MATEMATICKÉ METODY V MECHANICE NEWTONSKÝCH TEKUTIN

Matematické metody v mechanice nenewtonských tekutin

2|0 Zk

Zk { 50% DÚ
50% ÚSTNÍ POHOVOR

Prerekvisity
- PDR I, PDR II
- Mechanika & termodynamika nenewtonských tekutin

L1 Úvod

1) Rovnice a úlohy
2) NSR vs rovnice pro proudění nenewtonských tekutin
3) Cíle

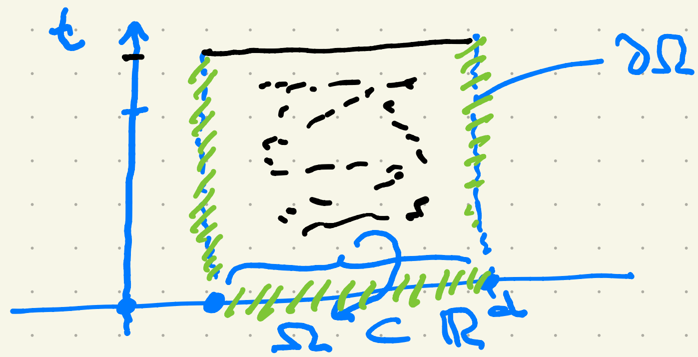
① Rovnice vs úlohy

- Bilanční rce
- Konstitutivní rce

} SYSTÉM NELINEÁRNÍCH PDR

- POČÁTEČNÍ PODMÍNKY
- OKRAJOVÉ PODMÍNKY
  vesměs představují konstitutivní rce na hranici

IBVP
počáteční & okrajová úloha



OTEVŘENÉ

- vtoky / výtoky
- $\theta = \theta_{given}$ na $(0,T)\times\partial\Omega$

těžší

VNITŘNÍ PROUDĚNÍ

UZAVŘENÉ

$\vec{v}\cdot\vec{n} = 0 \quad (0,T)\times\partial\Omega$

$\vec{q}\cdot\vec{n} = 0$

lehčí

kontrola celkové energie

$$E = \frac{1}{2}|v|^2 + e$$

$$\int_\Omega \rho E(t,x)\,dx = \int_\Omega \rho_0 E_0(x)$$

$E_0 = \frac{1}{2}|v_0|^2 + e_0$

$\boxed{\text{ROUNICE}}$ – **Bilanční**

$$\left.\begin{array}{l} \text{hmoty} \\ \text{hybnosti} \\ \text{energie} \end{array}\right.$$

$$\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \vec{v}) = 0 \qquad \vec{m} \qquad \nabla\varrho\cdot\vec{v} + \varrho \operatorname{div}\vec{v}$$

$$\frac{\partial(\varrho\vec{v})}{\partial t} + \operatorname{div}(\varrho\vec{v}\otimes\vec{v}) = \operatorname{div}\mathbb{T} + \varrho\vec{b}$$

$$\frac{\partial(\varrho E)}{\partial t} + \operatorname{div}(\varrho E\vec{v}) - \operatorname{div}\vec{j} = \operatorname{div}(\mathbb{T}\vec{v}) + \varrho\vec{b}\cdot\vec{v}$$

$E := \frac{1}{2}|v|^2 + e$

– ...

2. zákon $\qquad \dfrac{\partial(\varrho \eta)}{\partial t} + \ldots$

$\vec{v} = (v_1, v_2, v_3)$ rychlost

ϱ hustota

e vnitřní energie $\qquad$ (θ teplota)

$$\vec{a}, \vec{b} \in \mathbb{R} \qquad (\vec{a}\otimes\vec{b})_{ij} = a_i b_j$$

$\mathbb{T}$ Cauchyův tensor

$\vec{j}$ tok energie $\qquad \vec{b}$ dané objemové síly

⊕ Celková energie se zachovává !!

