{\partial\vec{\varphi}}{\partial u}\times\frac{\partial\vec{\varphi}}{\partial v}\right) du\,dv}$

POZOR NA ZNAMÉNKO!

Někdy je též zápis

$$\int_S j_x\,dy\,dz + j_y\,dx\,dz + j_z\,dx\,dy := \int_S (j_x, j_y, j_z)\cdot\vec{n}\, dS$$

(1) $\int_S (y-z)\,dydz + (z-x)\,dzdx + (x-y)\,dxdy$

kde S je 'vnější' strana kuželе $x^2+y^2=z^2$, $z\in(0,h)$

Máme tedy spočítat $\int_S (y-z, z-x, x-y)\cdot\vec{n}\,dS$

kde $\vec{n}$ je vnější normála $\to \int_{S_1} + \int_{S_2}$ – kde S_1 je plášť a S_2 je podstava

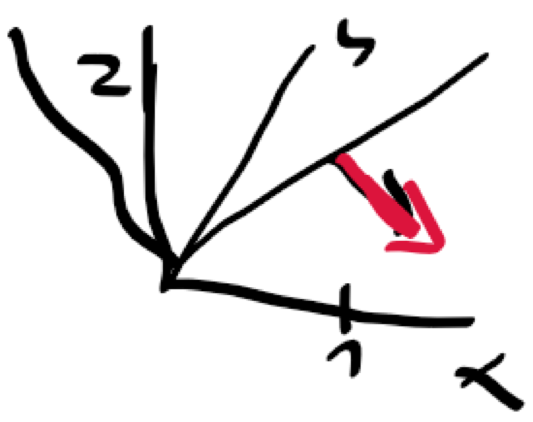
a) S_1 - plášť

$x = x$ $\quad\quad M = x^2+y^2 \le h^2$

$y = y$

$z = \sqrt{x^2+y^2}$

$\pm\vec{dS} = \left(1, 0, \frac{x}{\sqrt{x^2+y^2}}\right)\times\left(0, 1, \frac{y}{\sqrt{x^2+y^2}}\right) = \underbrace{\left(-\frac{x}{\sqrt{x^2+y^2}}, -\frac{y}{\sqrt{x^2+y^2}}, 1\right)}$



má směr normály?: v rovině $0 < x = z$ a $y=0$ $(-1, 0, 1)$ což je opačný směr než ↘

a tedy $\vec{dS} = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1\right)$

$\int_{S_1} (y-z, z-x, x-y)\cdot\vec{n}\,dS = \int_M \left(y-\sqrt{x^2+y^2}, \sqrt{x^2+y^2}-x, x-y\right)\cdot\left(\frac{x}{\sqrt{\ }}, \frac{y}{\sqrt{\ }}, -1\right) dxdy$

$= \int_M \frac{xy}{\sqrt{\ }} - 2x + 2y - \frac{xy}{\sqrt{\ }}\,dy = 0$ (M je kruh – ze symetrie

b) S_2 $\quad x = x \quad\quad x^2+y^2 \le h^2$

$y = y$

$z = h$

$\int_{S_2} (y-z, z-x, x-y)\cdot\vec{n}\,dS = \int_M x-y\,dxdy = \underline{\underline{0}}$

$\vec{dS} = (0, 0, 1)$

② $\iint_S x^2\,dy\,dz + y^2\,dx\,dz + z^2\,dx\,dy$

S je vnější strana sféry

$= \iint_S (x^2, y^2, z^2)\cdot\vec{n}\,dS$

$(x-a)^2+(y-b)^2+(z-c)^2=R^2$

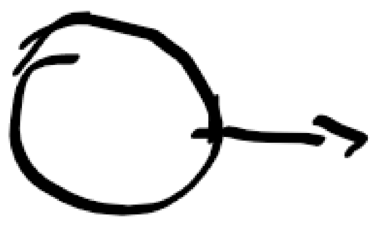
$x = a + R\cos\varphi\cos\psi$ $\quad \varphi\in(-\frac{\pi}{2},\frac{\pi}{2})$

$y = b + R\cos\varphi\sin\psi$ $\quad \psi\in(0,2\pi)$

$z = c + R\sin\varphi$

$\vec{dS} = R^2(-\sin\varphi\cos\psi, -\sin\varphi\sin\psi, \cos\varphi)\times(-\cos\varphi\sin\psi, \cos\varphi\cos\psi, 0)$

