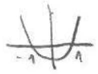
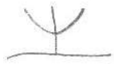
Opakování , 1/1

Opakování ze SŠ , $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$

Komplexní čísla $\mathbb{C}$ jsou rozšířením oboru reálných čísel ($x \in \mathbb{R}$ se dá zapsat $(x,0) \in \mathbb{C}$)
tak, aby každá algebraická rovnice měla příslušný počet řešení dle základní věty algebr

$x^2 - 1 = 0$ 
$x = \pm 1$ dve riešenia v $\mathbb{R}$ aj v $\mathbb{C}$

$x^2 = 0$ 
$x = 0$ dvojnásobné r. v $\mathbb{R}$ aj v $\mathbb{C}$

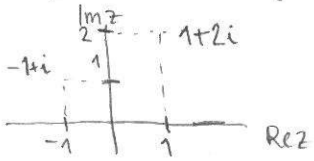
$x^2 + 1 = 0$ 
nemá rieš. v $\mathbb{R}$, $x = \pm i$ 2 rieš. v $\mathbb{C}$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{\pm\sqrt{-4}}{2} = \frac{\pm\sqrt{i^2 4}}{2} = \pm i$$

Príklad: $(x+1)^2 (x-2) = 0$
pol. 3 stupňa $\Rightarrow$ 3 riešenia v $\mathbb{R}$ (aj v $\mathbb{C}$): $-1, -1, 2$
$(x^2+1)(x-2) = 0$
pol. 3. stupňa $\Rightarrow$ 3 riešenia v $\mathbb{C}$: $i, -i, 2$

Algebraický tvar k.č. : $\underset{\text{kompl. č}}{z} = \underset{\text{Re} z}{a} + \underset{\text{imaginárna jednotka}}{i}\overset{\text{Im} z}{b}$, $i^2 = -1$

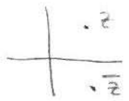
$i^1 = i$
$i^2 = -1$
$i^3 = i^2 \cdot i^1 = -i$
$i^4 = i^2 i^2 = 1$
$i^5 = i^4 \cdot i = i$
...

geometrická interpretace: 

počítání: • $z_1 \pm z_2 = (a_1 + ib_1) \pm (a_2 + ib_2) = (a_1 \pm a_2) + i(b_1 \pm b_2)$

• $z_1 \cdot z_2 = (a_1 + ib_1) \cdot (a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$

• $\dfrac{z_1}{z_2} = \dfrac{a_1 + ib_1}{a_2 + ib_2} = \dfrac{a_1 + ib_1}{a_2 + ib_2} \cdot \dfrac{a_2 - ib_2}{a_2 - ib_2} = \dfrac{(a_1 a_2 + b_1 b_2) + i(a_2 b_1 - a_1 b_2)}{a_2^2 + b_2^2}$

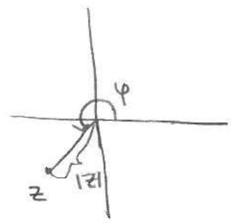
• $\bar{z} := a - ib$ komplexně združené k $z = a + ib$ 

~~Geometr~~

Goniometrický tvar k.č. : $z = |z| (\cos\varphi + i \sin\varphi) = |z| e^{i\varphi}$

kde • $|z| = \sqrt{a^2 + b^2}$ modul, velikost z

• $\varphi = \arg z$ argument z , $\varphi \in \langle 0, 2\pi)$ ($\langle -\pi, \pi)$, ...)



1. Nalezněte reálnou a imaginární část

$$a) z = \frac{2}{1-3i} = \frac{2+6i}{1+9} = \underbrace{\frac{1}{5}}_{\mathrm{Re}z} + \underbrace{\frac{3}{5}}_{\mathrm{Im}z} i$$

$$b) z = (1+i\sqrt{3})^3 = 1 + 3i\sqrt{3} + 3(i\sqrt{3})^2 + i^3(\sqrt{3})^3$$

$$= 1 + i3\sqrt{3} - 9 - i3\sqrt{3} = -8 = \mathrm{Re}z, \quad \mathrm{Im}z = 0$$

2. Nalezněte velikosti a argumenty následujících komplexních čísel

a) $z = -2-2i$

φ	$30°$ $\pi/6$	$45°$ $\pi/4$	$60°$ $\pi/3$
$\sin\varphi$	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$
$\cos\varphi$	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$

$$|z| = \sqrt{(-2)^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$$

$$\mathrm{Re}z = |z|\cos\varphi, \quad \mathrm{Im}z = |z|\sin\varphi$$

$$\left.\begin{aligned} \cos\varphi &= \frac{\mathrm{Re}z}{|z|} = \frac{-2}{\sqrt{8}} = -\frac{\sqrt{2}}{2} \\ \sin\varphi &= \frac{\mathrm{Im}z}{|z|} = \frac{-2}{\sqrt{8}} = \frac{-\sqrt{2}}{2} \end{aligned}\right\} \Rightarrow \varphi = \frac{5}{4}\pi$$

$$z = \sqrt{8}\, e^{i\frac{5}{4}\pi}$$

b) $z = 1 + i^{123} = 1 + i^{120} \cdot i^3 = 1 - i$

$$|z| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

$$\left.\begin{aligned} \cos\varphi &= \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \\ \sin\varphi &= -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2} \end{aligned}\right\} \Rightarrow \varphi = \frac{7}{4}\pi$$

$$z = \sqrt{2}\, e^{i\frac{7}{4}\pi}$$

3. Dokažte

a) $z + \bar{z} = a + ib + a - ib = 2a = 2\mathrm{Re}z$

$z + \bar{z} = \mathrm{Re}z + i\mathrm{Im}z + \mathrm{Re}z - i\mathrm{Im}z = 2\mathrm{Re}z$

b) $z - \bar{z} = \mathrm{Re}z + i\mathrm{Im}z - \mathrm{Re}z + i\mathrm{Im}z = 2i\,\mathrm{Im}z$

c) $\overline{(\bar{z})} = \overline{(\mathrm{Re}z - i\mathrm{Im}z)} = \mathrm{Re}z + i\mathrm{Im}z = z$

d) $|\bar{z}| = \sqrt{(\mathrm{Re}z)^2 + (-\mathrm{Im}z)^2} = \sqrt{(\mathrm{Re}z)^2 + (\mathrm{Im}z)^2} = |z|$

e) $|z_1 z_2| = |z_1||z_2|$

$$|z_1z_2| = \sqrt{(\mathrm{Re}\,z_1z_2)^2 + (\mathrm{Im}\,z_1z_2)^2} = \sqrt{(\mathrm{Re}z_1\mathrm{Re}z_2 - \mathrm{Im}z_1\mathrm{Im}z_2)^2 + (\mathrm{Re}z_1\mathrm{Im}z_2 + \mathrm{Re}z_2\mathrm{Im}z_1)^2}$$

$$= \sqrt{\mathrm{Re}z_1^2\mathrm{Re}z_2^2 \cancel{- 2\mathrm{Re}z_1\mathrm{Re}z_2\mathrm{Im}z_1\mathrm{Im}z_2} + \mathrm{Im}z_1^2\mathrm{Im}z_2^2 + \mathrm{Re}z_1^2\mathrm{Im}z_2^2 \cancel{+ 2\mathrm{Re}z_1\mathrm{Re}z_2\mathrm{Im}z_1\mathrm{Im}z_2} + \mathrm{Re}z_2^2\mathrm{Im}z_1^2}$$

$$= \sqrt{\mathrm{Re}z_1^2(\mathrm{Re}z_2^2 + \mathrm{Im}z_2^2) + \mathrm{Im}z_1^2(\mathrm{Re}z_2^2 + \mathrm{Im}z_2^2)} = \sqrt{(\mathrm{Re}z_1^2 + \mathrm{Im}z_1^2)(\mathrm{Re}z_2^2 + \mathrm{Im}z_2^2)}$$

$$= |z_1||z_2|$$

f) $\arg(z_1 z_2) = \arg z_1 + \arg z_2 \quad (\text{mod } 2\pi) \qquad z_1, z_2 \neq 0$

$$z_1 \cdot z_2 = |z_1|\, e^{i\varphi_1} \cdot |z_2|\, e^{i\varphi_2} = |z_1||z_2|\, e^{i(\varphi_1+\varphi_2)}$$, mod 2π , aby $\varphi_1+\varphi_2 \in \langle 0, 2\pi)$

nebo:

$$z_1 \cdot z_2 = |z_1|(\cos\varphi_1 + i\sin\varphi_1)\,|z_2|(\cos\varphi_2 + i\sin\varphi_2)$$

$$= |z_1||z_2|\left(\cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2 + i(\sin\varphi_1\cos\varphi_2 + \cos\varphi_1\sin\varphi_2)\right)$$

$$= |z_1 z_2|\left(\cos(\varphi_1+\varphi_2) + i\sin(\varphi_1+\varphi_2)\right)$$

g) $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2 \quad (\text{mod } 2\pi) \qquad z_1, z_2 \neq 0$

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{|\bar{z}|(\cos\varphi - i\sin\varphi)}{|z|^2} = \frac{1}{|z|}\left(\cos(-\varphi) + i\sin(-\varphi)\right)$$

$$\Rightarrow \left|\frac{1}{z}\right| = \frac{1}{|z|} \quad \& \quad \arg\bar{z} = -\arg z$$

$$\frac{z_1}{z_2} = \frac{z_1 \cdot \bar{z}_2}{|z_2|^2} = \frac{|z_1|}{|z_2|}\, e^{i\varphi_1} \cdot e^{-i\varphi_2} = \frac{|z_1|}{|z_2|}\, e^{i(\varphi_1-\varphi_2)}$$ mod 2π

nebo:

$$\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}(\cos\varphi_1 + i\sin\varphi_1)(\cos\varphi_2 - i\sin\varphi_2)$$

$$= \frac{|z_1|}{|z_2|}\left(\cos\varphi_1\cos\varphi_2 + \sin\varphi_1\sin\varphi_2 + i(\sin\varphi_1\cos\varphi_2 - \sin\varphi_2\cos\varphi_1)\right)$$

$$= \frac{|z_1|}{|z_2|}\left(\cos(\varphi_1-\varphi_2) + i\sin(\varphi_1-\varphi_2)\right)$$

4. Řešte v $\mathbb{C}$:

a) $x^6 + 1 = 0$

$$z^6 = |z|^6 e^{i6\varphi} = |z|^6(\cos(6\varphi) + i\sin(6\varphi)) = -1 \quad \Rightarrow \quad |z| = 1$$

$$6\varphi = \pi + 2k\pi \Rightarrow \varphi = \frac{\pi + 2k\pi}{6}, \quad k \in \mathbb{Z}$$ mod 2π

$x_0 = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{i}{2}$

$x_1 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$

$x_2 = \cos\frac{5}{6}\pi + i\sin\frac{5}{6}\pi = -\frac{\sqrt{3}}{2} + \frac{i}{2}$

$x_3 = \cos\frac{7}{6}\pi + i\sin\frac{7}{6}\pi = -\frac{\sqrt{3}}{2} - \frac{i}{2}$

$x_4 = -i$

$x_5 = \frac{\sqrt{3}}{2} - \frac{i}{2}$

$\varphi_0 = \frac{\pi}{6}$

$\varphi_1 = \frac{\pi + 2\pi}{6} = \frac{\pi}{2}$

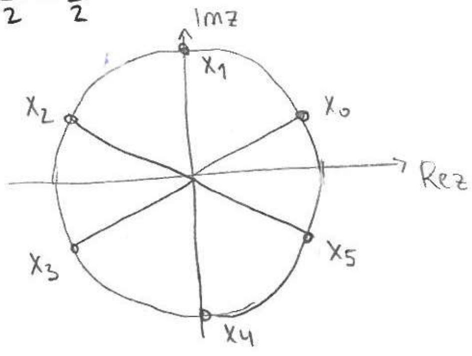
$\varphi_2 = \frac{\pi + 4\pi}{6} = \frac{5}{6}\pi$

$\varphi_3 = \frac{\pi + 6\pi}{6} = \frac{7}{6}\pi$

$\varphi_4 = \frac{\pi + 8\pi}{6} = \frac{3}{2}\pi$

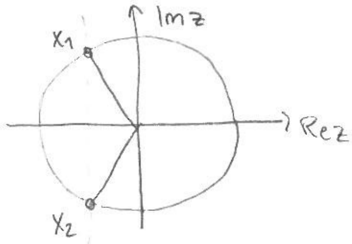
$\varphi_5 = \frac{\pi + 10\pi}{6} = \frac{11}{6}\pi$

$\varphi_6 = \frac{13}{6}\pi \underset{\text{mod } 2\pi}{=} \frac{\pi}{6} = \varphi_0$



b) $x^2 + x + 1 = 0$

$$x_{1,2} = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{i^2 3}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$



5) Řešte v $\mathbb{R}$:

a) $|x+1| + |x-1| \geq 2$

$x+1 \geq 0$, $x-1 \geq 0$: $x \in \langle 1, \infty)$

$x+1+x-1 \geq 2$

$2x \geq 2$

$x \geq 1$

$x+1 \leq 0$, $x-1 \geq 0$: $\emptyset$ $\quad \boxed{x \in \langle -1, 1\rangle}$

$x+1 \geq 0$, $x-1 \leq 0$:

$x+1-x+1 \geq 2$

$2 \geq 2$

$x+1 \leq 0$, $x-1 \leq 0$: $x \in (-\infty, -1\rangle$

$-x-1-x+1 \geq 2$

$-2x \geq 2$

$x \leq -1$

$x \in \mathbb{R}$ je řešení (geometricky: súčet vzdialeností od -1 a od 1)

b) $|x-3| + |x+2| \leq 0$

abs. hodnota je vždy ≥ 0, možná len rovnosť: $x = 3 \wedge x = -2$

Nemá řešení v $\mathbb{R}$.

Výroky, množiny, zobrazení

Zabýváme se pouze výroky, o kterých má smysl říci, zda jsou pravdivé (1 nebo 0)

Výroková funkce (predikát) je předpis, který každému prvku z daného pole objektů přiřadí výrok.

Př. Výrok: Matematika je krásná.

Výroková fce: $P(x)$: x je krásná. $x \in M = \{$matematika, fyzika, ...$\}$

Logické spojky: negace $\neg$ (non), konjunkce $\wedge$, disjunkce $\vee$, implikace $\Rightarrow$, ekvivalence $\Leftrightarrow$.

Pravdivostní tabulka.

A	B	$\neg A$	$A \wedge B$	$A \vee B$	$A \Rightarrow B$	$A \Leftrightarrow B$
1	1	0	1	1	1	1
1	0	0	0	1	0	0
0	1	1	0	1	1	0
0	0	1	0	0	1	1

6. Dokažte, že platí

a) $A \Rightarrow A$, c) $A \Leftrightarrow A$, f) non (non A) $\Leftrightarrow A$

A	A⇒A	A⇔A	¬(¬A)	⇔A
1	1	1	1	1
0	1	1	0	1

a) ✓ c) ✓ f) ✓

d) $(A \Leftrightarrow B) \Leftrightarrow (B \Leftrightarrow A)$, g) $(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$, h) $(A \Leftrightarrow B) \Leftrightarrow (\neg B \Leftrightarrow \neg A)$

A	B	$d_1=h_1$: A⇔B	d_2: B⇔A	d_1⇔d_2	g_1: A⇒B	¬B	¬A	g_2: ¬B⇒¬A	g_1⇔g_2	h_2: ¬B⇔¬A	h_1⇔h_2
1	1	1	1	1	1	0	0	1	1	1	1
1	0	0	0	1	0	1	0	0	1	0	1
0	1	0	0	1	1	0	1	1	1	0	1
0	0	1	1	1	1	1	1	1	1	1	1

d) ✓ g) ✓ h) ✓

b) $(\overbrace{A \Rightarrow B}^{b_1} \wedge \overbrace{B \Rightarrow C}^{b_2}) \Rightarrow (\overbrace{A \Rightarrow C}^{b_3})$, e) $(\overbrace{A \Leftrightarrow B}^{e_1} \wedge \overbrace{B \Leftrightarrow C}^{e_2}) \Rightarrow (\overbrace{A \Leftrightarrow C}^{e_3})$

A	B	C	A⇒B	B⇒C	b_1∧b_2	A⇒C	(b_1∧b_2)⇒b_3	A⇔B	B⇔C	e_1∧e_2	A⇔C	(e_1∧e_2)⇒e_3
1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	0	1	0	0	0	1	1	0	0	0	1
1	0	1	0	1	0	1	1	0	0	0	1	1
1	0	0	0	1	0	0	1	0	1	0	0	1
0	1	1	1	1	1	1	1	0	1	0	1	1
0	1	0	1	0	0	1	1	0	0	0	0	1
0	0	1	1	1	1	1	1	1	0	0	0	1
0	0	0	1	1	1	1	1	1	1	1	1	1

b) ✓ e) ✓

i) $(\overbrace{\neg(A \vee B)}^{i_1}) \Leftrightarrow (\overbrace{\neg A \wedge \neg B}^{i_2})$, j) $(\overbrace{\neg(A \wedge B)}^{j_1}) \Leftrightarrow (\overbrace{\neg A \vee \neg B}^{j_2})$

A	B	A∨B	i_1	¬A	¬B	i_2	i_1⇔i_2	A∧B	j_1	j_2	j_1⇔j_2
1	1	1	0	0	0	0	1	1	0	0	1
1	0	1	0	0	1	0	1	0	1	1	1
0	1	1	0	1	0	0	1	0	1	1	1
0	0	0	1	1	1	1	1	0	1	1	1

i) ✓ j) ✓

k) $(\overbrace{\neg(A \Rightarrow B)}^{k_1}) \Leftrightarrow (\overbrace{A \wedge \neg B}^{k_2})$, l) $(\overbrace{\neg(A \Leftrightarrow B)}^{l_1}) \Leftrightarrow ((\overbrace{A \wedge \neg B}^{l_2}) \vee (\overbrace{B \wedge \neg A}^{l_3}))$

A	B	A⇒B	k_1	¬B	k_2	k_1⇔k_2	A⇔B	l_1	l_2	¬A	l_3	l_2∨l_3	l_1⇔(l_2∨l_3)
1	1	1	0	0	0	1	1	0	0	0	0	0	1
1	0	0	1	1	1	1	0	1	1	0	0	1	1
0	1	1	0	0	0	1	0	1	0	1	1	1	1
0	0	1	0	1	0	1	1	0	0	1	0	0	1

k) ✓ l) ✓

Obecný kvantifikátor: $\forall x \in M : P(x)$ (pro každý prvek x z množiny M platí výrok $P(x)$)

Existenční kvantifikátor: $\exists x \in M : P(x)$ (existuje prvek x z množiny M takový, že $P(x)$ platí)

Pozn. $\exists!$ (existuje právě jeden)

Platí: 1, $\neg(\forall x \in M \; P(x)) \Leftrightarrow (\exists x \in M \; \neg P(x))$

2, $\neg(\exists x \in M \; P(x)) \Leftrightarrow (\forall x \in M \; \neg P(x))$

7, Zapište negaci výroku

$\exists x \in \mathbb{R} : \cos x = \sqrt{1-\sin^2 x}$

a rozhodněte, který z výroků je pravdivý.

Negace: $\forall x \in \mathbb{R} : \cos x \neq \sqrt{1-\sin^2 x}$

$x = \frac{\pi}{2}$: $\cos x = 0 = \sqrt{1-\sin^2 x}$ $\Rightarrow$ první výrok je pravdivý např. pro $x = \frac{\pi}{2}$

($\forall x : \cos x \geq 0$: $\cos x = \sqrt{1-\sin^2 x}$

$\cos^2 x = 1 - \sin^2 x$

$1 = \cos^2 x + \sin^2 x$ ✓)

8, Platí následující výroky?

a, $\forall a \in \mathbb{R} \; \exists \varepsilon > 0 \; \exists \alpha \in \mathbb{R} \; \forall x \in (a, a+\varepsilon) : x \in (a, a+\varepsilon) \Leftrightarrow |x-\alpha| < 1$

Výrok je pravdivý: $(a, a+\varepsilon) = (\alpha-1, \alpha+1)$

fix $a \in \mathbb{R}$, vol $\varepsilon = 2$ (délka intervalu) a $\alpha = a+1$ (střed)

b, $\exists a \in \mathbb{R} \; \forall \varepsilon > 0 \; \forall \alpha \in \mathbb{R} \; \exists x \in (a, a+\varepsilon) : x \in (a, a+\varepsilon) \Leftrightarrow |x-\alpha| < 1$

Negace: $\forall a \in \mathbb{R} \; \exists \varepsilon > 0 \; \exists \alpha \in \mathbb{R} \; \forall x \in (a, a+\varepsilon) : \neg \underbrace{(x \in (a, a+\varepsilon) \Leftrightarrow |x-\alpha| < 1)}_{P(x, a, \varepsilon, \alpha)}$

Už víme z a, že výrok $\forall a \in \mathbb{R} \; \exists \varepsilon > 0 \; \exists \alpha \in \mathbb{R} \; \forall x \in (a, a+\varepsilon) : P(x, a, \varepsilon, \alpha)$ je pravdivý.

To ale nestačí (př. $\exists x : P(x)$ i $\exists x : \neg P(x)$ oboje mohou být pravda)

$\neg P(x, a, \varepsilon, \alpha) = (x \in (a, a+\varepsilon) \wedge |x-\alpha| \geq 1) \vee (x \notin (a, a+\varepsilon) \wedge |x-\alpha| < 1)$

fix $a \in \mathbb{R}$, vol $\varepsilon = 1$, $\alpha = a+3$: $x \in (a, a+1) \wedge |x-(a+3)| > 2 \; (\geq 1)$

Tj výrok nie je pravdivý.

9) Dokažte:

a) $C \setminus (A \cup B) = (C \setminus A) \cap (C \setminus B)$

$$x \in C \setminus (A \cup B) \iff x \in C \wedge x \notin (A \cup B)$$
$$\iff (x \in C \wedge x \notin A) \wedge (x \in C \wedge x \notin B)$$
$$\iff (x \in C \setminus A) \wedge (x \in C \setminus B) \iff x \in (C \setminus A) \cap (C \setminus B)$$

b) $C \setminus (A \cap B) = (C \setminus A) \cup (C \setminus B)$

$$x \in C \setminus (A \cap B) \iff x \in C \wedge x \notin (A \wedge B)$$
$$\iff (x \in C \wedge x \notin A) \vee (x \in C \wedge x \notin B) \iff x \in (C \setminus A) \cup (C \setminus B)$$

c) Nechť A_i, $i = 1, 2, \dots$ je systém libovolných množin a nechť $B_n = \bigcup_{i=1}^{n} A_i$. Potom $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$.

Dk: $B_N = \bigcup_{i=1}^{N} A_i \Rightarrow B_1 \subset B_2 \subset \dots$

$$\bigcup_{i=1}^{N} A_i = B_N = \bigcup_{i=1}^{N} B_i \quad \forall N \in \mathbb{N}$$

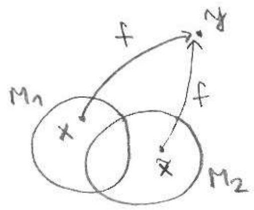
nebo: $[(A = B) \iff (A \subset B \wedge B \subset A)]$

„$\subset$" $x \in \bigcup_{n=1}^{\infty} A_n \Rightarrow \exists i \in \mathbb{N} : x \in A_i \Rightarrow x \in B_i \Rightarrow x \in \bigcup_{n=1}^{\infty} B_n$

„$\supset$" $x \in \bigcup_{n=1}^{\infty} B_n \Rightarrow \exists j \in \mathbb{N} : x \in B_j = \bigcup_{i=1}^{j} A_i \Rightarrow x \in \bigcup_{n=1}^{\infty} A_n$

10) Dokažte, že je-li f zobrazení, pak $f(M_1) \setminus f(M_2) \subset f(M_1 \setminus M_2)$.
Kdy platí rovnost? (M_1, M_2 jsou podmnožiny $\mathcal{D}_f$)

Dk: $y \in f(M_1) \setminus f(M_2) \iff y \in f(M_1) \wedge y \notin f(M_2)$



$$\iff (\exists x \in M_1 : y = f(x)) \wedge (\forall x \in M_2 : y \neq f(x))$$
$$! \quad \Rightarrow \exists x \in M_1 \setminus M_2 : y = f(x)$$
$$\iff y \in f(M_1 \setminus M_2)$$

Rovnost? f prosté: $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ $\quad ((\iff (f(x_1) = f(x_2) \Rightarrow x_1 = x_2)))$

! „$\Leftarrow$": $(\exists x \in M_1 \setminus M_2 : y = f(x)) \overset{\text{prostota}}{\wedge} (\forall \tilde{x} \in \mathcal{D}_f ; \tilde{x} \neq x \Rightarrow f(\tilde{x}) \neq f(x) = y)$

$$\Rightarrow (\exists x \in M_1 : y = f(x)) \wedge (\forall x \in M_2 : f(x) \neq y)$$

Množina je soubor prvků, u kterých lze rozhodnout, zda do dané množiny patří, či nikoliv.

Množiny zadáváme:
- výčtem prvků ($M = \{1, 2, 3\}$)
- zadáním vlastností prvků ($M = \{n : n$ je prvočíslo$\}$)
- pomocí už známé množiny ($M = \mathbb{N} \cap \langle 1, 3 \rangle$)

Značení: (i) $x \in M$ (x patří do množiny M) (je prvkem M)

$x \notin M$ nepatří není

(ii) $P \subset M \iff \forall p \in P \;\; p \in M$ (podmnožina)

(iii) $P = M \iff (P \subset M) \wedge (M \subset P)$ (rovnost množin)

(iv) $P \cup M = \{x : (x \in M) \vee (x \in P)\}$ (sjednocení množin)

(v) $P \cap M = \{x : (x \in M) \wedge (x \in P)\}$ (průnik množin)

(vi) $P \subsetneq M \iff (P \subset M) \wedge (P \neq M)$ (vlastní podmnožina)

(vii) $\emptyset$ prázdná množina (je podmnožinou každé množiny)

(viii) $M \setminus P = \{x \in M : x \notin P\}$

Zobrazení: Nechť A, B, D jsou množiny a $D \subset A$. Nechť každému prvku $x \in D$ je přiřazeno právě jedno $y_x \in B$. Označme $\varphi(x) := y_x$.

Pak říkáme, že $\varphi : A \to B$ je zobrazení z množiny A do množiny B.

Množinu D nazýváme definičním oborem φ a značíme ji $\mathcal{D}_\varphi$.

Množina $\varphi(D) := \{\varphi(x) : x \in D\}$ se nazývá obor hodnot φ (R_φ).

Množina $\varphi^{-1}(\{y\}) := \{x \in D : \varphi(x) = y\}$ se nazývá vzorem prvku y.

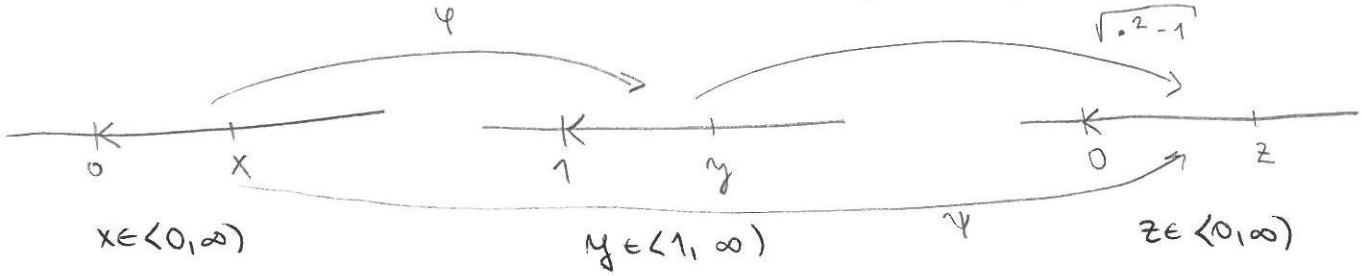
Jestliže $x_1 \neq x_2 \Rightarrow \varphi(x_1) \neq \varphi(x_2)$, φ se nazývá prosté (injektivní).

Je-li $R_\varphi = B$, řekneme, že $\varphi : A \to B$ je na (surjektivní).

Je-li φ prosté a na, říkáme, že φ je vzájemně jednoznačné (bijektivní).

11) Nechť $\varphi: [0,\infty) \to [1,\infty)$ je bijekce a nechť $\psi(x) = \sqrt{\varphi(x)^2 - 1}$.

Dokažte, že existuje inverzní funkce ψ^{-1} a vyjádřete ji pomocí φ^{-1}. Určete $D_{\psi^{-1}}$.



φ bijekce $\iff \forall y \in \langle 1, \infty) \ \exists!\, x \in \langle 0, \infty) : y = \varphi(x)$ (1)

$y \geq 1 \implies \varphi^2(x) - 1 \geq 0 \quad \forall x \in \langle 0, \infty) \implies \sqrt{\varphi(x)^2 - 1}$ definovaná jednoznačně $\forall x \in \langle 0, \infty)$

$\implies \forall z \in \langle 0, \infty) \ \exists!\, y \in \langle 1, \infty) : z = \sqrt{y^2 - 1}$ (2)

$\overset{z \geq 0}{\iff} z^2 = y^2 - 1 \iff z^2 + 1 = y^2 \overset{y \geq 0}{\iff} y = \sqrt{z^2 + 1}$

$(1) + (2) \implies \forall z \in \langle 0, \infty) \ \exists!\, x \in \langle 0, \infty) : z = \sqrt{\varphi(x)^2 - 1}$.

$z = \psi(x)$ je tedy bijekce a proto má inverzní zobrazení.

$\varphi(x) = y = \sqrt{z^2 + 1}$

$x = \varphi^{-1}(\varphi(x)) = \varphi^{-1}(\sqrt{z^2 + 1}) = \psi^{-1}(z)$ je hledaná inverze.

$D_{\psi^{-1}} = R_{\psi} = \langle 0, \infty)$.

Důkazy. Tvrzení: Nechť platí A, potom platí B.

Přímý důkaz. $A \Rightarrow B$

~~Důkaz sporem.~~ $\neg B \Rightarrow \neg A$ (6g1 $\Leftrightarrow A \Rightarrow B$) Nepřímý důkaz.

Důkaz matematickou indukcí se používá pro tvrzení typu "ukažte, že pro všechna $n \in \mathbb{N}$ platí $P(n)$". Postup:

1, ukažte, že platí $P(1)$,

2, ukažte, že pokud platí $P(n)$, potom platí i $P(n+1)$ pro všechny $n \geq 1$.

(Může se stát, že tvrzení dokazujeme jen pro $n \geq n_0$; $n_0 \in \mathbb{N}$.)

Důkaz sporem. $A \wedge \neg B \Rightarrow$ nepravdu, spor.

Supremum, infimum

Nechť X je množina. Relací na $X \times X$ rozumíme libovolnou podmnožinu $X \times X$.

Relaci R na $X \times X$ nazveme (částečným) uspořádáním, jestliže:

(i) $x \in X \Rightarrow (x,x) \in R$ (reflexivita)

(ii) $(x,y) \in R \wedge (y,z) \in R \Rightarrow (x,z) \in R$ (tranzitivita)

(iii) $(x,y) \in R \wedge (y,x) \in R \Rightarrow x = y$ (antisymetrie).

Pokud navíc platí: $x,y \in X \Rightarrow (x,y) \in R \vee (y,x) \in R$, relaci R nazveme úplným uspořádáním a značíme jí symbolem $\leq$ ($x \leq y \Leftrightarrow (x,y) \in R$).

Nechť A je množina s úplným uspořádáním a $B \subset A$. Řekneme, že množina B je omezená shora, jestliže existuje její horní závora v A ($\exists y \in A \ \forall x \in B: x \leq y$).
omezená zdola dolní $x \geq y$

Řekneme, že množina B je omezená, jestliže je omezená shora i zdola.

Prvek $M \in B$ nazveme maximem množiny B ($M = \max B$), je-li horní závorou B ($\forall x \in B: x \leq M$).
$m \in B$ minimem ($m = \min B$) dolní ($\forall x \in B: x \geq m$).

Prvek $S \in A$ nazveme supremem B ($S = \sup B$), jestliže je nejmenší horní závorou množiny B:

1, $\forall x \in B: x \leq S$ (horní závora)

2, $(y \in A \wedge y < S) \Rightarrow (\exists x \in B: x > y)$ (nejmenší, tj když ji zmenšíme, už nebude závorou).

Prvek $s \in A$ nazveme infimem B ($s = \inf B$), jestliže je ~~největší~~ největší dolní závorou B:

1, $\forall x \in B: x \geq s$ (dolní závora)

2, $(y \in A \wedge y > s) \Rightarrow (\exists x \in B: x < y)$ (největší).

Omezená množina nemusí nabývat max nebo min. Sup a inf má ale vždy.

Př. $B = \langle 0,1)$. $\max B \nexists$, $\min B = \inf B = 0$, $\sup B = 1$.

Opakování II , 1/2

Matematická indukce

Dokažte matematickou indukcí následující rovnosti a nerovnosti

1. $P(n): \quad 1^2+2^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$

$P(1): \quad 1^2 \overset{?}{=} \frac{1(1+1)(2+1)}{6} = \frac{1.2.3}{6} = 1$ ✓

$P(n+1): \quad \underbrace{1^2+2^2+\dots+n^2}+(n+1)^2 \overset{?}{=} \frac{(n+1)(n+2)(2n+3)}{6}$, $P(n)$ (=IP) ✓

$$\frac{n(n+1)(2n+1)}{6}+(n+1)^2 = (n+1)\left(\frac{n(2n+1)}{6}+n+1\right) = \frac{(n+1)}{6}\left(2n^2+n+6n+6\right) = \frac{n+1}{6}(n+2)(2n+3)$$

2. $P(n): \quad 1^3+2^3+\dots+n^3 = (1+2+\dots+n)^2$

$P(1): \quad 1^3 = 1^2$ ✓

$P(n+1): \quad \underbrace{1^3+2^3+\dots+n^3}+(n+1)^3 \overset{?}{=} (\underbrace{1+2+\dots+n}_{a}+\underbrace{n+1}_{b})^2$, IP

$(1+2+\dots+n)^2+(n+1)^{\cancel{3}\,2} \overset{?}{=} (1+2+\dots+n)^2+2(1+2+\dots+n)\cancel{(n+1)}+(n+1)^2$, $1+2+\dots+n = \frac{n(n+1)}{2}$

$(n+1)^2 = n(n+1)+n+1 = (n+1)(n+1)$ ✓

3. $P(n): \quad \prod_{i=1}^{n}(1+x_i) \ge 1+\sum_{i=1}^{n}x_i$, $x_i \ge -2$, x_i mají stejná znaménka

$P(1): \quad 1+x_1 \ge 1+x_1 \qquad \forall x_1 \ge -2$

$P(n+1): \quad \prod_{i=1}^{n+1}(1+x_i) \ge 1+\sum_{i=1}^{n+1}x_i$

a) $x_i = 0 \quad \forall i$ jasné

b) $x_i \ge 0 \quad \forall i$

$$\prod_{i=1}^{n+1}\underbrace{(1+x_i)}_{\ge 0\ \forall i} = \prod_{i=1}^{n}(1+x_i)\underbrace{(1+x_{n+1})}_{\ge 0} \overset{IP}{\ge} \left(1+\sum_{i=1}^{n}x_i\right)(1+x_{n+1}) = 1+\sum_{i=1}^{n}x_i + x_{n+1} + \left(\sum_{i=1}^{n}x_i\right)\cdot x_{n+1}$$

$\overset{(*)}{\ge} 1+\sum_{i=1}^{n+1}x_i$, x_i mají st. zn. $\Rightarrow (\sum_{i=1}^{n}x_i)$ má st. zn. jako $x_{n+1} \Rightarrow (\sum_{i=1}^{n}x_i)x_{n+1} \ge 0$ (*)

c) $x_i \in \langle -1,0) \quad \forall i$ rovnako ako b) , $1+x_i \ge 0 \quad \forall i$ & (*)

d) $x_i \in \langle -2,-1) \quad \forall i$, $1+x_i \in \langle -1,0) \quad \forall i$

$P(1): \quad 1+x_1 \ge 1+x_1$

$P(2): \quad (1+x_1)(1+x_2) = 1+x_1+x_2+\overbrace{x_1x_2}^{\ge 0} \ge 1+x_1+x_2$

$P(n+2): \quad \prod_{i=1}^{n+2}(1+x_i) = \prod_{i=1}^{n}(1+x_i)\underbrace{(1+x_{n+1})(1+x_{n+2})}_{\ge 0} \ge \left(1+\sum_{i=1}^{n}x_i\right)(1+x_{n+1}+x_{n+2}+x_{n+1}x_{n+2})$

$$= 1+\sum_{i=1}^{n}x_i + x_{n+1} + x_{n+2} + x_{n+1}x_{n+2} + \left(\sum_{i=1}^{n}x_i\right)(x_{n+1}+x_{n+2}+x_{n+1}x_{n+2})$$

$$= 1+\sum_{i=1}^{n+2}x_i + \underbrace{x_{n+1}x_{n+2}}_{\ge 0} + \underbrace{\left(\sum_{i=1}^{n}x_i\right)}_{\le 0}\Big[\underbrace{\underbrace{(1+x_{n+1})(1+x_{n+2})}_{\in(0,1\rangle}-1}_{\in(-1,0\rangle}\Big] \ge 1+\sum_{i=1}^{n+2}x_i$$

e) $x_i \in \langle -2,0) \quad \forall i$, $M_1 := \{i: x_i \in \langle -2,-1)\}$, $M_2 := \{j: x_j \in \langle -1,0)\}$

$P(n): \quad \prod_{i=1}^{n}(1+x_i) = \prod_{i\in M_1}(1+x_i)\underbrace{\prod_{j\in M_2}(1+x_j)}_{\ge 0} \overset{d)}{\ge} \left(1+\underbrace{\sum_{i\in M_1}x_i}_{\le 0}\right)\prod_{j\in M_2}(1+x_j)$

$\forall n \in \mathbb{N}$

$$\overset{c)}{\ge} 1+\sum_{j\in M_2}x_j + \underbrace{\left(\sum_{i\in M_1}x_i\right)}_{\le 0}\underbrace{\prod_{j\in M_2}(1+x_j)}_{\in\langle 0,1\rangle \ \& \le 1} \ge 1+\sum_{j\in M_2}x_j + \left(\sum_{i\in M_1}x_i\right).1 = 1+\sum_{i=1}^{n}x_i$$

4. $P(n): (a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$ (binomická věta)

$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$, $P(1):\ a+b = \binom{1}{0} a^1 b^0 + \binom{1}{1} a^0 b^1$ ✓

$P(n+1)$ LHS: $(a+b)^{n+1} = (a+b)^n (a+b) \overset{IP}{=} \underbrace{\sum_{k=0}^{n} \binom{n}{k} a^{n-k+1} b^k}_{=:A} + \underbrace{\sum_{k=0}^{n} a^{n-k} b^{k+1}}_{=:B}$

RHS: $\sum_{k=0}^{n+1} \binom{n+1}{k} a^{n+1-k} b^k =: (*)$

$$\binom{n+1}{k} = \frac{(n+1)!}{(n+1-k)!\,k!} = \frac{n+1}{n+1-k}\,\frac{n!}{(n-k)!\,k!} = \frac{n+1}{n+1-k}\binom{n}{k} = \left(1 + \frac{k}{n+1-k}\right)\binom{n}{k} \qquad (1)$$

$$(*) = \sum_{k=0}^{n} \binom{n+1}{k} a^{n+1-k} b^k + \binom{n+1}{n+1} a^0 b^{n+1}$$

$$= \underbrace{\sum_{k=0}^{n} \binom{n}{k} a^{n+1-k} b^k}_{=A} + \underbrace{\sum_{k=0}^{n} \binom{n}{k} \frac{k}{n+1-k} a^{n+1-k} b^k + b^{n+1}}_{=:\tilde{B}} = A + \tilde{B}$$

RHS = LHS $\iff A + B = A + \tilde{B} \iff B = \tilde{B}$

$$B = \binom{n}{0} a^n b^1 + \binom{n}{1} a^{n-1} b^2 + \dots + \binom{n}{n-1} a^1 b^n + \binom{n}{n} a^0 b^{n+1}$$

$$\tilde{B} = \binom{n}{0}\cdot 0 \cdot a^{n+1} b^0 + \binom{n}{1}\frac{1}{n} a^n b^1 + \binom{n}{2}\frac{2}{n-1} a^{n-1} b^2 + \dots + \binom{n}{n}\cdot\frac{n}{1} a^1 b^n + b^{n+1} \qquad (2)$$

$B = \tilde{B} \iff \binom{n}{0} = \binom{n}{1}\frac{1}{n} \iff \binom{n}{k} = \binom{n}{k+1}\cdot\frac{k+1}{n+1-(k+1)} = \binom{n}{k+1}\cdot\frac{k+1}{n-k} \quad \forall k \le n-1$

$\binom{n}{1} = \binom{n}{2}\frac{2}{n-1}$ $\qquad \frac{n!}{(n-k)!\,k!} = \frac{n!}{(n-k-1)!\,(k+1)!}\cdot\frac{k+1}{n-k}$ ✓

$\dots$

$\binom{n}{n-1} = \binom{n}{n}\cdot n$

5. $\sum_{k=0}^{n} \binom{n}{k} = 2^n \ : P(n)$

$P(1):\ \binom{1}{0} + \binom{1}{1} = 1 + 1 = 2 = 2^1$ ✓

$P(n+1):\ \sum_{k=0}^{n+1} \binom{n+1}{k} \overset{?}{=} 2^{n+1}$

$$\sum_{k=0}^{n} \binom{n+1}{k} + \binom{n+1}{n+1} \overset{(1)}{=} \sum_{k=0}^{n} \binom{n}{k} + \overbrace{\sum_{k=0}^{n} \frac{k}{n+1-k}\binom{n}{k}}^{=0 \text{ pro } k=0} + 1 = 1 + \sum_{k=0}^{n}\binom{n}{k} + \sum_{k=1}^{n} \frac{k}{n+1-k}\binom{n}{k} \qquad l = k-1$$

$$= \underbrace{1}_{=\binom{n}{n}} + \sum_{k=0}^{n}\binom{n}{k} + \sum_{l=0}^{n-1}\binom{n}{l+1}\frac{l+1}{n-l} \overset{(2)}{=} \sum_{k=0}^{n}\binom{n}{k} + \sum_{l=0}^{n-1}\binom{n}{l} + \binom{n}{n} = 2\sum_{k=0}^{n}\binom{n}{k} \overset{IP}{=} 2^{n+1}$$ ✓

6. $P(n):\ \underbrace{\sqrt[n]{x_1 \dots x_n}}_{\text{geom.pr.}} \le \underbrace{\frac{1}{n}(x_1 + \dots + x_n)}_{\text{aritmetický priemer}}$, $x_i \ge 0\ \forall i$ (AG nerovnost)

a) niektoré z nich $= 0$ .. jasné, $0 \le \dots$

b) $x_i \ne 0\ \forall i$, def. $y_i := \frac{x_i}{\sqrt[n]{x_1 \dots x_n}}$, potom $y_1 \dots y_n = \frac{x_1 \dots x_n}{(\sqrt[n]{x_1 \dots x_n})^n} = 1$

$P(n) \iff y_1 + \dots + y_n = \frac{x_1 + \dots + x_n}{\sqrt[n]{x_1 \dots x_n}} \overset{?}{\ge} n$

$P(1):\ \frac{x_1}{x_1} = 1 \ge 1$ ✓

$P(n+1):\ y_1 \dots y_{n-1} \underbrace{(y_n y_{n+1})}_{y_n'} = 1$, IP:

všetky $= 1$ .. jasné, $\frac{n}{1} \ge n$

nech $\exists\, i:\ y_i < 1 \Rightarrow \exists\, j:\ y_j > 1$

BÚNO: usporiadajme: $i = n,\ j = n+1$, $(1 - y_n)(y_{n+1} - 1) \ge 0$

$y_{n+1} + y_n - y_n y_{n+1} - 1 > 0$

$y_{n+1} + y_n > y_n y_{n+1} + 1 \qquad (+)$

$y_1 + \dots + y_{n+1} - 1 \overset{(+)}{\ge} y_1 + y_2 + \dots + y_{n-1} + y_n y_{n+1} \ge n$

$\Rightarrow y_1 + \dots + y_{n+1} \ge n+1$

7. P(n): $n! \le \left(\frac{n+1}{2}\right)^n$

P(1): $1! = 1 \le \left(\frac{1+1}{2}\right)^1 = 1$ ✓

P(n+1): $(n+1)! = (n+1)\,n! \overset{IP}{\le} (n+1)\left(\frac{n+1}{2}\right)^n \overset{?}{\le} \left(\frac{n+2}{2}\right)^{n+1}$

LHS: $(n+1)! \overset{IP}{\le} 2\left(\frac{n+1}{2}\right)^{n+1} = \frac{(n+1)^{n+1}}{2^n}$

$\frac{(n+1)^{n+1}}{2^n} \le \frac{(n+2)^{n+1}}{2^{n+1}} \iff \left(\frac{n+1}{n+2}\right)^{n+1} \le \frac{1}{2}$ (*)   sama o sobě důležitá posloupnost

Tvrzení: $a_n := \left(\frac{n}{n+1}\right)^n$ je klesající posloupnost.