Mechanika vs **Termodynamika**

$\boxed{\varrho, \vec{v}}$ $\quad \vec{z} = (z_1, z_2, z_3, z_4)$

- $(\varrho, \varrho v_1, \varrho v_2, \varrho v_3) =$

$$\mathrm{Div}_{t,x} := \frac{\partial}{\partial t} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i}$$

- Bilance hmoty

$$\boxed{\mathrm{Div}_{t,x}(\varrho, \varrho v_1, \varrho v_2, \varrho v_3) = 0}$$

- Bilance hybnosti $k = 1, 2, 3$

$$\boxed{\mathrm{Div}_{t,x}\begin{pmatrix} \varrho v_k, \ \varrho v_k v_1 + \mathbb{T}_{k1}, \\ \varrho v_k v_2 + \mathbb{T}_{k2}, \\ \varrho v_k v_3 + \mathbb{T}_{k3} \end{pmatrix} = 0}$$

stlačitelné

Nestlačitelné

$$\boxed{\operatorname{div} \vec{v} = 0}$$

$\boxed{\text{homogenní}}$ $\quad \varrho(t,x) = \varrho_* > 0$ (daná)

nehomogenní

BHm $\Rightarrow$ $\dfrac{\partial \varrho}{\partial t} + \nabla \varrho \cdot \vec{v} = 0$

transportní rovnice pro ϱ

$$\underbrace{\mathbb{T}}_{\text{tenzor napětí}} = \underbrace{\phi \mathbb{I}}_{\text{sférické napětí}} + \underbrace{\mathbb{S}}_{\text{konstitutivní část}}$$

ϕ (skalár) neznámá veličina, která vstupuje mezi neznámé veličiny PDR

$(t, x) \longmapsto \phi(t,x)$, $t \in (0,T)$, $x \in \Omega$

často: $\phi = -p$ $\ne$ termodynamický tlak!

často: $\frac{1}{3}\operatorname{tr}\mathbb{T} = \phi$ $\quad (\operatorname{tr}\mathbb{S} = 0)$

průměr normálových napětí

$$(*)\quad \boxed{\begin{aligned} &\operatorname{div} v = 0 \\ &\varrho_*\left(\partial_t \vec{v} + \operatorname{div}(\vec{v}\otimes\vec{v})\right) = \nabla\phi + \operatorname{div} \mathbb{S} + \varrho_* \vec{b} \end{aligned}}\qquad \mathbb{S} = \mathbb{S}^T$$

4 rovnice pro $\vec{v} = (v_1, v_2, v_3)$, ϕ

$\mathbb{S}$ vstupuje do dalších materiálových rovnic, kterým se říká KONSTITUTIVNÍ ROVNICE

Energetická bilance pro $(*)$

$\boxed{(*)_2 \cdot \vec{v}}$

$$|\vec{v}|^2 = \sum_{i=1}^{3} v_i v_i$$

- $\partial_t \vec{v}\cdot\vec{v} = \frac{1}{2}\partial_t|\vec{v}|^2 = \partial_t\left(\frac{|\vec{v}|^2}{2}\right)$

- $\vec{v}\cdot\operatorname{div}(\vec{v}\otimes\vec{v}) \underset{(*)_1}{=} \vec{v}\cdot\sum_{k=1}^{3} v_k \frac{\partial\vec{v}}{\partial x_k} = \sum_{k=1}^{3} v_k \frac{\partial}{\partial x_k}\left(\frac{|v|^2}{2}\right)$

$$\underset{(*)_1}{=} \operatorname{div}\left(\frac{|\vec{v}|^2}{2}\vec{v}\right)$$