$= R^2(-\cos^2\varphi\cos\psi, -\cos^2\varphi\sin\psi, -\sin\varphi\cos\varphi)$

 normála v $\varphi=0, \psi=0$ $(1,0,0)$ $\Rightarrow$ opačný směr

ale my $\vec{dS} = (\cos^2\varphi\cos\psi, \cos^2\varphi\sin\psi, \sin\varphi\cos\varphi)R^2$

$\Rightarrow \iint_S (x^2,y^2,z^2)\cdot n\,dS = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}\Big[\big((a+R\cos\varphi\cos\psi)^2, (b+R\cos\varphi\sin\psi)^2, (c+R\sin\varphi)^2\big)\cdot(\cos^2\varphi\cos\psi, \cos^2\varphi\sin\psi, \sin\varphi\cos\varphi)\Big]R^2\,d\psi\,d\varphi$

$= R^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi} a^2\cos^2\varphi\cos\psi + R^2\cos^4\varphi\cos^3\psi + 2aR\cos^3\varphi\cos^2\psi\,d\psi\,d\varphi$

$+ R^2\iint b^2\cos^2\varphi\sin\psi + R^2\cos^4\varphi\sin^3\psi + 2bR\cos^3\varphi\sin^2\psi\,d\psi\,d\varphi$

$+ R^2\iint c^2\sin\varphi\cos\varphi + R^2\sin^3\varphi\cos\varphi + 2cR\sin^2\varphi\cos\varphi\,d\psi\,d\varphi$

$= 2aR^3\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^3\varphi\,d\varphi + 2bR^3\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^3\varphi\,d\varphi + 2\pi R^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} c^2\sin\varphi\cos\varphi + R\sin^3\varphi\cos\varphi + 2cR\sin^2\varphi\cos\varphi$

$= \left[\sin\varphi - \frac{\sin^3\varphi}{3}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2R^3\pi(a+b) + \left[\frac{c^2\sin^2\varphi}{2} + \frac{R\sin^4\varphi}{4} + 2cR\frac{\sin^3\varphi}{3}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

$= \frac{8}{3}\pi R^3(a+b+c)$

③ $\iint_S (z-R)\,dx\,dy$ $\quad$ S je část plochy $x^2+y^2+z^2=2Rz$

a $z\in(R,2R)$

$\iff x^2+y^2+(z-R)^2=R^2$

$= \iint_S (0,0,z-R)\cdot\vec{n}\,dS$

z ② máme

$x = R\cos\varphi\cos\psi$

$y = R\cos\varphi\sin\psi$

$z = R+R\sin\varphi$

$\varphi\in(0,\frac{\pi}{2})\Leftarrow$ ZMĚNA OPROTI ② ať $z\in(R,2R)$

$\psi\in(0,2\pi)$

$\vec{dS}=R^2(\cos^2\varphi\cos\psi,\ \cos^2\varphi\sin\psi,\ \sin\varphi\cos\varphi)$

$\Rightarrow \iint_S (z-R)\,dx\,dy = \int_0^{\frac{\pi}{2}}\int_0^{2\pi} R\sin\varphi\cdot R^2\sin\varphi\cos\varphi\,d\psi\,d\varphi = 2\pi R^3\int_0^{\frac{\pi}{2}}\sin^2\varphi\cos\varphi\,d\varphi$

$= \frac{2\pi R^3}{3}$

④ $\iint_S z\,dy\,dz + x\,dz\,dx + y\,dx\,dy$, $\quad$ S je část $x-y+z=1$

$x,z\geq 0,\ y\leq 0$

orientace tak, že S vektorem $(0,1,0)$ svírá ostrý úhel $\Rightarrow \vec{n}\cdot(0,1,0)>0$

S: $x=x$ $\quad x\in(0,1)$

$y=y$ $\quad y\in(-1,0)$

$z=1+y-x$

$\pm\vec{dS}=\underbrace{(1,0,-1)}_{\partial_x\varphi}\times\underbrace{(0,1,1)}_{\partial_y\varphi}=(1,-1,1)$

protože $(1,-1,1)\cdot(0,1,0)<0$

máme $\vec{dS}=(-1,1,-1)$

$\iint_S z\,dy\,dz + x\,dz\,dx + y\,dx\,dy = \int_0^1\int_{-1}^0 (1+y-x,x,y)\cdot(-1,1,-1)\,dx\,dy$

$= \int_0^1\int_{-1}^0 x-y-1+x-y\,dx\,dy = \int_0^1 2x\,dx - \int_{-1}^0 2y\,dy - 2 = -2$