DK: AG nerovnost, $x_1,\dots,x_n = 1+\frac{1}{n}$, $x_{n+1} = 1$

$\sqrt[n+1]{x_1\cdots x_{n+1}} = \sqrt[n+1]{\left(1+\frac{1}{n}\right)^n} \le \frac{1}{n+1}\left(n\left(1+\frac{1}{n}\right)+1\right) = 1+\frac{1}{n+1}$ $/^{n+1}$

$\left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n \le \left(1+\frac{1}{n+1}\right)^{n+1} = \left(\frac{n+2}{n+1}\right)^{n+1} \iff \underbrace{\left(\frac{n}{n+1}\right)^n}_{=a_n} \ge \underbrace{\left(\frac{n+1}{n+2}\right)^{n+1}}_{=a_{n+1}} \Rightarrow$ kles.

$\frac{1}{2} = a_1 \ge a_2 \ge \dots \ge a_{n+1} = \left(\frac{n+1}{n+2}\right)^{n+1}$

8. P(n): $(2n)! < 2^{2n}(n!)^2$

P(1): $2 = 2! < 2^2(1!)^2 = 4$ ✓

P(n+1): $(2n+2)! \overset{?}{<} 2^{2n+2}((n+1)!)^2$

LHS: $(2n+2)! = (2n+2)(2n+1)(2n)! \overset{IP}{<} 2^{2n}(n!)^2\cdot 2(n+1)\cdot(2n+1)$

RHS: $2^{2n+2}((n+1)!)^2 = 4\cdot 2^{2n}\cdot (n+1)^2(n!)^2$

LHS < RHS $\iff \cancel{2}\cancel{(n+1)}(2n+1)\,\cancel{2^{2n}}\cancel{(n!)^2} \le \overset{2}{\cancel{4}}\cdot\cancel{2^{2n}}\cdot(n+1)^{\cancel{2}}\cancel{(n!)^2}$

$\cancel{2n^2+3n+1 \overset{?}{<} 2n^2+4n+2}$ $2n+1 < 2n+2$ ✓

9. P(n): $\left|\sin\left(\sum_{k=1}^{n} x_k\right)\right| \le \sum_{k=1}^{n} \sin x_k$, $x_k \in \langle 0,\pi\rangle$, $k = 1,\dots,n$

P(1): $|\sin x_1| \le \sin x_1$ $\forall x \in \langle 0,\pi\rangle$ platí $|\sin x| = \sin x$

P(n+1): $\left|\sin\left(\sum_{k=1}^{n+1} x_k\right)\right| = \left|\sin\left(\sum_{k=1}^{n} x_k + x_{n+1}\right)\right|$ $\sin(\varphi_1+\varphi_2) = \sin\varphi_1\cos\varphi_2 + \cos\varphi_1\sin\varphi_2$

$= \left|\sin\left(\sum_{k=1}^{n} x_k\right)\cos x_{n+1} + \sin x_{n+1}\cos\left(\sum_{k=1}^{n} x_k\right)\right|$

$\le \left|\sin\left(\sum_{k=1}^{n} x_k\right)\right|\cdot\underbrace{|\cos x_{n+1}|}_{\le 1} + |\sin x_{n+1}|\cdot\underbrace{\left|\cos\left(\sum_{k=1}^{n} x_k\right)\right|}_{\le 1} \overset{IP}{\le} \sum_{k=1}^{n}\sin x_k + \sin x_{n+1} = \sum_{k=1}^{n+1}\sin x_k$ ✓

10. P(n): $\frac{1}{2}\frac{3}{4}\frac{5}{6}\cdots\frac{2n-1}{2n} < \frac{1}{\sqrt{2n+1}}$

P(1): $\frac{1}{2} < \frac{1}{\sqrt{3}}$ ✓ $(\sqrt{3} < 2 = \sqrt{4})$

P(n+1): $\frac{1}{2}\frac{3}{4}\cdots\frac{2n+1}{2n+2} \overset{IP}{<} \frac{1}{\sqrt{2n+1}}\frac{2n+1}{2n+2} \overset{?}{<} \frac{1}{\sqrt{2n+3}} \iff \underbrace{\frac{2n+1}{2n+2}}_{<1} < \sqrt{\frac{2n+1}{2n+3}} \left(< \sqrt{\frac{2n+1}{2n+2}}\right)$

$\forall a < 1,\ a < \sqrt{a}$

11. $P(n): \quad n^{n+1} > (n+1)^n \quad , n \geq 3$

$P(3): \quad 3^4 \overset{?}{>} 4^3 \qquad 3^4 = (3^2)^2 \overset{?}{>} (2^2)^3 = (2^3)^2 = 4^3$

$81 = 9^2 > 8^2 = 64$ ✓

$P(n+1): \quad (n+1)^{(n+2)} > (n+2)^{n+1} \iff (n+1)(n+1)^{n+1} > (n+2)^{n+1} \iff n+1 > \left(\frac{n+2}{n+1}\right)^{n+1} = \left(1+\frac{1}{n+1}\right)^{n+1}$

IP: $\quad n > \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n > \left(1+\frac{1}{n+1}\right)^n \qquad / \cdot \left(1+\frac{1}{n+1}\right)$

$n+1 > n + \underbrace{\frac{n}{n+1}}_{<1} > \left(1+\frac{1}{n+1}\right)^{n+1}$ ✓

12. U následujících množin naleznete sup, inf, max a min (pokud $\exists$). Ověřte z definice!

a) $M = (0,1\rangle$. $m = \min M \in M: \quad \forall x \in M, x \geq m$

$\tilde{M} = \max M \in M: \quad \forall x \in M, x \leq \tilde{M}$

$S = \sup M: \ (\forall x \in M: x \leq S) \ \& \ (y \in \mathbb{R} \wedge y < S \Rightarrow \exists x \in M, x > y)$

$s = \inf M: \ (\forall x \in M: x \geq s) \ \& \ (y \in \mathbb{R} \wedge y > s \Rightarrow \exists x \in M: y \geq x)$

~~$\min M \not\exists: \neg(\forall x \in M \ x \geq m)$~~

$\min M \not\exists: \neg(\exists m \in M \ \forall x \in M: x \geq m) \iff \forall m \in M \ \exists x \in M: x < m$ (napr. $x = \frac{m}{2}$)

$\max M = 1: \quad \forall x \in M, x \leq 1$

$\inf M = 0: \ (\forall x \in M: x \geq 0) \ \& \ (y \in \mathbb{R}, y > 0 \Rightarrow \exists x \in M: y \geq x)$ (napr. $x = \frac{y}{2}$)

$\sup M = 1: \ (\forall x \in M: x \leq 1) \ \& \ (y \in \mathbb{R}, y < 1 \Rightarrow \exists x \in M: x \geq y)$ ($y < 1 \iff y = 1 - \varepsilon$, $x := 1 - \frac{\varepsilon}{2}$)

alebo: M omez. & $\max M \ \exists \Rightarrow \sup M \ \exists \wedge \sup M = \max M$ alebo $x = 1$

b) $M = \langle 0,1\rangle: \quad \inf M = \min M = 0, \quad \sup M = \max M = 1$

c) $M = (0,\infty): \quad \inf M = 0, \ \min M \not\exists, \ \sup M \not\exists, \ \max M \not\exists$

d) $M = \{\frac{m}{n}; m,n \in \mathbb{N}\}: \quad \inf M = 0, \ \min M \not\exists, \ \sup M \not\exists, \ \max M \not\exists$

e) $M = \{0{,}5; 0{,}55; 0{,}555; \dots\}: \ \sup M = 0{,}\overline{5} = \max M, \ \inf M = 0{,}5 = \min M$

f) $M = \{x \in \mathbb{Q}, x^2 < 3\}: \ \sup M = \sqrt{3}, \ \max M \not\exists \ (\sqrt{3} \notin \mathbb{Q}), \ \inf M = -\sqrt{3}, \ \min M \not\exists$

$\sqrt{3} \notin \mathbb{Q}$: keby áno, $\sqrt{3} = \frac{m}{n}$, BÚNO $\frac{m}{n}$ prvočíselný rozklad

potom $3 = \frac{m^2}{n^2}$, $\frac{m^2}{n^2}$ všetky prvočísla 2x vedľa seba (neopakujú sa), 3 je tiež prvočíslo ale v lichej mocnine, spor

13. Nechť A, B jsou neprázdné omezené podmnožiny $\mathbb{R}^*$. Dokažte:

a) $\inf(-A) = -\sup A$

$-A := \{-x : x \in A\}$

DK: $S := \sup A \Rightarrow \forall x \in A: x \leq S \iff -S \leq -x$

$(y \in \mathbb{R}, y < S \Rightarrow \exists x \in A, x > y) \iff (z \in \mathbb{R}, z > -S \ (z = -y) \Rightarrow \exists x \in A, \underset{-x \in -A}{-x} < -y = z)$

b) $\sup(A+B) = \sup A + \sup B$

$A+B := \{z: z = x+y, \ x \in A, y \in B\}$

1. vlastnosť: $\quad x + y \leq \sup A + \sup B \qquad \forall x \in A, \forall y \in B \Rightarrow$ aj pre sup z $x+y$

$\sup(A+B) \leq \sup A + \sup B$

2. vlastnosť: $A: \quad y_A := \sup A - \frac{\varepsilon}{2} \Rightarrow \exists x_A \in A: x_A > \sup A - \frac{\varepsilon}{2}$

$B: \quad y_B := \sup B - \frac{\varepsilon}{2} \Rightarrow \exists x_B \in B: x_B > \sup B - \varepsilon/2$

$+$

$\exists x_A, x_B: \quad x_A + x_B > \sup A + \sup B - \varepsilon$

$\Rightarrow \exists z \in A+B, \quad z = x_A + x_B > \sup A + \sup B - \varepsilon \qquad / \sup(x_A + x_B)$

$\sup(A+B) \geq \sup A + \sup B$

RSJ

c) $\inf(A-B) = \inf A - \sup B$

$\inf(A-B) \overset{a)}{=} -\sup(B-A) = -\sup(B+(-A)) \overset{b)}{=} -\sup B - \sup(-A) \overset{a)}{=} \inf A - \sup B$

d) $\sup(A.B) = \sup A . \sup B$, A, B obs. pouze nezáp. prvky

$A.B = \{z : z = x.y,\ x\in A,\ y\in B\}$

$\sup A . \sup B = 0 \Rightarrow A = \{0\} \vee B = \{0\} \Rightarrow A.B = \{0\} \Rightarrow \sup(A.B) = 0$

$\forall x\in A\ \forall y\in B:\ \underset{A.B}{\underset{\cap}{x.y}} \le \sup A . \sup B \Rightarrow$ 1. vl. $\sup A.\sup B =: \bar{S}$

$\Rightarrow \bar{S} = \sup(A.B)$

buď $\varepsilon > 0$, $S' := \sup A.\sup B - \varepsilon$, $S' < \sup A . \sup B$

$\Rightarrow \exists \bar{x}\in A:\ \bar{x} > \sup A - \frac{\varepsilon}{\sup A + \sup B} \ge 0$

$\exists \bar{y}\in B:\ \bar{y} > \sup B - \frac{\varepsilon}{\sup A + \sup B} \ge 0$

$\Rightarrow \underset{A.B}{\underset{\cap}{\bar{x}.\bar{y}}} > \sup A \sup B - \frac{\varepsilon}{\sup A + \sup B}(\sup A + \sup B) + \frac{\varepsilon^2}{(\sup A + \sup B)^2} \ge \sup A . \sup B - \varepsilon = S'$

$\Rightarrow \sup(A.B) \ge \sup A . \sup B$

14. Nechť A, B jsou neprázdné omezené podmnožiny $\mathbb{R}$. Lze obecně vyjádřit $\sup(A\cup B)$ a $\sup(A\cap B)$ pomocí $\sup A$ a $\sup B$?

$\sup(A\cup B) = \max\{\sup A, \sup B\}$ Úloha 2.2.15

$\sup(A\cap B) \le \min\{\sup A, \sup B\}$

15. Nechť M je neprázdná množina a nechť $f: M\to\mathbb{R}$, $g: M\to\mathbb{R}$ jsou omezené fce. Dokažte, že

a) $\sup_{x\in M}(f(x)+g(x)) \le \sup_{x\in M} f(x) + \sup_{x\in M} g(x)$. Musí platit rovnost?

rovnost např. pro konstanty, $\ne$ např pro $f(x) = x$, $g(x) = -x$

$f(x) + g(x) \le \sup f(x) + \sup g(x)$, aj pre sup na LHS

b) $\sup_{x\in M}(f(x)+g(x)) \ge \sup_{x\in M} f(x) + \inf_{x\in M} g(x)$

$\Leftrightarrow \sup f(x) \le \sup(f(x)+g(x)) - \inf g(x)$

$\Leftrightarrow \sup f(x) \le \sup(f(x)+g(x)) + \sup(-g(x))$

$\sup(f(x)) = \sup(f(x) + g(x) + (-g(x))) \overset{a)}{\le} \sup(f(x)+g(x)) + \sup(-g(x))$ ✓

c) $\sup_{x\in M}(f(x)-g(x)) \le \sup_{x\in M} f(x) - \inf g(x)$

b) pre $f(x) := f(x)+g(x)$

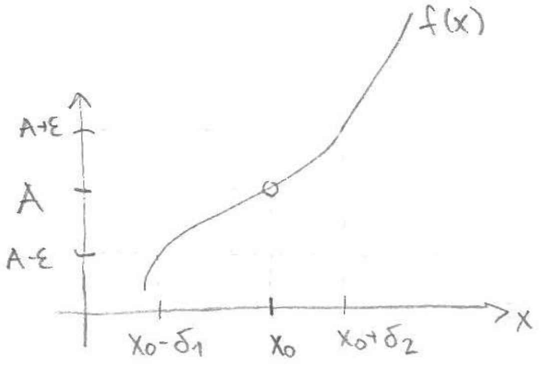
Def: $\sup_{x\in M} f(x) = \sup\{z : z = f(x),\ x\in M\}$

Limita funkce

Def.: Buď $f: \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R}^*$ a $A \in \mathbb{R}^*$.
Řekneme, že A je limitou funkce f pro x jdoucí k x_0, jestliže pro každé $\varepsilon > 0$ existuje $\delta > 0$ takové, že

$$x \in P_\delta(x_0) \Rightarrow f(x) \in \mathcal{U}_\varepsilon(A).$$

Píšeme: $\lim\limits_{x \to x_0} f(x) = A$, $f(x) \overset{x \to x_0}{\to} A$, ...



$\delta := \min\{\delta_1, \delta_2\}$

Def.: Nechť $\varepsilon > 0$ a $x \in \mathbb{R}^*$. Epsilonové okolí bodu x definujeme jako

$$\mathcal{U}_\varepsilon(x) := \begin{cases} (x-\varepsilon, x+\varepsilon), & x \in \mathbb{R}, \\ (\varepsilon, +\infty], & x = +\infty, \\ [-\infty, -\varepsilon), & x = -\infty. \end{cases}$$

Levé epsilonové okolí bodu $x \in (-\infty, \infty]$ definujeme jako

$$\mathcal{U}_\varepsilon^-(x) := \begin{cases} (x-\varepsilon, x], & x \in \mathbb{R}, \\ (\varepsilon, +\infty], & x = +\infty. \end{cases}$$

Analogicky definujeme pravé epsilonové okolí $\mathcal{U}_\varepsilon^+(x)$ bodu $x \in [-\infty, \infty)$.

Def.: Nechť $\varepsilon > 0$ a $x \in \mathbb{R}^*$. Epsilonové prstencové okolí bodu x definujeme jako

$$P_\varepsilon(x) := \mathcal{U}_\varepsilon(x) \setminus \{x\}.$$

Analogicky definujeme jednostranná prstencová okolí.

Vlastní limita ve vlastním bodě ($x_0 \in \mathbb{R}$, $A \in \mathbb{R}$)

$x_0 \in \mathbb{R}$: $\quad x \in P_\delta(x_0) \iff 0 < |x - x_0| < \delta$

$A \in \mathbb{R}$: $\quad f(x) \in \mathcal{U}_\varepsilon(A) \iff |f(x) - A| < \varepsilon$

Stručný zápis def. limity: $\quad \forall \varepsilon > 0 \ \exists \delta > 0: \ x \in P_\delta(x_0) \Rightarrow f(x) \in \mathcal{U}_\varepsilon(A)$.

Věta (jednoznačnost limity): Nechť $f: \mathbb{R} \to \mathbb{R}$ a $x_0 \in \mathbb{R}$. Pak existuje nejvýše jedna limita funkce f v bodě x_0.

Def.: Nechť $f: \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R}$ a $A \in \mathbb{R}$. Řekneme, že A je limitou fce f pro x jdoucí k x_0 zprava/zleva, jestliže pro každé $\varepsilon > 0$ existuje $\delta > 0$ takové, že

$$x \in P_\delta^{+/-}(x_0) \Rightarrow f(x) \in \mathcal{U}_\varepsilon(A),$$

kde $P_\delta^+(x_0) = (x_0, x_0 + \delta)$ a $P_\delta^-(x_0) = (x_0 - \delta, x_0)$.

Píšeme: $\lim\limits_{x \to x_0+} f(x) = A$, resp. $\lim\limits_{x \to x_0-} f(x) = A$.

Věta (vztah limity k jednostranným limitám): Buď $f: \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R}$ a $A \in \mathbb{R}$. Pak

$$\lim_{x \to x_0} f(x) = A \iff \lim_{x \to x_0+} f(x) = \lim_{x \to x_0-} f(x) = A.$$

Věta (vztah limity a omezenosti): Nechť $f: \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R}$ a $A \in \mathbb{R}$, $\lim_{x \to x_0} f(x) = A$.
Pak na jistém prstencovém okolí bodu x_0 je fce $f(x)$ omezená.

Věta (nenulová limita a odraženost od nuly): Nechť $f: \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R}$ a $\lim_{x \to x_0} f(x) = A \in \mathbb{R} \setminus \{0\}$. Pak je f na jistém prstencovém okolí bodu x_0 odražená od nuly ($f(x) \neq 0 \ \forall x \in P_\delta(x_0)$). Speciálně lze nalézt takové prstencové okolí x_0, že $|f|$ na něm je odražená od nuly hodnotou $\frac{|A|}{2}$.

Věta (Aritmetika limit). Nechť $f, g: \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R}$, $A, B \in \mathbb{R}$, $\lim_{x \to x_0} f(x) = A$ a $\lim_{x \to x_0} g(x) = B$.

Pak: a) $\lim_{x \to x_0} (f(x) + g(x)) = A + B$

b) $\lim_{x \to x_0} (f(x) . g(x)) = A . B$

c) pokud navíc $B \neq 0$, $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{A}{B}$.

Tvrzení: Nechť $f, g: \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R}$, $A \in \mathbb{R}$ a $\lim_{x \to x_0} f(x) = A$. Pak

i) $\lim_{x \to x_0} (f(x) + g(x)) = A + \lim_{x \to x_0} g(x)$,

ii) pokud navíc $A \neq 0$, $\lim_{x \to x_0} (f(x) . g(x)) = A \lim_{x \to x_0} g(x)$,

kde v obou případech rovnost chápeme tak, že limita na levé straně existuje a je konečná právě tehdy, když existuje a je konečná $\lim_{x \to x_0} g(x)$.

Věta (zachování (neostré) nerovnosti při limitním přechodu): Nechť $f, g: \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R}$, $A, B \in \mathbb{R}$, $\lim_{x \to x_0} f(x) = A$ a $\lim_{x \to x_0} g(x) = B$. Jestliže $f \leq g$ na nějakém $P_\delta(x_0)$, pak $A \leq B$.

Věta (o dvou policajtech). Nechť $f, g, h: \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R}$, $A \in \mathbb{R}$ a nechť $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = A$.
Jestliže $f \leq g \leq h$ na jistém $P_\delta(x_0)$, pak i $\lim_{x \to x_0} g(x) = A$.

Věta (omezená krát jdoucí k nule): Nechť $f, g: \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R}$, $\lim_{x \to x_0} f(x) = 0$ a nechť g je omezená na jistém $P_\delta(x_0)$. Pak $\lim_{x \to x_0} f(x) . g(x) = 0$.

Věta (o limitě složené fce). Nechť $f, g: \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R}$, $\lim_{x \to x_0} f(x) = A \in \mathbb{R}$, $\lim_{y \to A} g(y) = B \in \mathbb{R}$ a nechť navíc $f(x) \neq A$ na jistém $P_\delta(x_0)$.

Pak $\lim_{x \to x_0} g(f(x)) = B$.

Základní limity:

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{1-\cos^2 x}{x^2(1+\cos x)} = \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^2 \frac{1}{1+\cos x} = \frac{1}{2}$$

$$\lim_{x\to 0} \frac{e^x-1}{x} = 1$$

$$\lim_{x\to 0} \frac{a^x-1}{x} = \lim_{x\to 0} \frac{e^{\ln a \cdot x}-1}{\ln a \cdot x} \cdot \ln a = \ln a$$

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{y\to 1} \frac{\ln y}{y-1} = 1$$

limity typu "1^∞": $\lim_{x\to a} (f(x))^{g(x)} = e^{\lim_{x\to a} g(x)\ln(f(x))}$

Goniometrické vzorce:

$$\sin(x+y) = \sin x\cos y + \cos x\sin y$$
$$\sin(x-y) = \sin x\cos y - \cos x\sin y$$
$$\cos(x+y) = \cos x\cos y - \sin x\sin y$$
$$\cos(x-y) = \cos x\cos y + \sin x\sin y$$

$$\sin 2x = 2\sin x\cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x$$

$$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$$
$$\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$$
$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$$
$$\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$$

Součtové vzorce :

$$a^2 - b^2 = (a-b)(a+b)$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

$$(a-b)^n = a^n + \binom{n}{1} a^{n-1} b + \dots + \binom{n}{n-1} a b^{n-1} + b^n$$

$$x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \dots + x + 1)$$

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \dots + ab^{n-2} + b^{n-1})$$

Limity funkcí I , 1/3

1. Dokažte z definice, že

a) $\lim\limits_{x\to 1} \left(\frac{x}{2}\right)^3 = \frac{1}{8}$

chceme: $\forall \varepsilon > 0\ \exists \delta > 0 : 0 < |x-1| < \delta \Rightarrow \left|\left(\frac{x}{2}\right)^3 - \frac{1}{8}\right| < \varepsilon$

vol $\varepsilon > 0$ (pevné), hledáme $\delta > 0$ (BÚNO $\delta < 1$), $\delta < 1 \Rightarrow 0 < |x-1| < 1 \Rightarrow 1 < |x| < 2$

$$\left|\left(\frac{x}{2}\right)^3 - \frac{1}{8}\right| = \left|\frac{x^3-1}{8}\right| \leq \left|\frac{(x-1)(x^2+x+1)}{8}\right| \leq \frac{|x-1|}{8}|x^2+x+1| < \frac{\delta}{8}(|x|^2+|x|+1) = \frac{7}{8}\delta < \delta$$

stačí zvolit $\delta := \varepsilon$

b) $\lim\limits_{x\to 1+} [x] = 1$

$[x]$ celá část: $x = z + r$, $z \in \mathbb{Z}$, $r \in \langle 0,1)$, potom $z = [x]$

chceme: $\forall \varepsilon > 0\ \exists \delta > 0 : 1 < x < 1+\delta \Rightarrow |[x]-1| < \varepsilon$

ale $[x] = 1\ \forall x \in (1, 1+\delta) \Rightarrow [\forall \delta \in (0,1) : 1 < x < 1+\delta \Rightarrow 0 = |[x]-1| < \varepsilon]$

c) $\lim\limits_{x\to 1-} [x] = 0$

chceme: $\forall \varepsilon > 0\ \exists \delta > 0 : 1-\delta < x < 1 \Rightarrow |[x]-0| < \varepsilon$

ale $[x] = 0\ \forall x \in (1-\delta, 1) \Rightarrow [\forall \delta \in (0,1) : 1-\delta < x < 1 \Rightarrow 0 = |[x]| < \varepsilon]$

Spočtěte:

2. a) $$\lim_{x\to 0} \frac{x^2-1}{2x^2-x-1} = \frac{0-1}{0-0-1} = 1$$

b) $$\lim_{x\to 1} \frac{x^2-1}{2x^2-x-1} = \lim_{x\to 1} \frac{(x-1)(x+1)}{(x-1)(2x+1)} = \lim_{x\to 1} \frac{x+1}{2x+1} = \frac{1+1}{2\cdot 1+1} = \frac{2}{3}$$

3. $$\lim_{x\to 2}\left(\frac{1}{x^2-2x} - \frac{x}{x^2-4}\right) = \lim_{x\to 2}\left(\frac{1}{x(x-2)} - \frac{x}{(x-2)(x+2)}\right) = \lim_{x\to 2} \frac{x+2-x^2}{x(x-2)(x+2)} = \lim_{x\to 2} \frac{\overset{-1}{\cancel{(2-x)}}(1+x)}{x\cancel{(x-2)}(x+2)}$$

$$= -1 \cdot \lim_{x\to 2} \frac{1+x}{x(x+2)} = -\frac{3}{8}$$

4. $$\lim_{x\to 0} \frac{(1+x)(1+2x)\dots(1+nx)-1}{x}, \quad n \in \mathbb{N}$$

$$= \lim_{x\to 0} \frac{\cancel{1} + \cancel{x}(1+2+\dots+n) + x^{\cancel{2}}(\dots) + \dots \cancel{-1}}{\cancel{x}} = 1+2+\dots+n = \frac{n(n+1)}{2}$$

5. $$\lim_{x\to 1} \frac{x^{100}-2x+1}{x^{50}-2x+1} = \lim_{x\to 1} \frac{x^{100}-1-2x+2}{x^{50}-1-2x+2} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x^{99}+\dots+1)-2\cancel{(x-1)}}{\cancel{(x-1)}(x^{49}+\dots+1)-2\cancel{(x-1)}}$$

$$= \lim_{x\to 1} \frac{\overbrace{x^{99}+\dots+1}^{100\ \text{členov}}-2}{\underbrace{x^{49}+\dots+1}_{50\ \text{členov}}-2} = \frac{100-2}{50-2} = \frac{49}{24}$$

6. $\lim\limits_{x\to 0} \dfrac{(1+mx)^n-(1+nx)^m}{x^2}$ $\quad m,n\in\mathbb{N}$

$$= \lim_{x\to 0} \frac{\cancel{1}+\cancel{n.mx}+\binom{n}{2}m^2\cancel{x^2}+\dots+\binom{n}{n-1}m^{n-1}x^{n-1}+m^nx^n-\cancel{1}-\cancel{m.nx}-\binom{m}{2}n^2\cancel{x^2}-\dots}{\cancel{x^2}}$$

$$= \lim_{x\to 0} \binom{n}{2}m^2-\binom{m}{2}n^2+x(\dots) = \binom{n}{2}m^2-\binom{m}{2}n^2 \left(= \frac{n!\,m^2}{(n-2)!\,2!}-\frac{m!\,n^2}{(m-2)!\,2!}=\dots=\frac{mn}{2}(n-m)\right)$$

7. $\lim\limits_{x\to 1} \dfrac{x^{n+1}-(n+1)x+n}{(x-1)^2}$ $\quad n\in\mathbb{N}$

$$= \lim_{x\to 1}\frac{x(x^n-1)-n(x-1)}{(x-1)^2} = \lim_{x\to 1}\frac{x\cancel{(x-1)}(x^{n-1}+x^{n-2}+\dots+1)-n\cancel{(x-1)}}{(x-1)^{\cancel{2}}} = \lim_{x\to 1}\frac{x^n+x^{n-1}+\dots+x-n}{x-1} = (*)$$

$$= \lim_{x\to 1}\frac{(x^n-1)+(x^{n-1}-1)+\dots+(x-1)}{x-1} = \lim_{x\to 1}\underbrace{x^{n-1}+\dots+1}_{n}+\underbrace{x^{n-2}+\dots+1}_{n-1}+\dots+1 = \frac{n(n+1)}{2}$$

8. $\lim\limits_{x\to 1} \dfrac{x+x^2+\dots+x^n-n}{x-1}$ $\quad n\in\mathbb{N}$

$$= \frac{n(n+1)}{2} \qquad \text{z 7. } (*)$$

9. $\lim\limits_{x\to 1}\left(\dfrac{m}{1-x^m}-\dfrac{n}{1-x^n}\right)$ $\quad m,n\in\mathbb{N}$

$$= \lim_{y\to 0}\left(\frac{m}{1-(1+y)^m}-\frac{n}{1-(1+y)^n}\right) = \lim_{y\to 0}\left(\frac{m}{1-(1+my+\dots+y^m)}-\frac{n}{1-(1+ny+\dots+y^n)}\right)$$

$$= \lim_{y\to 0}\left(\frac{n}{ny+\dots+y^n}-\frac{m}{my+\dots+y^m}\right) = \cancel{\lim_{y\to 0}\ mny+n\tbinom{m}{2}+ny^m-mny+m\tbinom{n}{2}y^2-\dots}$$

$$= \lim_{y\to 0}\left(\frac{\cancel{nmy}+n\binom{m}{2}y^2+\dots-\cancel{mny}-m\binom{n}{2}y^2-\dots}{mny^2+y^3\dots}\right) = \frac{n\binom{m}{2}-m\binom{n}{2}}{mn} = \frac{m!}{m(m-2)!\,2}-\frac{n!}{n(n-2)!\,2} = \frac{m-n}{2}$$

10. $\lim\limits_{x\to 0}\dfrac{\frac{2}{x^2}+1}{\sqrt{\frac{3}{x^4}-\frac{6}{x^2}+5}} = \lim\limits_{x\to 0}\dfrac{2+x^2}{x^2\sqrt{\frac{3}{x^4}-\frac{6}{x^2}+5}} = \lim\limits_{x\to 0}\dfrac{2+x^2}{\sqrt{3-6x^2+5x^4}} = \dfrac{2}{\sqrt{3}} = \dfrac{2\sqrt{3}}{3}$

11. $\lim\limits_{x\to 0+}\dfrac{\left(\sqrt{\frac{1}{x^2}+1}-\sqrt{\frac{1}{x^2}-1}\right)}{x} = \lim\limits_{x\to 0+}\dfrac{\left(\sqrt{1+x^2}-\sqrt{1-x^2}\right)}{x^2}\cdot\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}$

$$= \lim_{x\to 0+}\frac{1+x^2-1+x^2}{x^2\left(\sqrt{1+x^2}+\sqrt{1-x^2}\right)} = \lim_{x\to 0+}\frac{2}{\sqrt{1+x^2}+\sqrt{1-x^2}} = 1$$

12. $$\lim_{x\to 0+}\left(\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}}}}-\sqrt{\frac{1}{x}-\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}}}}\right)=\lim_{x\to 0+}\left(\sqrt{\frac{1}{x}+\sqrt{\frac{1+\sqrt{x}}{x}}}-\sqrt{\frac{1}{x}-\sqrt{\frac{1+\sqrt{x}}{x}}}\right)$$

$$=\lim_{x\to 0+}\left(\sqrt{\frac{1+\sqrt{x}\sqrt{1+\sqrt{x}}}{x}}-\sqrt{\frac{1-\sqrt{x}\sqrt{1+\sqrt{x}}}{x}}\right)\cdot\frac{\sqrt{1+\sqrt{x}\sqrt{1+\sqrt{x}}}+\sqrt{1-\sqrt{x}\sqrt{1+\sqrt{x}}}}{\sqrt{1+\sqrt{x}\sqrt{1+\sqrt{x}}}+\sqrt{1-\sqrt{x}\sqrt{1+\sqrt{x}}}}$$

$$=\lim_{x\to 0+}\frac{1}{\cancel{\sqrt{x}}}\cdot\frac{\cancel{1}+\cancel{\sqrt{x}}\sqrt{1+\sqrt{x}}-\cancel{1}+\cancel{\sqrt{x}}\sqrt{1+\sqrt{x}}}{\sqrt{1+\sqrt{x}\sqrt{1+\sqrt{x}}}+\sqrt{1-\sqrt{x}\sqrt{1+\sqrt{x}}}}=\frac{\sqrt{1+0}+\sqrt{1+0}}{\sqrt{1+0}+\sqrt{1-0}}=1$$

13. a) $$\lim_{x\to 16}\frac{\sqrt[4]{x}-2}{\sqrt{x}-4}=\lim_{x\to 16}\frac{\cancel{\sqrt[4]{x}-2}}{(\sqrt[4]{x}-2)(\sqrt[4]{x}+2)}=\frac{1}{\sqrt[4]{16}+2}=\frac{1}{4}$$

b) $$\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}=\lim_{x\to 0}\frac{\cancel{x+1-1}}{\cancel{x}(\sqrt{x+1}+1)}=\frac{1}{2}$$

14. $$\lim_{x\to 0}\frac{\sqrt{1-2x-x^2}-(1-x)}{x}=\lim_{x\to 0}\frac{\cancel{1}\cancel{-2x}-x^2\cancel{-1}\cancel{+2x}-x^2}{\cancel{x}\left(\sqrt{1-2x-x^2}+(1-x)\right)}=\lim_{x\to 0}\frac{-2x}{\left(\sqrt{1-2x-x^2}+(1-x)\right)}=0$$

15. $$\lim_{x\to 0}\frac{\sqrt[3]{27+x}-\sqrt[3]{27-x}}{x+2\sqrt[3]{x^4}}=\lim_{x\to 0}\frac{27+x-27+x}{(x+2\sqrt[3]{x^4})\left((\sqrt[3]{27+x})^2+\sqrt[3]{27+x}\sqrt[3]{27-x}+(\sqrt[3]{27-x})^2\right)}$$

$$=\lim_{x\to 0}\frac{2\cancel{x}}{\cancel{x}(1+2\sqrt[3]{x})(\dots)}=\frac{2}{1(3^2+3.3+3^2)}=\frac{2}{27}$$

16. $$\lim_{x\to 0}\frac{\sqrt[m]{1+x}-\sqrt[n]{1+x}}{x}\qquad m,n\in\mathbb{N}$$

$$=\lim_{x\to 0}\frac{\sqrt[m]{1+x}-1}{x}-\frac{\sqrt[n]{1+x}-1}{x}\overset{(*)}{=}\frac{1}{m}-\frac{1}{n}=\frac{n-m}{mn}$$

(*): $$\lim_{x\to 0}\frac{\sqrt[m]{1+x}-1}{x}\cdot\frac{(1+x)^{\frac{m-1}{m}}+\dots+1}{(1+x)^{\frac{m-1}{m}}+\dots+1}=\lim_{x\to 0}\frac{1+x-1}{x(\dots)}=\frac{1}{m}$$ $\qquad \frac{a}{b}\cdot\frac{b}{a}=1$

17. $$\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt[3]{1-x}}{\sqrt[3]{1+x}-\sqrt{1-x}}=\lim_{x\to 0}\frac{\sqrt[6]{(1+x)^3}-\sqrt[6]{(1-x)^2}}{\sqrt[6]{(1+x)^2}-\sqrt[6]{(1-x)^3}}\cdot\frac{\left(\sqrt[6]{(1+x)^3}\right)^5+\left(\sqrt[6]{(1+x)^3}\right)^4\left(\sqrt[6]{(1-x)^2}\right)+\dots+\left(\sqrt[6]{(1-x)^2}\right)^5}{\left(\sqrt[6]{(1+x)^2}\right)^5+\dots+\left(\sqrt[6]{(1-x)^3}\right)^5}$$

$$=\lim_{x\to 0}\frac{(1+x)^3-(1-x)^2}{(1+x)^2-(1-x)^3}\cdot\frac{\left(\sqrt[6]{(1+x)^2}\right)^5+\dots+\left(\sqrt[6]{(1-x)^3}\right)^5}{\left(\sqrt[6]{(1+x)^3}\right)^5+\dots+\left(\sqrt[6]{(1-x)^2}\right)^5}$$ $\qquad$ 6 členov, $\xrightarrow{x\to 0} 6$

$$=\lim_{x\to 0}\frac{\cancel{1}+3x+3x^2+x^3\cancel{-1}+2x-x^2}{\cancel{1}+2x+x^2\cancel{-1}+3x-3x^2+x^3}\cdot\frac{\dots}{\dots}=\lim_{x\to 0}\frac{5+2x+x^2}{5-2x+x^2}\cdot\underbrace{\frac{\dots}{\dots}}_{\xrightarrow{x\to 0}6}=\frac{5}{5}\cdot\frac{6}{6}=1$$

18. $$\lim_{x\to a+} \frac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{\sqrt{x^2-a^2}} \quad , \quad a \in \mathbb{R}_0^+$$

$$\cancel{= \lim_{x\to a+} \left(\frac{\sqrt{x}-\sqrt{a}}{\sqrt{x^2-a^2}} + \frac{\sqrt{x-a}}{\sqrt{(x-a)}\sqrt{x+a}} \right) = \lim_{x\to a+} \frac{x-a}{\sqrt{x^2-a^2}}}$$

$$= \lim_{x\to a+} \left(\frac{\sqrt{x}-\sqrt{a}}{\sqrt{x^2-a^2}} \cdot \frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}} + \frac{\sqrt{x-a}}{\sqrt{x-a}\sqrt{x+a}} \right)$$

$$= \lim_{x\to a+} \left(\frac{x-a}{\sqrt{x-a}\sqrt{x+a}\,(\sqrt{x}+\sqrt{a})} + \frac{1}{\sqrt{x+a}} \right)$$

$$= \lim_{x\to a+} \left(\frac{\sqrt{x-a}}{\sqrt{x+a}\,(\sqrt{x}+\sqrt{a})} + \frac{1}{\sqrt{x+a}} \right) = \frac{1}{\sqrt{2a}} = \frac{\sqrt{2a}}{2a}$$