- $\nabla\phi\cdot\vec{v} \underset{(*)_1}{=} \operatorname{div}(\phi\vec{v})$

- $\operatorname{div}\mathbb{S}\cdot\vec{v} = \operatorname{div}(\mathbb{S}\vec{v}) + \underbrace{\mathbb{S}\cdot\nabla\vec{v}}_{(\mathbb{S})_{ij}\frac{\partial v_i}{\partial x_j}}$ symetrický

$$= \operatorname{div}(\mathbb{S}\vec{v}) + \mathbb{S}\cdot \mathbb{D}v$$

$$\mathbb{D}\vec{v} := \frac{\nabla v + (\nabla v)^T}{2}$$

- $\varrho_*\vec{b}\cdot\vec{v}$

Celkově

$$(E_{kin})\quad \partial_t\left(\rho_* \frac{|v|^2}{2}\right) + \mathrm{div}\left(\rho_* \frac{|v|^2}{2}\vec v\right) - \mathrm{div}(S\vec v) + S : Dv = \mathrm{div}(\phi \vec v) + \rho_* \vec b\cdot\vec v$$

Vnitřní proudění: $\vec v\cdot\vec n = 0$ na $(0,t)\times\partial\Omega$

$\int_\Omega (E_{kin})\,dx$ + Gaussova věta

$$(E_{glob})\quad \frac{d}{dt}\int_\Omega \rho_*\frac{|\vec v|^2}{2}\,dx + \cancel{\int_{\partial\Omega}\rho_*\frac{|v|^2}{2}\underbrace{(\vec v\cdot\vec n)}_{=0}\,dS} + \int_{\partial\Omega}(-S\vec v)\cdot\vec n\,dS + \int_\Omega S:Dv\,dx = \cancel{\int_{\partial\Omega}\phi\underbrace{\vec v\cdot\vec n}_{=0}\,dS} + \int_\Omega\rho_*\vec b\cdot\vec v\,dx$$

$$-S\vec v\cdot\vec n = -S\cdot(\vec v\otimes\vec n) \overset{\text{sym } S}{=} -S\cdot(\vec n\otimes\vec v) = -S\vec n\cdot\vec v =$$

$\vec z \longrightarrow (\vec z\cdot\vec n)\vec n$ normálový směr

$\vec z \longrightarrow \vec z_\tau := \vec z - (\vec z\cdot\vec n)\vec n \quad\dashrightarrow\quad \vec z_\tau\cdot\vec n = 0$

$$= \left[-(S\vec n)_\tau - (S\vec n\cdot\vec n)\vec n\right]\overset{0}{\cdot}\left[\vec v_\tau + \underbrace{(\vec v\cdot\vec n)}_{=0}\vec n\right] = \underbrace{-(S\vec n)_\tau}_{=\vec s}\cdot\vec v_\tau$$

DÚ Ukažte, že platí

$$(\mathbb{T}\vec n)_\tau = (S\vec n)_\tau \quad !$$

Do pondělí 12.10. 12:00

$$(E)\quad \boxed{\frac{d}{dt}\int_\Omega \rho_* \frac{|\vec v|^2}{2} + \int_\Omega \mathbb{S}\cdot \mathbb{D}v + \int_{\partial\Omega} \vec s\cdot \vec v_\tau \, dS = \int_\Omega \rho \vec b\cdot \vec v}$$

Konstitutivní rovnice

v objemu : spojují $\mathbb{S}$ a $\mathbb{D}v$

na hranici —\|\|— $\vec s$ a $\vec v_\tau$

Příklad **Lineární vztahy**

$\rightarrow$ $\boxed{\mathbb{S} = 2\nu\, \mathbb{D}v}$ Navier–Stokesova tekutina

