HOMEWORK PDE2

Problem 1: Consider the problem

$$\begin{aligned} -\Delta u + \ln u &= f \quad \text{in } \Omega, \\ u &= u_d \quad \text{on } \partial\Omega, \end{aligned}$$

where $f \in L^2(\Omega)$ is nonnegative, and $u_d \in W^{1,2}(\Omega)$ fulfills $u_d \geq \varepsilon > 0$ a.e. in Ω.
GOAL: Show that there exists unique positive $u \in W^{1,2}(\Omega)$ solving the problem.
GOAL (not obligatory): Prove the same statement but assume only $f \in L^2(\Omega)$, $u_d \in W^{1,2}(\Omega)$, $u_d > 0$ a.e. in Ω and $\int_\Omega |\ln u_d| < \infty$.
DEADLINE: April 1

Problem 2: Consider the following problem

$$\begin{aligned} -\Delta_p u + \sinh u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where $f \in L^\infty(\Omega)$ and $p \in (1, \infty)$.
GOAL: Define a proper notion of a weak solution and prove the existence and the uniqueness.
DEADLINE: April 30

Problem 3: Let $\Omega \subset \mathbb{R}^d$ be Lipschitz. Consider the sequences v^n and u^n such that for some $p, q \in (1, \infty)$ there holds

$$\begin{aligned} u^n &\rightharpoonup u \quad \text{weakly in } L^2(0,T; W^{1,p}(\Omega)), \\ v^n &\rightharpoonup v \quad \text{weakly in } L^q(0,T; L^q(\Omega)). \end{aligned}$$

In addition, assume that for all $\varphi \in \mathcal{C}_0^2((0,T) \times \Omega)$ there holds

$$\int_0^T \int_\Omega u^n \partial_t \varphi + v^n \Delta \varphi = 0.$$

GOAL: Show that there exists a subsequence such that

$$u^{n_k} \to u \text{ strongly in } L^1(0,T; L^p(\Omega)).$$

DEADLINE: May 24

Problem 4: Consider the evolutionary parabolic problem

$$\begin{aligned} \partial_t u - \Delta u - \alpha \Delta_p u &= 0 \quad \text{in } (0,T)\times\Omega,\\ (\nabla u + \alpha|\nabla u|^{p-2}\nabla u)\cdot\vec{\nu} + \beta|u|^{q-2}u &= 0 \quad \text{on } (0,T)\times\partial\Omega,\\ u(0) &= u_0 \quad \text{in } \Omega. \end{aligned}$$

with $\Omega\subset\mathbb{R}^d$ Lipschitz, $\alpha,\beta\ge 0$ and $p,q\in(1,2)$ and $\vec{\nu}$ being the outer noraml vector on $\partial\Omega$.

GOAL: For any $u_0\in L^2(\Omega)$ show that there exists unique weak solution. In case that $\alpha,\beta>0$, show that there exists $t_0\in(0,\infty)$ such that the weak solution satisfies $u(t)=0$ for all $t\ge t_0$. (In other words, prove the extinction in finite time).

DEADLINE: May 24

Problem 5: Let Ω be Lipschitz and $\Gamma_1\subset\partial\Omega$ be such that $|\Gamma_1|>0$. Assume that $\vec{g}\in L^3(\partial\Omega;\mathbb{R}^N)$, where $N\in\mathbb{N}$ is given and define the set S as[1]

$$S := \left\{\mathbf{A}\in L^3(\Omega;\mathbb{R}^{d\times N});\ \int_\Omega \mathbf{A}:\nabla\vec{v} = \int_{\partial\Omega\setminus\Gamma_1}\vec{g}\cdot\vec{v} \text{ for all } \vec{v}\in V\right\}$$

and

$$V := \left\{\vec{v}\in W^{1,\frac{3}{2}}(\Omega;\mathbb{R}^N);\ \vec{v}=0 \text{ on } \Gamma_1\right\}.$$

Consider the problem: Find $\mathbf{A}\in S$ such that for all $\mathbf{B}\in S$ there holds

$$\int_\Omega \frac{|\mathbf{A}|^3}{3} + |\mathbf{A}-\mathbf{I}|^2 \le \int_\Omega \frac{|\mathbf{B}|^3}{3} + |\mathbf{B}-\mathbf{I}|^2,$$

where $\mathbf{I}\in\mathbb{R}^{d\times N}$ is given.

GOAL: 1) Show that there exists unique $\mathbf{A}$ solving the problem.
2) Derive the Euler–Lagrange equations.
3) Find the corresponding primary formulation - the corresponding system of PDE's and show the equivalence of primary and dual formulation.

DEADLINE: three days before exam

[1]Here $\mathbf{A}$ denotes the $(d\times N)$-matrix-valued function and " : " is the scalar product in the space of matrices, i.e., for any $\mathbf{A},\mathbf{B}\in\mathbb{R}^{d\times N}$ there holds

$$\mathbf{A}:\mathbf{B} := \sum_{i=1}^{d}\sum_{\nu=1}^{N}(\mathbf{A})_{i,\nu}(\mathbf{B})_{i,\nu}, \qquad |\mathbf{A}|^2 := \mathbf{A}:\mathbf{A}.$$