19. $$\lim_{x\to 0} \frac{\sqrt[m]{1+ax}\,\sqrt[n]{1+bx}-1}{x} \qquad m,n \in \mathbb{N}, \quad a,b \in \mathbb{R}$$

$$= \lim_{x\to 0} \frac{\sqrt[mn]{(1+ax)^n(1+bx)^m}-1}{x} \cdot \frac{\left(\sqrt[m]{1+ax}\,\sqrt[n]{1+bx}\right)^{mn-1} + \ldots + 1}{\left(\sqrt[m]{1+ax}\,\sqrt[n]{1+bx}\right)^{mn-1} + \ldots + 1}$$

$$= \lim_{x\to 0} \frac{(1+ax)^n(1+bx)^m - 1}{x\left(\left(\sqrt[m]{\ }\,\sqrt[n]{\ }\right)^{mn-1} + \ldots + 1\right)}$$

$$= \lim_{x\to 0} \frac{(1+nax+x^2\ldots)(1+mbx+x^2\ldots)-1}{x.\ (\ldots)}$$

$$= \lim_{x\to 0} \frac{1+x(na+mb)+x^2\ldots-1}{x.\ \underbrace{(\ \ldots\ )}_{mn \text{ členov},\ \xrightarrow{x\to 0} mn}} = \frac{na+mb}{mn}$$

Limity funkcí II

1. $\lim\limits_{x\to a} \dfrac{\mathrm{tg}x - \mathrm{tg}a}{x-a}$, $a \in \mathbb{R}$

$$= \lim_{x\to a} \frac{\sin x\cos a - \sin a \cos x}{(x-a)\cos x \cos a} = \lim_{x\to a} \underbrace{\frac{\sin(x-a)}{x-a}}_{\to 1} \cdot \frac{1}{\cos x \cos a} = \frac{1}{\cos^2 a}$$

2. $$\lim_{x\to 0} \frac{\sqrt{1-\cos x^2}}{1-\cos x} = \lim_{x\to 0} \underbrace{\sqrt{\frac{1-\cos x^2}{x^4}}}_{\to \sqrt{\frac{1}{2}}} \cdot \underbrace{\frac{x^2}{1-\cos x}}_{\to 2} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

3. $$\lim_{x\to 0} \frac{\mathrm{tg}x - \sin x}{x^3} = \lim_{x\to 0} \underbrace{\frac{\sin x}{x}}_{\to 1} \underbrace{\frac{(1-\cos x)}{x^2}}_{\to \frac{1}{2}} \cdot \underbrace{\frac{1}{\cos x}}_{\to 1} = \frac{1}{2}$$

4. $$\lim_{x\to 0} \frac{1-\cos x\cos 2x \cos 3x}{1-\cos x} = \lim_{x\to 0} \frac{1-\cos x + \cos x(1-\cos 2x\cos 3x)}{1-\cos x}$$

$$= \lim_{x\to 0} \left(1 + \frac{\cos x}{1-\cos x}\left(1-\cos 2x + \cos 2x(1-\cos 3x)\right)\right)$$

$$= \lim_{x\to 0} \left(1 + \cos x \cdot \frac{1-\cos 2x}{1-\cos x} + \cos x \cos 2x \cdot \frac{1-\cos 3x}{1-\cos x}\right) \underset{\substack{a=2\\a=3}}{=} 1 + 1.2^2 + 1.1.3^2 = 14$$

$$\left[\lim_{x\to 0} \frac{1-\cos ax}{1-\cos x} = \lim_{x\to 0} \frac{1-\cos ax}{(ax)^2} \cdot \frac{x^2}{1-\cos x} \cdot a^2 = \frac{1}{2} \cdot 2 \cdot a^2 = a^2\right]$$

5. $\lim\limits_{x\to \pi} \dfrac{\sin nx}{\sin mx}$ $n, m \in \mathbb{N}$, $y = x - \pi$, $x = y + \pi$

$$= \lim_{y\to 0} \frac{\sin(n(y+\pi))}{\sin(m(y+\pi))} = \lim_{y\to 0} \frac{\sin(ny)\overbrace{\cos(n\pi)}^{=(-1)^n} + \cos(ny)\overbrace{\sin(n\pi)}^{=0}}{\sin(my)\underbrace{\cos(m\pi)}_{=(-1)^m} + \cos(my)\underbrace{\sin(m\pi)}_{=0}} = \lim_{y\to 0} (-1)^{n+m} \frac{\sin(ny)}{ny} \cdot \frac{my}{\sin(my)} \cdot \frac{ny}{my}$$

$$= (-1)^{n+m} \frac{n}{m}$$

6. $$\lim_{x\to 1} \frac{\sin(\pi x)}{1-x} \underset{\sin x = \sin(\pi - x)}{=} \lim_{x\to 1} \frac{\sin(\pi(1-x))}{\pi(1-x)} \cdot \pi = \lim_{y\to 0} \frac{\sin(\pi y)}{\pi y} \cdot \pi = \pi$$

7. $$\lim_{x\to \frac{\pi}{4}} \mathrm{tg}(2x)\,\mathrm{tg}\left(\tfrac{\pi}{4} - x\right) = \left|\tfrac{\pi}{4} - x = y\right| = \lim_{y\to 0} \mathrm{tg}\left(2\left(\tfrac{\pi}{4} - y\right)\right)\mathrm{tg}\,y = \lim_{y\to 0} \mathrm{tg}\left(\tfrac{\pi}{2} - 2y\right)\mathrm{tg}\,y = \left|\begin{matrix}\mathrm{cotg}\,x \\ = \mathrm{tg}\left(\frac{\pi}{2} - x\right)\end{matrix}\right|$$

$$= \lim_{y\to 0} \mathrm{cotg}(2y)\,\mathrm{tg}\,y = \lim_{y\to 0} \frac{\cos 2y}{\sin 2y} \cdot \frac{\sin y}{\cos y} = \lim_{y\to 0} \frac{\cos^2 y - \sin^2 y}{2\cancel{\sin y}\cos y} \cdot \frac{\cancel{\sin y}}{\cos y} = \frac{1}{2} \lim_{y\to 0} (1 - \mathrm{tg}^2 y) = \frac{1}{2}$$

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8. $$\lim_{x\to 0} \frac{\sin(a+2x)-2\sin(a+x)+\sin a}{x^2}, \quad a\in\mathbb{R}$$

$$= \lim_{x\to 0} \frac{2\sin\left(\frac{a+2x+a}{2}\right)\cos\left(\frac{a+2x-a}{2}\right)-2\sin(a+x)}{x^2} = \lim_{x\to 0} 2\sin(a+x)\left(\frac{\cos x-1}{x^2}\right) = -\sin a$$

9. $$\lim_{x\to 0} \frac{\operatorname{cotg}(a+2x)-2\operatorname{cotg}(a+x)+\operatorname{cotg} a}{x^2}, \quad \sin a \neq 0$$

$$= \lim_{x\to 0}\left(\frac{\operatorname{cotg}(a+2x)-\operatorname{cotg}(a+x)}{x^2} - \frac{\operatorname{cotg}(a+x)-\operatorname{cotg} a}{x^2}\right)$$

$$= \lim_{x\to 0}\left(\frac{\cos(a+2x)\sin(a+x)-\sin(a+2x)\cos(a+x)}{x^2\sin(a+x)\sin(a+2x)} - \frac{\cos(a+x)\sin a-\sin(a+x)\cos a}{x^2\sin a\sin(a+x)}\right)$$

$$= \lim_{x\to 0}\left(\frac{\sin(a+x-a-2x)}{x^2\sin(a+x)\sin(a+2x)} - \frac{\sin(a-a-x)}{x^2\sin a\sin(a+x)}\right)$$

$$= \lim_{x\to 0} \frac{\sin(-x)}{(-x)}\cdot\frac{\sin(a+2x)-\sin a}{x}\cdot\frac{1}{\sin a\sin(a+x)\sin(a+2x)}$$

$$= \lim_{x\to 0} \frac{\sin(-x)}{(-x)}\cdot\frac{2\cos\left(\frac{a+2x+a}{2}\right)\sin\left(\frac{a+2x-a}{2}\right)}{x}\cdot\frac{1}{\sin a\sin(a+x)\sin(a+2x)}$$

$$= \lim_{x\to 0} \frac{\sin(-x)}{(-x)}\cdot\frac{\sin x}{x}\cdot\frac{2\cos(a+x)}{\sin a\sin(a+x)\sin(a+2x)} = \frac{2\cos a}{\sin^3 a} = \frac{2\operatorname{cotg} a}{\sin^2 a}$$

10. $$\lim_{x\to 0+} \frac{\arccos(1-x)}{\sqrt{x}} = \left|\begin{array}{l} 1-x=\cos y \\ x\to 0+ \Rightarrow y\to 0+ \end{array}\right| = \lim_{y\to 0+}\frac{y}{\sqrt{1-\cos y}} = \lim_{y\to 0+}\sqrt{\frac{y^2}{1-\cos y}} = \sqrt{2}$$

11. $$\lim_{x\to 0+} \frac{\left(\frac{\pi}{2}-\arcsin\frac{1}{\sqrt{x^2+1}}\right)}{x} = \left|\begin{array}{l} \frac{1}{\sqrt{x^2+1}} = \sin y \\ x = \sqrt{\frac{1}{\sin^2 y}-1} \\ x\to 0+ \Rightarrow y\to\frac{\pi}{2}- \end{array}\right| = \lim_{y\to\frac{\pi}{2}-} \frac{\left(\frac{\pi}{2}-y\right)\cdot\sin y}{\cos y}$$

$\cos x = \sin(\frac{\pi}{2}-x)$

$$= \lim_{y\to\frac{\pi}{2}-} \frac{\left(\frac{\pi}{2}-y\right)}{\sin\left(\frac{\pi}{2}-y\right)}\cdot\sin y = 1$$

12. $$\lim_{x\to 0} \frac{\ln(\cos(ax))}{\ln(\cos(bx))} \qquad a,b\in\mathbb{R},\ b\neq 0$$

$$= \lim_{x\to 0} \frac{\ln(\cos(ax))}{\cos(ax)-1}\cdot\frac{\cos(bx)-1}{\ln(\cos(bx))}\cdot\frac{\cos(ax)-1}{(ax)^2}\cdot\frac{(bx)^2}{\cos(bx)-1}\cdot\frac{a^2}{b^2} = 1\cdot 1\cdot\left(-\frac{1}{2}\right)\cdot(-2)\cdot\frac{a^2}{b^2}$$

$$= \frac{a^2}{b^2}$$

13. $\lim\limits_{x\to 0} \dfrac{\ln(a+x)+\ln(a-x)-2\ln a}{x^2}$ $a>0$

$= \lim\limits_{x\to 0} \dfrac{\ln\left((a+x)(a-x)\right)+\ln(a^{-2})}{x^2} = \lim\limits_{x\to 0} \dfrac{\ln\left(\frac{a^2-x^2}{a^2}\right)}{x^2} = \lim\limits_{x\to 0} \dfrac{\ln\left(1-\frac{x^2}{a^2}\right)}{\left(-\frac{x^2}{a^2}\right)} \cdot \dfrac{1}{(-a^2)} = -a^{-2}$

14. $\lim\limits_{x\to 0} \dfrac{\ln\left(tg\left(\frac{\pi}{4}+ax\right)\right)}{\sin(bx)}$ $a,b\in\mathbb{R},\ b\neq 0,\ tg\left(\frac{\pi}{4}+ax\right)=\dfrac{\sin\left(\frac{\pi}{4}+ax\right)}{\cos\left(\frac{\pi}{4}+ax\right)}=\dfrac{\cos(ax)+\sin(ax)}{\cos(ax)-\sin(ax)}$

$= \lim\limits_{x\to 0} \dfrac{\ln\left(tg\left(\frac{\pi}{4}+ax\right)\right)}{tg\left(\frac{\pi}{4}+ax\right)-1} \cdot \dfrac{\cos(ax)+\sin(ax)-\cos(ax)+\sin(ax)}{\cos(ax)-\sin(ax)} \cdot \dfrac{1}{\sin(bx)}$

$= \lim\limits_{x\to 0} \dfrac{\ln(\ \ )}{(\ \ )-1} \cdot \dfrac{2\sin(ax)}{ax} \cdot \dfrac{bx}{\sin(bx)} \cdot \dfrac{a}{b} \cdot \dfrac{1}{\cos(ax)-\sin(ax)} = \dfrac{2a}{b}$

15. $\lim\limits_{x\to 0+} \ln(x\ln a)\ \ln\left(\dfrac{\ln(ax)}{\ln\frac{x}{a}}\right)$ $,\ a>0$

$= \lim\limits_{x\to 0+} \ln(x\ln a) \cdot \dfrac{\ln\left(\frac{\ln a+\ln x}{\ln x-\ln a}\right)}{\frac{\ln a+\ln x}{\ln x-\ln a}-1} \cdot \dfrac{\ln a+\ln x-\ln x+\ln a}{\ln x-\ln a}$

$= \lim\limits_{x\to 0+} \dfrac{\ln(\ \ )}{(\ \ )-1} \cdot \dfrac{\ln x+\ln\ln a}{\ln x-\ln a} \cdot 2\ln a = 1\cdot 1\cdot 2\ln a = 2\ln a$

16. $\lim\limits_{x\to 0} \dfrac{\ln(1+xe^x)}{\ln(x+\sqrt{1+x^2})} = \lim\limits_{x\to 0} \dfrac{\ln(1+xe^x)}{xe^x} \cdot \dfrac{x+\sqrt{1+x^2}-1}{\ln(x+\sqrt{1+x^2})} \cdot \dfrac{xe^x}{x+\frac{(\sqrt{1+x^2}-1)(\sqrt{1+x^2}+1)}{(\sqrt{1+x^2}+1)}}$

$= \lim\limits_{x\to 0} \dfrac{\ln(1+xe^x)}{xe^x} \cdot \dfrac{(\ \ )-1}{\ln(\ \ )} \cdot \dfrac{\cancel{x}e^x}{\cancel{x}\left(1+\frac{x}{\sqrt{1+x^2}+1}\right)} = 1\cdot 1\cdot \dfrac{e^0}{1+0} = 1$

17. $\lim\limits_{x\to 1} (1-x)\log_x 2$ $,\ y=\log_x 2 \iff x^y=2 \iff y\ln x=\ln 2 \iff y=\dfrac{\ln 2}{\ln x}$

$= \lim\limits_{x\to 1} (1-x)\dfrac{\ln 2}{\ln x} = \lim\limits_{x\to 1} \dfrac{x-1}{\ln x}(-\ln 2) = -\ln 2$

18. $\lim\limits_{x\to 0+} (1+x)^{1/x} = e^{\lim\limits_{x\to 0+} \frac{1}{x}\ln(1+x)} = e$

19. $\lim\limits_{x\to \frac{\pi}{2}} (\sin x)^{tg x} = e^{\lim\limits_{x\to\frac{\pi}{2}} tg x\cdot\ln(\sin x)} = e^0 = 1$

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$\lim\limits_{x\to\frac{\pi}{2}} \dfrac{\ln(\sin x)}{\sin x-1} \cdot \dfrac{\sin^2 x-1}{\sin x+1} \cdot \dfrac{\sin x}{\cos x} = \lim\limits_{x\to\frac{\pi}{2}} \dfrac{\ln(\sin x)}{\sin x-1} \cdot \dfrac{\sin x}{\sin x+1} \cdot (-\cos x) = 1\cdot\frac{1}{2}\cdot 0 = 0$

20. $\lim\limits_{x\to 0}\left(\frac{1+tgx}{1+\sin x}\right)^{\frac{1}{\sin^3 x}} = e^{\lim\limits_{x\to 0}\ln\left(\frac{1+tgx}{1+\sin x}\right)\cdot\frac{1}{\sin^3 x}} = e^{1/2} = \sqrt{e}$

$$\lim_{x\to 0}\frac{\ln\left(\frac{1+tgx}{1+\sin x}\right)}{\frac{1+tgx}{1+\sin x}-1}\cdot\frac{\cancel{1}+tgx-\cancel{1}-\sin x}{1+\sin x}\cdot\frac{1}{\sin^3 x} = \lim_{x\to 0}\frac{\ln(\;)}{(\;)-1}\cdot\frac{\sin x\,(1-\cos x)}{\sin^3 x\,(1+\sin x)\cos x}$$

$$=\lim_{x\to 0}\frac{\ln(\;)}{(\;)-1}\cdot\frac{1-\cos x}{x^2}\cdot\frac{x^2}{\sin^2 x}\cdot\frac{1}{(1+\sin x)\cos x} = \frac{1}{2}$$

21. $\lim\limits_{x\to 1}(1+\sin(\pi x))^{cotg(\pi x)} = e^{\lim\limits_{x\to 1} cotg(\pi x)\ln(1+\sin(\pi x))} = \frac{1}{e}$

$$\lim_{x\to 1}\frac{\cos(\pi x)}{\sin(\pi x)}\cdot\ln(1+\sin(\pi x)) = -1$$

22. $\lim\limits_{x\to 0+}(\cos\sqrt{x})^{1/x} = e^{\lim\limits_{x\to 0+}\frac{1}{x}\ln(\cos\sqrt{x})} = e^{-1/2} = \frac{1}{\sqrt{e}}$

$$\lim_{x\to 0+}\frac{\ln(\cos\sqrt{x})}{\cos\sqrt{x}-1}\cdot\frac{\cos\sqrt{x}-1}{x} = -\frac{1}{2}$$

23. $\lim\limits_{x\to 0}(1+x^2)^{cotg(\pi x)} = e^{\lim\limits_{x\to 0} cotg(\pi x)\ln(1+x^2)} = e^0 = 1$

$$\lim_{x\to 0}\frac{\ln(1+x^2)}{x^2}\cdot\frac{\pi x}{\sin(\pi x)}\cdot\frac{x\cos(\pi x)}{\pi} = 0$$

24. $\lim\limits_{x\to\frac{\pi}{4}}(tgx)^{tg(2x)} = e^{\lim\limits_{x\to\frac{\pi}{4}} tg(2x)\ln(tgx)} = \frac{1}{e}$

$$\lim_{x\to\frac{\pi}{4}}\frac{\ln(tgx)}{tgx-1}\cdot\frac{\sin x-\cos x}{\cancel{\cos x}}\cdot\frac{2\sin x\,\cancel{\cos x}}{\cos^2 x-\sin^2 x}\cdot\frac{\sin x+\cos x}{\sin x+\cos x} = \lim_{x\to\frac{\pi}{4}}\frac{\ln(\;)}{(\;)-1}\cdot\frac{(-2\sin x)}{\sin x+\cos x} = -1$$

25. $\lim\limits_{x\to 1}\frac{\sin(\pi x^\alpha)}{\sin(\pi x^\beta)}$ $\quad\alpha,\beta\in\mathbb{R},\ \beta\neq 0$

$$=\lim_{x\to 1}\frac{\sin(\pi(1+x^\alpha-1))}{\sin(\pi(1+x^\beta-1))} = \lim_{x\to 1}\frac{\sin\pi\cos(\pi(x^\alpha-1))+\sin(\pi(x^\alpha-1))\cos\pi}{\sin\pi\cos(\pi(x^\beta-1))+\sin(\pi(x^\beta-1))\cos\pi} = \lim_{x\to 1}\frac{\sin(\pi(x^\alpha-1))}{\sin(\pi(x^\beta-1))}$$

$$=\lim_{x\to 1}\frac{\sin(\pi(x^\alpha-1))}{\pi(x^\alpha-1)}\cdot\frac{\pi(x^\beta-1)}{\sin(\pi(x^\beta-1))}\cdot\frac{x^\alpha-1}{x^\beta-1} = \frac{\alpha}{\beta}$$

$$\lim_{x\to 1}\frac{x^\alpha-1}{x^\beta-1} = \lim_{x\to 1}\frac{e^{\alpha\ln x}-1}{e^{\beta\ln x}-1} = \lim_{x\to 1}\frac{e^{\alpha\ln x}-1}{\alpha\ln x}\cdot\frac{\beta\ln x}{e^{\beta\ln x}-1}\cdot\frac{\alpha}{\beta} = \frac{\alpha}{\beta}$$

$$26.\quad \lim_{x\to 0} \frac{e^{\alpha x}-e^{\beta x}}{\sin(\alpha x)-\sin(\beta x)} \qquad \alpha,\beta\in\mathbb{R},\ \alpha\neq\beta$$

$$=\lim_{x\to 0} \frac{e^{\alpha x}-1-\left(e^{\beta x}-1\right)}{\alpha\beta x}\cdot\frac{\alpha\beta x}{2\cos\left(\frac{\alpha+\beta}{2}x\right)\sin\left(\frac{\alpha-\beta}{2}x\right)}$$

$$=\lim_{x\to 0}\left[\frac{e^{\alpha x}-1}{\alpha x}\cdot\frac{1}{\beta}-\frac{e^{\beta x}-1}{\beta x}\cdot\frac{1}{\alpha}\right]\cdot\frac{\frac{(\alpha-\beta)x}{2}}{\sin\left(\frac{(\alpha-\beta)x}{2}\right)}\cdot\frac{1}{2\cos\left(\frac{(\alpha+\beta)x}{2}\right)}\cdot\frac{2\alpha\beta}{\alpha-\beta}$$

$$=\left(\frac{1}{\beta}-\frac{1}{\alpha}\right)\cdot 1\cdot\frac{1}{2}\cdot\frac{2\alpha\beta}{\alpha-\beta}=1$$

$$27.\quad \lim_{x\to a}\frac{a^x-x^a}{x-a}\ ,\ a\in\mathbb{R}^+$$

$$=\lim_{x\to a}\frac{e^{x\cdot\ln a}-e^{a\cdot\ln x}}{x-a}=\lim_{x\to a} e^{a\ln a}\cdot\frac{e^{(x-a)\ln a}-e^{a(\ln x-\ln a)}}{x-a}$$

$$=\lim_{x\to a} e^{a\ln a}\cdot\left[\frac{e^{(x-a)\ln a}-1}{(x-a)\ln a}\cdot\ln a-\frac{e^{a(\ln x-\ln a)}-1}{a(\ln x-\ln a)}\cdot\frac{a(\ln x-\ln a)}{x-a}\right]$$

$$\left(\lim_{x\to a}\frac{\ln x-\ln a}{\frac{x-a}{a}}=\lim_{x\to a}\frac{\ln\frac{x}{a}}{\frac{x}{a}-1}=1\right)$$

$$=a^a\left(\ln a-1\right)$$

$$28.\quad \lim_{x\to 0}\left(\frac{1+x2^x}{1+x3^x}\right)^{\frac{1}{x^2}}=e^{\lim_{x\to 0}\frac{1}{x^2}\ln\left(\frac{1+x2^x}{1+x3^x}\right)}=e^{\ln\frac{2}{3}}=\frac{2}{3}$$

$$\lim_{x\to 0}\frac{\ln\left(\frac{1+x2^x}{1+x3^x}\right)}{\frac{1+x2^x}{1+x3^x}-1}\cdot\frac{\cancel{1}+\cancel{x}2^x-\cancel{1}-\cancel{x}3^x}{1+x3^x}\cdot\frac{1}{x^{\cancel{2}}}=\lim_{x\to 0}\frac{\ln(\ )}{(\ )-1}\cdot\frac{2^x-3^x}{x}\cdot\frac{1}{1+x3^x}=\ln\frac{2}{3}$$

$$\lim_{x\to 0}\frac{2^x-3^x}{x}=\lim_{x\to 0}\frac{e^{x\ln 2}-e^{x\ln 3}}{x}=\lim_{x\to 0}e^{x\ln 3}\cdot\frac{e^{x(\ln 2-\ln 3)}-1}{x(\ln 2-\ln 3)}\cdot(\ln 2-\ln 3)=\ln\frac{2}{3}$$

$$29.\quad \lim_{x\to 0}\left(\frac{a^{x^2}+b^{x^2}}{a^x+b^x}\right)^{\frac{1}{x}}\ ,\ a,b\in\mathbb{R}^+$$

$$=e^{\lim_{x\to 0}\frac{1}{x}\ln(-)}=e^{\ln\frac{1}{\sqrt{ab}}}=\frac{1}{\sqrt{ab}}=\frac{\sqrt{ab}}{ab}$$

$$\lim_{x\to 0}\frac{\ln\left(\frac{a^{x^2}+b^{x^2}}{a^x+b^x}\right)}{\frac{a^{x^2}+b^{x^2}}{a^x+b^x}-1}\cdot\frac{a^{x^2}+b^{x^2}-a^x-b^x}{a^x+b^x}\cdot\frac{1}{x}=\lim_{x\to 0}\frac{\ln(\ )}{(\ )-1}\cdot\frac{1}{a^x+b^x}\cdot\left[a^x\cdot\frac{a^{x^2-x}-1}{x^2-x}+b^x\cdot\frac{b^{x^2-x}-1}{x^2-x}\right]\cdot\frac{x^2-x}{x}$$

$$=1\cdot\frac{1}{1+1}\cdot\left[1\cdot\ln a+1\cdot\ln b\right](-1)=\left(-\frac{1}{2}\right)(\ln a+\ln b)=\ln\frac{1}{\sqrt{ab}}$$

Spojitost a derivace funkcí, 1/5

Def. (Spojitost v bodě): Nechť $f: \mathbb{R} \to \mathbb{R}$ a $x_0 \in \mathbb{R}$. Řekneme, že fce f je spojitá v bodě x_0, jestliže $\lim_{x \to x_0} f(x) = f(x_0)$. Řekneme, že funkce f je spojitá na $M \subset \mathbb{R}$, jestliže je spojitá v x_0 pro každé $x_0 \in M$.

$$\forall \varepsilon > 0 \ \exists \delta > 0 : \quad x \in \mathcal{U}_\delta(x_0) \Rightarrow f(x) \in \mathcal{U}_\varepsilon(f(x_0))$$

(def limity: $\forall \varepsilon > 0 \ \exists \delta > 0 : \quad x \in P_\delta(x_0) \Rightarrow f(x) \in \mathcal{U}_\varepsilon(A)$)

Věta (aritmetika spojitosti): Nechť $f, g: \mathbb{R} \to \mathbb{R}$ jsou spojité v $x_0 \in \mathbb{R}$. Pak

i) $f+g$ a $f \cdot g$ jsou spojité v x_0

ii) je-li $g(x_0) \neq 0$, je $\frac{f}{g}$ spojitá v x_0.

Věta (Spojitost složené fce): Nechť $f, g: \mathbb{R} \to \mathbb{R}$, f je spojitá v x_0 a g je spojitá v $f(x_0)$. Pak $g \circ f$ je spojitá v x_0.

Def. (derivace v bodě): Nechť $f: \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R}$ a $A \in \mathbb{R}^*$. Řekneme, že f má v bodě x_0 derivaci rovnou A, jestliže

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = A \qquad \left(= \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} \right)$$

Věta (vztah derivace a spojitosti):
Nechť $f: \mathbb{R} \to \mathbb{R}$ má v bodě $x_0 \in \mathbb{R}$ <u>vlastní</u> derivaci. Pak je $f(x)$ v bodě x_0 spojitá.

Věta (aritmetika derivací): Nechť $f, g: \mathbb{R} \to \mathbb{R}$ mají vlastní derivace v bodě $x_0 \in \mathbb{R}$. Potom

a) $(f+g)'(x_0) = f'(x_0) + g'(x_0)$

b) $(fg)'(x_0) = f'(x_0) g(x_0) + f(x_0) g'(x_0)$

c) je-li $g(x_0) \neq 0$, pak $\left(\frac{f}{g}\right)'(x_0) = \dfrac{f'(x_0) g(x_0) - f(x_0) g'(x_0)}{g^2(x_0)}$

Věta (derivace složené fce):
Nechť $f, g: \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R}$, $f'(x_0) \in \mathbb{R}$ a $g'(f(x_0)) \in \mathbb{R}$. Potom $(g \circ f)'(x_0) = g'(f(x_0)) f'(x_0)$.

Věta (derivace inverzní fce I): Nechť $x_0 \in \mathbb{R}$ a $f: \mathbb{R} \to \mathbb{R}$ splňuje:

i) existují $\alpha, \beta_1, \beta_2 > 0$ taková, že f je prostá na $(x_0 - \alpha, x_0 + \alpha)$ a zobrazuje na $(f(x_0) - \beta_1, f(x_0) + \beta_2)$

ii) existuje vlastní nenulová $f'(x_0)$

iii) f^{-1} je spojitá v bodě $y_0 = f(x_0)$.

Pak existuje derivace f^{-1} v bodě y_0 a platí $(f^{-1})'(y_0) = \dfrac{1}{f'(x_0)} = \dfrac{1}{f'(f^{-1}(y_0))}$.

Věta (derivace inverzní fce II):
Nechť $x_0 \in \mathbb{R}$ a $f: \mathbb{R} \to \mathbb{R}$ je spojitá a ryze monotónní na jistém okolí x_0. Nechť existuje vlastní nenulová $f'(x_0)$. Pak $(f^{-1})'(f(x_0)) = \dfrac{1}{f'(x_0)}$.

Věta (o derivaci inverzní fce III): Nechť $x_0 \in \mathbb{R}$ a $f: \mathbb{R} \to \mathbb{R}$ splňuje jednu z podmínek:

i) na jistém okolí bodu x_0 platí $f' \in (0, \infty)$

ii) — " — $f' \in (-\infty, 0)$

Pak $(f^{-1})'(f(x_0)) = \dfrac{1}{f'(x_0)}$

Spojitost funkcí.

1. Dodefinujte funkci v bodě 0 tak, aby byla spojitá: $f(x) = \frac{1-\cos x}{x^2}$, $x \in \mathbb{R} \setminus \{0\}$

$\lim\limits_{x\to 0} f(x) = \lim\limits_{x \to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$ $\Rightarrow$ $\hat{f}(x) = \begin{cases} f(x) & x \in \mathbb{R}\setminus\{0\} \\ \frac{1}{2} & x = 0 \end{cases}$

2. Zjistěte, kde jsou nespojité fce

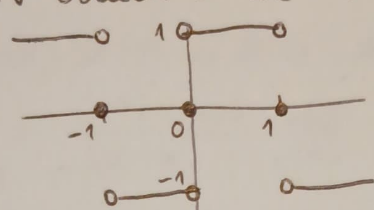
a) $f(x) = e^{-\frac{1}{x}}$ $\quad f(x) = g(h(x))$, $\left.\begin{array}{ll} g(x) = e^x & \text{sp. v } \mathbb{R} \\ h(x) = -\frac{1}{x} & \text{sp. v } \mathbb{R}\setminus\{0\} \end{array}\right\} \Rightarrow$ f(x) je nespojitá v $x=0$

b) $f(x) = \operatorname{sgn} \cos \frac{1}{x}$ $\quad f(x) = g(h(k(x)))$, $\left.\begin{array}{ll} g(x) = \operatorname{sgn} x & \text{nesp. v } x=0 \\ h(x) = \cos x & \text{sp. v } \mathbb{R} \\ k(x) = \frac{1}{x} & \text{nesp v } x = 0 \end{array}\right\}$ f(x) je nespojitá v $x=0$ a ve všech bodech, kde je $\cos\frac{1}{x} = 0$ $\left(x = \frac{2}{\pi + 2k\pi}\right)$

3. Vyšetřete spojitost složených fcí $f(g(x))$ a $g(f(x))$, je-li $f(x) = \operatorname{sgn} x$, $g(x) = x(1-x^2)$.

$f(g(x)) = \operatorname{sgn}(x(1-x^2))$ nesp. v bodech: $x(1-x^2) = 0 \Rightarrow x_1 = 0, x_2 = -1, x_3 = 1$.

	$(-\infty,-1)$	$(-1,0)$	$(0,1)$	$(1,\infty)$
x	$-$	$-$	$+$	$+$
$1-x$	$+$	$+$	$+$	$-$
$1+x$	$-$	$+$	$+$	$+$
	$+$	$-$	$+$	$-$



$\Rightarrow$ nesp. v x_1, x_2, x_3

$g(f(x)) = \operatorname{sgn} x\,(1 - \operatorname{sgn}^2 x)$ nesp. v $x = 0$

	$x<0$	$x>0$
$\operatorname{sgn} x$	-1	1
$1-\operatorname{sgn} x$	2	0
$1+\operatorname{sgn} x$	0	2

$\Rightarrow g(f(x)) \equiv 0$ $\Rightarrow$ sp. v $\mathbb{R}$

4. Zjistěte, zda jsou spojité fce

a) $f(x) = \begin{cases} \frac{\sin x}{|x|} & x \neq 0 \\ 1 & x = 0 \end{cases}$ $\quad \lim\limits_{x\to 0} \frac{\sin x}{|x|} = \begin{cases} \lim\limits_{x \to 0+} \frac{\sin x}{x} = 1 \\ \lim\limits_{x\to 0-} \frac{\sin x}{-x} = -1 \end{cases}$ $\Big\}$ lim + ≠ lim − není sp.

b) $f(x) = \begin{cases} x \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ $\quad \lim\limits_{x \to 0} x \underbrace{\sin\frac{1}{x}}_{\text{omez.}} = 0$ $\Rightarrow$ je spojitá

5. Dokažte, že jsou-li $f(x)$ a $g(x)$ sp. v x_0, pak jsou sp. v x_0 i fce

a) $\min\{f(x), g(x)\}$

1. $f(x_0) < g(x_0) \Rightarrow \exists \varepsilon > 0\ \forall x \in U_\varepsilon(x_0): f(x) < g(x) \Rightarrow \forall x \in U_\varepsilon(x_0): \min\{f(x), g(x)\} = f(x) \Rightarrow$ sp.
2. $g(x_0) < f(x_0) \Rightarrow$ −〃− $g(x) < f(x) \Rightarrow$ −〃− $= g(x) \Rightarrow$ sp.
3. $g(x_0) = f(x_0)$: A) a) $\exists \varepsilon > 0\ \forall x \in U_{\varepsilon+}(x_0): f(x) \leq g(x) \Rightarrow \forall x \in U_{\varepsilon+}(x_0): \min\{f(x), g(x)\} = f(x)$
   b) −〃− $g(x) \leq f(x)$ −〃− $= g(x)$
   B) a) $\exists \varepsilon > 0\ \forall x \in U_{\varepsilon-}(x_0): f(x) \leq g(x) \Rightarrow \forall x \in U_{\varepsilon-}(x_0): \min\{f(x), g(x)\} = f(x)$
   b) −〃− $g(x) \leq f(x) \Rightarrow$ −〃− $= g(x)$

& $\lim\limits_{-} = \lim\limits_{+}$ $\Rightarrow$ sp.

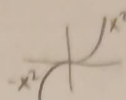
b) $\max\{f(x), g(x)\}$ rovnako

6. Uveďte příklad fce nespojité v každém $x \in \mathbb{R}$, jejíž 2. mocnina je spojitá na $\mathbb{R}$.

$$f(x) = \begin{cases} -1 & x \in \mathbb{R} \setminus \mathbb{Q} \\ 1 & x \in \mathbb{Q} \end{cases} \qquad \left(\text{Dirichletova fce: } D(x) = \begin{cases} 0 & x \in \mathbb{R} \setminus \mathbb{Q} \\ 1 & x \in \mathbb{Q} \end{cases}, \ f(x) = 2D(x) - 1 \right)$$

Derivace fcí

7. Existuje derivace fce $f(x) = x|x|$ v bodě 0?



$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x|x|}{x} = \lim_{x \to 0} |x| = 0 \quad \Rightarrow \text{Ano, existuje a rovná sa 0.}$$

8. Pro jaké α reálné má fce $f(x) = \begin{cases} |x|^\alpha \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ derivaci v bodě 0? Kdy je tato derivace v bodě 0 spojitá?

Nutná podmínka $\exists$ derivace: $f(x)$ je sp. pro $\alpha > 0$.

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{|x|^\alpha \sin \frac{1}{x} - 0}{x} = \lim_{x \to 0} \frac{|x|^\alpha}{x} \sin \frac{1}{x} = 0 \iff \alpha > 1$$

$\sin \frac{1}{x}$ omez., limita ex. jen pokud $\lim_{x \to 0} \frac{|x|^\alpha}{x} = 0 \iff \alpha > 1$

spojitost derivace:

$$\lim_{x \to 0+} (f(x))' = \lim_{x \to 0+} \left(x^\alpha \sin \tfrac{1}{x}\right)' = \lim_{x \to 0+} \alpha x^{\alpha-1} \sin \tfrac{1}{x} + x^\alpha \cos \tfrac{1}{x} \left(-\tfrac{1}{x^2}\right) = \lim_{x \to 0+} x^{\alpha-2} \underbrace{\left(\alpha x \sin \tfrac{1}{x} - \cos \tfrac{1}{x}\right)}_{\text{omez.}} = 0 \iff \alpha > 2$$

$$\lim_{x \to 0-} (f(x))' = \lim_{x \to 0-} \left((-x)^\alpha \sin \tfrac{1}{x}\right)' = \lim_{x \to 0-} -\alpha (-x)^{\alpha-1} \sin \tfrac{1}{x} + (-x)^\alpha \cos \tfrac{1}{x} \left(-\tfrac{1}{x^2}\right) = \lim_{x \to 0-} (-x)^{\alpha-2} \left(\alpha x \sin \tfrac{1}{x} - \cos \tfrac{1}{x}\right) = 0 \iff \alpha > 2$$

Fce f má v bodě 0 derivaci pro $\alpha > 1$ a táto je spojitá pro $\alpha > 2$.

9. Dokažte, že fce $f(x) = \begin{cases} x^2 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$ má derivaci pouze v nule.

$f(x) = x^2 \cdot D(x)$

nutná podmínka: spojitost: $D(x)$ nemá limitu v žádném $x \in \mathbb{R}$, ale je omezená

$\Rightarrow \lim_{x \to 0} x^2 \cdot D(x) = 0 \quad \Rightarrow x^2 D(x)$ je spojitá v 0

Jediný bod, ve kt. může mít derivaci je 0.

$\lim_{x \to 0} \frac{x^2 D(x) - 0}{x} = \lim_{x \to 0} x D(x) = 0$. Tedy $f(x)$ má der. v $x = 0$ a táto se rovná 0.

10. Ukažte, že derivace sudé fce (pokud existuje) je fce lichá.

sudá: $f(x) = f(-x)$, lichá: $f(x) = -f(-x) \iff -f(x) = f(-x)$

$$f'(-x) = \lim_{h \to 0} \frac{f(-x+h) - f(-x)}{h} = \lim_{-h \to 0} \frac{f(-x-h) - f(-x)}{-h} \overset{f \text{ sudá}}{=} \lim_{-h \to 0} \frac{f(x+h) - f(x)}{-h} = -f'(x)$$

11. Nechť $f(x) = \begin{cases} x^2 & x \leq 1 \\ ax + b & x > 1 \end{cases}$. Určete a, b tak, aby $f(x)$ měla v bodě 1 derivaci.

