5. Three Great Theorems of FA

There is a nice (thick) book on Linear & Nonlinear FA by Phillippe Ciarlet (*) SIAM, 2013). The author has worked in the fields like numerical mathematics, partial diff. equations, mathematics for deformation of elastic solids, calculus of variations. In Ciarlet's book, there is a special chapter called The „Great Theorems" of Linear FA, that are based on two fundamental results: Hahn-Banach theorem & Baire's theorem ***).
Besides H-B theorem, the list of great theorems includes:

- Banach - Steinhaus theorem alias uniform boundedness principle.
- Banach open mapping theorem
- Banach closed graph theorem

Theorem 5.1 (Banach-Steinhaus uniform boundedness principle)

Let X, Y be Banach spaces.
Let $\mathcal{F} \subset \mathcal{L}(X,Y)$ be arbitrary family of bdd linear operators.
Then,

Either $\mathcal{F}$ is uniformly bdd, i.e. $\boxed{\sup_{L\in\mathcal{F}} \|L\|_{\mathcal{L}(X,Y)} < \infty}$

Or There exists a dense**) set $S \subset X$ such that
$$\sup_{L\in\mathcal{F}} \|Lx\| = \infty \quad \forall x \in S$$

Pf For $m \in \mathbb{N}$ consider
$$S_m \overset{df.}{=} \{x \in X;\ \|Lx\|_Y > m \text{ for some } L \in \mathcal{F}\}.$$
Note S_m are open sets.

- If all these sets are dense in X, then their intersection is also dense in X by Baire's theorem B.2.
This proves part $\boxed{\text{OR}}$ as for all $x \in S := \bigcap_{m\in\mathbb{N}} S_m$ ~~and~~ for all $m \in \mathbb{N}$ there is $L \in \mathcal{F}$: $\|Lx\|_Y > m$.
- If at least one of S_m is not dense in X, let us denote such set S_k, then there is a ball $B_r(x_0)$ such that $\overline{B_r(x_0)} \cap S_k = \emptyset$.

**) A subset $S \subset X$ is dense in X if $\overline{S} = X$; i.e. for $\forall x \in X\ \forall \varepsilon > 0\ \exists y \in S\ \|y-x\|_X < \varepsilon$
It holds (verify!):
$$S \text{ is dense in } X \iff \forall\, O \subset X \text{ open}: O \cap S \neq \emptyset$$

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*) SIAM ... publisher ... Society for Industrial and Applied Mathematics

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***) See pages below

Hence, $\forall x \in B_r(x_0)$: $\|Lx\|_Y \le k$ for all $L \in \mathcal{F}$

It implies that $\boxed{\sup_{L\in\mathcal{F}} \|L\| < +\infty}$, which is the part $\boxed{\text{Either}}$

Indeed, if $\|x\|_X \le r$, then

$$\|Lx\|_Y \le \|L(x_0+x)\|_Y + \|Lx_0\|_Y \le 2k$$

and consequently

$$\|L\|_{\mathcal{L}(X,Y)} = \sup_{\|x'\|_X=1} \|Lx'\|_Y = \frac{1}{r} \sup_{\|rx'\|_X = r} \|L(rx')\|_Y \le \frac{2k}{r},$$

$x = rx'$

which implies .

NOTE Theorem 5.1 proves the following statement:

If $\mathcal{F} \subset \mathcal{L}(X,Y)$ is pointwise bounded, then it is uniformly bounded,

or written differently:

If $\boxed{\sup_{L\in\mathcal{F}} \|Lx\|_Y < +\infty}$ $\Longrightarrow$ for each $x \in X : \|x\|_X = 1$

then $\sup_{L\in\mathcal{F}} \|L\|_{\mathcal{L}(X,Y)} = \boxed{\sup_{L\in\mathcal{F}} \sup_{\|x\|_X=1} \|Lx\|_Y < \infty}$

This why Theorem 5.1 is called uniform boundedness principle

Two applications

(1) If $\phi_n \xrightarrow{*} \phi$ $*$-weakly in X^*, then $\sup_n \|\phi_n\|_{X^*} < +\infty$

i.e. $\{\phi_n\}_{n=1}^{\infty}$ is bdd.