⑤ $\iint_S (x^2,y^2,z^2)\cdot\vec{n}\,dS$ $\quad S: x^2+y^2+2az=a^2 \quad x<0;\ y,z\geq 0$

orient. $\vec{n}\cdot(0,1,0)>0$

$x=x$
$y=y$
$z=\frac{a^2-x^2-y^2}{2a}$ $\qquad \left.\begin{matrix} x^2+y^2\leq a^2 \\ x<0,\ y\geq 0\end{matrix}\right\} = M$

$\pm\vec{dS} = \partial_x\varphi\times\partial_y\varphi = (1,0,-\frac{x}{a})\times(0,1,-\frac{y}{a})$
$= (\frac{x}{a},\frac{y}{a},1)$ směr je ok.

$\iint_S (x^2,y^2,z^2)\cdot\vec{n}\,dS = \iint_M \frac{x^3}{a}+\frac{y^3}{a}+\frac{(a^2-x^2-y^2)^2}{4a^2}\,dx\,dy$

$x=r\cos\varphi \quad r\in(0,a)$
$y=r\sin\varphi \quad \varphi\in(\frac{\pi}{2},\pi)$

$=\int_0^a\int_{-\frac{\pi}{2}}^{\pi}\left(\frac{r^3\cos^3\varphi}{a}+\frac{r^3\sin^3\varphi}{a}+\frac{(a^2-r^2)^2}{4a^2}\right)r\,d\varphi\,dr$

$=\frac{\pi}{8a^2}\int_0^a a^4-2a^2r^2+r^4\,dr = \frac{\pi}{8a^2}\left(a^5-\frac{2a^5}{3}+\frac{a^5}{5}\right)$
$=\frac{\pi}{15}a^3$

⑥ $\iint_S \left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right)\cdot\vec{n}\,dS$ $\quad S$ je elipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$
normála orientace vně $(a,b,c>0)$

$x=a\cos\varphi\cos\psi$
$y=b\cos\varphi\sin\psi \qquad \varphi\in(-\frac{\pi}{2},\frac{\pi}{2}) \quad \psi\in[0,2\pi)$
$z=c\sin\varphi$

$\pm\vec{dS} = (-a\sin\varphi\cos\psi,-b\sin\varphi\sin\psi,c\cos\varphi)\times(-a\cos\varphi\sin\psi,b\cos\varphi\cos\psi,0)$
$=(-cb\cos^2\varphi\cos\psi,\ -ac\cos^2\varphi\sin\psi,\ -ab\sin\varphi\cos\varphi)$

opačný směr!

$\iint_S\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right)\cdot\vec{n}\,dS = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}\frac{cb\cos^2\varphi\cos\psi}{a\cos\varphi\cos\psi}+\frac{ac\cos^2\varphi\sin\psi}{b\cos\varphi\sin\psi}+\frac{ab\sin\varphi\cos\varphi}{c\sin\varphi}\,d\psi\,d\varphi$

$=2\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{cb}{a}+\frac{ac}{b}+\frac{ab}{c}\right)\cos\varphi\,d\varphi$

$=\frac{4\pi}{abc}\left(c^2b^2+a^2c^2+a^2b^2\right)$

VĚTY O INTEGRACI PER PARTES VE VÍCE DIMENZÍCH

(= různé verze obecné Stokesovy věty pro formy)

a) Gauss – Green – Ostrogradski

$\Omega \subseteq \mathbb{R}^n$ s "pěknou" hranicí – C^1-hladká

↑ otevřená, omezená

$\forall \vec{j} \in C^1(\bar{\Omega}; \mathbb{R}^n)$ $\quad \int_\Omega \operatorname{div} \vec{j}\, dx = \underbrace{\int_{\partial\Omega} \vec{j} \cdot \vec{n}\, dS}_{\text{integrál II. druhu}}$

$\vec{n}$ – vnější normála

Dá se použít k výpočtu objemu či plochy...