Nutná podmínka: spojitost: $\lim_{x \to 1-} x^2 = 1 = \lim_{x \to 1+} ax + b = a + b \quad \Rightarrow a + b = 1$

$$f'_+(1) = \lim_{x \to 1+} \frac{f(x) - 1}{x - 1} = \lim_{x \to 1+} \frac{ax + b - 1}{x - 1} = \lim_{x \to 1+} \frac{ax + 1 - a - 1}{x - 1} = \lim_{x \to 1+} \frac{a(x-1)}{x - 1} = a$$

$$f'_-(1) = \lim_{x \to 1-} \frac{f(x) - 1}{x - 1} = \lim_{x \to 1-} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1-} x + 1 = 2$$

$\Rightarrow a = 2, \ b = -1$

12. Určete rovnici tečny a normály ke grafu fce $f(x) = x^3 + 2x^2 - 4x - 3$ v bodě $[-2, ?]$ grafu.

$x_0 = -2$

$y_0 = f(x_0) = (-2)^3 + 2(-2)^2 - 4(-2) - 3 = -8 + 8 + 8 - 3 = 5$

$f'(x) = 3x^2 + 4x - 4$

tečna: $y = kx + q$: $y_0 = kx_0 + q \Rightarrow k = f'(x_0)$

$f'(x_0) = 3 \cdot (-2)^2 + 4(-2) - 4 = 12 - 8 - 4 = 0 = k \Rightarrow q = 5$

tečna v $[-2, 5]$: $y = 5$

normála: $x = -2$

tečna: $y - y_0 = f'(x_0)(x - x_0)$ normála: $y - y_0 = -\frac{1}{f'(x_0)}(x - x_0)$

$y - 5 = 0 \cdot (x + 2) \Rightarrow y = 5$ $0 = x + 2 \Rightarrow x = -2$

Elementární fce

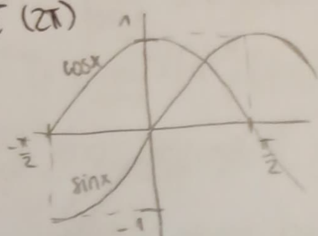
• **sinx, cosx**

omezené, periodické (2π)

$\sin x = \cos(x - \frac{\pi}{2})$

$\sin' x = \cos x$

$\cos' x = -\sin x$



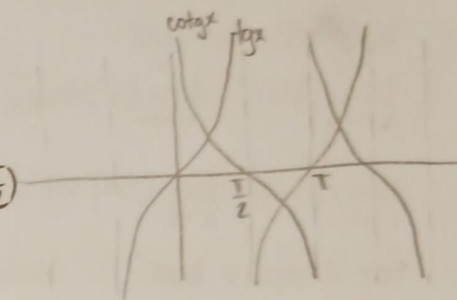
• **tgx, cotgx**

π-periodické

$\mathrm{tg}\,x = -\mathrm{cotg}(x - \frac{\pi}{2})$

$\mathrm{tg}' x = \frac{1}{\cos^2 x}$

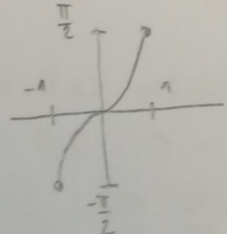
$\mathrm{cotg}' x = -\frac{1}{\sin^2 x}$



• **arcsinx**

$D_f = \langle -1, 1 \rangle$, $H_f = \langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle$

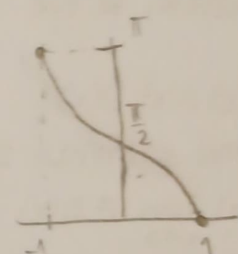
lichá, prostá, rostoucí



• **arccosx**

$D_f = \langle -1, 1 \rangle$, $H_f = \langle 0, \pi \rangle$

klesající, prostá



derivace arcsinx:

pro $x \in (-1, 1)$, $y = \arcsin x \Rightarrow y \in (-\frac{\pi}{2}, \frac{\pi}{2})$ a $x = \sin y$ má na $(-\frac{\pi}{2}, \frac{\pi}{2})$ vlastní nenulovou derivaci & můžeme použít větu o derivaci inverzní fce:

$$(\arcsin x)' = \frac{1}{\sin'(\arcsin x)} = \frac{1}{\cos(\arcsin x)} \underset{\text{platí pro } x \in (-1,1)}{=} \frac{1}{\sqrt{1 - \sin^2(\arcsin x)}} = \frac{1}{\sqrt{1 - x^2}}$$

pro $x = -1$: $\lim\limits_{x \to (-1)+} \frac{\arcsin x - \arcsin(-1)}{x + 1} = \lim\limits_{x \to (-1)+} \frac{\arcsin x + \frac{\pi}{2}}{x + 1} = \lim\limits_{y \to (-\frac{\pi}{2})+} \frac{\arcsin(\sin y) + \frac{\pi}{2}}{\sin y + 1}$

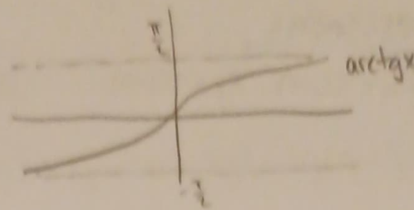
$= \lim\limits_{h \to 0+} \frac{h}{\sin(h - \frac{\pi}{2}) + 1} = \lim\limits_{h \to 0+} \frac{h^2}{1 - \cos h} \cdot \frac{1}{h} = +\infty$, podobně pro $x \to 1-$

derivace arccosx:

pro $x \in (-1, 1)$: $\arccos' x = \frac{1}{\cos'(\arccos x)} = \frac{-1}{\sin(\arccos x)} = \frac{-1}{\sqrt{1 - \cos^2(\arccos x)}} = \frac{-1}{\sqrt{1 - x^2}}$

$\arccos'(-1) = \arccos'(1) = -\infty$

- **arctg x**



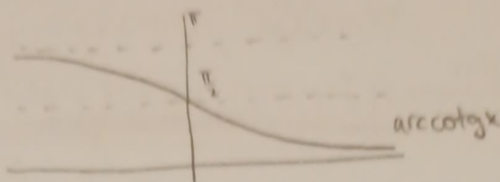
$\mathcal{D}_f = \mathbb{R},\ \mathcal{H}_f = (-\frac{\pi}{2}, \frac{\pi}{2})$

lichá, rostoucí, omezená

$$\text{arctg}'x = \frac{1}{\text{tg}'(\text{arctg}x)} = \frac{1}{\frac{1}{\cos^2(\text{arctg}x)}} = \frac{1}{\frac{\sin^2(\,)+\cos^2(\,)}{\cos^2(\,)}} = \frac{1}{1+\text{tg}^2(\text{arctg}x)} = \frac{1}{1+x^2}$$

$\text{arctg}x \to 0 \quad x \to \pm\infty$

- **arccotgx**

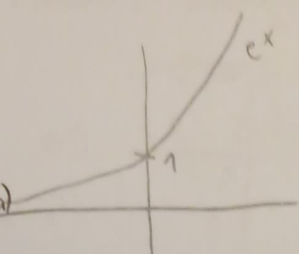


$\mathcal{D}_f = \mathbb{R},\ \mathcal{H}_f = (0, \pi)$

klesající, omezená

$$\text{arccotg}'x = \frac{1}{\text{cotg}'(\text{arccotg}x)} = \frac{-1}{\frac{1}{\sin^2(\text{arccotg}x)}} = \frac{-1}{\frac{\sin^2(\,)+\cos^2(\,)}{\sin^2(\,)}} = \frac{-1}{1+\text{cotg}^2(\text{arccotg}x)} = \frac{-1}{1+x^2}$$

- **e^x** $= \exp(x)$



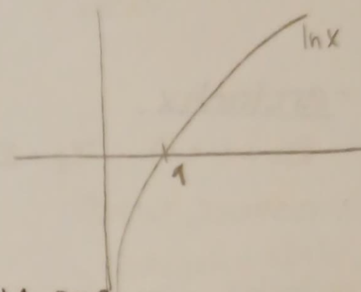
$\mathcal{D}_f = \mathbb{R},\ \mathcal{H}_f = \mathbb{R}_+$

rostoucí, neomezená (shora)

$(e^x)' = e^x$

$(a^x)' = a^x \ln a$

- **ln(x)**



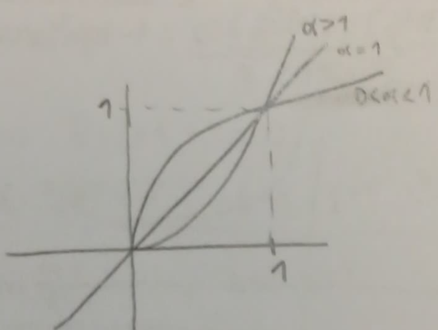
$\mathcal{D}_f = \mathbb{R}_+,\ \mathcal{H}_f = \mathbb{R}$

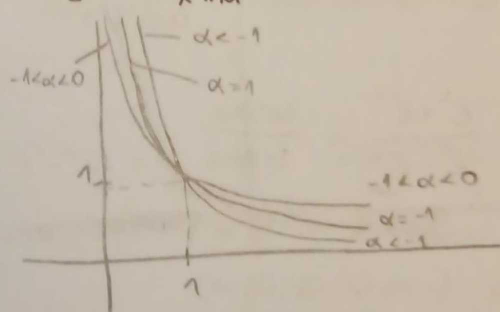
rostoucí, neomezená

$\ln' x = \frac{1}{x}$

$\log_a' x = \frac{1}{x \ln a}$, $a \neq 1, a > 0$

- **x^α**





$(x^\alpha)' = \alpha x^{\alpha-1}$

na $x \in \mathbb{R}_+$:

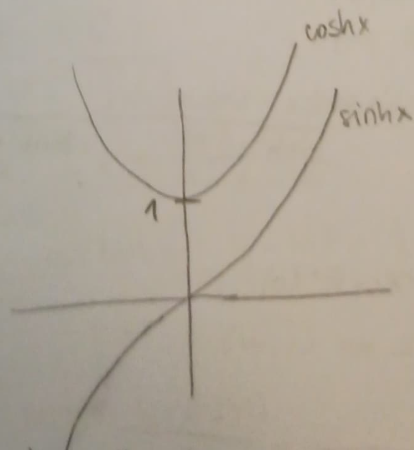
rostoucí pro $\alpha > 0$

konstantní pro $\alpha = 0$

klesající pro $\alpha < 0$

Hyperbolické fce

- **sinh x** $= \frac{e^x - e^{-x}}{2}$



$\mathcal{D}_f = \mathbb{R},\ \mathcal{H}_f = \mathbb{R}$

monotónní, rostoucí, lichá

- **coshx** $= \frac{e^x + e^{-x}}{2}$

$\mathcal{D}_f = \mathbb{R},\ \mathcal{H}_f = \langle 1, \infty)$

klesající na $(-\infty, 0\rangle$

rostoucí na $\langle 0, \infty)$

sudá

$$\sinh' x = \left(\frac{e^x - e^{-x}}{2}\right)' = \frac{e^x + e^{-x}}{2} = \cosh x$$

$$\cosh' x = \left(\frac{e^x + e^{-x}}{2}\right)' = \frac{e^x - e^{-x}}{2} = \sinh x$$

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{e^{2x} + 2e^x e^{-x} + e^{-2x}}{4} - \frac{e^{2x} - 2e^x e^{-x} + e^{-2x}}{4} = 1$$

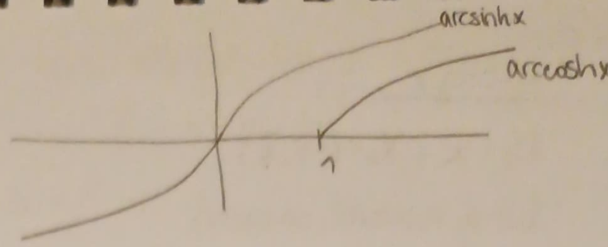
- **arcsinh x**

$D_f = \mathbb{R}$, $H_f = \mathbb{R}$

lichá

- **arccosh x**

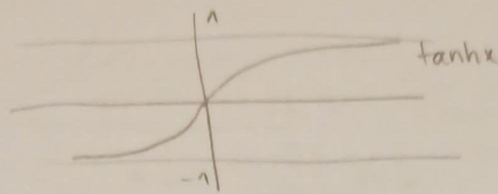
$D_f = \langle 1, \infty)$, $H_f = \langle 0, \infty)$



$$\operatorname{arcsinh}' x = \frac{1}{\sinh'(\operatorname{arcsinh} x)} = \frac{1}{\cosh(\operatorname{arcsinh} x)} = \frac{1}{\sqrt{1+\sinh^2(\;)}} = \frac{1}{\sqrt{1+x^2}}$$

$$\operatorname{arccosh}' x = \frac{1}{\cosh'(\operatorname{arccosh} x)} = \frac{1}{\underbrace{\sinh}_{>0}(\underbrace{\operatorname{arccosh} x}_{\in \langle 1,\infty)})} = \frac{1}{\sqrt{\cosh^2(\;)-1}} = \frac{1}{\sqrt{x^2-1}}$$

- **tanh x** $= \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \dfrac{\sinh x}{\cosh x}$



$D_f = \mathbb{R}$, $H_f = (-1, 1)$

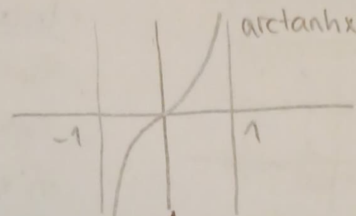
rostoucí, lichá, omezená

$$\tanh' x = \left(\frac{\sinh x}{\cosh x}\right)' = \frac{\sinh' x \cosh x - \sinh x \cosh' x}{\cosh^2 x} = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$$

- **arctanh x**

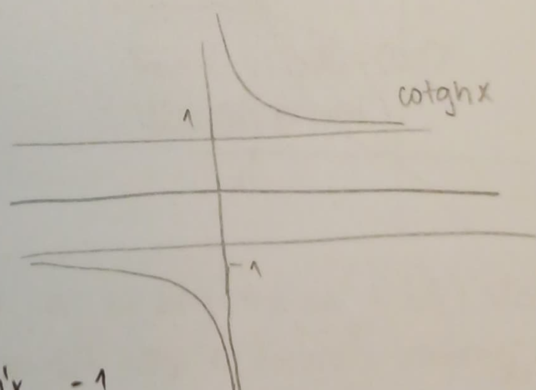
$D_f = (-1, 1)$, $H_f = \mathbb{R}$

rostoucí, lichá



$$\operatorname{arctanh}' x = \frac{1}{\tanh'(\operatorname{arctanh} x)} = \frac{1}{\frac{1}{\cosh^2(\;)}} = \frac{1}{\frac{\cosh^2(\;) - \sinh^2(\;)}{\cosh^2(\;)}} = \frac{1}{1 - \operatorname{tgh}^2(\operatorname{arctgh} x)} = \frac{1}{1 - x^2}$$

- **cotgh x** $= \dfrac{e^x + e^{-x}}{e^x - e^{-x}} = \dfrac{\cosh x}{\sinh x}$



$D_f = \mathbb{R} \setminus \{0\}$, $H_f = (-\infty, -1) \cup (1, \infty)$

klesající na $(-\infty, 0)$ a $(0, \infty)$

shora omezená na $(-\infty, 0)$

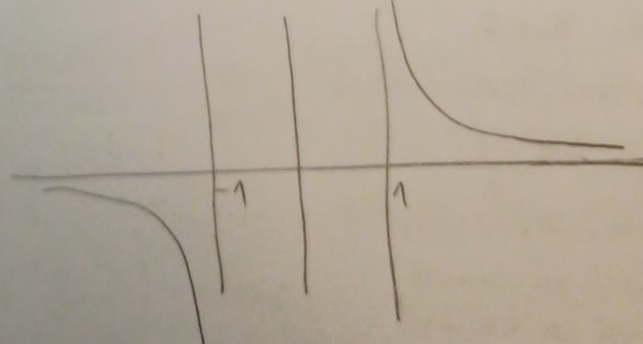
zdola omezená na $(0, \infty)$

$$\operatorname{cotgh}' x = \left(\frac{\cosh x}{\sinh x}\right)' = \frac{\cosh' x \sinh x - \cosh x \sinh' x}{\sinh^2 x} = \frac{-1}{\sinh^2 x}$$

- **arccotgh x**

$D_f = (-\infty, -1) \cup (1, \infty)$, $H_f = \mathbb{R} \setminus \{0\}$

klesající na $(-\infty, -1)$ a na $(1, \infty)$



$$\operatorname{arccotgh}' x = \frac{1}{\operatorname{cotgh}'(\operatorname{arccotgh} x)} = \frac{-1}{\frac{1}{\sinh^2 x}}$$

$$= \frac{-1}{\frac{\cosh^2(\;) - \sinh^2(\;)}{\sinh^2(\;)}} = \frac{1}{1 - x^2}$$

Dokažte, že

13. $\operatorname{arctg} x + \operatorname{arccotg} x = \frac{\pi}{2}$, $x \in \mathbb{R}$

$$\operatorname{arctg} x + \operatorname{arccotg} x = \operatorname{arctg}(\operatorname{cotg} y) + y = \operatorname{arctg}(-\operatorname{tg}(y-\tfrac{\pi}{2})) + y = \operatorname{arctg}(\operatorname{tg}(\tfrac{\pi}{2}-y)) + y$$

$$= \frac{\pi}{2} - y + y = \frac{\pi}{2}$$

$y = \operatorname{arccotg} x \iff x = \operatorname{cotg} y$, $y \in (0,\pi)$ & $x \in (-\infty,\infty)$

14. $\arcsin x + \arccos x = \frac{\pi}{2}$, $x \in \langle -1,1\rangle$

$y = \arccos x \iff x = \cos y$, $x \in \langle -1,1\rangle$, $y \in \langle 0,\pi\rangle$

$$\arcsin x + \arccos x = \arcsin(\cos y) + y = \arcsin(-\sin(y-\tfrac{\pi}{2})) + y = \arcsin(\sin(\underbrace{\tfrac{\pi}{2}-y}_{\in\langle -\frac{\pi}{2},\frac{\pi}{2}\rangle})) + y = \frac{\pi}{2}$$

15. $\operatorname{arcsinh} x = \ln(x + \sqrt{x^2+1})$, $x \in \mathbb{R}$

$y = \operatorname{arcsinh} x \iff x = \sinh y = \dfrac{e^y - e^{-y}}{2} \overset{(**)}{=} \dfrac{t - \frac{1}{t}}{2}$, $t \neq 0$, $x, y \in \mathbb{R}$

$t = e^y \iff y = \ln t$: (*)

(*): $t \overset{?}{=} x + \sqrt{x^2+1}$

$(t-x)^2 \overset{?}{=} x^2+1$

$t^2 - 2tx + x^2 \overset{?}{=} x^2 + 1$

$t^2 - 2tx - 1 \overset{?}{=} 0$ $\Leftarrow$

(**): $x = \dfrac{t - \frac{1}{t}}{2}$

$2x = t - \frac{1}{t}$

$t^2 - 2tx - 1 = 0$

16. $\operatorname{arccosh} x = \ln(x + \sqrt{x^2-1})$, $x \geq 1$

$y = \operatorname{arccosh} x \iff x = \cosh y$, $y \in \langle 1,\infty)$, $x \in \langle 0,\infty)$

$= \dfrac{e^y + e^{-y}}{2} = \dfrac{t + \frac{1}{t}}{2}$, $t \neq 0$, $y \in \mathbb{R}$, $x \geq 1$ $\iff$ $y = \operatorname{arccosh} x = \ln t$

$x = \dfrac{t + \frac{1}{t}}{2} \iff t^2 - 2xt + 1 = 0$

$\ln(x + \sqrt{x^2-1}) = \ln t \iff \sqrt{x^2-1} = t - x \iff x^2 - 1 = t^2 - 2xt + x^2 \iff t^2 - 2xt + 1 = 0$

17. $\operatorname{arctgh} x = \frac{1}{2} \ln \frac{1+x}{1-x}$, $x \in (-1,1)$

$y = \operatorname{arctgh} x \iff x = \operatorname{tgh} y$, $y \in \mathbb{R}$, $x \in (-1,1)$

$= \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \dfrac{t - \frac{1}{t}}{t + \frac{1}{t}} = \dfrac{t^2-1}{t^2+1}$, $t > 0$, $y = \ln t$

$x = \dfrac{t^2-1}{t^2+1} \iff t^2 x + x = t^2 - 1 \iff t^2 = \dfrac{1+x}{1-x}$

$t = \left(\frac{1+x}{1-x}\right)^{\frac{1}{2}} \iff t^2 = \dfrac{1+x}{1-x} \iff \operatorname{arctgh} x = y = \ln t = \frac{1}{2} \ln \frac{1+x}{1-x}$

18. $\operatorname{arccotgh} x = \frac{1}{2} \ln \frac{x+1}{x-1}$, $|x| > 1$

$y = \operatorname{arccotgh} x \iff x = \operatorname{cotgh} y$, $|x| > 1$, $y \in \mathbb{R} \setminus \{0\}$

$= \dfrac{e^x + e^{-x}}{e^x - e^{-x}} = \dfrac{t + \frac{1}{t}}{t - \frac{1}{t}} = \dfrac{t^2+1}{t^2-1}$, $t > 0$, $y = \ln t$

$x = \dfrac{t^2+1}{t^2-1} \iff t^2 x - x = t^2 + 1 \iff t^2 = \dfrac{x+1}{x-1}$

$\operatorname{arccotgh} x = y = \ln t = \frac{1}{2} \ln \frac{x+1}{x-1} \iff t^2 = \dfrac{x+1}{x-1}$

Derivace elementárních funkcí

Vypočtěte derivace následujících fcí v libovolném bodě x, kde derivace existuje:

20. $f(x) = \frac{2x}{1-x^2}$

$$f'(x) = \frac{2(1-x^2)-2x(-2x)}{(1-x^2)^2} = \frac{2(1+x^2)}{(1-x^2)^2}$$

21. $f(x) = \sqrt[3]{\frac{1+x^3}{1-x^3}}$

$$f'(x) = \frac{1}{3}\left(\frac{1+x^3}{1-x^3}\right)^{-2/3} \frac{3x^2(1-x^3)-(1+x^3)(-3x^2)}{(1-x^3)^2} = 2x^2(1-x^3)^{-\frac{4}{3}}(1+x^3)^{-\frac{2}{3}}$$

22. $f(x) = \frac{\sin^2 x}{\sin x^2}$

$$f'(x) = \frac{2\sin x \cos x \sin x^2 - \sin^2 x \cos x^2 \, 2x}{(\sin x^2)^2}$$

23. $f(x) = \sin\sin\sin x$

$$f'(x) = \cos\sin\sin x \cdot \cos\sin x \cdot \cos x$$

24. $f(x) = 2^{\mathrm{tg}\frac{1}{x}}$

$$f'(x) = 2^{\mathrm{tg}\frac{1}{x}} . \ln 2 \cdot \frac{1}{\cos^2\frac{1}{x}}\left(-\frac{1}{x^2}\right)$$

25. $f(x) = x^{a^a} + a^{x^a} + a^{a^x}$

$$f'(x) = a^a x^{(a^a-1)} + \ln a . a x^{a-1} . a^{x^a} + a^x \ln^2 a \, a^{a^x}$$

26. $f(x) = \sin x^{\cos x} + \cos x^{\sin x} = e^{\ln(\sin x) \cdot \cos x} + e^{\ln(\cos x) . \sin x}$

$$f'(x) = \sin x^{\cos x} . \left(\frac{\cos^2 x}{\sin x} - \ln(\sin x) . \sin x\right) + \cos x^{\sin x}\left(-\frac{\sin^2 x}{\cos x} + \ln(\cos x) . \cos x\right)$$

27. $f(x) = \mathrm{arctg}\frac{1+x}{1-x}$

$$f'(x) = \frac{1}{1+\left(\frac{1+x}{1-x}\right)^2} \cdot \frac{1(1-x)-(1+x)(-1)}{(1-x)^2} = \frac{1-x+1+x}{(1-2x+x^2+1+2x+x^2)} = \frac{1}{1+x^2}$$

28. $f(x) = x \arcsin^2 x + 2\sqrt{1-x^2} \arcsin x - 2x$

$$f'(x) = \arcsin^2 x + 2x \arcsin x \cdot \frac{1}{\sqrt{1-x^2}} + 2 . \frac{1}{2} \cdot \frac{1}{\sqrt{1-x^2}}(-2x)\arcsin x + \frac{2\sqrt{1-x^2}}{\sqrt{1-x^2}} - 2 = \arcsin^2 x$$

29. $f(x) = \ln(e^x + \sqrt{1+e^{2x}})$

$$f'(x) = \frac{e^x + \frac{e^{2x}}{\sqrt{1+e^{2x}}}}{e^x + \sqrt{1+e^{2x}}} = \frac{e^x + \sqrt{1+e^{2x}}\left(\frac{e^{2x}}{1+e^{2x}}\right)}{e^x + \sqrt{1+e^{2x}}} = 1 - \frac{\sqrt{1+e^{2x}}}{(e^{2x}+1)(e^x+\sqrt{1+e^{2x}})}$$

$$= \frac{e^x(\sqrt{1+e^{2x}} + e^x)}{e^x + \sqrt{1+e^{2x}}} = \frac{e^x}{\sqrt{1+e^{2x}}}$$

Derivace vyšších řádů. Parciální derivace

30. Ověřte, že funkce $u(x) = \frac{1}{|x|}$, kde $|x|^2 = x_1^2 + x_2^2 + x_3^2$, splňuje v $\mathbb{R}^3 \setminus \{0\}$ Laplaceovu rovnici $\Delta u = \sum_{i=1}^{3} \frac{\partial^2 u}{\partial x_i^2} = 0$.

$$|x| = \sqrt{x_1^2 + x_2^2 + x_3^2} \Rightarrow \frac{\partial |x|}{\partial x_i} = \frac{1}{2|x|} \cdot 2x_i = \frac{x_i}{|x|}$$

$$\frac{\partial u}{\partial x_i} = -\frac{1}{|x|^2} \cdot \frac{\partial |x|}{\partial x_i} = -\frac{x_i}{|x|^3}, \quad \frac{\partial^2 u}{\partial x_i^2} = \frac{(-1)|x|^3 + x_i 3|x|^2 \frac{x_i}{|x|}}{|x|^6} = \frac{3x_i^2|x| - |x|^3}{|x|^6}$$

$$\sum_{i=1}^{3} \frac{3x_i^2 - |x|^2}{|x|^5} = \frac{3|x|^2 - 3|x|^2}{|x|^5} = 0$$

31. Ověřte, že fce $v(t,x) = \frac{1}{t^{3/2}} e^{-\frac{|x|^2}{4t}}$, kde $|x|^2 = x_1^2 + x_2^2 + x_3^2$, splňuje v $(0,\infty) \times (\mathbb{R}^3 \setminus \{0\})$ rovnici vedení tepla $\frac{\partial v}{\partial t} - \Delta v = 0$.

$$\frac{\partial v}{\partial t} = -\frac{3}{2} t^{-5/2} e^{-\frac{|x|^2}{4t}} + t^{-3/2} e^{-\frac{|x|^2}{4t}} \frac{|x|^2}{4t^2} = \frac{e^{-\frac{|x|^2}{4t}}}{2t^{5/2}} \left(\frac{|x|^2}{2t} - 3 \right)$$

$$\frac{\partial v}{\partial x_i} = t^{-3/2} e^{-\frac{|x|^2}{4t}} \left(-\frac{2|x|}{4t} \cdot \frac{x_i}{|x|} \right) = t^{-3/2} e^{-\frac{|x|^2}{4t}} \left(\frac{-2x_i}{4t} \right)$$

$$\sum_{i=1}^{3} \frac{\partial^2 v}{\partial x_i^2} = \sum_{i=1}^{3} \left(t^{-3/2} e^{-\frac{|x|^2}{4t}} \left(\frac{-2x_i}{4t} \right)^2 + t^{-3/2} e^{-\frac{|x|^2}{4t}} \left(-\frac{1}{2t} \right) \right) = \frac{e^{-\frac{|x|^2}{4t}}}{2t^{5/2}} \sum_{i=1}^{3} \left(\frac{x_i^2}{2t} - 1 \right) = \frac{e^{-\frac{|x|^2}{4t}}}{2t^{5/2}} \left(\frac{|x|^2}{2t} - 3 \right) = \frac{\partial v}{\partial t}$$

32. Spočtěte $f^{(10)}(x)$, je-li $f(x) = \sqrt{x}$.

$$f'(x) = \frac{1}{2} x^{-\frac{1}{2}}, \quad f''(x) = \frac{1}{2} \cdot \left(-\frac{1}{2}\right) x^{-\frac{3}{2}}, \dots, f^{(10)}(x) = \frac{1}{2} \cdot \frac{(-1)}{2} \cdot \dots \cdot \frac{1-9 \cdot 2}{2} x^{\frac{1-10 \cdot 2}{2}}$$

33. Spočtěte $f^{(50)}(x)$, je-li $f(x) = x^2 \sin 2x$.

$$f'(x) = 2x \sin 2x + x^2 \cos 2x \cdot 2 = 2x(\sin 2x + x\cos 2x) = \sin 2x \cdot 2x + \cos 2x \cdot 2x^2$$

$$f''(x) = 2(\sin 2x + x\cos 2x) + 2x(\cos 2x + \cos 2x - 2x \sin 2x) = \sin 2x(2 - 4x^2) + \cos 2x \, 6x$$

$$f'''(x) = 2\cos 2x(2 - 4x^2) + \sin 2x(-8x) + 6\cos 2x - 12x \sin 2x = \sin 2x(-20x) + \cos 2x(10 - 8x^2)$$

$$f^{(IV)}(x) = 2\cos 2x(-20x) + \sin 2x(-20) + \cos 2x(-16x) + \sin 2x(-2 \cdot (10 - 8x^2)) = \sin 2x($$

$$(f \cdot g)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}$$

$$f^{(50)}(x) = (x^2 \sin 2x)^{(50)} = \binom{50}{0} x^2 (\sin 2x)^{(50)} + \binom{50}{1} 2x (\sin 2x)^{(49)} + \binom{50}{2} 2 (\sin 2x)^{(48)}$$

$$= -x^2 2^{50} \sin 2x + 50 \cdot 2 \cdot x \cdot 2^{49} \cos 2x + \frac{50!}{48! \, 2} \cdot 2 \cdot 2^{48} \sin 2x$$

$$= \sin 2x \cdot 2^{48} (50 \cdot 49 - 4x^2) + \cos 2x \cdot 100 \cdot 2^{49} x$$

RSJ

Primitivní fce

Def. Nechť $f, F: \mathbb{R} \to \mathbb{R}$. Řekneme, že F je primitivní fcí k f na (a,b), jestliže $F' = f$ na (a,b). Píšeme $F = \int f(x)\,dx$

Věta (o nejednoznačnosti):

1. Je-li F prim. fce k f na (a,b) a $c \in \mathbb{R}$, pak $F + c$ je také prim. fce k f na (a,b).
2. Jsou-li F a G prim. fce k f na (a,b), pak existuje $c \in \mathbb{R}$ takové, že $G = F + c$.

Věta (spojitost prim. fce). Je-li F prim. fce k f na (a,b), pak je na (a,b) spojitá.

Věta (prim. fce ke spoj. fci). Nechť $f: \mathbb{R} \to \mathbb{R}$ je spojitá na (a,b). Pak zde má primitivní fci.

Def. (Darbouxova vlastnost): Nechť $f: \mathbb{R} \to \mathbb{R}$ a $(a,b) \subset \mathbb{R}$. Řekneme, že f má na (a,b) Dar. vl., jestliže platí:

$$a < x < y < b \ \wedge \ c \in (\min\{f(x), f(y)\}, \max\{f(x), f(y)\}) \Rightarrow \exists z \in (x,y): f(z) = c.$$ (zobr. interval na interval)

Věta (Dar. vl. derivace): Nechť $F: \mathbb{R} \to \mathbb{R}$ má vlastní derivaci f na (a,b). Pak zde má f Darbouxovu vlastnost.

Věta (o úspěšnosti "lepení" pro spojitý interval): Nechť $a < b < c$, f je spojitá na (a,c), F_1 je prim. fce k f na (a,b) a F_2 je prim. fce k f na (b,c). Pak ex. vlastní limity $\lim_{x \to b-} F_1(x)$ a $\lim_{x \to b+} F_2(x)$ a fce

$$F(x) = \begin{cases} F_1(x) & x \in (a,b) \\ \lim\limits_{x \to b-} F_1(x) & x = b \\ F_2(x) - \lim\limits_{x \to b+} F_2(x) + \lim\limits_{x \to b-} F_1(x) & x \in (b,c), \end{cases}$$

je prim. fce k F na (a,c).

Věta (prim. fce součtu a nás. konstantou). Nechť $\alpha \in \mathbb{R}$, F je prim. fce k f na (a,b) a G je prim. fce k g na (a,b). Pak αF je prim. fce k αf na (a,b) a $F+G$ je prim. fce k $f+g$ na (a,b).

Věta (metoda per partes). Nechť f, g mají na (a,b) vlastní derivace. Pak

$$\int f'g\,dx = fg - \int fg'\,dx,$$ jestliže alespoň jedna z prim. fcí existuje.

Věta (1. věta o substituci). Nechť F je prim. fce k f na (a,b) a $\varphi: (\alpha, \beta) \to (a,b)$ má všude v (α, β) vlastní derivaci. Pak $F \circ \varphi$ je prim. fce k $(f \circ \varphi) \cdot \varphi'$ na (α, β).

Věta (2. věta o substituci). Nechť $f: (a,b) \to \mathbb{R}$ a nechť fce $\varphi: (\alpha, \beta) \to (a,b)$ je bijekce, která má všude v (α, β) vlastní nenulovou derivaci a Φ je prim. fce k $(f \circ \varphi) \cdot \varphi'$ na (α, β). Pak $\Phi \circ \varphi^{-1}$ je prim. fce k f na (a,b).

Pozn. U obou metod používáme: $F(x) = \int f(x)\,dx = \int f(\varphi(t)) \cdot \varphi'(t)\,dt = \Phi(t)$.

U první $\longleftarrow$ $x = \varphi(t)$, dostanu $\hat{F}(t) = F(\varphi(x))$

U druhé $\longrightarrow$ $t = \varphi^{-1}(t)$, dostanu $\hat{\Phi}(t) = \Phi(\varphi^{-1}(x))$

Primitivní fce - racionální lomené fce

$\int R(x)\,dx$, kde $R(x) = \frac{P(x)}{Q(x)}$, P, Q polynomy s reálnými koef.

1. částečné podělení: $R(x) = P_1(x) + \frac{P_2(x)}{Q(x)}$, P_1, P_2 polynomy & $\operatorname{st} P_2 < \operatorname{st} Q$
2. rozklad $Q(x)$ na ireducibilní polynomy (ve smyslu polynomů s reálnými kořeny).

Lemma: Nechť Q je polynom s reálnými kořeny a $a+ib$, $a,b \in \mathbb{R}$ je jeho komplexní kořen. Pak je $a-ib$ také kořenem Q a má stejnou násobnost jako $a+ib$.

Důsledek: $Q(x) = c(x-\alpha_1)^{r_1} \cdots (x-\alpha_k)^{r_k}(x^2+p_1x+q_1)^{s_1} \cdots (x^2+p_\ell x+q_\ell)^{s_\ell}$ (*)

kde $c \in \mathbb{R}$, α_i jsou reálné kořeny, r_i jejich násobnosti, $x^2 + p_j x + q_j$ jsou ireducibilní polynomy vzniklé z dvojic komplexně združených kořenů a s_j jsou jejich násobnosti.

Primitivní fce 1   1/6

Nalezněte následující prim. fce na maximálních možných intervalech. Určete i tyto intervaly.

1. $\int \left(\frac{1-x}{x}\right)^2 dx = \int \frac{1-2x+x^2}{x^2} dx = \int \frac{1}{x^2} dx - \int \frac{2}{x} dx + \int 1 \, dx = \frac{-1}{x} - 2\ln|x| + x + C, \quad x \neq 0$

2. $\int \frac{2^{x+1} - 5^{x-1}}{10^x} dx = \int \frac{e^{(x+1)\ln 2} - e^{(x-1)\ln 5}}{e^{x \ln 10}} dx = \int e^{x(\ln 2 - \ln 10) + \ln 2} dx - \int e^{x(\ln 5 - \ln 10) - \ln 5} dx$

$\int \frac{2 \cdot 2^x}{2^x 5^x} - \frac{5^x}{5 \cdot 5^x 2^x} = e^{\ln 2} \int e^{x \ln \frac{1}{5}} dx - e^{-\ln 5} \int e^{x \ln \frac{1}{2}} dx = 2 \cdot \frac{e^{x \ln \frac{1}{5}}}{\ln \frac{1}{5}} - \frac{1}{5} \frac{e^{x \ln \frac{1}{2}}}{\ln \frac{1}{2}} = \frac{2^{-x}}{5 \cdot \ln 2} - \frac{2 \cdot 5^{-x}}{\ln 5} + C \quad \forall x \in \mathbb{R}$

3. $\int \mathrm{tg}^2 x \, dx = \int \frac{\sin^2 x}{\cos^2 x} dx = \int \frac{1-\cos^2 x}{\cos^2 x} dx = -\int 1 \, dx + \int \frac{1}{\cos^2 x} dx = \mathrm{tg}\, x - x + C, \quad x \neq \frac{\pi}{2} + k\pi$

4. $\int \frac{1}{x^2 - x + 2} dx = \int \frac{1}{(x-\frac{1}{2})^2 + \frac{7}{4}} dx = \int \frac{1}{\frac{7}{4}\left(1 + \left(\frac{2}{\sqrt{7}} \frac{(2x-1)}{2}\right)^2\right)} dx = \frac{4}{7} \int \frac{1}{1 + \left(\frac{2x-1}{\sqrt{7}}\right)^2} dx$

$= \left| \begin{array}{l} t = \frac{2x-1}{\sqrt{7}} \\ dt = \frac{2}{\sqrt{7}} dx \end{array} \right| = \frac{2\sqrt{7}}{7} \int \frac{1}{1+t^2} dt = \frac{2\sqrt{7}}{7} \mathrm{arctg}\, t + C = \frac{2\sqrt{7}}{7} \mathrm{arctg}\left(\frac{2x-1}{\sqrt{7}}\right) + C \quad \forall x \in \mathbb{R}$

5. $\int \max\{1, x^2\} dx, \quad f(x) := \max\{1, x^2\}$

$x \in (-\infty, -1): f(x) = x^2, \quad \int f(x) = \frac{x^3}{3} + C_1 = F_1, \quad \lim_{x \to -1-} F_1(x) = -\frac{1}{3} + C_1$

$x \in (-1, 1): f(x) = 1, \quad \int f(x) = x + C_2 = F_2 \quad \lim_{x \to -1+} F_2(x) = -1 + C_2, \ \lim_{x \to 1-} F_2(x) = 1 + C_2$

$x \in (1, \infty): f(x) = x^2, \quad \int f(x) = \frac{x^3}{3} + C_3 = F_3 \quad \lim_{x \to 1+} F_3(x) = \frac{1}{3} + C_3$

integrand spojitý ⇒ lepení „projde"

$$\int \max\{1, x^2\} dx = \begin{cases} \frac{x^3}{3} + C & x \in (-\infty, -1) \\ x + \frac{2}{3} + C & x \in \langle -1, 1 \rangle \\ \frac{x^3}{3} + \frac{4}{3} + C & x \in (1, \infty) \end{cases}$$

6. $\int x e^{-x^2} dx = \left| \begin{array}{l} x^2 = t \\ 2x dx = dt \end{array} \right| = \frac{1}{2} \int e^{-t} dt = -\frac{1}{2} e^{-t} + C = -\frac{e^{-x^2}}{2} + C \quad x \in \mathbb{R}$

7. $\int \frac{1}{e^x + e^{-x}} dx = \left| \begin{array}{l} e^x = t \\ e^x dx = dt \end{array} \right| = \int \frac{1}{t + \frac{1}{t}} \cdot \frac{1}{t} dt = \int \frac{1}{t^2+1} dt = \mathrm{arctg}\, e^x + C \quad x \in \mathbb{R}$

8. $\int e^{3x} \cos 2x \, dx =: I = \left| \begin{array}{ll} f' = e^{3x} & f = \frac{e^{3x}}{3} \\ g = \cos 2x & g' = -2\sin 2x \end{array} \right| = \frac{e^{3x} \cos 2x}{3} + \int \frac{2}{3} e^{3x} \sin 2x = \left| \begin{array}{ll} f' = e^{3x} & f = \frac{e^{3x}}{3} \\ g = \sin 2x & g' = 2\cos 2x \end{array} \right|$

$= \frac{e^{3x} \cos 2x}{3} + \frac{2}{9} e^{3x} \sin 2x - \int \frac{4}{9} e^{3x} \cos 2x \, dx = \frac{1}{3} e^{3x} \cos 2x + \frac{2}{9} e^{3x} \sin 2x - \frac{4}{9} I \quad = I$

$\Rightarrow \frac{13}{9} I = \frac{1}{3} e^{3x} \cos 2x + \frac{2}{9} e^{3x} \sin 2x$

$\Rightarrow I = \int e^{3x} \cos 2x \, dx = e^{3x} \left(\frac{3}{13} \cos 2x + \frac{2}{13} \sin 2x \right) + C$

$= \left| \begin{matrix} u' = \frac{1}{x} & u = \ln|x| = \ln x \\ v = \ln^2 x & v' = \frac{2\ln x}{x} \end{matrix} \right| = \ln^3 x - 2\int \frac{\ln^3 x}{x}dx$

9. $\int \frac{\ln^2 x}{x}dx = \left| \begin{matrix} t=\ln|x| \\ dt=\frac{1}{x}dx\end{matrix}\right| = \int t^2 dt = \frac{\ln^3|x|}{3} + c = \frac{\ln^3 x}{3} + c, \quad x>0$

10. $\int \frac{1}{\sqrt{1-x^2}\arcsin^2 x}dx = \left| \begin{matrix} t=\arcsin x \\ dt = \frac{1}{\sqrt{1-x^2}}dx \end{matrix}\right| = \int \frac{1}{t^2}dt = -\frac{1}{\arcsin x}+c \quad x\in(-1,0)\cup(0,1)$

11. $\int\frac{1}{1+\cos x}dx = \int \frac{1-\cos x}{\sin^2 x}dx = \int\frac{1}{\sin^2 x}dx - \int\frac{\cos x}{\sin^2 x}dx = \left| \begin{matrix} t=\sin x \\ dt=\cos x\, dx \end{matrix}\right| \quad x\neq \pi+2k\pi, \ x\in((2k+1)\pi,(2k+3)\pi)$

$= -\operatorname{cotg} x + \frac{1}{\sin x}+c$

12. $\int\frac{1}{\sin x}dx = \int\frac{\sin x}{1-\cos^2 x}dx = \left| \begin{matrix} t=\cos x \\ dt=-\sin x\, dx \end{matrix}\right| = \int \frac{1}{t^2-1}dt = \frac{-1}{2}\int \frac{1}{1-t}+\frac{1}{t+1}dt \quad x\neq k\pi, \ x\in(k\pi,(k+1)\pi)$

$= +\ln\sqrt{|-\cos x+1|} - \ln\sqrt{|\cos x+1|} + c = +\ln\sqrt{1-\cos x} - \ln\sqrt{1+\cos x}+c = \ln\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)+c = \ln\left|\operatorname{tg}\frac{x}{2}\right|+c$