Pf By definition of $*$-weak convergence,
$\langle \phi_n, x\rangle = \phi_n(x) \to \phi(x) = \langle \phi, x\rangle$, hence $\sup_{n\in\mathbb{N}} |\phi_n(x)| < \infty$.

By Theorem 5.1, $\sup_{n\in\mathbb{N}} \|\phi_n\|_{Y^*} = \sup_{n\in\mathbb{N}} \sup_{\|x\|=1} |\phi_n(x)| < +\infty$. ▨

Recall that we have already proved that (1) implies:

If $x_n \to x$ weakly in X, then $\sup_{n\in\mathbb{N}} \|x_n\|_X < +\infty$

i.e. $\{x_n\}_{n=1}^{\infty}$ is bdd.

(2)
- Let X, Y be Banach
- Let $\{L_n\}_{n=1}^{\infty} \subset \mathcal{L}(X,Y)$
- Let $\lim_{n\to\infty} L_n x$ exist for all $x \in X$.

Then L defined as $Lx = \lim_{n\to\infty} L_n(x)$ is bounded, linear operator, i.e. $L \in \mathcal{L}(X,Y)$.

"Pointwise limit of bdd lin. operators is bdd lin. operator."

Pf Since $Lx := \lim_{n\to\infty} L_n x$ exists for all x, then by Theorem 5.1.
$\sup_{n\in\mathbb{N}} \|L_n\|_{\mathcal{L}(X,Y)} < \infty$. Hence

$$\|L\|_{\mathcal{L}(X,Y)} = \sup_{\substack{\|x\|=1 \\ X}} \|Lx\|_{\mathcal{L}(X,Y)} = \sup_{\substack{\|x\|=1 \\ X}} \left(\lim \|L_n x\|_{\mathcal{L}(X,Y)}\right)$$
$$\leq \sup_{\|x\|=1} \left(\lim_{n\to\infty} \|L_n\|_{\mathcal{L}(X,Y)} \|x\|_X\right)$$
$$\leq \sup_{n\in\mathbb{N}} \|L_n\|_{\mathcal{L}(X,Y)} < +\infty .$$

Thus L is bdd (and linearity follows from pointwise convergence of linear operators). ☺

Before formulating the Open mapping theorem, we recall the definition of open mapping: this is a mapping that maps open sets on open sets, i.e. $\boxed{\forall\, \mathcal{U} \subset X \text{ open: } F(\mathcal{U}) \subset Y \text{ is open}}$

This condition is the same as: $(\forall B_r(x))\ (\exists B_\delta(f(x)))\ B_\delta(f(x)) \subset F(B_r(x))$

Note $L: \mathbb{R}\to\mathbb{R}$ $Lx = ax$ maps (A,B) onto $(aA; aB)$, which is open interval if $a \neq 0$.

Theorem 5.2 (Open mapping) Let X, Y be Banach spaces.
Let $L \in \mathcal{L}(X,Y)$ be surjective (onto). Then L is open.

Pf **Step 1** As L is linear, we have
$$L(B_r(x)) = Lx + L(B_r(0)) = Lx + rL(B_1(0)).$$
Thus, to prove Open mapping theorem it suffices to show that (as $L0=0$) there exists $B_\rho(0)$ such that $\boxed{B_\rho(0) \subset L(B_1(0))}$

$\boxed{\text{Step 2}}$ Since L is onto, $Y = \bigcup_{n=1}^{\infty} L(B_n(0))$. As Y is complete, by Baire's theorem 5.3 at least one of the closures $\overline{L(B_n(0))}$ has nonempty interior. By rescaling, as $\overline{L(B_1(0))} = \frac{1}{n}\overline{L(B_n(0))}$, $\overline{L(B_1(0))}$ has an unempty interior.