Vol $\vec{j} = \frac{(x,y,z)}{3} \Rightarrow \operatorname{div} \vec{j} = 1$

$|\Omega| = \int_\Omega 1\, dx = \int_{\partial\Omega} \vec{j} \cdot \vec{n}\, dS$ $\qquad$ Pokud $\Omega = B_R$ koule

$\frac{4}{3}\pi R^3 = \int_{B_R} 1\, dx = \frac{1}{3} \int_{\partial\Omega} (x,y,z) \cdot \vec{n}\, dS = \frac{1}{3} \int_{\partial B_R} \sqrt{x^2+y^2+z^2}\, dS = \frac{R}{3} |\partial B_R|$

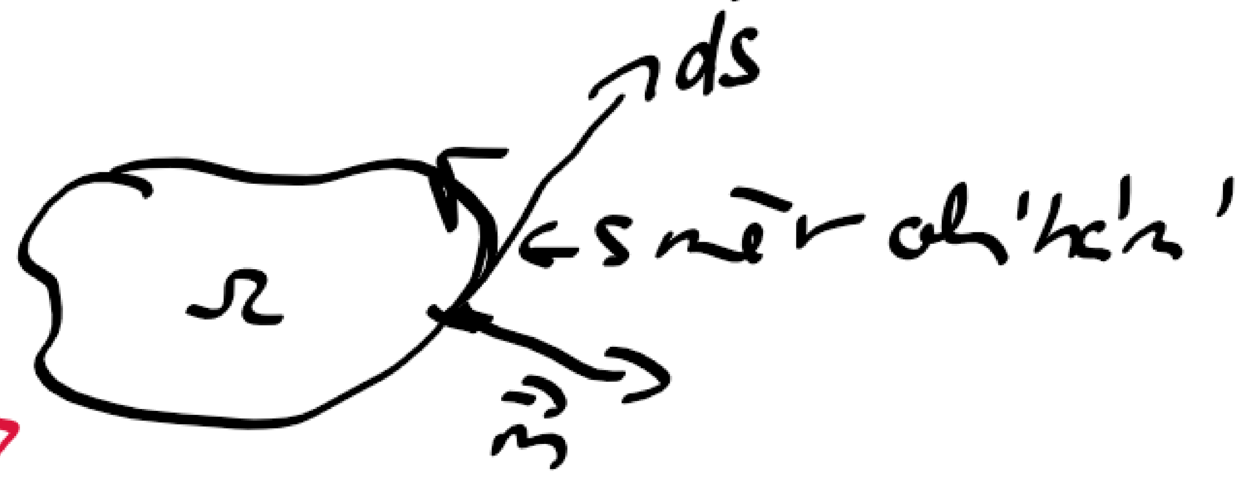
$\vec{n} = \frac{(x,y,z)}{\sqrt{x^2+y^2+z^2}}$

b) Gauss ve 2D

$\Omega \subseteq \mathbb{R}^2$ $\quad \int_\Omega \operatorname{div} \vec{j}\, dx = \int_{\partial\Omega} \vec{j} \cdot \vec{n}\, \underbrace{dS}_{\text{délkový element křivky}}$

POZOR NA ORIENTACI!

ZNAMÉNKO!