13. $\int\frac{1}{\sin x\cos^3 x}dx = \int\frac{1}{\operatorname{tg}x\cos^4 x}dx \quad \left| \operatorname{tg}^2 x = \frac{\sin^2 x}{\cos^2 x} = \frac{1-\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}-1 \Rightarrow \frac{1}{\cos^2 x} = \operatorname{tg}^2 x+1 \right|$

$= \int \frac{\operatorname{tg}^2x+1}{\operatorname{tg}x}\cdot\frac{1}{\cos^2 x}dx = \left| \begin{matrix} t=\operatorname{tg}x \\ dt=\frac{1}{\cos^2 x}dx \end{matrix}\right| = \int\frac{t^2+1}{t}dt = \frac{\operatorname{tg}^2 x}{2}+\ln|\operatorname{tg}x|+c \quad x\neq\frac{k\pi}{2}, \ x\in\left(\frac{k\pi}{2},\frac{(k+1)\pi}{2}\right)$

14. $\int \ln x\, dx = \left| \begin{matrix} f'=1 & f=x \\ g=\ln x & g'=\frac1x \end{matrix}\right| = x\ln x - \int 1\, dx = x(\ln x -1)+c \quad x>0$

15. $\int x^3 a^{-x^2}dx = \left| \begin{matrix} t=x^2 \\ dt=2x\, dx \end{matrix}\right| = \frac12\int t a^{-t}dt = \left| \begin{matrix} f'=a^{-t} & f=\frac{-a^{-t}}{\ln a} \\ g=t & g'=1 \end{matrix}\right| = \frac{-ta^{-t}}{2\ln a} + \frac{1}{2\ln a}\int a^{-t}dt = \frac{-x^2a^{-x^2}}{2\ln a} - \frac{a^{-x^2}}{2\ln^2 a}+c \quad x\in\mathbb{R}$

16. $\int x\operatorname{arctg}(x+1)dx = \left| \begin{matrix} f'=x & f=\frac{x^2}{2} \\ g=\operatorname{arctg}(x+1) & g'=\frac{1}{1+(x+1)^2} \end{matrix}\right| = \frac{x^2}{2}\operatorname{arctg}(x+1) - \frac12\int\frac{x^2}{2+2x+x^2}dx = \frac{x^2\operatorname{arctg}(x+1)}{2} - \frac{x}{2} + \int\frac{x+1}{1+(x+1)^2}dx$

$= \left| \begin{matrix} t=(x+1)^2 \\ dt=2(x+1)dx \end{matrix}\right| = \frac{x}{2}(x\operatorname{arctg}(x+1)-1) + \frac12\int\frac{1}{1+t}dt = \frac{x}{2}(x\operatorname{arctg}(x+1)-1)+\ln\sqrt{1+(x+1)^2}+c \quad x\in\mathbb{R}$

17. $\int x^2\arccos x\, dx = \left| \begin{matrix} f'=x^2 & f=\frac{x^3}{3} \\ g=\arccos x & g'=\frac{-1}{\sqrt{1-x^2}} \end{matrix}\right| = \frac{x^3\arccos x}{3} + \frac13\int\frac{x^3}{\sqrt{1-x^2}}dx = \left| \begin{matrix} t=1-x^2 \\ dt=-2x\, dx \end{matrix}\right| =$

$= \frac{x^3\arccos x}{3} - \frac16\int\frac{1-t}{\sqrt t}dt = \frac{x^3\arccos x}{3} - \frac13\sqrt{1-x^2} + \frac19(\sqrt{1-x^2})^3 + c = \frac{x^3\arccos x}{3} - \frac{\sqrt{1-x^2}}{9}(x^2+2)+c$

18. $\int\frac{x}{\cos^2 x}dx = \left| \begin{matrix} f'=\frac{1}{\cos^2 x} & f=\operatorname{tg}x \\ g=x & g'=1 \end{matrix}\right| = x\operatorname{tg}x - \int\operatorname{tg}x\, dx = \left| \begin{matrix} t=\cos x \\ dt=-\sin x\, dx \end{matrix}\right| = x\operatorname{tg}x + \ln|\cos x|+c, \quad x\neq\frac\pi2+k\pi, \ x\in\left(\frac{(2k+1)\pi}{2},\frac{(2k+3)\pi}{2}\right)$

19. $\int\sin(\ln x)dx = \left| \begin{matrix} e^t=x \\ e^t dt=dx \end{matrix}\right| = \int\sin t\, e^t dt = \left| \begin{matrix} f'=e^t & f=e^t \\ g=\sin t & g'=\cos t \end{matrix}\right| = e^t\sin t - \int e^t\cos t\, dt = \left| \begin{matrix} f'=e^t & f=e^t \\ g=\cos t & g'=-\sin t \end{matrix}\right|$

$= e^t(\sin t-\cos t) - \int e^t\sin t\, dt \Rightarrow \int\sin(\ln x)dx = \frac{x}{2}(\sin(\ln x)-\cos(\ln x))+c \quad x>0$

20. $\int\sin^7 x\, dx = \int(1-\cos^2 x)^3\sin x\, dx = \left| \begin{matrix} t=\cos x \\ dt=-\sin x\, dx \end{matrix}\right| = \int(t^2-1)^3 dt = \frac{\cos^7 x}{7} - \frac{3\cos^5 x}{5}+\cos^3 x - \cos x + c \quad x\in\mathbb{R}$

21. $\int\cos^2 x\, dx = \int\frac{1+\cos 2x}{2}dx = \frac x2 + \frac{\sin 2x}{4}+c \quad x\in\mathbb{R}$

22. Nalezněte rekurentní vztah pro $I_n = \int\cos^n x\, dx$, $n\in\mathbb{N}$.

$I_n = \int\cos x\cos^{n-1}x\, dx = \left| \begin{matrix} f'=\cos x & f=\sin x \\ g=\cos^{n-1}x & g'=-(n-1)\cos^{n-2}x\sin x \end{matrix}\right| = \sin x\cos^{n-1}x + (n-1)\int\underbrace{\sin^2 x}_{1-\cos^2 x}\cos^{n-2}x\, dx = \sin x\cos^{n-1}x + (n-1)(I_{n-2}-I_n)$

$I_n(1+n-1) = \sin x\cos^{n-1}x + (n-1)I_{n-2} \Rightarrow I_n = \frac{\sin x\cos^{n-1}x}{n}+\frac{n-1}{n}I_{n-2}, \ I_0=x+c, \ I_1=\sin x+c$

Věta (o rozkladu na parciální zlomky): Nechť $P(x)$ a $Q(x)$ jsou dva polynomy s reálnými koeficienty, $st P < st Q$ a platí (*). Pak existují sady reálných konstant A_i^m, B_j^n a C_j^n, že platí:
$$\frac{P(x)}{Q(x)} = \frac{A_1^1}{(x-\alpha_1)} + \frac{A_1^2}{(x-\alpha_1)^2} + \dots + \frac{A_1^{r_1}}{(x-\alpha_1)^{r_1}} + \dots + \frac{A_k^1}{(x-\alpha_k)} + \dots + \frac{A_k^{r_k}}{(x-\alpha_k)^{r_k}} +$$
$$+ \frac{B_1^1 x + C_1^1}{x^2+p_1x+q_1} + \dots + \frac{B_1^{s_1}x + C_1^{s_1}}{(x^2+p_1x+q)^{s_1}} + \dots + \frac{B_\ell^{s_\ell}x + C_\ell^{s_\ell}}{(x^2+p_\ell x+q_\ell)^{s_\ell}}.$$

Věta (Ostrogradského metoda): Nechť P, Q jsou dva polynomy s reálnými koeficienty, $st P < st Q$ a platí (*) a $c=1$. Definujme
$$Q_1(x) := (x-\alpha_1)^{r_1-1} \dots (x-\alpha_k)^{r_k-1}(x^2+p_1x+q_1)^{s_1-1} \dots (x^2+p_\ell x+q_\ell)^{s_\ell-1} \quad \text{a}$$
$$Q_2(x) := (x-\alpha_1) \dots (x-\alpha_k)(x^2+p_1x+q_1)\dots(x^2+p_\ell x+q_\ell).$$ Potom $Q = Q_1Q_2$ a existují polynomy P_1, P_2 s reálnými koeficienty takové, že $st P_1 < st Q_1$, $st P_2 < st Q_2$ a
$$\int \frac{P(x)}{Q(x)}\,dx = \frac{P_1(x)}{Q_1(x)} + \int \frac{P_2(x)}{Q_2(x)}\,dx. \qquad (**)$$

Použití: Zderivujme (**):
$$\frac{P(x)}{Q(x)} = \frac{d}{dx}\left(\frac{P_1(x)}{Q_1(x)}\right) + \frac{P_2(x)}{Q_2(x)} = \frac{P_1'(x)Q_1(x) - P_1(x)Q_1'(x)}{Q_1^2(x)} + \frac{P_2(x)}{Q_2(x)} \Rightarrow P(x) = P_1'Q_2 - P_1\frac{Q_1'}{Q_1}Q_2 + P_2Q_1.$$

Shrnutí (Integrace racionálních funkcí – "KUCHAŘKA")

1. částečné podílení polynomů $\frac{P}{Q} \Rightarrow \frac{P}{Q} = P_1 + \frac{P_2}{Q}$, $st P_2 < st Q$
2. rozklad Q na ireducibilní polynomy
3. rozklad na parciální zlomky
4. • pro zlomky kde jmenovatel st. 1 $\Rightarrow \ln$
   • pro zlomky typu $\frac{Bx+C}{x^2+px+q}$ vytvoříme v čitateli derivaci jmenovatele + zbytkový člen
   $\Rightarrow \ln|x^2+px+q| + \int \frac{1}{x^2+px+q} \rightarrow$ doplnění na čtverec $\rightarrow \operatorname{arctg}()$
   • pro zlomky typu $\frac{Bx+C}{(x^2+px+q)^k}$ vytvoříme v čitateli derivaci (x^2+px+q) + zbytek
   $\Rightarrow -\frac{1}{k+1}\frac{1}{(x^2+px+q)^{k+1}} + \int \frac{D}{(x^2+px+q)^k}dx \rightarrow$ doplnění na čtverec $\rightarrow \int \frac{1}{(1+t^2)^k}dt$

spočteme: $I_n = \int \frac{1}{(1+x^2)^n}dx$, $I_1 = \arctan x + C$

$$n \geq 2: \quad I_n = \int \frac{1}{(1+x^2)^n}dx \overset{PP}{=} \left| \begin{matrix} f' = 1 & f = x \\ g = \frac{1}{(1+x^2)^n} & g' = \frac{-2nx}{(1+x^2)^{n+1}} \end{matrix} \right| = \frac{x}{(1+x^2)^n} + 2n\int \frac{x^2+1-1}{(1+x^2)^{n+1}}dx$$
$$= \frac{x}{(1+x^2)^n} + 2n\left(\int \frac{1}{(1+x^2)^n}dx - \int \frac{1}{(1+x^2)^{n+1}}dx\right)$$
$$\Rightarrow I_n(1-2n) = \frac{x}{(1+x^2)^n} - 2n\, I_{n+1}$$
$$\Rightarrow I_{n+1} = \frac{x}{2n(1+x^2)^n} + \frac{2n-1}{2n}\cdot I_n \qquad \text{rekurentní formule}$$

Substituce vedoucí na racionální lomené funkce

$R(x) = \frac{P(x)}{Q(x)}$, $R(u,v) = \frac{P(u,v)}{Q(u,v)}$, $P(u,v) = \sum_{0 \le i+j \le n} a_{ij} u^i v^j$

$a_{ij} \in \mathbb{R}$

- **Exponenciální substituce** $\int R(e^{\alpha x})\,dx \rightarrow t = e^{\alpha x}$

$$\int R(e^{\alpha x})\,dx = \left| \begin{matrix} t = e^{\alpha x} \\ dt = \alpha e^{\alpha x} dx \end{matrix} \right| = \int R(t) \frac{1}{\alpha t}\,dt$$

- **Logaritmická substituce** $\int \frac{1}{x} R(\ln x)\,dx \longrightarrow t = \ln x$

$$\int \frac{1}{x} R(\ln x)\,dx = \left| \begin{matrix} t = \ln x \\ dt = \frac{1}{x} dx \end{matrix} \right| = \int R(t)\,dt$$

- **Odmocninová substituce** $\int R\left(x, \sqrt[s]{\frac{ax+b}{cx+d}}\right) dx \rightarrow t = \sqrt[s]{\frac{ax+b}{cx+d}}$

Nechť $s \in \mathbb{N} \setminus \{1\}$ a $a,b,c,d \in \mathbb{R}$ splňují $ad-bc \neq 0$. $t^s = \frac{ax+b}{cx+d} \Rightarrow t^s(cx+d) = ax+b$

pak $x = \frac{b-dt^s}{ct^s-a}$ a $\frac{dt}{dx} = \frac{1}{s}\left(\frac{ax+b}{cx+d}\right)^{\frac{1}{s}-1} \cdot \frac{ad-bc}{(cx+d)^2} = \frac{1}{s} t^{1-s} \frac{ad-bc}{(cx+d)^2}$

$$= \frac{1}{s} \frac{ad-bc}{t^{s-1}} \cdot \frac{1}{\left(\frac{cb-cdt^s}{ct^s-a} + d\right)^2} = \frac{ad-bc}{s\,t^{s-1}} \cdot \frac{(ct^s-a)^2}{(cb-cdt^s+cdt^s-ad)^2} = \frac{(ct^s-a)^2}{(ad-bc)\,s t^{s-1}},$$

proto $$\int R\left(x, \sqrt[s]{\frac{ax+b}{cx+d}}\right) dx = \left| t = \sqrt[s]{\frac{ax+b}{cx+d}} \right| = \int R\left(\frac{b-dt^s}{ct^s-a}, t\right) \cdot \frac{(ad-bc)st^{s-1}}{(ct^s-a)^2}\,dt$$

- **Eulerovy substituce**

Nechť $a,b,c \in \mathbb{R}$. Studujme $\int R(x, \sqrt{ax^2+bx+c})\,dx$.

Případy: 1. polynom ax^2+bx+c má 2 reálné kořeny (ne nutně různé)

2. platí $a > 0$

3. platí $c > 0$

1. $x_1 = x_2$: $\sqrt{ax^2+bx+c} = \sqrt{a(x-x_1)^2} = a|x_1 - x|$ → rac. lom. fce na $(-\infty, x_1)$ a (x_1, ∞) - umíme + lepení

$x_1 < x_2$: $ax^2+bx+c = a(x-x_1)(x-x_2)$, pro $a<0$ je $\sqrt{\ }$ def na $\langle x_1, x_2 \rangle$

$a > 0$ na $(-\infty, x_1\rangle \cup \langle x_2, \infty)$

Přepis pro aplikaci odmocninové subst.:

$$\sqrt{ax^2+bx+c} = \begin{cases} -\sqrt{a}\sqrt{\frac{x-x_1}{x-x_2}}(x-x_2) & \text{pro } a>0 \text{ a } x \in (-\infty, x_1\rangle \\ \sqrt{a}\sqrt{\frac{x-x_1}{x-x_2}}(x-x_2) & \text{pro } a>0 \text{ a } x \in \langle x_2, \infty) \\ \sqrt{-a}\sqrt{\frac{x-x_1}{x_2-x}}(x_2-x) & \text{pro } a<0 \text{ a } x \in \langle x_1, x_2 \rangle \end{cases}$$

2. $a > 0$, první Eulerova substituce $\sqrt{ax^2+bx+c} = \pm\sqrt{a}\,x + t$

3. $c > 0$, druhá Eulerova substituce $\sqrt{ax^2+bx+c} = \sqrt{c} \pm xt$

(použití 3. ovlivní délku a náročnost výpočtu)

- <u>Goniometrické substituce</u> $\int R(\cos x, \sin x)\,dx$

i) $t = \sin x$ pokud $R(-\cos x, \sin x) = -R(\cos x, \sin x)$

ii) $t = \cos x$ pokud $R(\cos x, -\sin x) = -R(\cos x, \sin x)$

iii) $t = \tan x$ pokud $R(-\cos x, -\sin x) = R(\cos x, \sin x)$

iv) $t = \tan \frac{x}{2}$ vždy, použití:

$$1 + t^2 = 1 + \frac{\sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}} = \frac{\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}} = \frac{1}{\cos^2 \frac{x}{2}}$$

$$\sin x = 2 \sin \tfrac{x}{2} \cos \tfrac{x}{2} = 2t \cos^2 \tfrac{x}{2} = \frac{2t}{1+t^2}$$

$$\cos x = \cos^2 \tfrac{x}{2} - \sin^2 \tfrac{x}{2} = 2\cos^2 \tfrac{x}{2} - 1 = \frac{2}{1+t^2} - 1 = \frac{1-t^2}{1+t^2}$$

$$t = \tan \tfrac{x}{2} \Rightarrow x = 2\arctan t \Rightarrow dx = \frac{2}{1+t^2}\,dt$$

<u>Další speciální substituce</u>

- $\int \sqrt{1-x^2}\,dx$ | $x = \sin t$ nebo $x = \cos t$

- $\int \sqrt{1+x^2}\,dx$ | $x = \sinh t$

- $\int \sqrt{x^2-1}\,dx$ | $x = \cosh t$

Primitivní fce II 1/7

Nalezněte následující primitivní fce na maximálních možných intervalech, určete je.

1. $\int \frac{x^3+1}{x^3-5x^2+6x}\,dx = \int 1 + \underbrace{\frac{5x^2-6x+1}{x^3-5x^2+6x}}_{(*)}\,dx = \int 1 + \frac{1}{6x} + \frac{28}{3(x-3)} - \frac{9}{2(x-2)}\,dx$

$(*) = \frac{5x^2-6x+1}{x(x-2)(x-3)} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{x-2} \Rightarrow 5x^2 - 6x + 1 = Ax^2 - 5Ax + 6A + Bx^2 - 2Bx + Cx^2 - 3Cx$

$$\Rightarrow \left.\begin{aligned} 5 &= A+B+C \\ +6 &= +5A+2B+3C \\ 1 &= 6A \end{aligned}\right\} \Rightarrow \left.\begin{aligned} A &= \tfrac{1}{6} \\ 11 &= 6A+3B+4C \end{aligned}\right\} \Rightarrow \left.\begin{aligned} 10 &= 3B+4C \\ 29 &= 6B+6C \end{aligned}\right\} \Rightarrow \left.\begin{aligned} 9 &= (6-8)C \\ -28 &= -3B \end{aligned}\right\} \Rightarrow \begin{aligned} C &= -\tfrac{9}{2} \\ B &= \tfrac{28}{3} \end{aligned}$$

$\int \frac{x^3+1}{x^3-5x^2+6x}\,dx = x + \frac{1}{6}\ln|x| + \frac{28}{3}\ln|x-3| - \frac{9}{2}\ln|x-2| + c$, $x \in (-\infty, 0) \cup (0,2) \cup (2,3) \cup (3, +\infty)$

$(x^2-x+1)^2 = x^4 - x^3 + x^2 - x^3 + x^2 - x + x^2 - x + 1$
$= x^4 - 2x^3 + 3x^2 - 2x + 1$

2. $\int \frac{1}{(x^3+1)^2}\,dx = \int \frac{1}{[(x+1)(x^2-x+1)]^2}\,dx = \int \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2-x+1} + \frac{Ex+F}{(x^2-x+1)^2}\,dx$

$1 = A(x+1)(x^2-x+1)^2 + B(x^2-x+1)^2 + (Cx+D)(x+1)^2(x^2-x+1) + (Ex+F)(x+1)^2$

$x=0$: $1 = A + B + D + F$

$x=-1$: $1 = 9B$ $\Rightarrow B = \frac{1}{9}$

$x=1$: $1 = 2A + B + 4C + 4D + 4E + 4F$

$1 = 6A + 9B - 4F \Rightarrow 3A = 2F$

$1 = A + \frac{1}{9} - \frac{1}{9} + 3A + \frac{3}{2}A = \frac{11}{2}A \Rightarrow A = \frac{2}{11}$

$\Rightarrow C = -\frac{2}{11}$, $D = -\frac{1}{9} + \frac{6}{11}$, $E = \frac{5}{9} - \frac{14}{11}$, $F = \frac{3}{11}$

x^1: $0 = A(1-2) - 2B + C + D(2-1) + E + 2F = -A - 2B + C + D + E + 2F$

x^2: $0 = A(-2+3) + 3B + C + D(1+3) + 2E + F = A + 3B + C + 4D + 2E + F$

x^3: $0 = A(3-2) - 2B + D(-2+6-2) + E = A - 2B + 2D + E$

x^4: $0 = A(1-2) + B + 2C + D = -A + B + 2C + D \Rightarrow 3C + D = -\frac{1}{9}$

x^5: $0 = A + C \Rightarrow -A = C$

$0 = -C - \frac{2}{9} - \frac{2}{9} - 6C + E$

$\Rightarrow -7C + E = \frac{4}{9}$

Ostrogradský: $\frac{1}{(1+x^3)^2} = \frac{P}{Q} = \frac{1}{(1+x)^2(x^2-x+1)^2} \Rightarrow Q_1 = (1+x)(x^2-x+1) = (1+x^3)$

OM: $\int \frac{P(x)}{Q(x)} = \frac{P_1}{Q_1} + \int \frac{P_2}{Q_2} \quad \Big| '$

$Q_2 = (1+x)(x^2-x+1) = (1+x^3)$

$P_1 = a_1x^2 + b_1x + c_1$

$P_2 = a_2x^2 + b_2x + c_2$

$\left.\right\} \quad P_1' = 2a_1x + b_1, \quad Q_1' = 3x^2$

$\frac{P}{Q} = \left(\frac{P_1}{Q_1}\right)' + \frac{P_2}{Q_2} \quad / Q_1Q_2 = Q$

$\Rightarrow P = P_1'Q_2 - P_1\frac{Q_1'}{Q_1}Q_2 + P_2Q_1$

$\Rightarrow 1 = (2a_1x + b_1)(1+x^3) - (a_1x^2 + b_1x + c_1)3x^2 \cdot \frac{1+x^3}{1+x^3} + (a_2x^2 + b_2x + c_2)(1+x^3)$

$1 = x^5a_2 + x^4(2a_1 - 3a_1 + b_2) + x^3(b_1 - 3b_1 + c_2) + x^2(-3c_1 + a_2) + x(2a_1 + b_2) + b_1 + c_2$

$\Rightarrow a_2 = 0, \; +a_1 = b_2, \; c_2 = 2b_1, \; a_2 = 3c_1 = 0, \; 2a_1 = -b_2, \; b_1 + c_2 = 1$

$0 = a_2 = c_1 = a_1 = b_2, \; c_2 = 1 - b_1 = 2b_1 \Rightarrow b_1 = \frac{1}{3}, \; c_2 = \frac{2}{3}$

$\Rightarrow \int \frac{1}{(1+x^3)^2}\,dx = \frac{1}{3}\frac{x}{1+x^3} + \frac{2}{3}\int \frac{1}{1+x^3}\,dx \quad , \frac{1}{1+x^3} = \frac{A}{1+x} + \frac{Bx+C}{x^2-x+1} \Rightarrow \begin{matrix} B+A=0=-A+B+C=2B+C \\ 1=A+\cancel{B}\cancel{B}=-3B \end{matrix}$

$\int \frac{1}{(1+x^3)^2}\,dx = \frac{x}{3(1+x^3)} + \frac{2}{9}\ln|1+x| + \frac{2}{3}\int \frac{-\frac{1}{3}x + \frac{2}{3}}{x^2-x+1}\,dx$

$\int \frac{x-2}{x^2-x+1}\,dx = \frac{1}{2}\int \frac{2x-1}{x^2-x+1}\,dx - \frac{3}{2}\int \frac{1}{x^2-x+1}\,dx = \frac{1}{2}\ln|x^2-x+1| - \frac{3}{2}\int \frac{1}{(x-\frac{1}{2})^2 + \frac{3}{4}}\,dx$

$\frac{4}{3}\int \frac{1}{1+\left(\frac{2}{\sqrt{3}}\cdot\frac{2x-1}{2}\right)^2}\,dx = \frac{4}{3}\int \frac{1}{1+\left(\frac{2x-1}{\sqrt{3}}\right)^2}\,dx = \frac{4}{3}\frac{\sqrt{3}}{2}\operatorname{arctg}\left(\frac{2x-1}{\sqrt{3}}\right) + c$

$x \in (-\infty,-1) \cup (-1,\infty)$

$\int \frac{1}{(1+x^3)^2}\,dx = \frac{x}{3(1+x^3)} + \frac{2}{9}\ln|1+x| - \frac{2}{9}\left(\frac{1}{2}\ln|x^2-x+1| - \sqrt{3}\operatorname{arctg}\left(\frac{2x-1}{\sqrt{3}}\right)\right) + c$

Vhodnou substitucí převeďte integrály na integrály z racionálních fcí a ty se pokuste vyřešit.

3. $\int \frac{1}{x(1+2\sqrt[6]{x}+\sqrt[3]{x})}\,dx = \int \frac{1}{x}\frac{1}{1+2x^{1/6}+x^{2/6}}\,dx = R(x, x^{\frac{1}{6}})$

$\left|\begin{matrix} t = x^{1/6} \\ 6t^5\,dt = dx \end{matrix}\right| = \int \frac{1}{t^6}\frac{6t^5}{(1+2t^3+t^2)}\,dt = \int \frac{6}{t(t+1)(2t^2-t+1)}\,dt = 6\int \frac{A}{t} + \frac{B}{t+1} + \frac{Ct+D}{2t^2-t+1}\,dt$

$1t^0 = 1 = A$

$0 \cdot t^1 = 0 = B + D$

$0 \cdot t^2 = 0 = A(2-1) - B + C + D$

$0 \cdot t^3 = 0 = 2A + 2B + C$

$\Rightarrow \left\{\begin{matrix} A = 1, \; C = -\frac{3}{2}, \; D = \frac{1}{4}, \; B = -\frac{1}{4} \\ 0 = B + D \\ 0 = 1 + 2D + C \\ 0 = 2 + 2B + C \end{matrix}\right. \quad 0 = 3 + 2C$

$\int \frac{-\frac{3}{2}t + \frac{1}{4}}{2t^2-t+1}\,dt = -\frac{3}{8}\int \frac{4t-1}{2t^2-t+1}\,dt - \frac{1}{8}\frac{1}{2}\int \frac{1}{(t-\frac{1}{4})^2 + \frac{7}{16}}\,dt = -\frac{3}{8}\ln|2t^2-t+1| - \frac{1}{7}\int \frac{1}{\left(\frac{4t-1}{\sqrt{7}}\right)^2 + 1}\,dt$

$\int \frac{1}{x(1+2\sqrt[6]{x}+\sqrt[3]{x})}\,dx = \frac{6}{6}\ln|x| - \frac{3}{2}\ln|\sqrt[6]{x}+1| - \frac{9}{4}\ln|2\sqrt[3]{x} - \sqrt[6]{x} + 1| - \frac{3}{2}\frac{\sqrt{7}}{7}\operatorname{arctg}\left(\frac{4\sqrt[6]{x}-1}{\sqrt{7}}\right) + c \quad x \in (0,\infty)$

4. $\int x\sqrt{x^2-2x+2}\,dx = \frac{1}{2}\int (2x-2)\sqrt{x^2-2x+2}\,dx + \int \sqrt{x^2-2x+2}\,dx = \frac{1}{2}\int \sqrt{y}\,dy + \int \sqrt{x^2-2x+2}\,dx$

$\int \sqrt{x^2-2x+2}\,dx = \left|\begin{matrix} f = \sqrt{x^2-2x+2} & f' = \frac{x-1}{\sqrt{x^2-2x+2}} \\ g' = 1 & g = x \end{matrix}\right| = x\sqrt{x^2-2x+2} - \int \frac{x(x-1)}{\sqrt{x^2-2x+2}}\,dx$

$-\operatorname{arcsinh}(x-1) + c$

$\int \frac{x(x-1)}{\sqrt{x^2-2x+2}}\,dx = \int \frac{x^2-2x+2}{\sqrt{x^2-2x+2}}\,dx + \int \frac{x-2}{\sqrt{x^2-2x+2}}\,dx = \int \sqrt{x^2-2x+2}\,dx + \frac{1}{2}\int \frac{2x-2}{\sqrt{\quad}}\,dx - \int \frac{1}{\sqrt{(x-1)^2+1}}\,dx = \int \sqrt{x^2-2x+2}\,dx + \sqrt{x^2-2x+2}$

$\Rightarrow \int \sqrt{x^2-2x+2}\,dx = \frac{1}{2}\left((x-1)\sqrt{x^2-2x+2} + \operatorname{arcsinh}(x-1)\right) + c$

$\Rightarrow \int x\sqrt{x^2-2x+2}\,dx = \sqrt{x^2-2x+2}\left(\frac{x^2-2x+2}{3} + \frac{x-1}{2}\right) + \frac{1}{2}\operatorname{arcsinh}(x-1) + c$

$= \frac{\sqrt{x^2-2x+2}}{6}(2x^2 - x + 1) + \frac{1}{2}\operatorname{arcsinh}(x-1) + c \qquad x \in \mathbb{R} \qquad (x^2-2x+2 > 0)$

5. $\int \frac{x+\sqrt{1+x+x^2}}{1+x+\sqrt{1+x+x^2}}\,dx = \left| \begin{array}{l} \text{Euler: } \\ \sqrt{x^2+x+1} = t-x \\ x^2+x+1 = t^2-2xt+x^2 \\ x=\frac{t^2-1}{1+2t},\ dx = 2\frac{t^2+t+1}{(1+2t)^2}dt \end{array}\right| = 2\int \frac{t}{t+1}\cdot\frac{t^2+t+1}{(1+2t)^2}\,dt = 2\int \frac{t^3+t^2+t}{t+1+4t^2+4t+4t^3+4t^2}\,dt$

$= \frac12\int \frac{4t^3+8t^2+5t+1}{4t^3+8t^2+5t+1}\,dt - \frac12\int\frac{4t^2+t+1}{(t+1)(1+2t)^2}\,dt = (*)$

$\frac{4t^2+t+1}{(t+1)(1+2t)^2} = \frac{A}{t+1}+\frac{B}{(1+2t)}+\frac{C}{(1+2t)^2} \Rightarrow 4t^2+t+1 = A+4At+4At^2+Bt+B+2Bt^2+2Bt+Ct+C$

$4=4A+2B,\quad 1=4A+3B+C,\quad 1=A+B+C$

$4t^2+t+1 = A(1+2t)^2+B(1+t)(1+2t)+C(1+t),\quad t=-1:\ 4=A,\quad t=-\frac12:\ \frac32=\frac C2,\quad t=0:\ 1=A+B+C \Rightarrow B=1-4-3=-6$

$(*) = \frac t2 - \frac12\left(\int\frac{4}{t+1} - \frac{6}{1+2t}+\frac{3}{(1+2t)^2}\,dt\right) = \frac{x+\sqrt{x^2+x+1}}{2} - 2\ln\left|\sqrt{x^2+x+1}+x+1\right| + 3\ln\left|1+2x+2\sqrt{x^2+x+1}\right| + \frac34\frac{1}{1+2x+2\sqrt{x^2+x+1}} + C$

$1+x = -\sqrt{x^2+x+1}\quad /^2 \iff 1+2x+x^2 = 1+x+x^2 \iff x=0 \Rightarrow$ ~~$x\in(-\infty,0)\cup(0,\infty)$~~ $\quad x\in\mathbb{R}$

6. $\int \frac{x-\sqrt{x^2+3x+2}}{x+\sqrt{x^2+3x+2}}\,dx = \left|\begin{array}{l}\text{Euler: } \sqrt{x^2+3x+2}=-x+t \\ x^2+3x+2 = x^2-2xt+t^2 \\ x=\frac{t^2-2}{2t+3},\ dx=\frac{2t(2t+3)-2(t^2-2)}{(2t+3)^2}dt\end{array}\right| = \int\frac{2\left(\frac{t^2-2}{2t+3}\right)-t}{t}\cdot\frac{2t^2+6t+4}{(2t+3)^2}\,dt = -2\int\frac{3t^3+13t^2+18t+8}{t(8t^3+36t^2+54t+27)}\,dt$

$= -2\int\frac{(4+3t)(t+2)(t+1)}{t(2t+3)^3}\,dt:\quad \frac{(4+3t)(t+2)(t+1)}{t(2t+3)^3} = \frac At+\frac B{2t+3}+\frac C{(2t+3)^2}+\frac D{(2t+3)^3} = A(2t+3)^3+Bt(2t+3)^2+Ct(2t+3)+Dt$

$t=0:\ 8=27A,\quad t=-\frac32:\ -\frac12\cdot\frac12\cdot-\frac12=\frac18=-\frac{3D}{2},\quad t=-2:\ 0=-A-2B+2C-2D,\quad t=-1:\ 0=A-B-C-D$

$A=\frac{27}{8},\ D=\frac{-1}{12},\quad 0=-3B+C+\frac14,\quad 0=\frac{27}{8}-B-3B+\frac14+\frac1{12} \Rightarrow B=\frac{3^4+6+2}{2.2.2.3}=\frac{89}{24}$

$= \left|\begin{array}{l}\text{Euler: } \sqrt{x^2+3x+2}=x+t \\ x^2+3x+2=x^2+2xt+t^2 \\ x=\frac{t^2-2}{3-2t}\quad dx=\frac{2t(3-2t)+2(t^2-2)}{(3-2t)^2}dt\end{array}\right| = \int\frac{-t}{2\left(\frac{t^2-2}{3-2t}\right)+t}\cdot\frac{(-2)(t-1)(t-2)}{(3-2t)^2}\,dt = \int\frac{2t(t-1)(t-2)}{(3t-4)(3-2t)^2}\,dt = \frac16\int\frac{12t^3-36t^2+24t}{12t^3-52t^2+75t-36}\,dt$

$= \frac16\int 1+\frac{16t^2-51t+36}{(3t-4)(3-2t)^2}\,dt = \frac t6 - \frac{16}{9}\ln|3t-4|+2\ln|3-2t| - \frac34\frac{1}{3-2t}+C,\quad t=\sqrt{x^2+3x+2}-x,$

$x^2+3x+2=(x+1)(x+2)\geq 0 \wedge x=-\frac{2}{3} \Rightarrow x\in(-\infty,-2)\cup(-1,-\frac23)\cup(-\frac23,\infty)$

$\frac{16t^2-51t+36}{(3t-4)(3-2t)^2} = \frac{4(3-2t)^2-3t}{(3t-4)(3-2t)^2} = \frac{A}{3t-4}+\frac{B}{3-2t}+\frac{C}{(3-2t)^2} \Rightarrow 4(3-2t)^2-3t = A(3-2t)^2+B(3t-4)(3-2t)+C(3t-4)$

$t=\frac43:\ \frac49-4=\frac A9 \Rightarrow A=-32,\quad t=\frac32:\ -\frac92=\frac C2 \Rightarrow C=-9,\quad t=0:\ 36=9A-12B-4C=-9\cdot32-12B+36 \Rightarrow B=-24$

Nalezněte následující primitivní fce:

7. $\int\frac{\sin^2x}{1+\sin^2x}\,dx = \left|\begin{array}{l}R(-\cos x,-\sin x)=R(\cos x,\sin x) \\ y=\tan x,\ x\in(-\frac\pi2,\frac\pi2)+k\pi \\ y^2=\frac{\sin^2x}{1-\sin^2x}\Rightarrow \sin^2x=\frac{y^2}{1+y^2}\end{array}\right| \begin{array}{l}x=\arctan y \\ dx=\frac{dy}{1+y^2}\end{array}\Bigg| = \int\frac{y^2}{1+y^2}\cdot\frac{1}{1+\frac{y^2}{1+y^2}}\cdot\frac1{1+y^2}\,dy = \int\frac{y^2}{(1+y^2)(1+2y^2)}\,dy$

$= \int\frac1{1+y^2}-\frac1{1+2y^2}\,dy = \arctan y - \frac1{\sqrt2}\arctan(\sqrt2\, y)+c = x-\frac1{\sqrt2}\arctan(\sqrt2\tan x)+C_k,\quad x\in(-\frac\pi2,\frac\pi2)+k\pi$

Lepení: $x\to(\frac\pi2+k\pi)_-,\quad x-\frac1{\sqrt2}\arctan(\sqrt2\tan x)\to\frac\pi2+k\pi-\frac1{\sqrt2}\frac\pi2 \qquad \lim_+-\lim_- = \frac{\pi}{\sqrt2}$

$x\to(\frac\pi2+k\pi)_+,\quad x-\frac1{\sqrt2}\arctan(\sqrt2\tan x)\to\frac\pi2+k\pi+\frac1{\sqrt2}\frac\pi2 \qquad C_k=k\frac{\pi}{\sqrt2}$ [illegible] $\forall k\in\mathbb{Z}$

$x\to(-\frac\pi2+(k+1)\pi)_+,\quad x-\frac1{\sqrt2}\arctan(\sqrt2\tan x)\to -\frac\pi2+(k+1)\pi+\frac1{\sqrt2}\frac\pi2$ ~~lim [illegible]~~

8. $\int\frac1{2\sin x-\cos x+5}\,dx = \left|\begin{array}{l}y=\tan\frac x2 \\ dx=\frac2{1+y^2}dy \\ x\neq\pi+2k\pi\end{array}\right| = \int\frac{1}{\frac{4y}{1+y^2}+\frac{y^2-1}{y^2+1}+5}\,\frac2{1+y^2}\,dy = \int\frac{2}{6y^2+4y+4}\,dy = \int\frac1{3y^2+2y+2}\,dy =$

$= \frac13\int\frac1{(y+\frac13)^2+\frac23-\frac19}\,dy = \frac35\int\frac1{\left(\frac{3y+1}{\sqrt5}\right)^2+1}\,dy = \frac{\sqrt5}{5}\arctan\left(\frac{3y+1}{\sqrt5}\right)+C_k = \frac1{\sqrt5}\arctan\left(\frac{3\tan\frac x2+1}{\sqrt5}\right)+k\frac{\pi}{\sqrt5},\quad x\in(-\pi,\pi)+2k\pi$

9. $\int \frac{\sin x \cos x}{1+\sin^3 x}dx = \left| \begin{matrix} R(-\cos x, \sin x) = -R(\cos x, \sin x) \\ y = \sin x \\ dy = \cos x\, dx \end{matrix} \right| = \int \frac{y}{1+y^3}dy = \frac{1}{3}\int \frac{(-1)}{1+y} + \frac{1}{2}\frac{2y-1}{y^2-y+1} + \frac{3}{2}\frac{1}{(y-\frac{1}{2})^2+\frac{3}{4}}\, dy$

$\frac{y}{1+y^3} = \frac{A}{1+y} + \frac{By+C}{y^2-y+1} \Rightarrow Ay^2 - Ay + A + By + By^2 + C + Cy = y$

$y^2: A+B=0 \quad , y^1: -A+B+C=1 \quad , y^0: A+C=0 \Rightarrow \left.\begin{matrix} 2B+C=1 \\ B+2C=1 \end{matrix}\right\} \Rightarrow -3C=-1 \quad C=\frac{1}{3} \quad B=\frac{1}{3} \quad A=-\frac{1}{3}$

$= -\frac{1}{3}\ln|1+\sin x| + \frac{1}{6}\ln|\sin^2 x - \sin x + 1| + \frac{\sqrt{3}}{3}\operatorname{arctg}\left(\frac{2\sin x - 1}{\sqrt{3}}\right) + C$ $\quad \sin x \neq -1$

$x \in (-\frac{\pi}{2}, \frac{3}{2}\pi) + 2k\pi$

10. $\int \frac{\sin^3 x}{\cos^4 x}dx = \int \frac{(1-\cos^2 x)\sin x}{\cos^4 x}dx = \left| \begin{matrix} \cos x = y \\ -\sin x\, dx = dy \end{matrix} \right| = \int \frac{y^2-1}{y^4}dy = \frac{1}{3y^3} - \frac{1}{y} + C$

$= \frac{1}{3\cos^3 x} - \frac{1}{\cos x} + C \quad , \quad x \in (-\frac{\pi}{2}, \frac{\pi}{2}) + k\pi$

11. $\int \frac{1}{(1-x^2)^{3/2}}dx = \left| \begin{matrix} \sqrt{1-x^2} = \cos t \\ x = \sqrt{1-\cos^2 t} = \sin t \\ dx = \cos t\, dt \end{matrix} \right| = \int \frac{\cos t}{\cos^3 t}dt = \operatorname{tg} t + C = \frac{\sin(\arcsin x)}{\cos(\arcsin x)} + C = \frac{x}{\sqrt{1-x^2}} + C$

$x \in (-1, 1)$

12. $\int \sqrt{a^2+x^2}dx = |a|\int \sqrt{1+(\frac{x}{a})^2}dx = \left| \begin{matrix} y = \frac{x}{a} \\ dy = \frac{dx}{a} \end{matrix} \right| = a|a| \int \sqrt{1+y^2}dy = \left| \begin{matrix} y = \sinh t \\ dy = \cosh t\, dt \end{matrix} \right|$

$= a|a| \int \sqrt{1+\sinh^2 t}\cosh t\, dt = a|a|\int \cosh^2 t\, dt$

$\int \cosh^2 t\, dt = \left| \begin{matrix} f = \cosh t & f' = \sinh t \\ g' = \cosh t & g = \sinh t \end{matrix} \right| = \sinh t \cosh t - \int \sinh^2 t\, dt = \sinh t \cosh t + \int 1 - \cosh^2 t\, dt$

$\Rightarrow \int \sqrt{a^2+x^2}dx = a|a|\left(\frac{\sinh t \cosh t}{2} + \frac{t}{2}\right) + C$

$= \frac{a|a|}{2}\left(y \cosh(\operatorname{arcsinh} y) + \operatorname{arcsinh} y\right) + C$

$= \frac{a|a|}{2}\left(\frac{x}{a}\cdot\sqrt{1+(\frac{x}{a})^2} + \operatorname{arcsinh}\frac{x}{a}\right) + C$

$= \frac{x}{2}\sqrt{a^2+x^2} + \frac{a|a|}{2}\operatorname{arcsinh}\frac{x}{a} + C \qquad x \in \mathbb{R}$

Limity funkcí podruhé

doposud: limita ve vlastním bodě : $\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in P_\delta(x_0)\ f(x) \in \mathcal{U}_\varepsilon(A)$

(Ne)vlastní limita v (ne)vlastním bodě:

- $A \in \mathbb{R},\ x_0 = +\infty$: $\lim\limits_{x \to +\infty} f(x) = A \iff \forall \varepsilon > 0\ \exists K > 0\ \forall x > K\ |f(x) - A| < \varepsilon$
- $A \in \mathbb{R},\ x_0 = -\infty$: $\lim\limits_{x \to -\infty} f(x) = A \iff \forall \varepsilon > 0\ \exists K > 0\ \forall x < -K\ |f(x) - A| < \varepsilon$
- $A = +\infty,\ x_0 \in \mathbb{R}$: $\lim\limits_{x \to x_0} f(x) = +\infty \iff \forall K > 0\ \exists \delta > 0\ \forall x \in P_\delta(x_0)\ f(x) > K$
- $A = +\infty,\ x_0 = +\infty$: $\lim\limits_{x \to +\infty} f(x) = +\infty \iff \forall K > 0\ \exists L > 0\ \forall x > L\ f(x) > K$

. . . .