Hence : $$\boxed{\exists\, y_0 \in Y \quad \exists\, r>0 \quad B_r(y_0) \subset \overline{L(B_1(0))}} \quad (*)$$

Since $B_1(0)$ is convex and symmetric (i.e. if $a \in B_1(0)$ then $-a \in B_1(0)$)
then $L(B_1(0))$ is —"—
and $\overline{L(B_1(0))}$ is —"—

In particular, it follows from $(*)$ and symmetry:
$$B_r(-y_0) \subset \overline{L(B_1(0))}$$

Due to convexity:
$$B_r(0) = \frac{1}{2}B_r(y_0) + \frac{1}{2}B_r(-y_0) \subset \overline{L(B_1(0))}$$

By rescaling, we get
$$(**) \quad \boxed{B_{\frac{r}{2^n}}(0) \subset \overline{L\left(B_{\frac{1}{2^n}}(0)\right)} \quad \text{for all } n \in \mathbb{N}}$$

$\boxed{\text{Step 3}}$ We wish to show that $B_{\frac{r}{2}}(0) \subset L(B_1(0))$, see Step 1

Let $y \in B_{\frac{r}{2}}(0)$ be arbitrary. Our aim is to find $x \in B_1(0)$: $y = Lx$.

By $(**)$, for $n=1$, $\exists\, x_1 \in B_{\frac{1}{2}}(0)$: $\|y - Lx_1\| < \frac{r}{2^2}$ (i.e. $y - Lx_1 \in B_{\frac{r}{2^2}}(0)$)

By $(**)$ again, for $n=2$, $\exists\, x_2 \in B_{\frac{1}{2^2}}(0)$: $\|(y - Lx_1) - Lx_2\|_Y < \frac{r}{2^3}$

$\vdots$

an so on, as $y - \sum_{j=1}^{m-1} Lx_j \in B_{\frac{r}{2^m}}(0)$ and $(**)$ holds,
there is $x_m \in B_{\frac{1}{2^m}}(0)$: $\left\|\left(y - \sum_{j=1}^{m-1} Lx_j\right) - Lx_m\right\|_Y < \frac{r}{2^{m+1}}$

Since X is Banach and $\sum_{n=1}^{\infty}\|x_n\| < +\infty$, then $\exists\, x \in X$:

$\boxed{\sum_{n=1}^{\infty} x_n = x}$ Furthermore: $\|x\| \leq \sum_{n=1}^{\infty}\|x_n\|_X < \sum_{n=1}^{\infty}\frac{1}{2^m} = 1$.

Evenmore : $\lim_{n\to\infty}\sum_{j=1}^{m} Lx_j = y$ and also Lx

Hence $y = Lx$ where $x \in B_1(0)$; i.e. $y \in L(B_1(0))$,

Q.E.D.

$\boxed{\text{Theorem 5.3}}$ (An important corollary of Open mapping theorem)

Let X, Y be Banach spaces and $L \in \underset{\boxed{\text{bdd + lin}}}{\mathcal{L}}(X,Y)$ be BIJECTION $\left(\begin{smallmatrix}\text{injective} \\ \& \\ \text{surjective}\end{smallmatrix}\right)$

Then $L^{-1} \in \mathcal{L}(Y, X)$.

[It is remarkable that one gets the continuity (boundedness) of L^{-1} for free!!]

