Criteria for the exam

Partial differential equations 1

Winter semester 2023/2024

The exam will be only written. In some exceptional cases there can be also an oral exam.

The exam consists of three parts. The first part **A** is obligatory in the sense that any mistake or imprecision automatically means that you do **not** pass the exam. The second part **B** will be theoretical - typically the proof of a lemma or a theorem presented during the semester. The last part **C** will be "practical", i.e., you should be able to use your knowledge to solve some problem. In part **C** you **must** provide at least the knowledge of the notion of weak solution otherwise you do **not** pass.

The evaluation will be the following: you can obtain $0 - 100\%$ from each part **A**, **B**, **C**. The final evaluation is the following:

$$E := \begin{cases} 0 & \text{if } A \neq 100 \text{ or } C < 10, \\ \dfrac{B+C}{2} & \text{if } A = 100 \text{ and } C \geq 10, \end{cases}$$

The corresponding mark M is then obtained from the following:

$$M := \begin{cases} 1 & \text{if } E \in (85, 100], \\ 2 & \text{if } E \in (67, 85], \\ 3 & \text{if } E \in (50, 67], \\ 4 & \text{if } E \leq 50. \end{cases}$$

What can be in the exam:

Part A: Precise knowledge of the statements of the following important definitions, lemmas and theorems (precise meaning of all assumptions, logical order of quantifiers, etc.):
weak derivative; Sobolev space $W^{k,p}(\Omega)$ with integer k (definition, norm, reflexivity, separability, completeness, scalar product (for $p = 2$)); local approximation theorem of Sobolev functions by smooth functions; global approximation theorem by $\mathcal{C}^\infty(\overline{\Omega})$ - for domains with continuous boundary; domain of the class $\mathcal{C}^{k,\alpha}$; extension theorem for $W^{1,p}(\Omega)$ on Lipschitz domains; continuous and compact embedding of Sobolev spaces into Lebesgue spaces or into Hölder spaces - for Lipschitz domains; characterization of Sobolev spaces with the help of difference quotients; Trace theorem for Lipschitz domains with range in Lebesgue spaces; linear and nonlinear Lax-Milgram lemma; Bochner integral and Bochner measurability and simple functions; the spaces $L^p(0,T;X)$ (definition, norm, reflexivity, separability, completeness, scalar product (for $p = 2$ and X being a Hilber space)); weak time derivative for Bochner spaces; Gelfand triple; integration by parts theorem for Sobolev-Bochner functions

Part B: Proof of the above mentioned theorems/lemmas (in case they were proven at the lecture) or their application to some theoretical aspects of PDE theory.

Part C: An example:

$$-\Delta u + v \cdot \nabla u = f \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega.$$

Define the notion of a weak solution. Under which assumptions on f and v can you say something about the existence and the uniqueness of a weak solution? (if you use some theorem, formulate it precisely and check all assumptions) Consider also the case when you control div v from below/above/in a suitable norm. Can you provide a "sharp" bound on $\|v\|_\infty$ for which you can get the existence of a solution?

Another example: Let $\Omega := (-1,1)^2$. Define $\Omega_1 := (-1,1) \times (0,1)$ and $\Omega_2 := (-1,1) \times (-1,0)$. Define $a(x) = i$ in Ω_i. Take $u_0 \in \mathcal{C}^\infty(\overline{\Omega})$ and consider the following problem

$$-\text{div } (a(x)\nabla u(x)) = 0 \text{ in } \Omega, \qquad u = u_0 \text{ on } \partial\Omega.$$

Define the notion of weak solution, prove its existence and uniqueness. Is u a minimizer to some variational problem? Can you find a dual formulation? Is $u \in W^{2,2}_{loc}(\Omega_i)$? Is $u \in W^{2,2}_{loc}(\Omega)$? Can you say something about $\nabla u(x_1, 0)$?

Another example: Let $\Omega \subset \mathbb{R}^d$ be a $\mathcal{C}^{0,1}$ domain and $T > 0$. Consider the following problem

$$\left.\begin{aligned} \partial_t u^1 - \Delta u^1 + \frac{\partial^2 u^2}{\partial x_1 \partial x_2} + u^2 &= f^1, \\ \partial_t u^2 - \Delta u^2 - \frac{\partial^2 u^1}{\partial x_1 \partial x_2} + u^1 &= f^2 \end{aligned}\right\} \text{ in } (0,T) \times \Omega, \qquad u^i(t,x) = x + t \text{ on } (0,T) \times \partial\Omega, \qquad u^i(0) = u^i_0.$$

Define the notion of weak solution (also proper function spaces for u_0^i and f^i), prove its existence and uniqueness. Find the optimal (=minimal) assumptions such that the weak solution satisfies $\partial_t u^i \in L^2_{loc}(0,T;L^2(\Omega))$ and/or $\partial_t u^i \in L^2(0,T;L^2(\Omega))$.