Definice: Posloupnost je zobrazení $\varphi: \mathbb{N} \to \mathbb{R}$. Hodnota $\varphi(n)$ se nazývá n-tý člen posloupnosti.

značení: $\varphi(n) = \varphi_n$, posl. $\{\varphi_n\}_{n=1}^{\infty}$, nebo $\{\varphi_n\}$

Definice: Limita posloupnosti. Nechť $\{\varphi_n\}$ je posloupnost a $A \in \mathbb{R}^*$. Řekneme, že A je limitou posloupnosti $\{\varphi_n\}$ pro n jdoucí do nekonečna, jestliže $\forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall n > n_0\ \varphi_n \in \mathcal{U}_\varepsilon(A)$.
Píšeme $\lim\limits_{n \to \infty} \varphi_n = A$, nebo $\varphi_n \to A$ $n \to \infty$.

Definice (obecná definice limity): Nechť $f: \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R}^*$ a $A \in \mathbb{R}^*$. Nechť pro každé $\eta > 0$ je $P_\eta(x_0) \cap \mathcal{D}_f \neq \emptyset$. Pak

$$\lim_{x \to x_0} f(x) = A \overset{\text{def}}{\iff} \forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in P_\delta(x_0) \cap \mathcal{D}_f\ f(x) \in \mathcal{U}_\varepsilon(A)$$

Definice: $\lim\limits_{x \to x_0} f(x) = 0_+ \overset{\text{df}}{\iff} \lim\limits_{x \to x_0} f(x) = 0 \wedge f > 0$ na $P_\delta(x_0) \cap \mathcal{D}_f$ pro jisté $\delta > 0$

Věta (vztahy mezi vlastními a nevlastními limitami): Nechť $f: \mathbb{R} \to \mathbb{R}$ a $x_0 \in \mathbb{R}^*$. Pak

i) $\lim\limits_{x \to x_0} f(x) = +\infty \iff \lim\limits_{x \to x_0} \dfrac{1}{f(x)} = 0+$

ii) $\lim\limits_{x \to x_0} f(x) = -\infty \iff \lim\limits_{x \to x_0} \dfrac{1}{f(x)} = 0-$

iii) $\lim\limits_{x \to +\infty} f(x) = \lim\limits_{y \to 0+} f\left(\dfrac{1}{y}\right)$

iv) $\lim\limits_{x \to -\infty} f(x) = \lim\limits_{y \to 0-} f\left(\dfrac{1}{y}\right)$

} limita vpravo $\exists$ právě tehdy pokud $\exists$ limita vlevo

Věta (jednoznačnost limity): Nechť $f: \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R}^*$ a $\lim\limits_{x \to x_0} f(x) \in \mathbb{R}^*$. Pak je tato limita jednoznačná.

Věta (vztah k jednostranným limitám): Nechť $f: \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R}$ a $A \in \mathbb{R}^*$. Nechť navíc pro všechna $\eta > 0$ jsou obě množiny $(x_0 - \eta, x_0) \cap \mathcal{D}_f$ a $(x_0, x_0 + \eta) \cap \mathcal{D}_f$ neprázdné. Pak

$$\lim_{x \to x_0} f(x) = A \iff \lim_{x \to x_0-} f(x) = A \wedge \lim_{x \to x_0+} f(x) = A.$$

Věta (limita absolutní hodnoty): Nechť $f: \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R}^*$ a nechť $\lim\limits_{x \to x_0} f(x) = A \in \mathbb{R}^*$. Pak $\lim\limits_{x \to x_0} |f(x)| = |A|$

Věta (limity a chování na okolí): Nechť $f: \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R}^*$ a $\lim\limits_{x \to x_0} f(x) = A \in \mathbb{R}^*$.

i) Je-li A vlastní, pak je f na průniku $\mathcal{D}_f$ a jistého prstencového okolí x_0 omezená.

ii) Je-li $A \neq 0$, — " — odražená od nuly.

Věta (aritmetika limit). Nechť $f, g: \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R}^*$, $D_f = D_g$, $\lim_{x \to x_0} f(x) = A \in \mathbb{R}^*$, $\lim_{x \to x_0} g(x) = B \in \mathbb{R}^*$. Pak

i) $\lim_{x \to x_0} (f(x) + g(x)) = A + B$,

ii) $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{A}{B}$,

iii) $\lim_{x \to x_0} (f(x) \cdot g(x)) = A \cdot B$,

má-li pravá strana smysl.
Smysl nemají výrazy $+\infty - \infty$, $0 \cdot \infty$, ∞^0, $\frac{cokoli}{0}$, $1^{\pm\infty}$
$0^0 \overset{def}{=} 1$

Věta (zesílená aritmetika nevlastních limit) Nechť $f, g: \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R}^*$, $D_f = D_g$, $\delta > 0$, $\alpha \in \mathbb{R}$, $\beta > 0$.

i) Je-li $\lim_{x \to x_0} f(x) = +\infty$ a $g \geq \alpha$ na $P_\delta(x_0) \cap D_f$, pak $\lim_{x \to x_0} (f(x) + g(x)) = +\infty$

ii) Je-li $\lim_{x \to x_0} f(x) = -\infty$ a $g \leq \alpha$ $\qquad = -\infty$

iii) Je-li $\qquad = \pm\infty \qquad g \geq \beta \qquad \lim_{x \to x_0} (f(x) \cdot g(x)) = \pm\infty$

iv) Je-li $\qquad = \pm\infty \qquad g \leq -\beta \qquad = \mp\infty$

Věta (zachování nerovnosti při limitním přechodu): Nechť $f, g: \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R}^*$, $A, B \in \mathbb{R}^*$, $f(x) \to A$ a $g(x) \to B$, $x \to x_0$. Jestliže $D_f = D_g$ a $f \leq g$ na jistém $P(x_0) \cap D_f$, pak $A \leq B$.

Věta (o jednom strážníkovi): Nechť $f, g: \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R}^*$, $D_f = D_g$.

i) Je-li $\lim_{x \to x_0} f(x) = +\infty$ a $f \leq g$ na jistém $P(x_0) \cap D_f$, pak $\lim_{x \to x_0} g(x) = +\infty$

ii) Je-li $\lim_{x \to x_0} f(x) = -\infty$ a $g \leq f$ na $\qquad \lim_{x \to x_0} g(x) = -\infty$.

Věta (o limitě složené fce). Nechť $f, g: \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R}^*$, pro každé $\eta > 0$ je $P_\eta(x_0) \cap D_{g \circ f} \neq \emptyset$, $f(x) \to A \in \mathbb{R}^*$, $x \to x_0$ a $g(y) \to B \in \mathbb{R}^*$, $y \to A$. Nechť platí alespoň 1 z podmínek:

i) vnitřní fce na jistém $P(x_0)$ nenabývá své limitní hodnoty v bodě x_0

ii) vnější fce je spojitá v A.

Pak $\lim_{x \to x_0} g(f(x)) = B$.

Věta (l'Hospitalovo pravidlo). Nechť $f, g: \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R}^*$. Nechť dále

a) na jistém $P(x_0)$ existuje vlastní f', g' a platí zde $g' \neq 0$

b) $\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = A \in \mathbb{R}^*$

c) platí jedna z podmínek: i) $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0$, ii) $\lim_{x \to x_0} |g(x)| = +\infty$.

Pak $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)} = A$.

Poznámka. Označme $\lim_{x \to +\infty} \frac{f(x)}{g(x)} = +\infty$ jako $g \ll f$, pak pro $0 < \alpha < \beta$ a $x \to +\infty$:

$$e^{-x} \ll x^{-\beta} \ll x^{-\alpha} \ll \frac{1}{\log x} \ll 1 \ll \log x \ll x^\alpha \ll x^\beta \ll e^x$$

Pro $0 < \alpha < \beta$ a $x \to 0+$:

$$x^\beta \ll x^\alpha \ll \frac{1}{\log(\frac{1}{x})} \ll 1 \ll \log(\tfrac{1}{x}) \ll x^{-\alpha} \ll x^{-\beta}$$

Klasifikace nekonečně malých a nekonečně velkých veličin, symboly o a O

Definice (symboly o a O, silná a slabá ekvivalence): Nechť $f, g : \mathbb{R} \to \mathbb{R}$ a $x_0 \in \mathbb{R}^*$.

i) Píšeme $f = o(g)$ pro $x \to x_0$, jestliže $\lim\limits_{x \to x_0} \frac{f(x)}{g(x)} = 0$.

ii) Píšeme $f = O(g)$ pro $x \to x_0$, jestliže existují $K, \delta > 0$ taková, že $|f(x)| \leq K|g(x)|$ na $P_\delta(x_0)$.

iii) Řekneme, že f je silně ekvivalentní s g pro $x \to x_0$, jestliže $\lim\limits_{x \to x_0} \frac{f(x)}{g(x)} = 1$.

Píšeme $f \cong g$ pro $x \to x_0$.

iv) Řekneme, že f je slabě ekvivalentní s g pro $x \to x_0$, jestliže $\lim\limits_{x \to x_0} \frac{f(x)}{g(x)} \in \mathbb{R} \setminus \{0\}$.

Píšeme $f \sim g$ pro $x \to x_0$.

Poznámka:
- $f \in o(g) \Rightarrow f \in O(g)$
- $f \cong g \Rightarrow f \sim g \Rightarrow f = O(g)$ a $g = O(f)$

Limity monotónních funkcí a posloupností

Věta (existence limity pro monotonní fci): Nechť $f : \mathbb{R} \to \mathbb{R}$ a $(a,b) \subset D_f \subset \mathbb{R}$.

i) Je-li f neklesající na (a,b), pak $\lim\limits_{x \to a+} f(x) = \inf\limits_{x \in (a,b)} f(x)$ a $\lim\limits_{x \to b-} f(x) = \sup\limits_{x \in (a,b)} f(x)$.

ii) Je-li f nerostoucí $\quad = \sup\limits_{x \in (a,b)} f(x) \quad\quad = \inf\limits_{x \in (a,b)} f(x)$.

Důsledek (existence limity pro monotónní posloupnost): Nechť $\{a_n\} \subset \mathbb{R}$ je mon. posl.
Pak $\{a_n\}$ má limitu a platí pro ni: $\lim\limits_{n \to \infty} a_n = \begin{cases} \sup\limits_{n \in \mathbb{N}} a_n, & \text{pokud je } \{a_n\} \text{ neklesající} \\ \inf\limits_{n \in \mathbb{N}} a_n, & \text{nerostoucí.} \end{cases}$

Věta (Heine): Nechť $f : \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R}^*$, $A \in \mathbb{R}^*$ a pro každé $\delta > 0$ platí $P_\delta(x_0) \cap D_f \neq \emptyset$.
Potom $\lim\limits_{x \to x_0} f(x) = A$ právě tehdy, když $\lim\limits_{n \to +\infty} f(x_n) = A$ pro každou $\{x_n\} \subset D_f \setminus \{x_0\}$, $x_n \to x_0$.

Věta (Heine-zesílená). Nechť $f : \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R}^*$ a pro každé $\delta > 0$ platí $P_\delta(x_0) \cap D_f \neq \emptyset$.
Pak $\lim\limits_{x \to x_0} f(x)$ existuje právě tehdy, když existuje $\lim\limits_{n \to +\infty} f(x_n)$ pro každou $\{x_n\} \subset D_f \setminus \{x_0\}$, $x_n \to x_0$.

Důsledek (Heineho definice spojitosti). Nechť $f : \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R}^*$ a pro každé $\delta > 0$, $P_\delta(x_0) \cap D_f \neq \emptyset$.
Pak f je spojitá právě tehdy, když $\lim\limits_{n \to \infty} f(x_n) = f(x_0)$ pro každou $\{x_n\} \subset D_f \setminus \{x_0\}$, $x_n \to x_0$.

Definice (podposloupnost): Nechť $\{a_n\} \subset \mathbb{R}$ je posloupnost a $\{n_k\}$ je rostoucí posl. přir. čísel.
Posloupnost $\{a_{n_k}\}$ nazveme podposloupností $\{a_n\}$, značíme $\{a_{n_k}\} \subset \{a_n\}$.

Věta (Weierstrassova): Z každé omezené posloupnosti reálných čísel lze vybrat konvergentní podposloupnost.

Věta (o vybírání z neomezené posloupnosti): i) Jestliže není reálná posl. omezená shora, obsahuje podposl., která diverguje do $+\infty$.

ii) – " – zdola, – " – do $-\infty$.

Definice (hromadný bod): Nechť $\{a_n\} \subset \mathbb{R}$ je posl. Číslo $A \in \mathbb{R}^*$ nazveme hromadným bodem, má-li $\{a_n\}$ podposl., která má za limitu A.

Definice (limes superior a limes inferior): Nechť $\{a_n\}$ je reálná posloupnost.

- $\limsup\limits_{n\to\infty} a_n := \lim\limits_{n\to\infty} \underbrace{\sup_k \{a_k, k \geq n\}}_{\text{nerostoucí}}$ $\quad \left(= \overline{\lim\limits_{n\to\infty}}\, a_n\right)$

- $\liminf\limits_{n\to\infty} a_n := \lim\limits_{n\to\infty} \underbrace{\inf_k \{a_k, k \geq n\}}_{\text{neklesající}}$ $\quad \left(= \underline{\lim\limits_{n\to\infty}}\, a_n\right)$

Věta: Nechť $\{a_n\}$ je reálná posl. Pak limsup a liminf jsou hromadnými body. Navíc pro libovolný hromadný bod A platí $\underline{\lim}\, a_n \leq A \leq \overline{\lim}\, a_n$.

Věta (vztah $\underline{\lim}$ a $\overline{\lim}$ k limitě): $\lim\limits_{n\to\infty} a_n$ existuje $\Leftrightarrow$ $\overline{\lim\limits_{n\to\infty}}\, a_n = \underline{\lim\limits_{n\to\infty}}\, a_n$.

Bolzano - Cauchyova podmínka

Věta (O B-C podmínce): Nechť $f: \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R}^*$ a pro každé $\eta > 0$, $P_\eta(x_0) \cap D_f \neq \emptyset$. Pak vlastní $\lim\limits_{x\to x_0} f(x)$ existuje právě tehdy, když je splněna B-C podmínka:

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x, y \in P_\delta(x_0) \cap D_f : |f(x) - f(y)| < \varepsilon.$$

Pozn. BC podmínka pro posloupnosti: $\forall \varepsilon > 0 \; \exists n_0 \in \mathbb{N} \; \forall n, m > n_0 : |a_n - a_m| < \varepsilon$.

Definice (Cauchyovská posloupnost). Posl. $\{a_n\}$ se nazývá cauchyovská, jestliže splňuje B-C podmínku.

Limity funkcí podruhé, 1/8

Limity funkcí v nevlastních bodech

1. $\lim\limits_{x\to\infty} \dfrac{a_n x^n + \dots + a_1 x + a_0}{A_m x^m + \dots + A_1 x + A_0}$, $a_n, A_m \neq 0$

$= \lim\limits_{x\to\infty} \dfrac{x^n(a_n + \dots + a_1 x^{1-n} + a_0 x^{-n})}{x^m(A_m + \dots + A_1 x^{1-m} + A_0 x^{-m})} = \begin{cases} n<m: & = 0 \\ n=m: & = \dfrac{a_n}{A_m} \\ n>m: & = \operatorname{sgn}\dfrac{a_n}{A_m}\cdot(+\infty) \end{cases}$

2. $\lim\limits_{x\to\infty} \dfrac{2x^2+1}{\sqrt{3x^4-6x^2+5}} = \lim\limits_{x\to\infty} \dfrac{2x^2+1}{x^2\sqrt{3-\frac{6}{x^2}+\frac{5}{x^4}}} = \dfrac{2}{\sqrt{3}}$

3. $\lim\limits_{x\to\infty} x(\sqrt{x^2+1}-\sqrt{x^2-1}) = \lim\limits_{x\to\infty} \dfrac{x(x^2+1-(x^2-1))}{(\sqrt{x^2+1}+\sqrt{x^2-1})} = \lim\limits_{x\to\infty} \dfrac{2x}{x(\sqrt{1+\frac{1}{x^2}}+\sqrt{1-\frac{1}{x^2}})} = 1$

4. $\lim\limits_{x\to\infty} x^{4/3}(\sqrt[3]{x^2+1}-\sqrt[3]{x^2-1}) = \lim\limits_{x\to\infty} \dfrac{x^{4/3}(x^2+1-(x^2-1))}{(x^2+1)^{2/3}+(x^2+1)^{1/3}(x^2-1)^{1/3}+(x^2-1)^{2/3}} = \lim\limits_{x\to\infty} \dfrac{2x^{4/3}}{x^{4/3}\left((1+\frac{1}{x^2})^{2/3}+(1+\frac{1}{x^2})^{1/3}(1-\frac{1}{x^2})^{1/3}+(1-\frac{1}{x^2})^{2/3}\right)} = \dfrac{2}{3}$

Limity funkcí l'Hospitalovým pravidlem

5. $\lim\limits_{x\to 0} \dfrac{\operatorname{tg}x - x}{x-\sin x} \overset{\text{8. l'H}}{=} \lim\limits_{x\to 0} \dfrac{\frac{1}{\cos^2 x}-1}{1+\cos x} = \lim\limits_{x\to 0} \dfrac{1-1}{1+1} = 0$

6. $\lim\limits_{x\to 0} \dfrac{x(e^x+1)-2(e^x-1)}{x^3} \overset{\text{l'H}}{=} \lim\limits_{x\to 0} \dfrac{e^x+1+xe^x-2e^x}{3x^2} \overset{\text{l'H}}{=} \lim\limits_{x\to 0} \dfrac{\cancel{e^x}+xe^x-\cancel{e^x}}{6x} = \dfrac{1}{6}$

7. $\lim\limits_{x\to 0} \dfrac{1-\cos x^2}{x^2\sin x^2} \overset{\text{l'H}}{=} \lim\limits_{x\to 0} \dfrac{+2x\sin x^2}{2x\sin x^2 + x^2 2x\cos x^2} = \lim\limits_{x\to 0} \dfrac{\sin x^2}{\sin x^2 + x^2\cos x^2}\cdot\dfrac{x^2}{x^2} = \lim\limits_{x\to 0} \dfrac{\sin x^2}{x^2}\cdot\dfrac{1}{\frac{\sin x^2}{x^2}+\cos x^2} = 1\cdot\dfrac{1}{1+1} = \dfrac{1}{2}$

8. $\lim\limits_{x\to 0^+} x^x = e^{\lim\limits_{x\to 0^+} x\ln x} \overset{\text{l'H}}{=} e^{\lim\limits_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}}} = e^{\lim\limits_{x\to 0^+} -x} = 1$

9. $\lim\limits_{x\to \frac{\pi}{4}} (\operatorname{tg}x)^{\operatorname{tg}2x} = e^{\lim\limits_{x\to\frac{\pi}{4}} \operatorname{tg}2x\,\ln(\operatorname{tg}x)} \overset{\text{l'H}}{=} e^{\lim\limits_{x\to\frac{\pi}{4}} \frac{\frac{1}{\operatorname{tg}x}\cdot\frac{1}{\cos^2 x}}{-\frac{2}{\operatorname{tg}^2 2x}\cdot\frac{1}{\cos^2 2x}}} = e^{\lim\limits_{x\to\frac{\pi}{4}} \frac{-\sin^2 2x}{2\operatorname{tg}x\cos^2 x}} = e^{-\frac{1}{2\cdot 1\cdot(\frac{\sqrt{2}}{2})^2}} = \dfrac{1}{e}$

Symboly $O, o, \sim, \simeq$

Dokažte platnost následujících tvrzení

10. $\operatorname{arctg}x = O(1)$, $x\to\infty$: $|\operatorname{arctg}x| \leq \frac{\pi}{2}\cdot 1 \Rightarrow K = \frac{\pi}{2}$

11. $x^2e^{-x} = o(x^a)$, $x\to\infty$, $a<0$ $\Leftrightarrow \lim\limits_{x\to\infty} \dfrac{x^2e^{-x}}{x^a} = 0$

$\lim\limits_{x\to\infty} \dfrac{x^{2-a}}{e^x} = |2-a>0| \overset{\text{l'H}}{=} \lim\limits_{x\to\infty} \dfrac{(2-a)x^{1-a}}{e^x} = \text{l'H } (2-a)\text{-krát} = \lim\limits_{x\to\infty} \dfrac{(2-a)\dots x^k}{e^x}, k<0 = 0$

12. $\sqrt{x+\sqrt{x+\sqrt{x}}} = O(\sqrt[8]{x})$, $x\to 0^+$ ukážeme $\sqrt{x+\sqrt{x+\sqrt{x}}} \simeq \sqrt[8]{x}$ $(\Rightarrow \sqrt{\sqrt{r}} = O(\ ))$

$\lim\limits_{x\to 0^+} \dfrac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{x^{1/2}} = \lim\limits_{x\to 0^+} \sqrt{x^{1-\frac{1}{4}}+\sqrt{x^{1-\frac{1}{2}}+\sqrt{x^{1-1}}}} = \lim\limits_{x\to 0^+}\sqrt{x^{3/4}+\sqrt{x^{1/2}+1}} = 1$

13. $\sqrt{x+\sqrt{x+\sqrt{x}}} \simeq \sqrt{x}$, $x\to\infty$

$\lim\limits_{x\to\infty} \dfrac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{\sqrt{x}} = \lim\limits_{x\to\infty} \sqrt{1+\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x^2}}}} = 1$

Najděte reálné a, aby platilo

14. $\dfrac{1+x}{1+x^4} \sim x^a$, $x\to\infty$: $\lim\limits_{x\to\infty} \dfrac{\frac{1+x}{1+x^4}}{x^a} = \lim\limits_{x\to\infty} \dfrac{1+x}{x^a+x^{4+a}} = \lim\limits_{x\to\infty} \dfrac{\frac{1}{x}+1}{x^{a-1}+x^{4+a-1}} = \begin{cases} a<-3 & \frac{0+1}{0+0} = \infty \\ a=-3 & \frac{0+1}{0+1} = 1 \quad \in \mathbb{R}\setminus\{0\} \\ a>-3 & \frac{0+1}{\infty} = 0 \end{cases} \Rightarrow a=-3$

15. $e^x - \cos x \sim x^a$, $x\to 0$: $\lim\limits_{x\to 0} \dfrac{e^x-\cos x}{x} \overset{\text{l'H}}{=} \lim\limits_{x\to 0} \dfrac{e^x+\sin x}{1} = 1$

$a\neq 1$: $\lim\limits_{x\to 0} \dfrac{e^x-\cos x}{x^a} = \lim\limits_{x\to 0} \dfrac{e^x-\cos x}{x}\cdot x^{1-a} = 1\cdot\lim\limits_{x\to 0} x^{1-a} = \begin{cases} a>1 & \pm\infty \\ a<1 & 0 \end{cases} \Rightarrow a=1$

Limita posloupnosti 1/9

Vypočítejte

1. $\lim\limits_{n\to\infty} \dfrac{\sqrt{n^3-2n^2+1}+\sqrt[3]{n^4+1}}{\sqrt[4]{n^6-6n^5+2}+\sqrt[5]{n^7+n^5+1}} = \lim\limits_{n\to\infty} \dfrac{n^{3/2}\sqrt{1-\frac{2}{n}+\frac{1}{n^3}}+n^{4/3}\sqrt[3]{1+\frac{1}{n^4}}}{n^{3/2}\sqrt[4]{1-\frac{6}{n}+\frac{2}{n^6}}+n^{7/5}\sqrt[5]{1+\frac{1}{n^4}+\frac{1}{n^7}}} = \lim\limits_{n\to\infty} \dfrac{\sqrt{1-\dots}+n^{\overbrace{\frac{4}{3}-\frac{3}{2}}^{-\frac{1}{6}}}\sqrt[3]{1+\dots}}{\sqrt[4]{1-\dots}+n^{\underbrace{\frac{7}{5}-\frac{3}{2}}_{-\frac{1}{10}}}\sqrt[5]{1+\dots}} = 1$

2. $\lim\limits_{n\to\infty} \dfrac{a^n}{n!}$, $a\in\mathbb{R}$ $= \lim\limits_{n\to\infty} \dfrac{a}{1}\cdot\dfrac{a}{2}\cdot\dfrac{a}{3}\cdot\ldots\cdot\dfrac{a}{n-1}\cdot\dfrac{a}{n}$; nájdem $n_0: \dfrac{a}{n_0}<\varepsilon$,

   potom $\forall n>n_0: \dfrac{a}{n}<\varepsilon \Rightarrow \lim\limits_{n\to\infty}\dfrac{a^n}{n!} = \text{omezená}\cdot 0 = 0$

3. $\lim\limits_{n\to\infty} \sqrt[n]{n} \overset{\text{Heine}}{=} \lim\limits_{x\to\infty} \sqrt[x]{x} = e^{\lim\limits_{x\to\infty}\frac{\ln x}{x}} \overset{\text{l'H}}{=} e^{\lim\limits_{x\to\infty}\frac{1}{x}} = 1$

4. $\lim\limits_{n\to\infty} \dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dots+\dfrac{1}{n(n+1)} = \lim\limits_{n\to\infty} \dfrac{1}{1}-\dfrac{1}{2}+\dfrac{1}{2}-\dfrac{1}{3}+\dots+\dfrac{1}{n}-\dfrac{1}{n+1} = \lim\limits_{n\to\infty} 1-\dfrac{1}{n+1} = 1$

5. $\lim\limits_{n\to\infty} a_n$, kde $a_1=\sqrt{2}$, $a_{n+1}=\sqrt{a_n+2}$, $n\geq 1$

   monotonie (rastúca): $\dfrac{a_{n+1}}{a_n} = \dfrac{\sqrt{\frac{a_n}{a_n}+\frac{2}{a_n}}}{\sqrt{a_n}} = \dfrac{\sqrt{1+\frac{2}{a_n}}}{\sqrt{a_n}} \overset{a_n<2}{>} \dfrac{\sqrt{1+\frac{2}{2}}}{\sqrt{2}} = 1 \Rightarrow a_{n+1}>a_n$

   omezenost shora: $a_1=\sqrt{2}<2$, IP: $a_n<2$: $a_{n+1}=\sqrt{a_n+2}<\sqrt{2+2}=\sqrt{4}=2$

   $\Rightarrow$ má limitu: $\lim\limits_{n\to\infty} a_n = a$: $\quad a_{n+1}=\sqrt{a_n+2}$
   $\downarrow \qquad \downarrow$
   $a = \sqrt{a+2} \Rightarrow a^2-a-2=0 \Rightarrow a_{1,2}=\dfrac{1\pm\sqrt{1+8}}{2} = \begin{cases} 2 \\ \cancel{-1} \end{cases}$

6. $\lim\limits_{n\to\infty} a_n$, $a_1>0$, $a_{n+1}=\frac{1}{2}\left(a_n+\frac{1}{a_n}\right)$, $n\geq 1$

   omezená zdola: $a_{n+1}=\frac{1}{2}\left(a_n+\frac{1}{a_n}\right) \overset{A-G}{>} \sqrt{a_n\cdot\frac{1}{a_n}} = 1 \Rightarrow a_n>1 \ \forall n$

   nerostoucí: $a_{n+1}=\frac{1}{2}\left(a_n+\frac{1}{a_n}\right) < \frac{1}{2}(a_n+a_n) = a_n$

   $\Rightarrow$ má $\lim\limits_{n\to\infty} a_n = a$. $\quad a_{n+1}=\frac{1}{2}\left(a_n+\frac{1}{a_n}\right) \Leftrightarrow a=\frac{1}{2}\left(\frac{a^2+1}{a}\right) \Leftrightarrow a^2=1 \Leftrightarrow a_{1,2}=\begin{cases} 1 \\ \cancel{-1} \end{cases}$

7. Zjistěte, pro která $x\in\mathbb{R}$ existuje $\lim\limits_{n\to\infty}\sin(nx)$.

   Nech existuje a $\lim\limits_{n\to\infty}\sin(nx)=L$. BÚNO, $\forall m\in\mathbb{N}$ platí: $\sin((n+m)x) = \sin(nx)\cos(mx)+\sin(mx)\cos(nx)$

   $(\sin((n+m)x)-\sin(nx)\cos(mx))^2 = \sin^2(mx)\cos^2(nx)$

   $\downarrow$ $(L - L\cos(mx))^2 = \sin^2(mx)(1-L^2)$

   $L^2(1-\cos(mx))(1-\cos(mx)) = (1-\cos(mx))(1+\cos(mx))(1-L^2) \quad /$ if $\cos(mx)\neq 1 \quad (*)$

   $L^2-L^2\cos(mx) = 1+\cos(mx)-L^2-L^2\cos(mx)$

   $\cos(mx) = 2L^2-1 \quad \forall m\in\mathbb{N}$

   to je možné, ak: $\boxed{\begin{matrix} x=0 \\ x=2k\pi \end{matrix}} \Rightarrow \begin{matrix} mx=0 \\ mx=2km\pi \end{matrix} \Rightarrow \begin{matrix} \cos(mx)=1 \\ \cos(mx)=1 \end{matrix} \overset{(*)}{\Rightarrow} 0=0 \ \forall m\in\mathbb{N}$

   $\sin(nx)=0 \ \forall n\in\mathbb{N} \Rightarrow L=0$

   $x=(2k+1)\pi \quad mx=m(2k+1)\pi \ \lightning$

   $x=k\frac{\pi}{2} \quad \lightning$

Najděte limsup a liminf

8. $a_n=\dfrac{n-1}{n+1}\cos\left(\frac{2}{3}n\pi\right)$ $\quad : \dfrac{n-1}{n+1} = 1-\dfrac{2}{n+1} \to 1 \ \forall n$

   $\cos\left(\frac{2}{3}n\pi\right) \in \left\{\cos\frac{2}{3}\pi, \cos\frac{4}{3}\pi, \cos 2\pi\right\} \ \forall n = \left\{-\frac{1}{2}, 1\right\} \ \forall n$

   $\Rightarrow \limsup a_n = 1$
   $\liminf a_n = -\frac{1}{2}$

10. $a_n=\cos^n\left(\frac{2}{3}n\pi\right)$ $\quad : \cos^n\left(\frac{2}{3}n\pi\right) = \begin{cases} 1 & n=3k \to 1 \\ \left(-\frac{1}{2}\right)^n & \text{inak} \to 0 \end{cases} \Rightarrow \begin{matrix} \limsup a_n = 1 \\ \liminf a_n = 0 \end{matrix}$

9. $a_n=n(2+(-1)^n)$

   $= \begin{cases} n & n=2k+1 \to \infty \\ 3n & n=2k \to \infty \end{cases} \Rightarrow \limsup a_n = \liminf a_n = \infty$

Najděte hromadné body následujících posloupností

11. $\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{7}{8}, \dots, \frac{1}{2^n}, \frac{2^n-1}{2^n}$

všetky body (členy) majú tvar

1. $\frac{1}{2^n} \to 0$
2. $\frac{2^n-1}{2^n} = 1 - \frac{1}{2^n} \to 1$

$\Rightarrow$ h.b. $= \{0, 1\}$

12. $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \dots$

$\forall x: x \in (0,1) \cap \mathbb{Q}$ sa v postupnosti nachádzajú

$\forall x: x \in (0,1)\ \forall \varepsilon > 0\ \exists x_1, x_2 \in (0,1) \cap \mathbb{Q} : |x_1 - x| < \varepsilon \wedge |x_2 - x| < \varepsilon$

podpostupnosť $\frac{1}{n} \to 0$

– " – $\frac{n-1}{n} \to 1$

každý člen postupnosti $\in (0,1)$

$\Rightarrow$ h.b. $= \langle 0, 1 \rangle$

Hlubší vlastnosti funkcí 1/10

Lokální a globální extrémy funkcí

Nalezněte lokální extrémy funkcí

1. $f(x) = x^3 - 6x^2 + 9x - 4 \quad , \quad x \in \mathbb{R}$

$f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3)$

$\Rightarrow$ stac. body $x_1 = 1$ a $x_2 = 3$

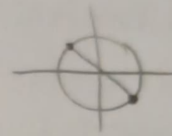
	$(-\infty,1)$	$(1,3)$	$(3,\infty)$
$x-1$	$-$	$+$	$+$
$x-3$	$-$	$-$	$+$
f'	$+$	$-$	$+$

Fce f roste k $x_1 = 1$, pak klesá k $x_2 = 3$ a pak roste $\Rightarrow$ v $x=1$ je lok. maximum a v $x=3$ je lok. minimum

2. $f(x) = e^x \sin x \quad , \quad x \in \mathbb{R}$

$f'(x) = e^x \cos x + e^x \sin x = e^x(\sin x + \cos x)$

$f'(x) = 0 \iff \sin x = -\cos x \iff x_1 = \frac{3}{4}\pi \; , \; x_2 = \frac{7}{4}\pi \quad (+2k\pi)$



	$(-\frac{\pi}{4}, \frac{3}{4}\pi)$	$(\frac{3}{4}\pi, \frac{7}{4}\pi)$
$f'(x)$	$+$	$-$

$\Rightarrow f(x)$ má v $x = \frac{3}{4}\pi + 2k\pi$ lok. maximum a v $x = \frac{7}{4}\pi + 2k\pi$ lok. minimum $\Big\} \forall k \in \mathbb{Z}$

3. $f(x) = x^{1/3}(1-x)^{2/3} \quad , \quad x \in \mathbb{R}$

$f'(x) = \frac{1}{3} x^{-2/3}(1-x)^{2/3} - \frac{2}{3} x^{1/3}(1-x)^{-1/3} = \frac{1}{3}\left(\left(\frac{1-x}{x}\right)^{2/3} - \left(\frac{x}{1-x}\right)^{1/3}\right) = \frac{1}{3}\frac{1-3x}{x^{2/3}(1-x)^{1/3}}$

$\Rightarrow$ stac. bod $x = \frac{1}{3}$, derivace neexistuje v $x=0$ a $x=1$

	$(-\infty,0)$	$(0,\frac{1}{3})$	$(\frac{1}{3},1)$	$(1,+\infty)$
$1-3x$	$+$	$+$	$-$	$-$
$(1-x)$	$+$	$+$	$+$	$-$
$f'(x)$	$+$	$+$	$-$	$+$

$\Rightarrow f(x)$ má v $x = \frac{1}{3}$ lok. maximum a v $x=1$ lok. minimum

Dokažte následující nerovnosti

4. $xy \leq \frac{x^p}{p} + \frac{y^q}{q} \quad , \quad x,y > 0 \; , \; 1 < p,q < \infty \; , \; \frac{1}{p} + \frac{1}{q} = 1$ (Youngova ner.)

$f(x) = xy - \frac{x^p}{p}$

$f'(x) = y - x^{p-1} = 0 \iff x = y^{\frac{1}{p-1}}$

$f''(x) = (1-p)x^{p-2} < 0 \quad \forall x \Rightarrow$ fce je konkávna a v $x = y^{\frac{1}{p-1}}$ má maximum

$\Rightarrow y^{\frac{1}{p-1}} y - \frac{y^{\frac{p}{p-1}}}{p} = y^{\frac{p}{p-1}}\left(\frac{p-1}{p}\right) \geq xy - \frac{x^p}{p} \quad \forall x,y > 0$

$\frac{1}{p} + \frac{1}{q} = 1 \iff q = \frac{p}{p-1} \; , \quad \frac{y^q}{q} \geq xy - \frac{x^p}{p} \iff \frac{x^p}{p} + \frac{y^q}{q} \geq xy$

5. $e^x > x + 1 \quad , \quad x \in \mathbb{R} \setminus \{0\}$

$f(x) = e^x - x - 1$

$f'(x) = e^x - 1 = 0 \iff x = 0$

$f''(x) = e^x > 0 \quad \forall x \Rightarrow$ fce je konvexná a v $x=0$ má minimum (ostré)

$\Rightarrow f(0) = e^0 - 0 - 1 = 0 < e^x - x - 1 \quad \forall x \in \mathbb{R} \setminus \{0\} \iff e^x > x+1$

6. Dokažte, že funkce $f(x)=\begin{cases} e^{-1/x^2}, & x\neq 0 \\ 0, & x=0 \end{cases}$ má v bodě 0 ostré lokální minimum a fce $g(x)=\begin{cases} xe^{-1/x^2}, & x\neq 0 \\ 0, & x=0 \end{cases}$ nemá v bodě 0 lokální extrém, přestože platí $f^{(n)}(0)=g^{(n)}(0)=0$, $n=1,2,\dots$

$f(x)=e^{-1/x^2}>0 \quad \forall x\in\mathbb{R}\setminus\{0\}$ (vlastnost ostrého lok. min. vyplývá už z toho)

$f'(x)=\frac{2}{x^3}e^{-1/x^2} \begin{cases} x>0: & >0 \quad \text{roste} \\ x<0: & <0 \quad \text{klesá} \end{cases} \Bigg\} \Rightarrow f(x)$ má v $x=0$ ostré lok. min.

$g(x)=xe^{-1/x^2} \begin{cases} x>0: & >0 \\ x<0: & <0 \end{cases} \Bigg\}$ v $x=0$ nemůže být extrém, $g(0)=0$

$g'(x)=e^{-1/x^2}+\frac{2}{x^2}e^{-1/x^2}=e^{-1/x^2}\left(1+\frac{2}{x^2}\right)>0 \quad \forall x\neq 0 \Rightarrow$ fce je rostoucí na celém $\mathbb{R}$

7. Nalezněte globální extrémy funkce $f(x)=x^2-4x+6$ na intervalu $[-3,10]$.

$f(x)=(x-2)^2+2 \Rightarrow$ parabola (konvexní)

$f'(x)=2(x-2)=0 \Leftrightarrow x=2$ tam má minimum

$|2-(-3)|=5<|2-10|=8 \Rightarrow$ v $x=10$ má maximum

nebo: $f(-3)=27<f(10)=66$

8. Nalezněte supremum a infimum funkce $f(x)=xe^{-0{,}01x}$ na intervalu $(0,\infty)$.

$f(x)=xe^{-\frac{x}{100}}$

$f'(x)=e^{-\frac{x}{100}}-\frac{x}{100}e^{-\frac{x}{100}}=e^{-\frac{x}{100}}\left(\frac{100-x}{100}\right) \begin{cases} =0 & \text{pro } x=100 \\ >0 & \text{pro } x\in(0,100) \\ <0 & \text{pro } x\in(100,\infty) \end{cases}$

$\lim\limits_{x\to 0+} f(x)=\lim\limits_{x\to 0+}\frac{x}{e^{\frac{x}{100}}}=\frac{0}{1}=0$

$f(100)=\frac{100}{e}$

$\lim\limits_{x\to+\infty} f(x)=\lim\limits_{x\to+\infty}\frac{x}{e^{\frac{x}{100}}}\overset{\text{l'H}}{=}\lim\limits_{x\to+\infty}\frac{1}{\frac{1}{100}e^{\frac{x}{100}}}=0$

Fce $f(x)$ má supremum (i maximum) v bodě $x=100$, $f(100)=\frac{100}{e}$ a infimum 0.