(Pf) L maps X onto Y. Since L is injective, the inverse mapping $\boxed{L^{-1} : Y \xrightarrow{\text{onto}} X \text{ exists}}$ and $L^{-1} \circ L = \text{Identity}\big|_X$. Given $y_1, y_2 \in Y$, there are unique $x_1, x_2 \in X$: $y_1 = Lx_1$ and $y_2 = Lx_2$. Hence for any $\alpha_1, \alpha_2 \in \mathbb{K}$:

$$L^{-1}(\alpha_1 y_1 + \alpha_2 y_2) = L^{-1}(\alpha_1 L x_1 + \alpha_2 L x_2) = L^{-1} L(\alpha_1 x_1 + \alpha_2 x_2)$$
$$= \alpha_1 x_1 + \alpha_2 x_2 = \alpha_1 L^{-1} y_1 + \alpha_2 L^{-1} y_2 .$$

Hence

$\boxed{L^{-1} \text{ is linear}}$

To prove that $\boxed{L^{-1} \text{ is continuous at } 0}$ $\left(\begin{smallmatrix}\text{lin. } L^{-1} \\ \Rightarrow \text{ continuity in all } Y\end{smallmatrix}\right)$

we need to show

$$\forall B_\varepsilon(0) \subset X \ \exists B_\delta(0) \subset Y : \Lambda^{-1}(B_\delta(0)) \subset B_\varepsilon(0).$$

But the last inclusion is equivalent to

$$B_\delta(0) \subset \Lambda(B_\varepsilon(0)),$$

and this is exactly what Banach open mapping theorem 5.2 states (for some $\delta > 0$).

[Theorem 5.4] (sufficient condition for the equivalence of two norms in infinite-dimensional vector space).

Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on the same vector space X with the following properties:

- $(X, \|\cdot\|_1)$ is complete
- $(X, \|\cdot\|_2)$ is complete
- $\exists\, c > 0 \quad \|x\|_2 \leq c_1 \|x\|_1 \quad \forall x \in X$

Then $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.

(Pf) The bijective linear identity mapping $i : (X, \|\cdot\|_1) \to (X, \|\cdot\|_2)$ is continuous (bdd) by the 3rd assumption. By Theorem 5.3, $i^{-1} : (X, \|\cdot\|_2) \to (X, \|\cdot\|_1)$ is linear & bounded. The boundedness implies the $\exists\, C_2 > 0$: $\|x\|_1 \leq C_2 \|x\|_2$ and the equivalence of the norm follows. $\square$

▷ Let X, Y be Banach spaces. Then the product $X \times Y$ is defined through

$$X \times Y := \{ (x,y) ;\ x \in X \ \& \ y \in Y \}$$

Setting $\|(x,y)\|_{X\times Y} := \|x\|_X + \|y\|_Y$, then $(X\times Y, \|\cdot\|_{X\times Y})$ is Banach space, as well. Prove it.

Let L be (possibly unbounded) linear operator with $\mathrm{Dom}\, L \subseteq X$; i.e. $L : \mathrm{Dom}\, L \to Y$. Then graph of L, Graph (L), is defined as

$$\mathrm{Graph}\, L := \{ (x,y) ;\ x \in \mathrm{Dom}\, L \subset X,\ y = Lx \}$$

We say that **L is closed** if its Graph L is closed subset of $X \times Y$.

It means that L is closed $\Leftrightarrow$ $\boxed{\underset{\subset \mathrm{Dom}\, L}{x_n} \to x \text{ in } X \ \& \ Lx_n \to y \text{ in } Y \Rightarrow \underset{x \in \mathrm{Dom}\, L}{y = Lx}}$

$\boxed{\text{Note}}$ Every continuous (bdd) lin. operator $L : X \to Y$ is closed.
$\boxed{\text{VERIFY IT!}}$
The Closed Graph Theorem shows that the converse implication hold if $\mathrm{dom}\, L = X$.

$\boxed{\text{Theorem 5.5}}$ (Closed Graph Theorem) Let X, Y be Banach spaces. Let $L : X \to Y$ be a closed linear operator defined on the whole X. Then L is continuous.

Proof. Define on X yet another norm: $\|x\|_2 := \|x\|_X + \|Lx\|_Y$ $\forall x \in X$.