$\rightarrow$ $\boxed{\vec s = \gamma_* \vec v_\tau}$ Navierův skluz

$$\vec s := -(\mathbb{S}n)_\tau = -(\mathbb{T}n)_\tau$$

NS tekutina
$\Updownarrow$
Newtonská tekutina

$$\boxed{S = 2\nu D \underset{\nu>0}{\iff} D = \frac{1}{2\nu} S}$$

viskozita — tekutost

$$\boxed{G(A,B) := A - 2\nu B}$$

Klasifikace nenewtonovských tekutin

1) $G(S,D) = 0$
2) $G(\overset{*}{S}, \overset{*}{D}, S, D) = 0$
3) $G(\overset{*}{S}, \overset{*}{D}, S, D) - \Delta S = 0$
4) $G(\overset{**}{S}, \overset{**}{D}, \overset{*}{S}, \overset{*}{D}, S, D) = 0$
5) —||— $-\Delta S = 0$

G nelineární spojitá funkce

$\overset{*}{A}$ nějaká objektivní derivace A

Ad 1)

$$G(S,D) = 0$$

implicitně konst. tekutiny

$$S = 2\nu(|D|^2) D \qquad D = \alpha(|S|^2) S$$

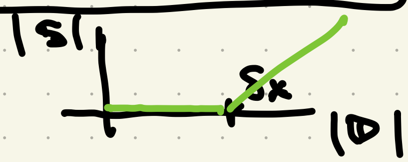
$$\boxed{D = Dv}$$

$$\boxed{\alpha(|S|^2, |D|^2) S = \beta(|S|^2, |D|^2) D}$$

$$S = |D|^{p-2} D, \quad p \in (1,+\infty) \quad \underset{\text{ekvi.}}{\iff} \quad D = |S|^{p'-2} S, \quad p' = p/(p-1)$$

$$S = (1+|D|^2)^{\frac{p-2}{2}} D \quad \longleftrightarrow \quad D = (1+|S|^2)^{\frac{p'-2}{2}} S$$

$$\boxed{\frac{1}{2\nu} S = \frac{(|D| - \delta_*)^+}{|D|} D} \quad \longleftrightarrow \quad \boxed{2\nu D = \frac{(|S| - \tau_*)^+}{|S|} S}$$



mají schopnost popsat jevy typu
- zesílení / zeslabení
- přítomnost aktivačních napětí kritérií.

$\boxed{\text{Ad 2) - 5)}}$ schopnost popsat · normal stress diff.
· napěťovou relaxaci
· nelineární creep.

a také zrychlení/zeslabení smyku.

2) Oldroyd B, Maxwell
4) Burgersův

3) s napěť. relax.
5) — " —

El-Kareh, Leal 1989

Pohled PDR-matematika na mat. teorii pro modely nestlač. tekutin

FYZ — MAT

NSR 1821-1845

Oseen 1921/22
formulace bilančních rovnic v integrálním tvaru

$\underline{C^2} \longrightarrow \underline{C^1}$

<u>Leray</u> 1929 - 1933

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \nu\|\nabla v\|_2^2 = 0$$

$$v \in L^\infty(0,T; L^2(\Omega)^3)$$
$$\nabla v \in L^2(0,T; L^2(\Omega)^{3\times 3})$$

$\exists$ slabého řeš.

· pro Cauchyho úlohu v 3D
· pro okraj. úlohu na ž omez. $\Omega \subset \mathbb{R}^2$ v 2D

· ⊕ ROBUSTNÍ TEORIE
· ⊕ základ num. metod
<u>MKP</u>, MKO, spektrální metody

1949 · Hopf (3D teorie v omez. oblastech)