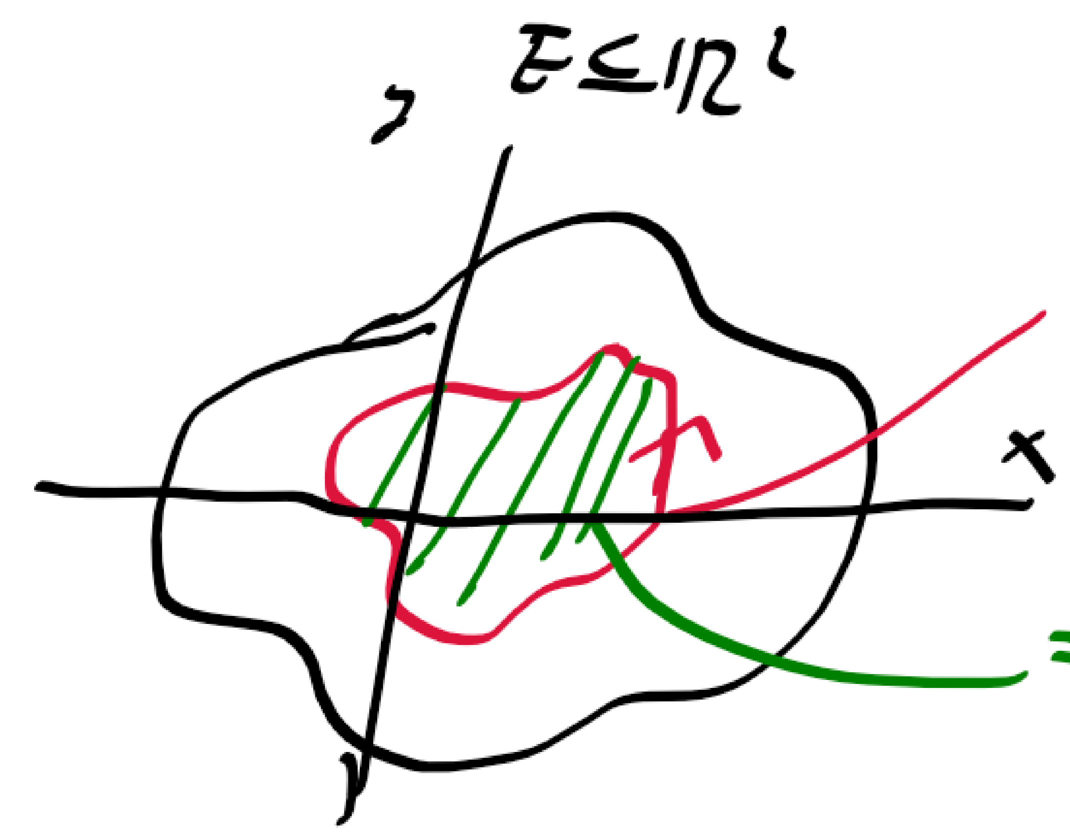


pokud máme kladný směr oběhu, pak $\frac{\vec{ds}}{ds} = (-n_2, +n_1)$

$\Rightarrow \int_\Omega \operatorname{div} \vec{j}\, dx = \int_{\partial\Omega} (-j_2, j_1) \cdot \vec{ds}$ $\quad$ křivkový integrál II. druhu

$\Rightarrow \int_\Omega \left(\frac{\partial T_2}{\partial x_1} - \frac{\partial T_1}{\partial x_2} \right) dx = \int_{\partial\Omega \to} \vec{T} \cdot \vec{ds}$

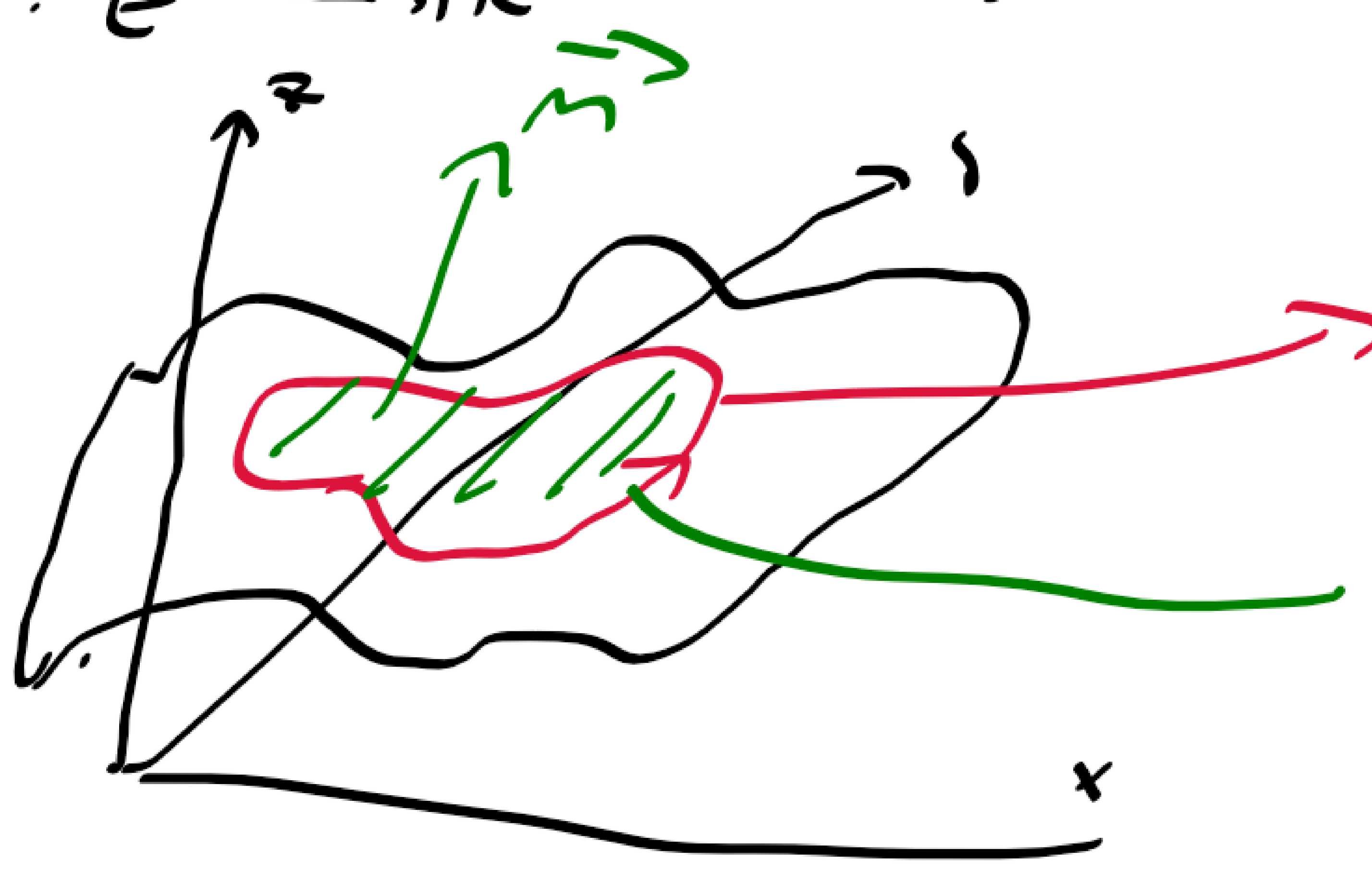
(c) STOKESOVA VĚTA pro 2-plochy ve 3 dimenzích