Then
- $(X, \|\cdot\|_X)$ is complete
- $(X, \|\cdot\|_2)$ is complete (verify it! It is a consequence of the closedness of L)
- $\exists\, C_1 > 0$ (in fact $C_1 = 1$): $\|x\|_X \le C_1 \|x\|_2$

By Theorem 5.4, $\exists\, C_2 > 0$: $\|x\|_2 \le C_2 \|x\|_X$ $\forall x \in X$.
which implies $\|Lx\|_Y \le C_2 \|x\|_X$ and L is bdd. ☺

Another proof. Call $\Gamma := \mathrm{Graph}\, L$. By ~~the~~ assumption, Γ is closed subspace of the Banach space $X \times Y$, hence it is a Banach space as well. Consider the projections $\pi_1 : \Gamma \to X$ and $\pi_2 : \Gamma \to Y$ defined $\pi_1(x, Lx) = x$ and $\pi_2(x, Lx) = Lx$
π_1 is bijection between Γ and X, by Theorem 5.3, π_1^{-1} is continuous.
Then $L = \pi_2 \circ \pi_1^{-1}$ is composition of two continuous maps. Hence continuous. ☒

Let us illustrate, by an $\boxed{\text{example}}$, that it is important to assume that $\operatorname{dom} L = X$ in the Closed Graph Theorem.

Consider $X = \underbrace{BC}(\mathbb{R})$ and $\|f\|_{BC} := \max_{x\in\mathbb{R}} |f(x)|$

bounded, continuous functions

Let $L: f \mapsto f'$

We know that L is unbounded operator on X (considering $f_n(x) = \sin nx$) and $\operatorname{Dom} L = \{f \in BC(\mathbb{R});\ f' \in BC(\mathbb{R})\}$.

On the other hand, L has closed graph on $\operatorname{Dom} L$:

If $f_n \to f$ in $BC(\mathbb{R})$ and $\underset{Lf_n}{f_n'} \to g$ in $BC(\mathbb{R})$

then $f' = g$ and $(f,g) \in X \times X$ belongs to Graph L.

- Another viewpoint on the similar case. Consider $X = (C^1([0,1]), \|f\|_\infty)$

Then $L: (X, \|\cdot\|_\infty) \to (C([0,1]), \|\cdot\|_\infty)$ is not continuous, but it is closed. This does not contradict to Closed Graph Theorem as X is not a Banach space. ☒

Application of Theorem 5.3 consider, for $a, b, c \in C([0,1])$,

$$Lu := \boxed{\begin{array}{l} a(x)\, u''(x) + b(x)\, u'(x) + c(x)\, u(x) = f(x), \quad 0 \le x \le 1, \\ u(0) = u(1) = 0 \end{array}} \qquad (P)$$

Assume that (P) has one and only one solution for each $f \in C([0,1])$

Show that there exists $C > 0$:

(◎) $\qquad \|u\|_\infty + \|u'\|_\infty + \|u''\|_\infty \le C\|f\|_\infty \qquad (\forall f)$

Pf Consider $X := C^2([0,1]) \cap C_0([0,1]) = \{\varphi \in C^2([0,1]);\ \varphi(0) = \varphi(1) = 0\}$

with $\|u\|_X := \|u\|_\infty + \|u'\|_\infty + \|u''\|_\infty$.

Then
- $(X, \|\cdot\|_X)$ is complete.
- $L: (X, \|\cdot\|_X) \to (C([0,1]), \|\cdot\|_\infty)$ is a linear bijection by the assumption
- L is also continuous:

$\|Lv\|_\infty \le \max(\|a\|_\infty + \|b\|_\infty, \|c\|_\infty)\, \|u\|_X$

by Theorem 5.3

Hence L^{-1} is continuous ; $\boxed{L^{-1}: f \mapsto u = L^{-1} f}$,

which means (◎). ☒

Note that (◎) means a continuous dependence on f.