1959 · Ladyzhenskaya, Kiselev

! ve 2D

· ! ve 3D OPEN

· hladkost ve 3D 2000 OPEN

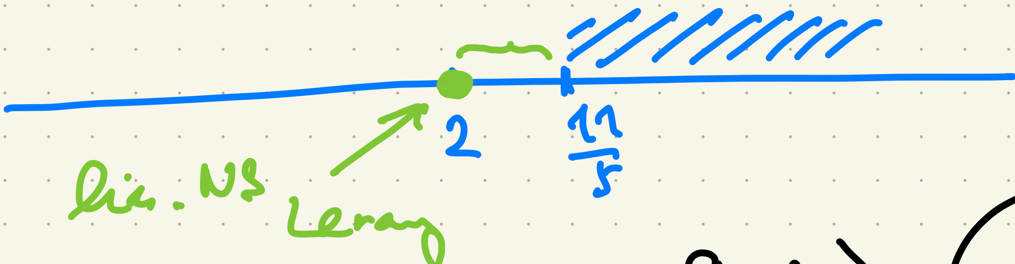
1965-70 $\mathbb{S} = 2(\nu_0 + \nu_1 |Dv|^{p-2}) Dv$

redukuje na NSR $p=2$ nebo $\nu_1 = 0$

$\exists!$ slabého řeš. $p \geq \frac{11}{5}$ $\quad p-2 = \frac{1}{5}$ (J.-L. Lions)

<u>1970</u> * Ladyzh. $p \geq \frac{5}{2}$

<u>2019</u> * Bulíček, Kaplický, Pražák $p \geq \frac{11}{5}$



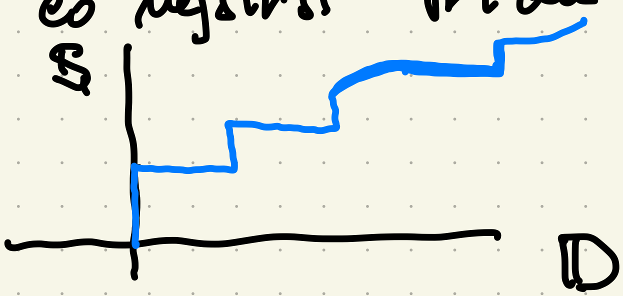
Aplikace (shear-thinning fluids) $\quad p < 2$?

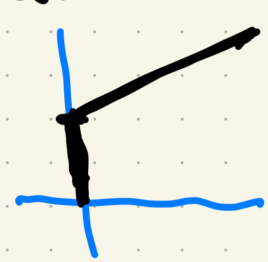
1988 Nečas & spol. <u>Téma.</u> 1985

Cíl: teorii à la Leray a) pro co nejkratší interval p

b) pro co nejširší třídu tekutin

$G(S, D) = 0$





Pokud $G(S, D) = 0$ generuje
maximální monotónní graf splňující
$$S : D \geq c_1 \left(|S|^{p'} + |D|^p \right) - c_2$$
$p \in (1, +\infty)$

$p > \frac{6}{5}$ ve 3D $\Rightarrow$ $\exists$ slabé řeš. à la Leray

$\boxed{\text{Bulíček, Málek, Rajagopal . SIAM J. Math. Anal. 2000}}$

- $p \in \langle 1, \frac{6}{5} \rangle$ obecnější pojem disipativního řeš.
  existuje $\forall$ data
  a je jediné ve třídě silných řeš.

Abbatiello, Feireisl
2000

- Burczak, Modena, Székelyhidi – ArXiv 2020
  Leray-Hopf řeš. je nejednoznačné pro $1 \leq p < \frac{6}{5}$

Teorie à la Leray pro modely 2) – 5) ve 3D

- 2000 Lions (PL), Masmoudi speciální model Oldroydova typu s korotační derivací
- 2011 Masmoudi Gieseku<u>s</u>ův a Fene-P model

až "doposud" žádný výsledek pro modely s napěť. relaxací

výsledek ‖ Bathory, Bulíček, Málek 2021 Adv. in Nonl. Analysis.