Monotonie funkcí

11. Nalezněte intervaly, na kterých je funkce $f(x)=x^ne^{-x}$, $n\in\mathbb{N}$ rostoucí a klesající.

$f'(x)=nx^{n-1}e^{-x}-x^ne^{-x}=e^{-x}x^n\left(\frac{n}{x}-1\right)$

$x>0$: $\left(\frac{n}{x}-1\right)>0 \Leftrightarrow n>x \dots (0,n)$

$\qquad <0 \Leftrightarrow n<x \dots (n,+\infty)$

$x<0$: $\left(\frac{n}{x}-1\right)>0 \Leftrightarrow n<x \dots \emptyset$

$\qquad <0 \Leftrightarrow n>x \dots (-\infty,0)$

n sudé, $f'(x)$: $-$ 0 $+$ n $-$

n liché, $f'(x)$: $+$ 0 $+$ n $-$

Konvexita, konkávnost

Nalezněte intervaly, na kterých je funkce konvexní/konkávní, a najděte inflexní body.

13. $f(x) = e^{-x^2}$, $x \in \mathbb{R}$

	$(-\infty, -\frac{\sqrt{2}}{2})$	$(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$	$(\frac{\sqrt{2}}{2}, +\infty)$
$f''(x)$	+	−	+

$f'(x) = -2x e^{-x^2}$

$f''(x) = -2e^{-x^2} + 4x^2 e^{-x^2} = 2e^{-x^2}(2x^2-1)$

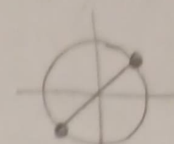
$f''(x) = 0 \iff x = \pm\frac{\sqrt{2}}{2}$ inflexní body

$\Rightarrow f(x)$ je konvexní na $(-\infty, -\frac{\sqrt{2}}{2}) \cup (\frac{\sqrt{2}}{2}, \infty)$

a konkávní na $(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$

14. $f(x) = x \sin \ln x$, $x \in \mathbb{R}^+$

$f'(x) = \sin \ln x + x \cdot \cos \ln x \cdot \frac{1}{x} = \sin \ln x + \cos \ln x$



$f''(x) = \frac{\cos \ln x}{x} - \frac{\sin \ln x}{x} = \frac{1}{x}(\cos \ln x - \sin \ln x)$

$f''(x) = 0 \iff \cos \ln x = \sin \ln x \iff \ln x = \frac{\pi}{4} + k\pi \iff x = e^{\frac{\pi}{4}(4k+1)}$, $k \in \mathbb{Z}$

	$(-\frac{3}{4}\pi, \frac{\pi}{4})$	$(\frac{\pi}{4}, \frac{5}{4}\pi)$
$f''(x)$	+	−

$\Rightarrow f(x)$ je konvexní na $\left(e^{-\frac{3}{4}\pi + 2k\pi}, e^{\frac{\pi}{4} + 2k\pi}\right)$ $\forall k \in \mathbb{Z}$

a konkávní na $\left(e^{\frac{\pi}{4} + 2k\pi}, e^{\frac{5\pi}{4} + 2k\pi}\right)$ $\forall k \in \mathbb{Z}$

15. Dokažte nerovnost $\frac{1}{2}(x^n + y^n) > \left(\frac{x+y}{2}\right)^n$, $x, y > 0$, $x \neq y$, $n > 1$ a vysvětlete její geometrický význam.

Význam: Aritmetický průměr objemů 2 krychlí s délkami hran x a y je větší než objem krychle, která má délku hran aritmetický průměr x a y.

Důkaz: $f(x) := \frac{x^n}{2} + \frac{y^n}{2} - \frac{(x+y)^n}{2^n}$, $f(y) = \frac{y^n}{2} + \frac{y^n}{2} - \frac{(2y)^n}{2^n} = 0$

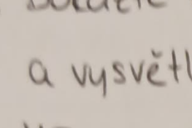
$f'(x) = \frac{n}{2}x^{n-1} - \frac{n}{2^n}(x+y)^{n-1} = \frac{n}{2}x^{n-1}\left(1 - \left(\frac{x+y}{2x}\right)^{n-1}\right)$

$x > y$: $f'(x) > 0$ $\quad \left(1 - \left(\frac{x+y}{2x}\right)^{n-1}\right) > \left(1 - \left(\frac{x+x}{2x}\right)^{n-1}\right) = 0$ $\quad (-y > -x)$

$x < y$: $f'(x) < 0$ $\quad \left(1 - \left(\frac{x+y}{2x}\right)^{n-1}\right) < \left(1 - \left(\frac{x+x}{2x}\right)^{n-1}\right) = 0$ $\quad (-y < -x)$

$\Rightarrow f(x)$ má ostré minimum v bodě $x = y$, kde $f(y) = 0$, proto $f(x) > 0 \ \forall x \neq y$.

9.



rychlost u výtoku: $\frac{1}{2}v^2 = \rho g x$

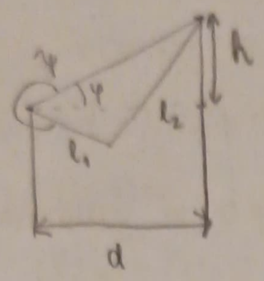
za jak dlouho spadne dolů $(h-x) = \frac{1}{2}gt^2 \Rightarrow t = \sqrt{\frac{2(h-x)}{g}}$

vzdálenost: $s = vt$

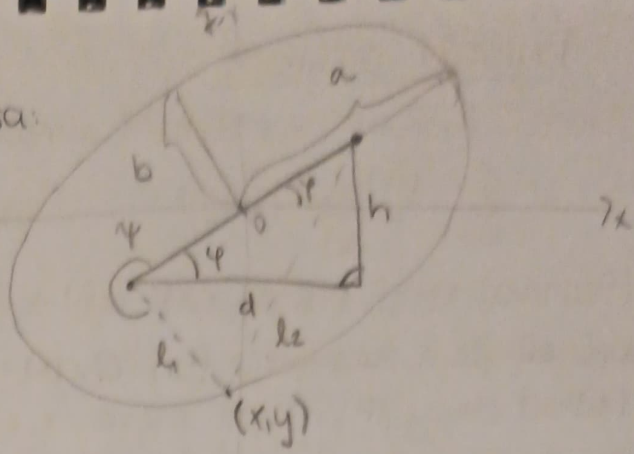
$s = vt = \sqrt{2\rho g x}\sqrt{\frac{2(h-x)}{g}} = 2\sqrt{x(h-x)}\sqrt{\rho}$

$s'(x) = 2\sqrt{\rho}\,\frac{1}{2}\,\frac{h-2x}{\sqrt{x(h-x)}} = \sqrt{\rho}\,\frac{h-2x}{\sqrt{x(h-x)}} = 0 \iff h - 2x = 0 \iff x = \frac{h}{2}$

10.



nakloněná elipsa:

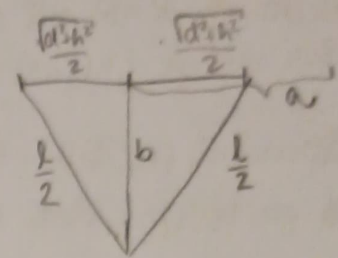


$x = a\cos\varphi\cos\psi - b\sin\varphi\sin\psi$

$y = a\sin\varphi\cos\psi + b\cos\varphi\sin\psi$

vzdálenost ohnisk $= \sqrt{d^2+h^2}$

$\operatorname{tg}\varphi = \frac{h}{d}$



vzdálenost (x,y) od ohnisk je konstantní, $l_1 + l_2 = l$

$a = \frac{l}{2}$

$b = \sqrt{\frac{l^2}{4} - \frac{d^2+h^2}{4}} = \frac{1}{2}\sqrt{l^2-d^2-h^2}$

tedy $y(\psi) = a\sin\varphi\cos\psi + b\cos\varphi\sin\psi$

$y'(\psi) = -a\sin\varphi\sin\psi + b\cos\varphi\cos\psi = 0 \iff a\sin\varphi\sin\psi = b\cos\varphi\cos\psi$

$\operatorname{tg}\psi = \frac{b}{a}\frac{1}{\operatorname{tg}\varphi} = \frac{\sqrt{l^2-d^2-h^2}}{l}\cdot\frac{d}{h}$

Potenciální energie je minimální, když $\psi = \operatorname{arctg}\left(\frac{\sqrt{l^2-d^2-h^2}}{l}\cdot\frac{d}{h}\right) + \pi$

Taylorův polynom

Def.: Nechť $f: \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R}$, $n \in \mathbb{N}$ a nechť $f^{(n)}(x_0) \in \mathbb{R}$. Pak se polynom

$$P_n(x) := \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$ nazývá Taylorův polynom st. n fce f v bodě x_0.

Věta (Peanova): Nechť $f: \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R}$, $n \in \mathbb{N}$ a $f^{(n)}(x_0) \in \mathbb{R}$. Pak existuje právě jeden polynom Q_n stupně nejvýše n takový, že $f(x) - Q_n(x) = o((x-x_0)^n)$. Navíc, $Q_n(x) = P_n(x)$

Věta (odhad chyby TP): Nechť $f: \mathbb{R} \to \mathbb{R}$, $x_0 < x$, $n \in \mathbb{N}$ a f má na $\langle x_0, x\rangle$ spojitou n-tou derivaci a na (x_0, x) má derivaci řádu $n+1$ (nemusí být vlastní).
Nechť $\Phi: \mathbb{R} \to \mathbb{R}$ má v (x_0, x) nenulovou vlastní derivaci a je spojitá na $\langle x_0, x\rangle$.
Pak existuje $\xi \in (x_0, x)$ takové, že $R_{n+1}(x) = \frac{(x-\xi)^n}{n!} \frac{\Phi(x) - \Phi(x_0)}{\Phi'(\xi)} f^{(n+1)}(\xi)$.
Speciálně pro $\Phi(t) = (x-t)^{n+1}$ platí $R_{n+1}(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$ ... Lagrangeův tvar zbytku,
a pro $\Phi(t) = t$ platí $R_{n+1}(x) = \frac{f^{(n+1)}(x_0 + \theta(x-x_0))}{n!}(1-\theta)^n (x-x_0)^{n+1}$, $\theta := \frac{\xi - x_0}{x - x_0} \in (0,1)$ ... Cauchyův tz.

Věta (Základní Taylorove rozvoje):

$e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + o(x^n)$ a $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ $\forall x \in \mathbb{R}$

$\cos x = \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!} + o(x^{2n+1})$ $\qquad \cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$ $\forall x \in \mathbb{R}$

$\sin x = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!} + o(x^{2n+2})$ $\qquad \sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$ $\forall x \in \mathbb{R}$

$\cosh x = \sum_{k=0}^{n} \frac{x^{2k}}{(2k)!} + o(x^{2n+1})$ $\qquad \cosh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$ $\forall x \in \mathbb{R}$

$\sinh x = \sum_{k=0}^{n} \frac{x^{2k+1}}{(2k+1)!} + o(x^{2n+2})$ $\qquad \sinh x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$ $\forall x \in \mathbb{R}$

$\log(1+x) = \sum_{k=1}^{n} (-1)^{k-1} \frac{x^k}{k} + o(x^n)$ $\qquad \log(1+x) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k}$ $\forall x \in (-1, 1\rangle$

$(1+x)^\alpha = \sum_{k=1}^{n} \binom{\alpha}{k} x^k + o(x^n)$ $\qquad (1+x)^\alpha = \sum_{k=1}^{\infty} \binom{\alpha}{k} x^k$ $\forall x \in (-1, 1)$.

Cvičení: Nechť $f, g, h: \mathbb{R} \to \mathbb{R}$ jsou fce definované na okolí $x_0 \in \mathbb{R}$ a $0 \neq \lambda \in \mathbb{R}$. Dokažte, že pro $x \to x_0$ platí:
(i) $o(f) \pm o(f) = o(f)$
(ii) $o(f) \cdot o(g) = o(f \cdot g)$
(iii) $h = o(g)$, $g = o(f) \Rightarrow h = o(f)$
(iv) $o(\lambda f) = o(f)$

Průběh funkcí

Kuchařka:
1. definiční obor
2. obor spojitosti
3. limity v krajních bodech D_f (nebo podintervalů) a v bodech nespojitosti
4. speciální vlastnosti (sudost, lichost, periodicita, symetrie)
5. významné body (průsečníky s osami) a jejich funkční hodnoty
6. $f'(x)$: množiny monotonie, extrémy (lok & glob), R_f, omezenost, jednostr. derivace
7. $f''(x)$: konvexita, konkávnost, inflexní body
8. asymptoty
9. náčrt grafu

Taylorův polynom 1/11

1. Napište TP fce $f(x) = e^{2x-x^2}$ stupně 3 v bodě 0.

$f(x) = e^{2x-x^2} \Rightarrow f(0) = 1$

$f'(x) = 2(1-x)e^{2x-x^2} \Rightarrow f'(0) = 2$

$f''(x) = 4(1-x)^2 e^{2x-x^2} - 2e^{2x-x^2} = 2e^{2x-x^2}(1-4x+2x^2) \Rightarrow f''(0) = 2$

$f'''(x) = 2e^{2x-x^2}\left((2-2x)(1-4x+2x^2) - 4 + 4x\right) = 2e^{2x-x^2}(-2-6x+12x^2-4x^3) \Rightarrow f'''(0) = -4$

$\Rightarrow P_3(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 = 1 + 2x + x^2 - \frac{2}{3}x^3$

2. Napište TP fce $f(x) = \sqrt{x}$ stupně 3 v bodě 1.

$f(x) = \sqrt{x} \Rightarrow f(1) = 1$

$f'(x) = \frac{1}{2\sqrt{x}} \Rightarrow f'(1) = \frac{1}{2}$

$f''(x) = -\frac{1}{4\sqrt{x^3}} \Rightarrow f''(1) = -\frac{1}{4}$

$f'''(x) = \frac{3}{8\sqrt{x^5}} \Rightarrow f'''(1) = \frac{3}{8}$

$\Rightarrow P_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 = 1 + \frac{x-1}{2} - \frac{(x-1)^2}{8} + \frac{(x-1)^3}{16}$

3. Spočtěte přibližně $\sqrt[5]{250}$.

$f(x) = x^{1/5}$, $f(250) = ?$

$\sqrt[5]{250} = \sqrt[5]{\frac{250}{3^5} \cdot 3^5} = 3\sqrt[5]{\frac{250}{243}} = 3\sqrt[5]{1 + \frac{7}{243}}$, $\frac{7}{243} \in (-1, 1)$

$f(250) = g\left(\frac{7}{243}\right)$, $g(x) = 3\sqrt[5]{1+x}$

$g\left(\frac{7}{243}\right) = 3 \cdot \left(\frac{1}{5} \cdot \frac{7}{243} - \frac{4}{5^2 \cdot 2}\left(\frac{7}{243}\right)^2 + \frac{6}{5^3}\left(\frac{7}{243}\right)^3\right) + R_4\left(\frac{7}{243}\right)$

$R_4\left(\frac{7}{243}\right) = 3 \cdot \frac{1}{5} \cdot \left(-\frac{4}{5}\right) \cdot \left(-\frac{9}{5}\right) \cdot \left(-\frac{14}{5}\right) \cdot \frac{1}{4!} \left(\frac{7}{243}\right)^4$

$f(250) = 3{,}017084 \pm \frac{21}{5^4}\left(\frac{7}{243}\right)^4$

přesně: $\sqrt[5]{250} = 3{,}01708816$

4. Spočtěte přibližně $\arcsin 0{,}45$.

$f(x) = \arcsin x \Rightarrow f\left(\frac{1}{2}\right) = \frac{\pi}{6}$

$f'(x) = \frac{1}{\sqrt{1-x^2}} \Rightarrow f'\left(\frac{1}{2}\right) = \frac{2}{\sqrt{3}} = \frac{2}{3}\sqrt{3}$

$f''(x) = \left(\frac{-1}{2}\right) \cdot \frac{(-2x)}{\sqrt{(1-x^2)^3}} = \frac{x}{\sqrt{(1-x^2)^3}} \Rightarrow f''\left(\frac{1}{2}\right) = \frac{\frac{1}{2}}{\frac{\sqrt{3}^3}{2^3}} = \frac{4}{9}\sqrt{3}$

$f'''(x) = \frac{(1-x^2)^{3/2} - x\frac{3}{2}(1-x^2)^{1/2}(-2x)}{(1-x^2)^3}$

$\arcsin x = \arcsin x_0 + \arcsin' x_0 \cdot (x-x_0) + \frac{\arcsin'' x_0}{2}(x-x_0)^2 + R_3(x-x_0)$, $x_0 = \frac{1}{2}$, $x = 0{,}45$, $x - x_0 = \frac{-1}{20}$

$\Rightarrow \underline{\arcsin 0{,}45} = \frac{\pi}{6} + \frac{2}{3}\sqrt{3} \cdot \left(-\frac{1}{20}\right) + \frac{4}{9}\sqrt{3} \cdot \frac{1}{400} + R_3$

$= \underline{0{,}466825 + R_3(x-x_0)}$

$R_3(x-x_0) = \frac{f'''(\xi)}{3!}(x-x_0)^3 = \frac{1}{6} \cdot \frac{(-1)}{8000} \cdot \frac{1+2\xi^2}{(1-\xi^2)^{\frac{5}{2}}}$, $\cancel{|\xi|}$ $\xi \in (0{,}45;\ 0{,}5)$

5. Energie volné částice je v teorii relativity dána vztahem $E = mc^2 = \frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}}$.
Ukažte, že pro $v \ll c$ představuje veličina $T = E - m_0 c^2$
kinetickou energii newtonské mechaniky.

$$f(x) = (1+x)^{-\frac{1}{2}} = 1 - \frac{1}{2}x + \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{x^2}{2!} + \dots$$

$$f\left(-\frac{v^2}{c^2}\right) = 1 + \frac{v^2}{2c^2} + \frac{3}{8}\frac{v^4}{c^4} + \dots \quad \Rightarrow \quad E = m_0 c^2\left(1 + \frac{v^2}{2c^2} + \frac{3}{8}\frac{v^4}{c^4} + \dots\right)$$

$$\Rightarrow T = E - m_0 c^2 = \frac{m_0 v^2}{2} + \underbrace{\frac{3}{8}\frac{m_0 v^4}{c^2}}_{\ll 1} = \frac{1}{2} m_0 v^2$$

Použitím Taylora spočtěte limity

6. $$\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^4} = \lim_{x\to 0} \frac{1 - \frac{x^2}{2!} + \frac{x^4}{4!} + o(x^4) - \left(1 - \frac{x^2}{2} + \frac{x^4}{4}\cdot\frac{1}{2} + o(x^4)\right)}{x^4}$$

$$= \lim_{x\to 0} \frac{x^4\left(\frac{1}{4!} - \frac{1}{8}\right)}{x^4} + \lim_{x\to 0} \frac{o(x^4)}{x^4} \overset{!}{=} \frac{1-3}{24} = -\frac{1}{12}$$

$\left[f = o(g) \text{ pro } x \to x_0 \iff \lim_{x\to x_0} \frac{f(x)}{g(x)} = 0 : \quad f(x) := o(x^4) \ (\Rightarrow g(x) = x^4) : \ \lim_{x\to 0} \frac{o(x^4)}{x^4} = 0 \right]$

7. $$\lim_{x\to 0} \frac{a^x + a^{-x} - 2}{x^2}, \ a \in \mathbb{R}^+ \qquad (a^x)' = \ln a \cdot a^x, \ (a^x)^{(k)} = (\ln a)^k a^x$$

$$= \lim_{x\to 0} \frac{1}{x^2}\left(1 + \ln a \cdot a^x \cdot x + (\ln a)^2 a^x \frac{x^2}{2} + o(x^2) + 1 - \ln a \cdot a^x \cdot x + (\ln a)^2 a^x \cdot \frac{x^2}{2} + o(x^2) - 2\right)$$

$$= \lim_{x\to 0} (\ln a)^2 a^x + \lim_{x\to 0} \frac{o(x^2)}{x^2} = (\ln a)^2$$

8. $$\lim_{x\to 0} \frac{e^x \sin x - x(x+1)}{x^3} = \lim_{x\to 0} \frac{1}{x^3}\left[\left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + o(x^3)\right)\left(x - \frac{x^3}{3!} + o(x^3)\right) - x(x+1)\right]$$

$$= \lim_{x\to 0} \frac{1}{x^3}\left[\underline{x} - \frac{x^3}{3!} + o(x^3) + \underline{x^2} - \cancel{\frac{x^4}{3!}} + o(x^3) + \frac{x^3}{2} - \cancel{\frac{x^5}{12}} + o(x^3) - \underline{x^2} - \underline{x}\right]$$

$$= \lim_{x\to 0} \frac{x^3}{x^3}\left(\frac{1}{2} - \frac{1}{3!}\right) + \lim_{x\to 0} \frac{o(x^3)}{x^3} = \frac{1}{3}$$

PRŮBĚH FUNKCE

① $f(x) = 3x - x^3$; $\lim\limits_{x\to\pm\infty} f(x) = \mp\infty$; f je lichá ; $f(\pm\sqrt{3}) = 0$

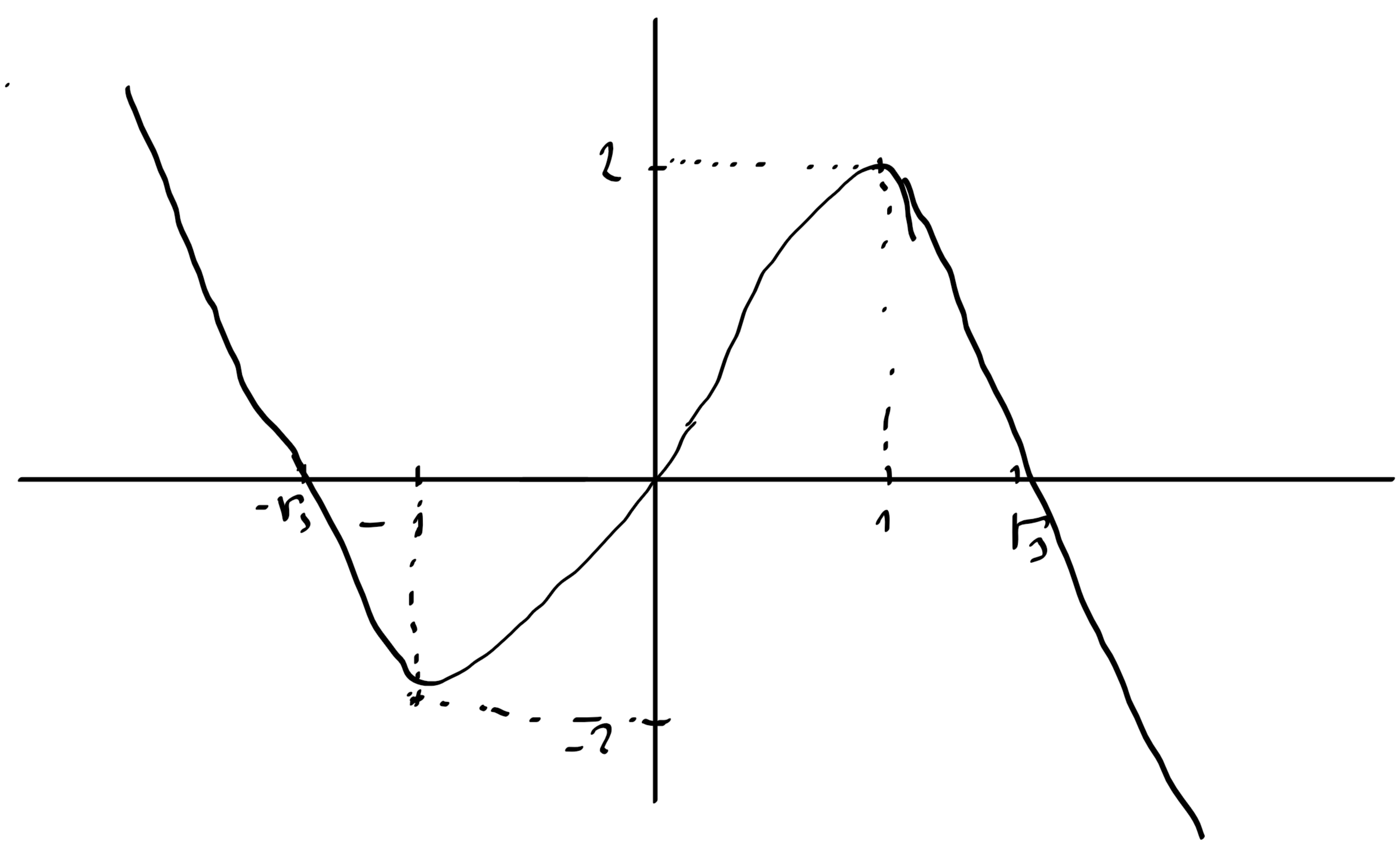
$D_f = \mathbb{R}$; $f(0) = 0$, $H_f = \mathbb{R}$ (spojitá)

$$f'(x) = 3 - 3x^2 = 3(1-x^2) \Rightarrow f'(x) \begin{cases} <0 & x \in (-\infty,-1) \cup (1,\infty) \\ >0 & x \in (-1,1) \end{cases}$$

$$f''(x) = -6x \Rightarrow f''(x) \begin{cases} >0 & x \in (-\infty,0) \\ <0 & x \in (0,\infty) \end{cases}$$

$\lim\limits_{x\to\pm\infty} f'(x) = -\infty \Rightarrow$ není asymptota

$\Rightarrow$ f je klesající na $(-\infty,-1) \cup (1,\infty)$ a rostoucí na $(-1,1)$. v $x=-1$ má f lokální minimum $f(-1) = -2$; v $x=1$ má lokální maximum $f(1) = 2$. Funkce je konvexní na $(-\infty,0)$ a konkávní na $(0,\infty)$. Bod $x=0$ je inflexní bod. Funkce nemá globální extrém.



② $f(x) = \dfrac{x^2-1}{x^2-5x+6} = \dfrac{x^2-1}{(x-3)(x-2)}$, není sudá, lichá, periodická

$D_f = \mathbb{R} \setminus \{2,3\}$

Limity v krajních bodech:

$\lim\limits_{x\to\pm\infty} f(x) = 1$ → asymptota v $\pm\infty$ $\Rightarrow$ $y = 1$

$\lim\limits_{x\to 2-} f(x) =$ "$\dfrac{4}{0+}$" $= \infty$; $\lim\limits_{x\to 2+} f(x) =$ "$\dfrac{4}{0-}$" $= -\infty$

$\lim\limits_{x\to 3-} f(x) =$ "$\dfrac{8}{0-}$" $= -\infty$; $\lim\limits_{x\to 3+} f(x) =$ "$\dfrac{8}{0+}$" $= \infty$

$$f'(x) = \left(\frac{x^2-1}{x^2-5x+6}\right)' = \left(1+\frac{5x-7}{x^2-5x+6}\right)' = \frac{5(x^2-5x+6)-(5x-7)(2x-5)}{(x^2-5x+6)^2}$$

$$= \frac{5x^2-25x+30-10x^2+25x+14x-35}{(\ )^2} = \frac{-5x^2+14x-5}{(x^2-5x+6)^2} \qquad x \neq 2,3$$

$$f''(x) = \frac{(-10x+14)(x^2-5x+6)^2-(-5x^2+14x-5)\,2\cdot(x^2-5x+6)(2x-5)}{(x^2-5x+6)^4}$$

$$= \frac{2}{(x^2-5x+6)^3}\Big[(-5x+7)(x^2-5x+6)+(2x-5)(5x^2-14x+5)\Big]$$

$$= \frac{2}{(\ )^3}\Big[-5x^3+25x^2-30x+7x^2-35x+42+10x^3-28x^2+10x-25x^2+70x-25\Big]$$

$$= \frac{2}{(\ )^3}\Big[5x^3-21x^2+15x+17\Big] = \frac{2}{25(\ )^3}\Big[(5x-7)^3-360x+468\Big]$$

$$= \frac{2}{25(\ )^3}\Big[(5x-7)^3-72(5x-7)+264\Big]$$

[ ] MÁ POUZE 1 REÁLNÝ KOŘEN

$$5x-7 = -2\cdot 3^{1/3}(2+3^{1/3})$$

$$\underline{x \doteq -0{,}6}$$

Určíme ještě nulové body f'. $\quad 5x^2-14x+5=0 \quad \underline{x = \frac{7}{5} \pm \frac{2\sqrt{6}}{5}}$

Vyšetříme znaménka f' a f''

	$(-\infty;-0{,}6)$	$(-0{,}6;\frac{7}{5}-\frac{2\sqrt6}{5})$	$(\frac{7}{5}-\frac{2\sqrt6}{5};2)$	$(2;\frac{7}{5}+\frac{2\sqrt6}{5})$	$(\frac{7}{5}+\frac{2\sqrt6}{5};3)$	$(3;\infty)$
f'	−	−	+	+	−	−
f''	−	+	+	−	−	+

Vidíme tedy, že

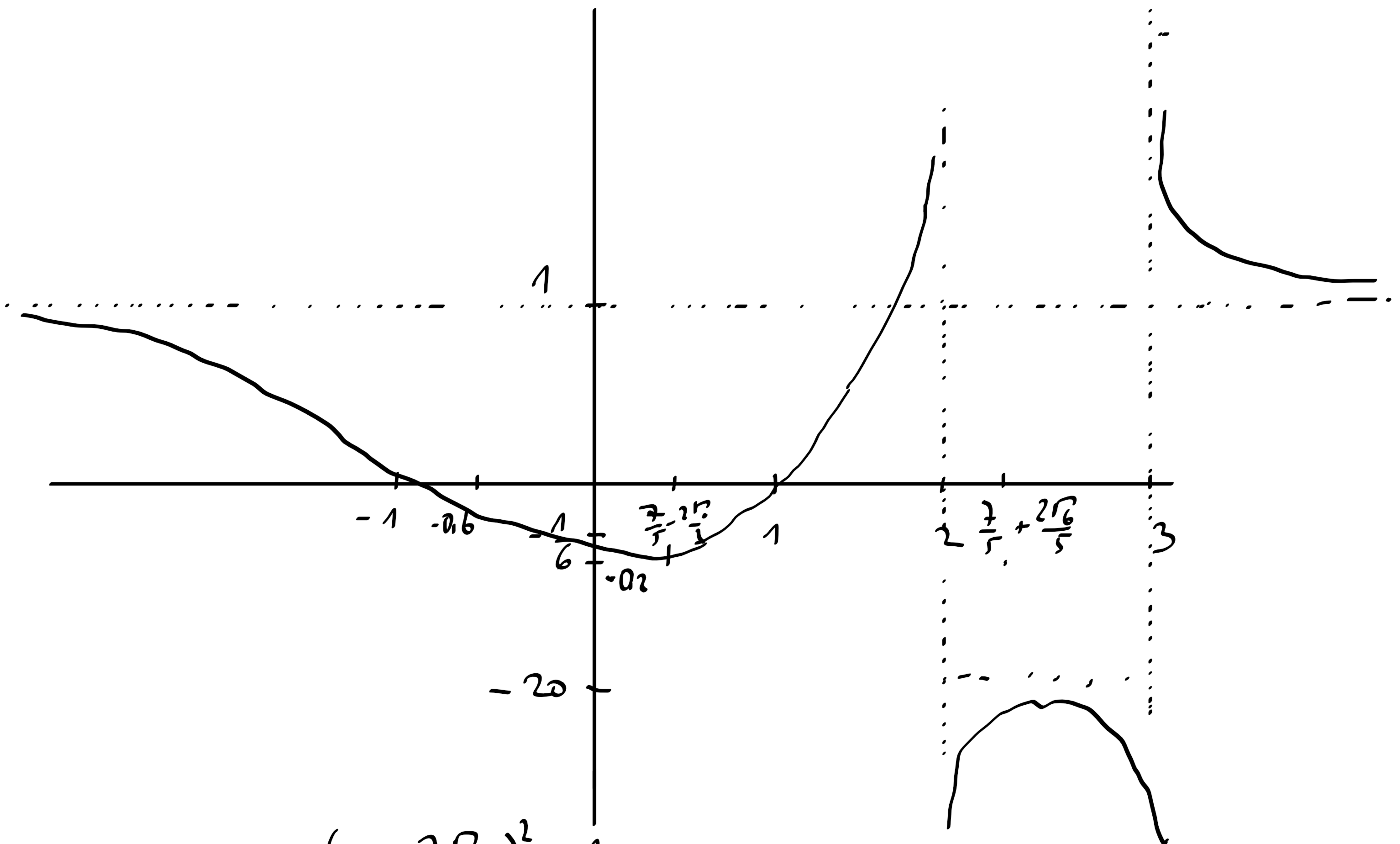
f je rostoucí na $\left(\frac{7}{5}-\frac{2\sqrt6}{5}, 2\right) \cup \left(2, \frac{7}{5}+\frac{2\sqrt6}{5}\right)$

klesající na $\left(-\infty, \frac{7}{5}-\frac{2\sqrt6}{5}\right) \cup \left(\frac{7}{5}+\frac{2\sqrt6}{5}, 3\right) \cup (3,\infty)$

konvexní na $(-0{,}6, 2) \cup (3,\infty)$

konkávní na $(-\infty, -0{,}6) \cup (2,3)$

lokální minimum v $x = \frac{7}{5}-\frac{2\sqrt6}{5}$, maximum v $\frac{7}{5}+\frac{2\sqrt6}{5}$, inflexní bod $x \doteq 0{,}6$



$$f\left(\frac{7}{5} \pm \frac{2\sqrt{6}}{5}\right) = \frac{\left(\frac{7}{5} \pm \frac{2\sqrt{6}}{5}\right)^2 - 1}{\left(\frac{7}{5} \pm \frac{2\sqrt{6}}{5}\right)^2 - 5\left(\frac{7}{5} \pm \frac{2\sqrt{6}}{5}\right) + 6}$$

$$= \frac{(7 \pm 2\sqrt{6})^2 - 25}{(7 \pm 2\sqrt{6})^2 - 25(7 \pm 2\sqrt{6}) + 150} = \frac{49 \pm 28\sqrt{6} + 24 - 25}{49 \pm 28\sqrt{6} + 24 - 175 \mp 50\sqrt{6} + 150}$$

$$= \frac{48 \pm 28\sqrt{6}}{48 \mp 22\sqrt{6}} = \frac{24 \pm 14\sqrt{6}}{24 \mp 11\sqrt{6}}$$

Preveri $-0{,}2 \doteq \frac{24 - 14\sqrt{6}}{24 + 11\sqrt{6}} > \frac{24 + 14\sqrt{6}}{24 - 11\sqrt{6}} \doteq -20$

Vidimo, da je zaloga vrednosti $\left(-\infty, f\left(\frac{7}{5} + \frac{2\sqrt{6}}{5}\right)\right] \cup \left[f\left(\frac{7}{5} - \frac{2\sqrt{6}}{5}\right), \infty\right)$

③ $f(x) = \sqrt{8x^2 - x^4}$ ~ SUDÁ

D_f: $8x^2 - x^4 \geq 0$ $\qquad x^2(8 - x^2) \geq 0$

$\iff x \in [-\sqrt{8}, \sqrt{8}]$

$f(0) = f(\pm\sqrt{8}) = 0$, f nezáporná $\Rightarrow$ f má v $\pm\sqrt{8}$ minima

$$f' = \frac{8x - 2x^3}{\sqrt{8x^2 - x^4}} = \frac{2x(4 - x^2)}{\sqrt{8x^2 - x^4}} = \frac{\operatorname{sign} x \, (4 - x^2) \cdot 2}{\sqrt{8 - x^2}}$$

$x \in (-\sqrt{8}, 0) \cup (0, \sqrt{8})$

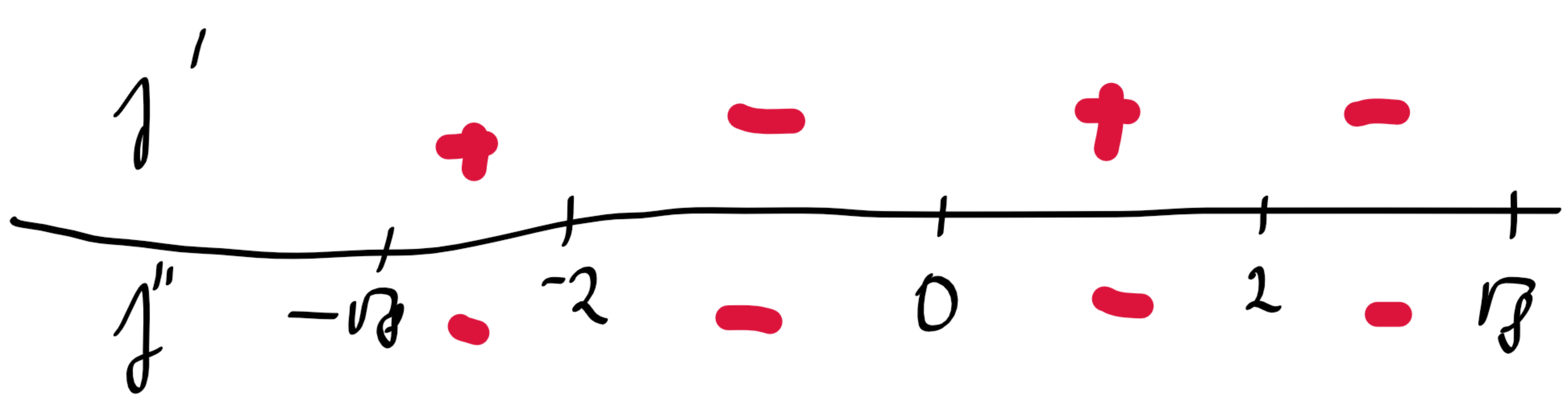
$f'(x) = 0 \iff x = \pm 2$

$f(\pm 2) = 4$

$\lim_{x \to 0\pm} f'(x) = \pm\sqrt{8}$; $\lim_{x \to \pm\sqrt{8}} f'(x) = \mp\infty$

$$f''(x) = 2 \operatorname{sign} x \, \frac{-2x\sqrt{8 - x^2} + (4 - x^2)(8 - x^2)^{-1/2} x}{8 - x^2} = \frac{2x \operatorname{sign} x}{(8 - x^2)^{3/2}} \left(-2(8 - x^2) + 4 - x^2 \right)$$

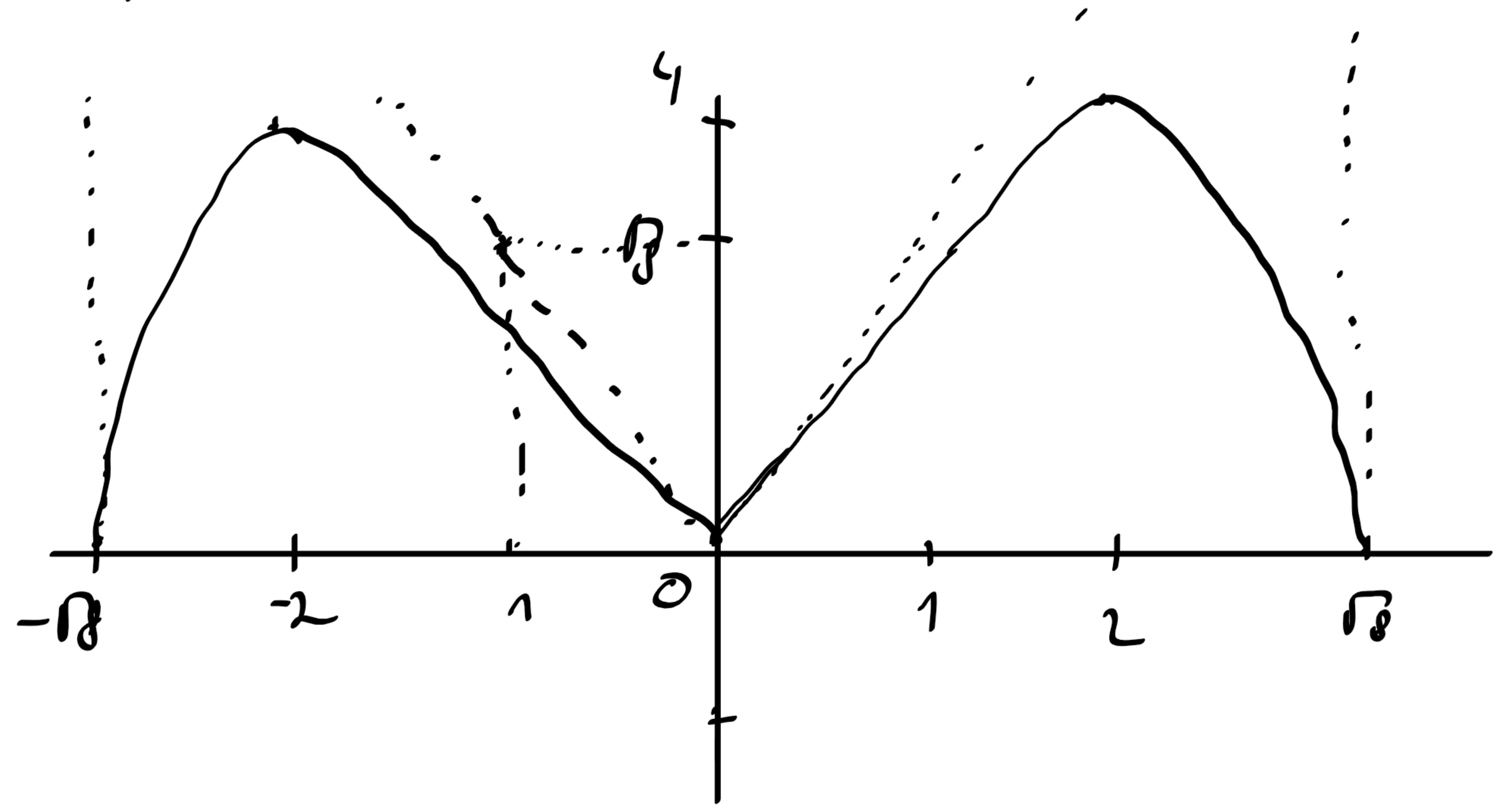
$$= \frac{2|x|}{(8 - x^2)^{3/2}} (x^2 - 12) \qquad \text{pro } x \neq 0$$



f je rostoucí na $(-\sqrt{8}, -2) \cup (0, 2)$

klesající na $(-2, 0) \cup (2, \sqrt{8})$

konkávní na $(-\sqrt{8}, 0) \cup (0, \sqrt{8})$



$H_f = [0, 4]$

4) $f(x) = \dfrac{\cos x}{\cos 2x}$ $2x \neq \dfrac{(2k+1)\pi}{2}$

$D_f = \mathbb{R} \setminus \left\{ \dfrac{(2k+1)\pi}{4} \right\}$

2π – periodická, sudá

VYŠETŘÍME TEDY POUZE $x \in [0, \pi]$, $f(0) = 1$, $f(\pi) = -1$, $f(\frac{\pi}{2}) = 0$

$$\lim_{x \to \frac{\pi}{4}\pm} f(x) = \mp \infty$$

$$\lim_{x \to \frac{3\pi}{4}\pm} f(x) = \pm \infty$$

$$f'(x) = \left(\frac{\cos x}{\cos^2 x - \sin^2 x} \right)' = \left(\frac{\cos x}{2\cos^2 x - 1} \right)' = \frac{-\sin x (2\cos^2 x - 1) + \cos x \cdot 4 \cos x \sin x}{(2\cos^2 x - 1)^2}$$

$$= \frac{2\cos^2 x \sin x + \sin x}{(2\cos^2 x - 1)^2} > 0 \quad x \neq \frac{\pi}{4}, \frac{3\pi}{4}$$

$$f''(x) = \frac{(2\cos^2 x - 1)^2 (-4\cos x \sin^2 x + 2\cos^3 x + \cos x) + (2\cos^2 x \sin x + \sin x)(2\cos^2 x - 1) \cdot 8 \cos x \sin x}{(2\cos^2 x - 1)^4}$$

$$= \frac{(2\cos^2 x - 1)(-4\cos x \sin^2 x + 2\cos^3 x + \cos x) + 8 \sin x \cos x (2\cos^2 x \sin x + \sin x)}{(2\cos^2 x - 1)^3}$$

$$= \frac{1}{()^3} \left(-8c^3 s^2 + 4c^5 + 2c^3 + 4cs^2 - 2c^3 - c + 16 s^2 c^3 + 8 s^2 c \right)$$

$$= \frac{c}{()^3} \left(8c^3 s^2 + 4c^5 + 4cs^2 - c + 8s^2 c \right)$$

$$= \frac{c}{()^3} \left(8c^2 s^2 + 4c^4 + 4s^2 - 1 + 8s^2 \right) = \frac{c}{()^3} \left(8c^2 - 8c^4 + 4c^4 + 4 - 4c^2 - 1 + 8 - 8c^2 \right)$$