$\vec\psi$ – křivka uzavřená Jordanova $[a,b]$ orientovaná kladně!

$= \tilde{A}$

$\vec\varphi : E \to \mathbb{R}^3$ $\quad \vec\varphi(E) = \tilde{S}$ – plocha



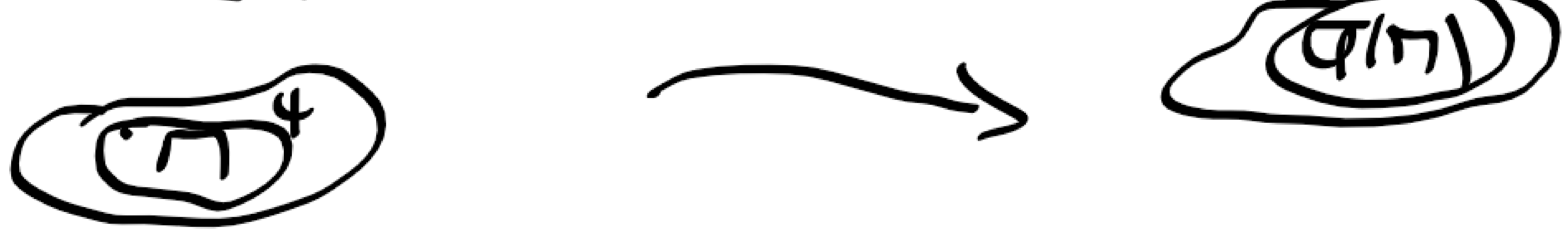
$\vec\varphi(\psi) = \vec\gamma$ – křivka ve 3D

$\vec\varphi(\tilde A)$ = "vnitřek" $\vec\gamma$ s orientací $\vec n$

$$\underbrace{\int_{\vec\varphi(A)} \mathrm{rot}\,\vec T \cdot d\vec S}_{\text{plošný integrál II. druhu}} = \underbrace{\int_{\vec\gamma} \vec T \cdot d\vec\gamma}_{\text{křivkový integrál II. druhu}}$$

Př: Greenova věta jako důsledek Stokesovy věty

$M \subseteq \mathbb{R}^2$



$\vec\varphi(\psi) = (\psi_1, \psi_2, 0)$

$\vec\varphi$: $x = x$, $y = y$, $z = 0$

$d\vec S = (1,0,0) \times (0,1,0)\, dx\,dy = (0,0,1)\, dx\,dy$

$\vec T = (T_1(x,y), T_2(x,y), 0)$

$\mathrm{rot}\,\vec T = \left(0, 0, \frac{\partial T_2}{\partial x_1} - \frac{\partial T_1}{\partial x_2}\right)$

$d\vec\varphi = (\psi_1', \psi_2', 0)\, dt = (\vec{d\psi}, 0)$

$$\left.\begin{aligned} \int_{\varphi(M)} \mathrm{rot}\,\vec T \cdot d\vec S &= \int_M \left(\frac{\partial T_2}{\partial x_1} - \frac{\partial T_1}{\partial x_2}\right) dx\,dy \\ \int_{\varphi(\psi)} \vec T \cdot d\vec\varphi &= \int_{\vec\psi} (T_1, T_2) \cdot \vec{d\psi} \end{aligned}\right\} \Rightarrow \text{Greenova věta}$$