MINULE

→ Nestlačitelné hom. tekutiny

$$\frac{1}{2}\frac{d}{dt}\|v\|_2^2 + \int_\Omega \mathbb{S}:\mathbb{D} + \int_{\partial\Omega} s\cdot v_\tau \, dS = 0$$

→ $\dot{\varrho} = -\varrho \operatorname{div}\vec{v}$ $E := \frac{|v|^2}{2} + e$

$\varrho\dot{v} = \operatorname{div}\mathbb{T}$, $\mathbb{T} = \mathbb{T}^T$

$\varrho\dot{E} = \operatorname{div}(\mathbb{T}v - \vec{j}_e)$ → $\boxed{\varrho\dot{e} = \mathbb{T}:\mathbb{D} - \operatorname{div}\vec{j}_e}$

$\varrho\dot{\eta} - \operatorname{div}\vec{j}_\eta =: \zeta$ a $\zeta \geq 0$

Isotermální proces : θ konst.

$$\boxed{\psi = e - \theta\eta}$$

$$\varrho\dot{\psi} = \varrho\dot{e} - \theta\varrho\dot{\eta} = \mathbb{T}:\mathbb{D} - \operatorname{div}\vec{j}_e - \theta\zeta + \operatorname{div}(\theta\vec{j}_\eta)$$
$$= \mathbb{T}:\mathbb{D} - \theta\zeta + \underbrace{\operatorname{div}(\theta\vec{j}_\eta - \vec{j}_e)}_{=0}$$

$$\boxed{\vec{j}_\eta = \frac{\vec{j}_e}{\theta}}$$

stress power

$$\boxed{\mathbb{T}:\mathbb{D} - \varrho\dot{\psi} = \xi}$$

rychlost disipace $\boxed{\xi := \theta\zeta}$

redukovaná termodynamická identita

KT : $\psi = \psi(\varrho)$ ⟹ složk. WSR + nestl. $\boxed{\operatorname{div} v = 0}$

ϱ konst. ⟹ $\dot{\psi} = 0$

$$\boxed{\text{RT1}} \implies \boxed{\mathbb{S}:\mathbb{D} = \xi}$$

Přednáška 3

Mat. teorie

$$(P)\quad \begin{cases} \dfrac{\partial v}{\partial t} + \operatorname{div}(v\otimes v) - \operatorname{div} \mathbb{S} = -\nabla p \\ \operatorname{div} v = 0 \end{cases} \quad \text{in } (0,T)\times\Omega$$

$$\mathbb{S} = \mathbb{S}^T \qquad G(\mathbb{S}, \mathbb{D}) = 0$$

$$\vec{v}\cdot\vec{n} = 0 \quad a \quad \boxed{\theta v_\tau + (1-\theta)\gamma_*(\mathbb{S}n)_\tau = 0} \quad \text{na } (0,T)\times\partial\Omega$$

$$\Downarrow \quad \boxed{(\mathbb{S}n)_\tau = \frac{\theta}{(1-\theta)\gamma_*} v_\tau} \quad \theta \in [0,1)$$

$$v(0,\cdot) = v_0 \quad \text{in } \Omega$$

$$(E)\quad \frac{1}{2}\frac{d}{dt}\int_\Omega |v|^2 + \int_\Omega \mathbb{S}:\mathbb{D}v + \int_{\partial\Omega} (-\mathbb{S}n)_\tau \cdot v_\tau \, dS = 0$$

$\Downarrow$ NS
Navier-slip

$$(E*)\quad \boxed{\frac{1}{2}\int_\Omega |v(t)|^2 + \nu\int_0^t\int_\Omega |\mathbb{D}v|^2 + \frac{\theta}{(1-\theta)\gamma_*}\int_0^t\int_{\partial\Omega} |v_\tau|^2 dS = \frac{1}{2}\|v_0\|_2^2}$$

$$\boxed{\nabla v \in L^2} \qquad L^2(0,T; L^2(\Omega)) \quad L^2(Q_T)$$

$$\boxed{v_0 \in L^2,\ \underset{\in L^2}{\operatorname{div} v_0} = 0 \ \ a \ \ v_0\cdot n = 0 \text{ na } \partial\Omega \ \ \in (W^{1/2,2}(\partial\Omega))^*}$$