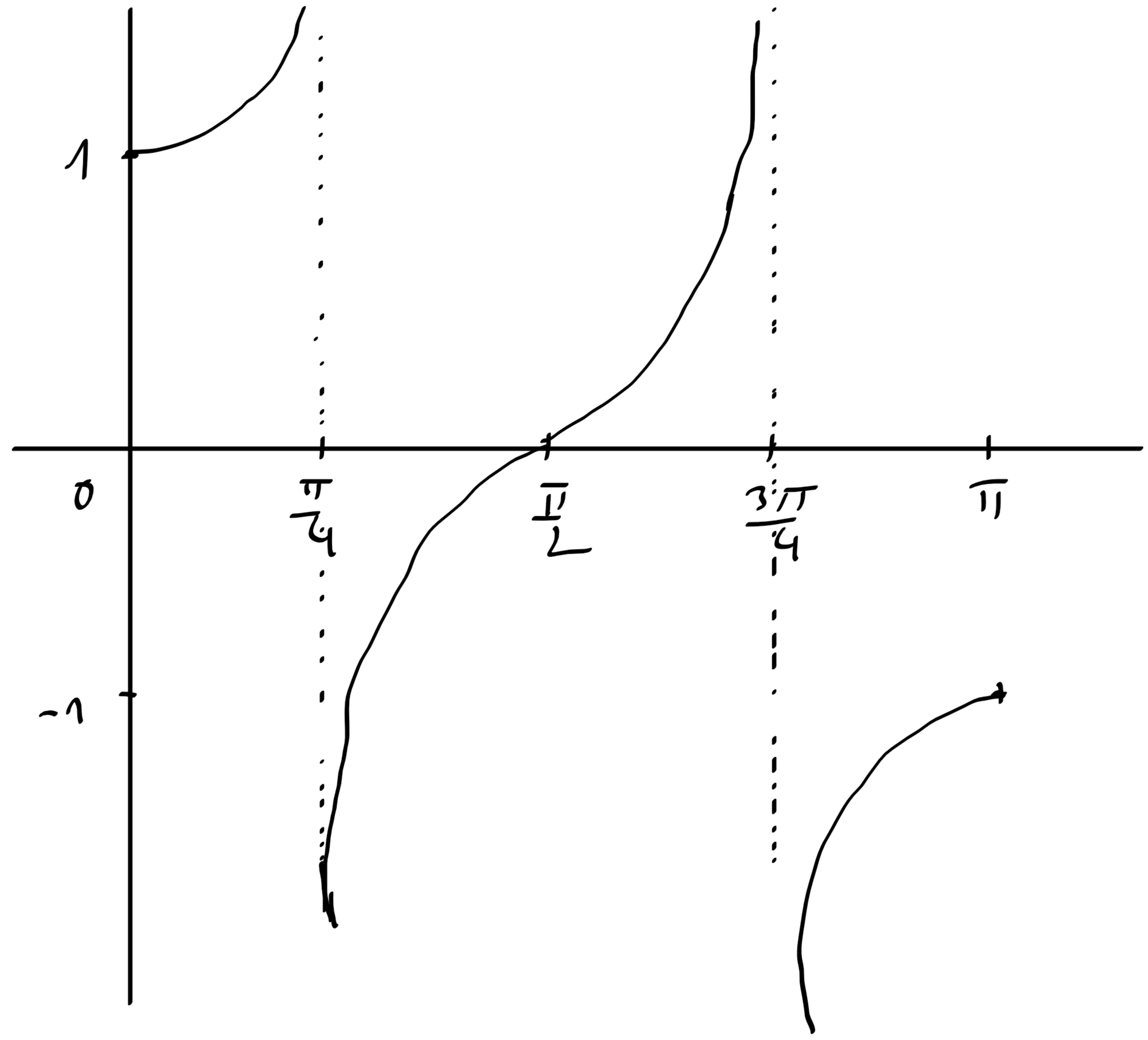
$$= \frac{c}{()^3} \left(-4c^4 - 4c^2 + 11 \right) = \frac{c}{()^3} \left(\underbrace{12 - (2c^2 + 1)^2}_{>0} \right)$$

$f' > 0 \Rightarrow f$ rostoucí na $(0, \frac{\pi}{4}) \cup (\frac{\pi}{4}, \frac{3\pi}{4}) \cup (\frac{3\pi}{4}, \pi)$

f'': 0 $+$ $\frac{\pi}{4}$ $-$ $\frac{\pi}{2}$ $+$ $\frac{3\pi}{4}$ $-$ π

f – konvexní na $(0, \frac{\pi}{4}) \cup (\frac{\pi}{2}, \frac{3\pi}{4})$ $x = \frac{\pi}{2}$ – inflexní bod

konkávní na $(\frac{\pi}{4}, \frac{\pi}{2}) \cup (\frac{3\pi}{4}, \pi)$



v $x=0$ lokální minimum

v $x=\pi$ lokální maximum

ZBYTEK SUDÉ A PERIODICKÁ

5) $f(x) = e^{-2x} \sin^2 x \qquad D_f = \mathbb{R}$

$\lim\limits_{x \to \infty} f(x) = 0 \qquad f$ - nezáporná $\quad f(k\pi) = 0$

$\limsup\limits_{x \to -\infty} f(x) = \infty \qquad \Rightarrow H_f = \mathbb{R}_+$

$\liminf\limits_{x \to -\infty} f(x) = 0$

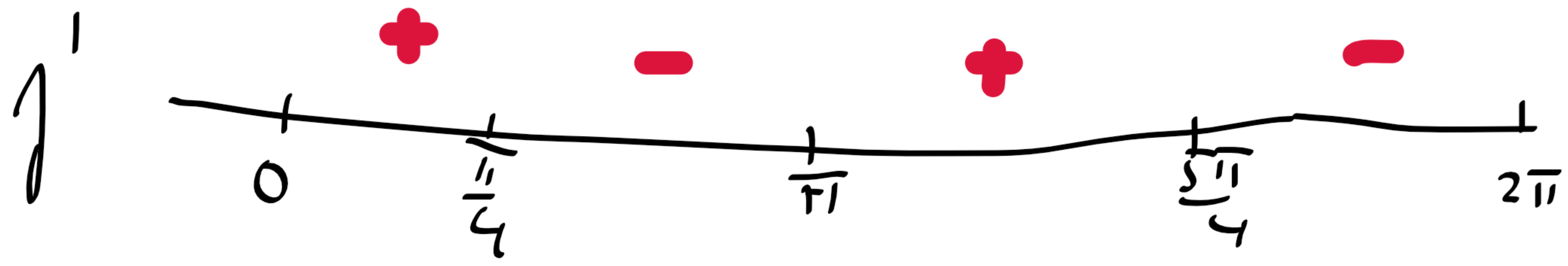
$$f' = -2e^{-2x}\sin^2 x + 2e^{-2x}\sin x \cos x = 2\sin x\, e^{-2x}(\cos x - \sin x)$$

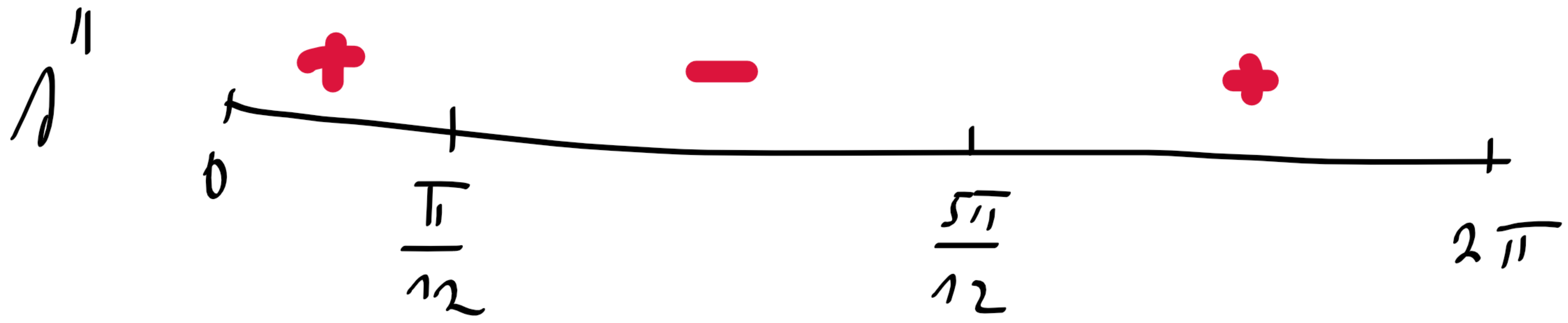
$$f'' = 4e^{-2x}\sin^2 x - 4e^{-2x}\sin x\cos x - 4e^{-2x}\sin x \cos x + 2e^{-2x}\cos^2 x - 2e^{-2x}\sin^2 x$$

$$= 2e^{-2x}(-4\sin x\cos x + 1) = 2e^{-2x}(1 - 2\sin 2x)$$

$f'(x) = 0 \iff x = k\pi, \quad x = \frac{\pi}{4} + k\pi \ ;$

$f''(x) = 0 \iff x = \frac{\pi}{12} + k\pi \ ; \ \frac{5\pi}{12} + k\pi$





f rostoucí na $\left(0, \frac{\pi}{4}\right) + k\pi$

klesající na $\left(\frac{\pi}{4}, \pi\right) + k\pi$

f má lokální maximum v $x = \frac{\pi}{4} + k\pi$

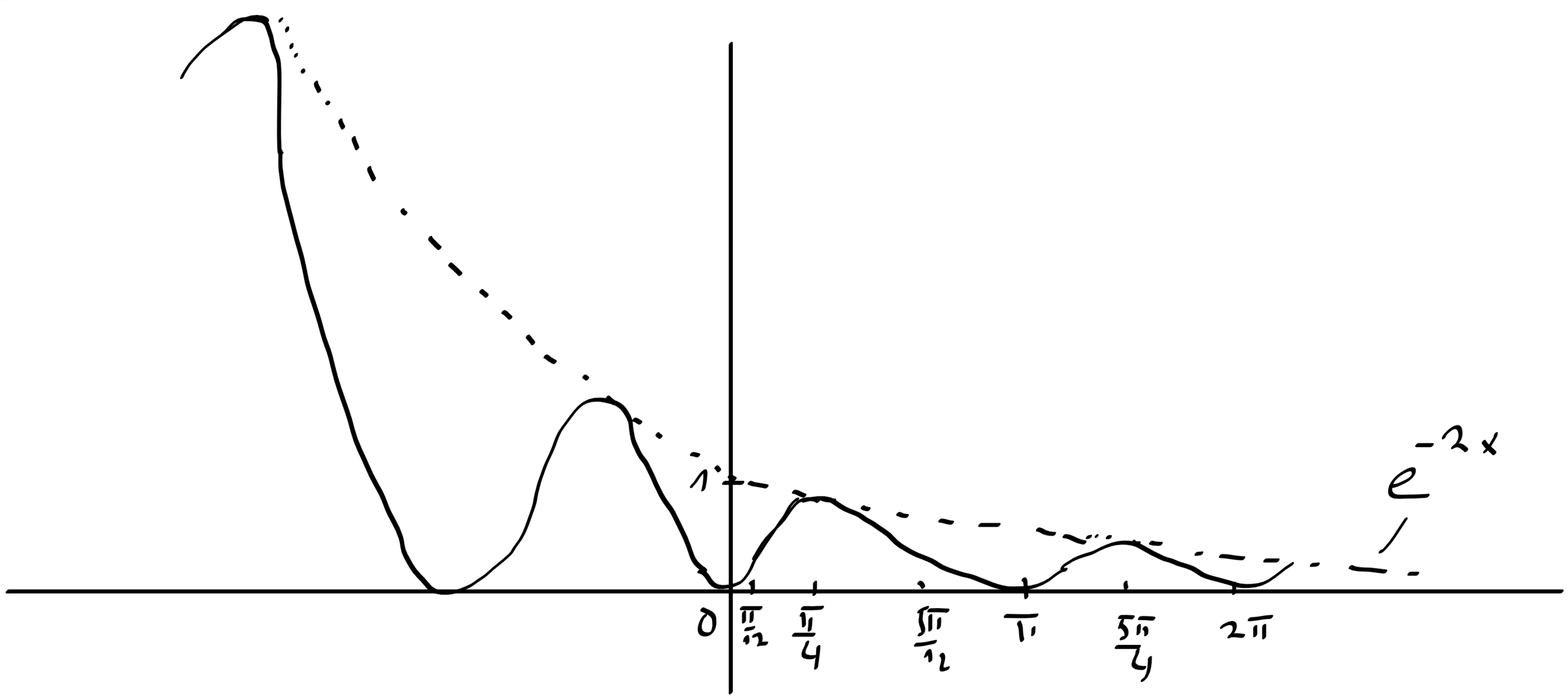
minimum $x = k\pi$

f konkávní na $\left(\frac{\pi}{12}, \frac{5\pi}{12}\right) + 2k\pi$

konvexní na $\left(\frac{5\pi}{12}, \frac{25\pi}{12}\right) + 2k\pi$

inflexní body $x = \frac{\pi}{12} + 2k\pi$

$x = \frac{5\pi}{12} + 2k\pi$



6) $f(x) = \arccos \dfrac{2x}{x^2+1}$

D_f: $-1 \leq \dfrac{2x}{x^2+1} \leq 1$ $\quad x \in \mathbb{R}$ $\qquad f(1) = \arccos 1 = 0$

$\qquad f(-1) = \arccos -1 = \pi$

spojitá funkce - obor hodnot arccos je $[0,\pi]$ $\Rightarrow$ $H_f = [0,\pi]$

minimum globální v $x=1$

maximum globální v $x=-1$

$$\lim_{x\to\pm\infty} f(x) = \arccos 0 = \frac{\pi}{2}$$

$$f' = -\frac{1}{\sqrt{1-\left(\frac{2x}{x^2+1}\right)^2}} \cdot \left(\frac{2x}{x^2+1}\right)' = -\frac{1}{\sqrt{1-\left(\frac{2x}{x^2+1}\right)^2}} \quad \frac{2(x^2+1)-4x^2}{(x^2+1)^2}$$

$$= 2\frac{x^2-1}{\sqrt{(x-1)^2(x+1)^2}} \quad \frac{1}{(x^2+1)} = 2\frac{(x-1)(x+1)}{|x-1||x+1|} \quad \frac{1}{x^2+1} \quad \Big| \; x \neq \pm 1$$

$$= \frac{2\,\mathrm{sig}(x-1)\,\mathrm{sign}(x+1)}{x^2+1}$$

$$f'' = -2\,\mathrm{sign}(x-1)\,\mathrm{sign}(x+1) \cdot \frac{2x}{(x^2+1)^2} \qquad x \neq \pm 1$$

$$\lim_{x\to 1_+} f'(x) = \lim_{x\to -1_-} f'(x) = 1 \; ; \; \lim_{x\to 1_-} f' = \lim_{x\to -1_+} f'(x) = -1$$

		-1		0		1	
f'	+		−		−		+
f''	+		−		+		−

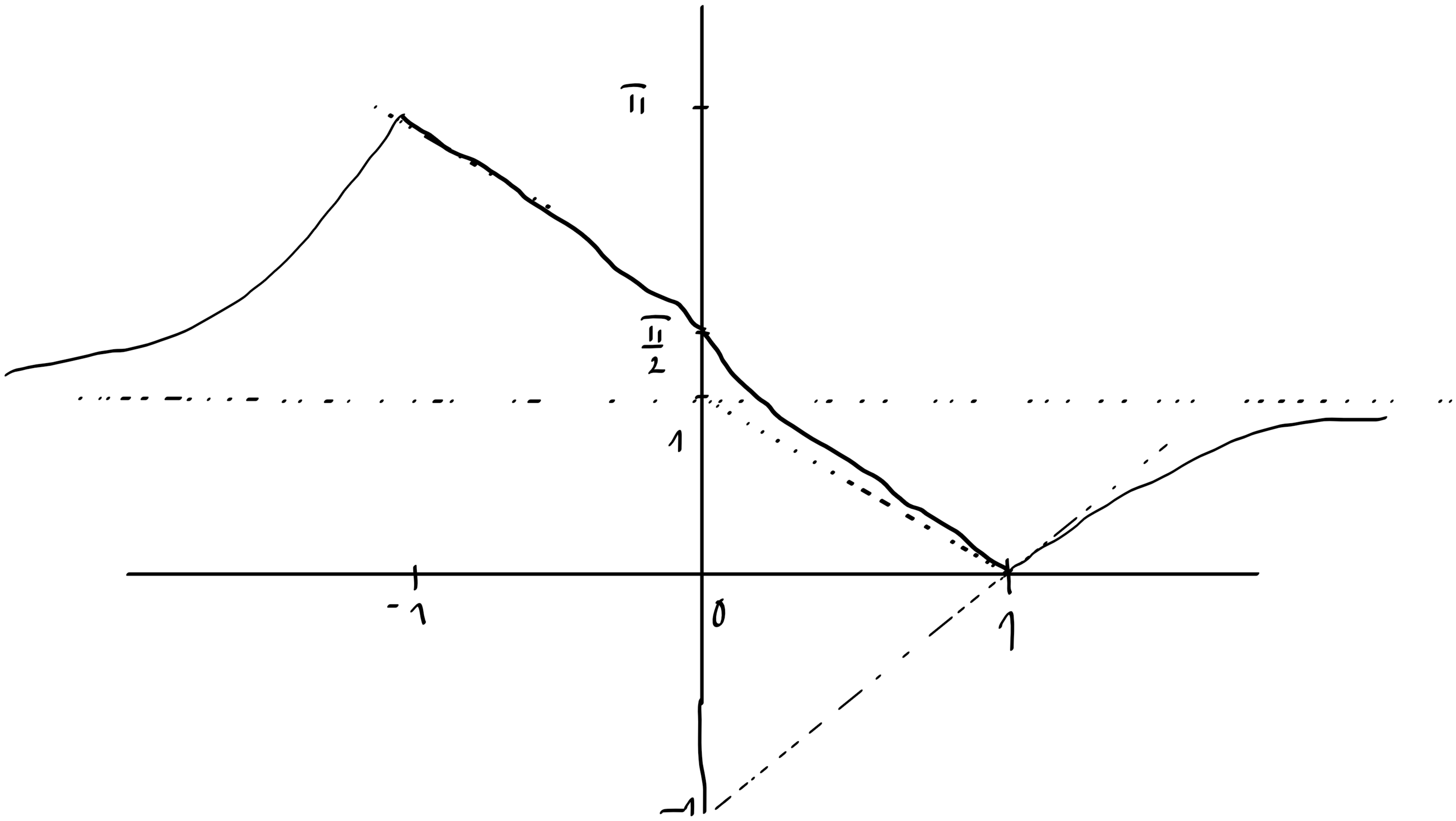
f je rostoucí na $(-\infty,-1)$ a na $(1,\infty)$

klesající na $(-1, 1)$

konvexní na $(-\infty,-1)$ a na $[0,1)$

konkávní na $(-1,0]$ a na $(1,\infty)$

0 je inflexní bod



π
π/2
1
-1
0
1
-1

NEWTONŮV A RIEMANNŮV INTEGRÁL

(1) $$\int_0^{\ln 2} \sqrt{e^x-1}\,dx \overset{\substack{e^x-1=t,\ x=\ln(t+1)\\ dx=\frac{1}{t+1}dt}}{=} \int_0^1 \frac{\sqrt{t}}{t+1}\,dt \overset{\substack{y=\sqrt{t},\ t=y^2\\ dt=2y\,dy}}{=} \int_0^1 \frac{2y^2}{y^2+1}\,dy$$

$e^0-1=0$

$e^{\ln 2}-1=1$

$$= \int_0^1 2-\frac{2}{y^2+1}\,dy = 2\left[y-\operatorname{arctg} y\right]_0^1$$

$$= 2(1-0-\operatorname{arctg}1+\operatorname{arctg}0) = 2\left(1-\frac{\pi}{4}\right)$$

(2) $$\int_0^1 1\cdot\arccos x\,dx \overset{PP}{=} \left[x\arccos x\right]_0^1 + \int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx = 0 - \left[\sqrt{1-x^2}\right]_0^1 = 1$$

$u'=1 \quad v=\arccos x$

$u=x \quad v'=-\frac{1}{\sqrt{1-x^2}}$

(3) $$\underbrace{\int_0^\infty x^{2k-1}e^{-\frac{x^2}{2}}\,dx}_{I_k}, \quad k\in\mathbb{N}$$

$$\lim_{x\to\infty}\left(-e^{-\frac{x^2}{2}}\right)-\left(-e^{-\frac{0^2}{2}}\right) \qquad 2k-3=2(k-1)-1$$

$$I_1 = \int_0^\infty xe^{-\frac{x^2}{2}}\,dx = -\left[e^{-\frac{x^2}{2}}\right]_0^\infty = 1$$

$$k>1 \quad I_k = \int_0^\infty x^{2k-1}e^{-\frac{x^2}{2}}\,dx = \int_0^\infty x^{2k-2}\,xe^{-\frac{x^2}{2}}\,dx \overset{PP}{=} -\left[x^{2k-2}e^{-\frac{x^2}{2}}\right]_0^\infty + (2k-2)\int_0^\infty x^{2k-3}e^{-\frac{x^2}{2}}\,dx$$

$u=x^{2k-2}$

$u'=(2k-2)x^{2k-3}$

$v'=xe^{-\frac{x^2}{2}}$

$v=-e^{-\frac{x^2}{2}}$

$$=(2k-2)I_{k-1} = 2(k-1)I_{k-1}$$

$$\Rightarrow I_k = 2^{k-1}(k-1)!$$

(4) $$\int_0^{4\pi}\frac{1}{1+\sin^2x}\,dx = 8\int_0^{\frac{\pi}{2}}\frac{1}{1+\sin^2x}\,dx = 8\int_0^{\frac{\pi}{2}}\frac{\cos^2x}{1+\sin^2x}\,\frac{dx}{\cos^2x}$$

$$= 8\int_0^{\frac{\pi}{2}}\frac{1}{\boxed{\frac{1+\sin^2x}{\cos^2x}}}\,\frac{dx}{\cos^2x} = 8\int_0^{\frac{\pi}{2}}\frac{1}{1+2\operatorname{tg}^2x}\,\frac{dx}{\cos^2x} \overset{\substack{y=\operatorname{tg}x\\ dy=\frac{dx}{\cos^2x}}}{=} 8\int_0^\infty\frac{1}{1+2y^2}\,dy$$

$\frac{\cos^2+\sin^2+\sin^2}{\cos^2}$

$$= 4\sqrt{2}\left[\operatorname{arctg}\sqrt{2}y\right]_0^\infty = 2\sqrt{2}\pi$$

(5) $$\int_0^{2\pi}\frac{1}{\sin^4x+\cos^4x}\,dx = 4\int_0^{\frac{\pi}{2}}\frac{1}{\sin^4x+\cos^4x}\,dx \overset{\substack{y=\operatorname{tg}x\\ dx=\frac{dy}{y^2+1}}}{=} 4\int_0^\infty\frac{1}{\left(\frac{y^2}{1+y^2}\right)^2+\left(\frac{1}{1+y^2}\right)^2}\cdot\frac{1}{1+y^2}\,dy$$

$$= 4\int_0^\infty\frac{1+y^2}{y^4+1}\,dy = 4\int_0^\infty\frac{y^2+1}{(y^2+1-\sqrt{2}y)(y^2+1+\sqrt{2}y)}\,dy$$

$$= 2\int_0^\infty\frac{1}{y^2+1-\sqrt{2}y}+\frac{1}{y^2+1+\sqrt{2}y}\,dy = 4\int_0^\infty\frac{1}{2\left(y-\frac{\sqrt{2}}{2}\right)^2+1}+\frac{1}{2\left(y+\frac{\sqrt{2}}{2}\right)^2+1}\,dy$$

$$= \frac{4}{\sqrt{2}}\left[\operatorname{arctg}\left(\sqrt{2}\left(y-\frac{\sqrt{2}}{2}\right)\right)+\operatorname{arctg}\left(\sqrt{2}\left(y+\frac{\sqrt{2}}{2}\right)\right)\right]_0^\infty = 2\sqrt{2}\pi$$

(6) $\int_2^\infty \frac{1}{x^2}dx = -\left[\frac{1}{x}\right]_2^\infty = \frac{1}{2} \longrightarrow -\left(\lim_{x\to\infty}\left(\frac{1}{x}-\frac{1}{2}\right)\right)$

(7) $\int_0^\infty e^{-3x}dx = -\frac{1}{3}\left[e^{-3x}\right]_0^\infty = \frac{1}{3}$

(8) $\int_0^1 x\ln x\,dx \overset{PP}{=} \left[\frac{x^2}{2}\ln x\right]_0^1 - \int_0^1 \frac{x}{2} = 0 - \left[\frac{x^2}{4}\right]_0^1 = -\frac{1}{4}$

$u'=x \quad v=\ln x$
$u=\frac{x^2}{2} \quad v'=\frac{1}{x}$

$\frac{1^2}{2}\ln 1 - \lim_{x\to 0+}\frac{x^2}{2}\ln x$

(9) $\int_0^\infty e^{-ax}\cos bx$ $\quad \boxed{a>0}$ $\quad b=0 \Rightarrow \int_0^\infty e^{-ax}dx = \left[-\frac{e^{-ax}}{a}\right]_0^\infty = \frac{1}{a}$

$b\neq 0 \quad I = \int_0^\infty e^{-ax}\cos bx = \left[-\frac{e^{-ax}}{a}\cos bx\right]_0^\infty - \int_0^\infty \frac{e^{-ax}}{a}b\sin bx$

$v'=e^{-ax} \quad u=\cos bx$
$v=-\frac{e^{-ax}}{a} \quad u'=-b\sin bx$

$v'=\frac{e^{-ax}}{a} \quad u=b\sin bx$
$u'=b^2\cos bx \quad v=-\frac{e^{-ax}}{a^2}$

$\left(\frac{a^2+b^2}{a^2}\right)I = \frac{1}{a}$

$= \frac{1}{a} - \left[-\frac{e^{-ax}}{a^2}b\sin bx\right]_0^\infty - \int_0^\infty \frac{e^{-ax}}{a^2}b^2\cos bx$

$= \frac{1}{a} - \frac{b^2}{a^2}I \Rightarrow I = \frac{a}{a^2+b^2}$

(10) $\int_0^{\frac{\pi}{2}} \operatorname{tg} x\,dx = \lim_{a\to\frac{\pi}{2}}\int_0^a \operatorname{tg} x\,dx = \lim_{a\to\frac{\pi}{2}} -\left[\ln(\cos x)\right]_0^a = \lim_{a\to\frac{\pi}{2}}(-\ln\cos a) = \infty$

$F'(\alpha) = \int_0^\pi \frac{2\alpha-2\cos x}{1-2\alpha\cos x+\alpha^2}dx \to$ Lmilm
$F(0)=0 \quad F(\alpha) = \int_0^\alpha F'(\alpha) = \int_0^\alpha\left(\int_0^\pi \frac{2\alpha-2\cos x}{1-2\alpha\cos x+\alpha^2}dx\right)d\alpha$

(11) $F(\alpha) = \int_0^\pi \ln(1-2\alpha\cos x+\alpha^2)\,dx \qquad |\alpha|\neq 1$

z definice Horní a dolní součet na $(0,\frac{\pi}{4})$ $\quad x_0=0,\ x_1=\frac{\pi}{n},\ \dots\dots\ x_n=\pi$

$I_n = \frac{1}{n}\sum_{k=1}^n \ln(1-2\alpha\cos x_k+\alpha^2)$ $\quad ;\ \frac{1}{n}\sum_{k=1}^n \ln(1-2\alpha\cos x_{k-1}+\alpha^2) = y^n$

Horní dolní součty

Spočteme pouze jeden součet:

$\frac{1}{n}\sum_{k=0}^{n-1}\ln\left(1-2\alpha\cos\frac{\pi k}{n}+\alpha^2\right) = \frac{1}{n}\sum_{k=0}^{n-1}\ln\left(\left(\alpha-\cos\frac{\pi k}{n}-i\sin\frac{\pi k}{n}\right)\left(\alpha-\cos\frac{\pi k}{n}+i\sin\frac{\pi k}{n}\right)\right)$

$= \frac{1}{n}\ln\left(\prod_{k=0}^{n-1}\left(\alpha-\cos\frac{\pi k}{n}-i\sin\frac{\pi k}{n}\right)\left(\alpha-\cos\frac{\pi k}{n}+i\sin\frac{\pi}{n}k\right)\right)$

$= \frac{1}{n}\ln\left(\prod_{k=0}^{n-1}\left(\alpha-\cos\frac{\pi k}{n}-i\sin\frac{\pi k}{n}\right)\prod_{k=0}^{n-1}\left(\alpha-\cos\frac{\pi k}{n}+i\sin\frac{\pi k}{n}\right)\right)$

$= \frac{1}{n}\ln\left(\prod_{k=0}^{n-1}\left(\alpha-\cos\frac{\pi k}{n}-i\sin\frac{\pi k}{n}\right)\prod_{k=0}^{n-1}\left(\alpha-\cos\frac{\pi(k-2n)}{n}+i\sin\frac{\pi(k-2n)}{n}\right)\right)$

$= \frac{1}{n}\ln(\quad \text{"} - \quad \prod_{k=n+1}^{2n}$

$= \frac{1}{n}\ln(\quad \text{"} -$

$\underbrace{\left(\alpha-\cos\left(\frac{-k\pi}{n}\right)+i\sin\left(-\frac{\pi k}{n}\right)\right)}_{\left(\alpha-\cos\left(\frac{\pi k}{n}\right)-i\sin\left(\frac{\pi k}{n}\right)\right)}$

$$= \frac{1}{n}\ln\left(\prod_{\substack{k=0\\k\neq n}}^{2n}\left(\alpha-\cos\frac{k\pi}{n}-i\sin\frac{k\pi}{n}\right)\right)$$

$$= \frac{1}{n}\ln\left(\left[\prod_{k=0}^{2n-1}\left(\alpha-\cos\frac{k\pi}{n}-i\sin\frac{k\pi}{n}\right)\right]\frac{\alpha-\cos\frac{2n\pi}{n}-i\sin\frac{2n\pi}{n}}{\alpha-\cos\frac{n\pi}{n}-i\sin\frac{n\pi}{n}}\right)$$

$$= \frac{1}{n}\ln\left(\frac{\alpha-1}{\alpha+1}\prod_{k=0}^{2n-1}(\alpha-z_k)\right)$$ kde $z_k = \cos\frac{k\pi}{n}+i\sin\frac{k\pi}{n}$

ALE $(z_k)^{2n} = \cos 2k\pi + i\sin 2k\pi = 1$

a tedy $\prod_{k=0}^{2n-1}(\alpha-z_k) = \alpha^{2n}-1$

$$\Rightarrow \; = \frac{1}{n}\ln\left(\frac{\alpha-1}{\alpha+1}(\alpha^{2n}-1)\right)$$ (máme $|\alpha|\neq 1$!)

a tedy $\int_0^\pi \ln(1-2\alpha\cos x+\alpha^2) = \lim_{n\to\infty}\frac{1}{n}\ln\left(\frac{\alpha-1}{\alpha+1}(\alpha^{2n}-1)\right)$

pokud $|\alpha|<1$ pak $= \lim_{n\to\infty}\frac{1}{n}\cdot\lim_{n\to\infty}\ln\left(\frac{1-\alpha}{\alpha+1}(1-\alpha^{2n})\right)$

$$= 0\cdot\ln\left(\frac{1-\alpha}{1+\alpha}\right) = 0$$

pokud $|\alpha|>1$ pak $= \lim_{n\to\infty}\left(\frac{1}{n}\ln\frac{\alpha-1}{\alpha-1}+\frac{1}{n}\ln(\alpha^{2n}-1)\right)$

$$= \lim_{n\to\infty}\frac{1}{n}\ln\left(\alpha^{2n}\frac{\alpha^{2n}-1}{\alpha^{2n}}\right) = \lim_{n\to\infty}\left(\frac{1}{n}\ln\alpha^{2n}+\frac{1}{n}\ln\left(\frac{\alpha^{2n}-1}{\alpha^{2n}}\right)\right)$$

$$= \underline{\underline{\ln\alpha^2}}$$

KONVERGENCE INTEGRÁLU – TJ "$\int_a^b f(x)\,dx < \infty$" ?

(12) $\int_0^\infty x^p\,dx = \lim_{\varepsilon\to0+}\int_\varepsilon^{1/\varepsilon}x^p\,dx = \lim_{\varepsilon\to0+}\left[\frac{x^{p+1}}{p+1}\right]_\varepsilon^{1/\varepsilon}$ ($p\neq1$) $= \lim_{\varepsilon\to0+}\frac{1}{p+1}\left(\varepsilon^{-(p+1)}-\varepsilon^{p+1}\right) = \infty$

$= \lim_{\varepsilon\to0+}[\ln x]_\varepsilon^{1/\varepsilon}$ ($p=1$) $= \lim_{\varepsilon\to0+}\ln\frac{1}{\varepsilon^2} = \infty$

NEKONVERGUJE!

(13) $\int_1^\infty x^p\,dx = \lim_{\varepsilon\to0+}\left[\frac{x^{p+1}}{p+1}\right]_1^{1/\varepsilon}$ ($p\neq1$) $= \begin{cases}-\frac{1}{p+1} & \text{PRO } p<-1\\ \infty & \text{PRO } p>-1\end{cases}$

$= \lim_{\varepsilon\to0+}[\ln x]_1^{1/\varepsilon} = \infty$

KONVERGUJE $\Leftrightarrow$ $p<-1$! Důležitý!

$\int_2^\infty \frac{1}{x\ln^p x}\,dx = \lim_{\varepsilon\to0+}\frac{1}{p-1}\left[\frac{1}{\ln^{p-1}x}\right]_2^{1/\varepsilon}$ ($p\neq1$) $\begin{cases}\text{KONVERGUJE} & p>1\\ \text{DIVERGUJE} & p<1\end{cases}$

$\lim_{\varepsilon\to0}\left[\frac{1}{\ln(\ln x)}\right]_2^{1/\varepsilon}$ ($p=1$) – DIVERGUJE

$\int_{10}^\infty \frac{1}{x\ln x(\ln\ln x)^p}$

(14) $\int_0^{10} x^p = \lim_{\varepsilon \to 0+} \left[\frac{x^{p+1}}{p+1} \right]_\varepsilon^{10} \begin{cases} \frac{10^{p+1}}{p+1} & p > -1 \\ \infty & p < -1 \end{cases}$

$p = -1 \Rightarrow [\ln x]_\varepsilon^{10} \Rightarrow \infty$

KONVERGUJE $\Leftrightarrow$ $p > -1$

[INTEGRÁL KONVERGUJE "u ∞" pokud $f(x) \sim x^p$ pro $p < -1$
"u 0" -||- $p > -1$]

(15) $\int_0^\infty \frac{x^{3/2}}{1+x^2}\, dx$ jediný problémový bod je "∞"

PROTOŽE $\frac{x^{3/2}}{1+x^2} \sim x^{-1/2}$ $x \to \infty$

a $\int_1^\infty x^{-1/2} dx = \infty$

$\boxed{\int_0^\infty \text{-||-} \geq \int_1^\infty \frac{x^{3/2}}{1+x^2} dx \geq \int_1^\infty \frac{x^{3/2}}{2x^2} = \frac{1}{2} \int_1^\infty x^{-\frac{1}{2}} dx = \infty}$

pak $\int_0^\infty \frac{x^{3/2}}{1+x^2} dx = \infty$

(16) $\int_0^1 \frac{1}{\sqrt{x(1-x^2)}} dx = \int_0^1 \underbrace{\frac{1}{\sqrt{1+x}} \frac{1}{\sqrt{x}} \cdot \frac{1}{\sqrt{1-x}}}_{f(x)} dx$ PROBLÉMOVÉ BODY $x = 0$, $x = 1$

v okolí "0" $f(x) \sim \frac{1}{\sqrt{x}}$ a $\int_0^1 \frac{1}{\sqrt{x}} < \infty$

v okolí "1" $f(x) \sim \frac{1}{\sqrt{1-x}}$ a $\int_0^1 \frac{1}{\sqrt{1-x}} dx = \int_0^1 \frac{1}{\sqrt{x}} dx < \infty$

$\Rightarrow \int_0^1 f(x) < \infty$

$\boxed{\int_0^1 \text{-||-} = \int_0^{1/2} + \int_{1/2}^1 \leq \int_0^{1/2} 1 \cdot \frac{1}{\sqrt{1/2}} \frac{1}{\sqrt{x}} + \int_{1/2}^1 1 \cdot \frac{1}{\sqrt{1/2}} \frac{1}{\sqrt{1-x}} < \infty}$

(17) $\int_0^2 \frac{1}{\ln x} dx$ PROBLÉMOVÝ BOD "$x = 1$"

$\ln x \sim (x-1)$ u "$x = 1$" jinak je $\frac{1}{\ln x}$ spojitá

a $\int_1^2 \frac{1}{x-1} dx = \infty$ $\Rightarrow$ INTEGRÁL NEKONVERGUJE!

(18) $\int_0^{1/2} \frac{\ln \sin x}{x^p} dx$ PROBLÉMOVÝ BOD $x = 0$

$\ln \sin x \sim \ln x$

POKUD $p \geq 1$ pak $\frac{|\ln x|}{x^p} \geq \frac{1}{x}$ v okolí "0" a integrál diverguje

pokud $p < 1$ pak $\frac{|\ln x|}{x^p} \lesssim \frac{1}{x^{1-\varepsilon}}$ pro nějaké $\varepsilon > 0$

$\Rightarrow$ KONVERGUJE

(19) $\int_0^\infty \frac{\operatorname{arctg} x}{x^{3/2}} = \int_1^\infty \frac{\operatorname{arctg} x}{x^{3/2}} + \int_0^1 \frac{\operatorname{arctg} x}{x^{3/2}} \leq \int_1^\infty \frac{\frac{\pi}{2}}{x^{3/2}} + \int_0^1 \frac{x}{x^{3/2}} < \infty$

$\int x^{3/2}$ $\int x^{1/2}$