[Constantin, Foias – NSEs, 1988
sekce 2,3,5,6,7,8 teorie pro NSEs]

Prostory funkcí

* $\boxed{\text{No-slip}}$ $C_0^\infty = \{v : \Omega \to \mathbb{R}^3,\ v \text{ hladké},\ \operatorname{supp} v \subset \Omega\}$

$C_{0,div}^\infty = \{v \in C_0^\infty;\ \operatorname{div} v = 0 \ v\ \Omega\}$

$\Omega \subset \mathbb{R}^3$ omezená, otevř., souvislé

$W_0^{1,p} := \overline{C_0^\infty}^{\|\cdot\|_{1,p}}$

$W_{0,div}^{1,p} := \overline{C_{0,div}^\infty}^{\|\cdot\|_{1,p}}$

$L_{n,div}^p := \overline{C_{0,div}^\infty}^{\|\cdot\|_p}$

Ω je lipschitz.

$W_0^{1,p} = \{v \in \{W^{1,p}(\Omega)\}^3;\ v = 0 \text{ na } \partial\Omega\}$

$W_{0,div}^{1,p} = \{v \in W_0^{1,p};\ \operatorname{div} v = 0 \text{ na } \Omega\}$

* $\boxed{\text{Skluzové podmínky}}$ Ω předpokládáme: ot., om., souvis. + Lipschitz hranici

$\Downarrow$

$n : \partial\Omega \to \mathbb{R}^3$ je definován s.v. na $\partial\Omega$.

$W_n^{3,2} := \{v \in (W^{3,2}(\Omega))^3;\ \underline{v} \cdot \underline{n} = 0 \text{ na } \partial\Omega\}$

$W_{n,div}^{3,2} = \{v \in W_n^{3,2};\ \operatorname{div} v = 0\}$

$W_{n,div}^{1,p} := \overline{W_{n,div}^{3,2}}^{\|\cdot\|_{1,p}}$

$W_n^{3,2} \cap L_{n,div}^2$

$\boxed{\text{V obou situacích platí Poincarého} \leq}$

$\boxed{\text{* No-slip}}$ $p \in (1,+\infty)$:

$$\underline{\|Dv\|_p \leq \|\nabla v\|_p} \leq \|v\|_{1,p} \underset{\uparrow \text{ Poincaré}}{\leq} c_p \|\nabla v\|_p \underset{\uparrow \text{ Korn}}{\leq} c_p c_K \|Dv\|_p$$

$\boxed{\text{* Okrajové podmínky}}$

Poincaré $\leq$ platí, ale stačí $\vec{v}\cdot\vec{n} = 0$ na $\partial\Omega$

$$\|\nabla v\|_p \leq \|v\|_{1,p} \leq c_p \|\nabla v\|_p$$

$\boxed{\text{Korn} \leq}$ Rozlišit evol. vs stacionární problém

- lze využít $\cdot \int_{\partial\Omega} |v|^2 dS < C$

$$\|\nabla v\|_p \leq C_K \left\{ \|Dv\|_p + \|v\|_{2,\partial\Omega} \right\}$$

- Pokud Ω není axisymetrická, $1 < p < +\infty$

$$\|\nabla v\|_p \leq C_p \|Dv\|_p$$

Ω je axisymetrická $\equiv$ $\exists\, v \in W^{1,\infty}_n$ tak, že $Dv = 0$ a $\nabla v \neq 0$

$x_0 \in \mathbb{R}^3$

$v(x) = Q(x - x_0)$

$\boxed{\text{cylinder}}$

Platí
$$\|\nabla v\|_{p,\Omega} \leq C_K \left\{ \|Dv\|_{p,\Omega} + \|v\|_{1,\Omega} \right\}$$

Rychlo-kurs k teorii NSR — Lerayův program

$(P)_{NS} := (P)$ kde $G(\mathbb{S},\mathbb{D})$ je nahrazena pomocí $\boxed{\mathbb{S} = 2\nu\,\mathbb{D}v}$ $\left[\nu = \frac{\mu_*}{\varrho_*} > 0\right]$

$$(\operatorname{div}\mathbb{S})_i = \nu\frac{\partial}{\partial x_j}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) = \nu\Delta v_i + \nu\frac{\partial}{\partial x_i}\operatorname{div} v = \nu\Delta v_i$$

$$\boxed{\text{NSR}\qquad \frac{\partial v}{\partial t} + \operatorname{div}(v\otimes v) - \nu\Delta v = -\nabla p}$$

NEVHODNÝ POČ. TVAR RCIC PRO SLIPOVÉ PODMÍNKY

Lépe

$$\frac{\partial v}{\partial t} + \operatorname{div}_j(v\otimes v) - \operatorname{div}_j(\underbrace{2\nu\mathbb{D}v}_{=\mathbb{S}}) = -\nabla p \qquad \Big|\cdot\varphi + \int_\Omega dx$$

NO SLIP $\varphi = 0$ MOHU NEMUSÍM $\operatorname{div}\varphi = 0$

SLIP $\varphi\cdot n = 0$

$$\int_\Omega \frac{\partial v}{\partial t}\cdot\varphi - \int_\Omega (v\otimes v)\cdot\nabla\varphi\,dx + \cancel{\int_{\partial\Omega} v_i \underbrace{v_j n_j}_{\vec{v}\cdot\vec{n}} \varphi_i\,dS}$$

$$+ \int_\Omega 2\nu\mathbb{D}v\cdot\nabla\varphi - \int_{\partial\Omega} S_{ij}n_j\varphi_i = \cancel{\int_\Omega p\operatorname{div}\varphi\,dx} - \cancel{\int_{\partial\Omega} p\underbrace{\varphi_i n_i}_{\vec{\varphi}\cdot\vec{n}}\,dS}$$

$= 0$ pokud no-slip neboť $\varphi = 0$ na $\partial\Omega$

$= (\mathbb{S}n)_\tau\cdot\varphi_\tau$ pokud slip neboť $\varphi\cdot n = 0$ na $\partial\Omega$

$= \frac{-\theta}{(1-\theta)\gamma_*}\,v_\tau\cdot\varphi_\tau$

$$\int_{\partial\Omega}\frac{\theta}{(1-\theta)\gamma_*}\,v_\tau\cdot\varphi_\tau\,dS$$

$$(E)_{NS}\left[\ \operatorname*{esssup}_{t}\frac{\|v(t)\|_2^2}{2}+2\nu\int_0^t\int_\Omega|Dv|^2+\frac{\theta}{(1-\theta)\gamma_*}\int_0^t\int_{\partial\Omega}|v_\tau|^2\leq\frac{\|v_0\|_2^2}{2}\right.$$

$$(WF)\quad -\int_0^t\int_\Omega v\cdot\frac{\partial\varphi}{\partial t}\,dx+\int_0^t\int_\Omega(v\otimes v)\cdot\nabla\varphi\qquad \boxed{\varphi(T,\cdot)=0}$$

$$+\int_0^t\int_\Omega 2\nu\, Dv\cdot D\varphi+\int_0^t\int_{\partial\Omega}c(\theta,\gamma_*)\,v_\tau\cdot\varphi_\tau\,dS$$

$$=\int_\Omega \underbrace{v(0,\cdot)}_{v_0(x)}\cdot\varphi(0,\cdot)\,dx$$

Def. slabé řešení $(P)_{NS}$

Řekneme, že v je slabé řeš. $(P)_{NS}$ pokud

- $v\in L^\infty(0,T;L^2)\cap L^2(0,T;W^{1,2})$ $\to$ $W^{1,2}_{0,div}\big|_{v=0}$, $W^{1,2}_{n,div}\big|_{v\cdot n=0}$
- WF platí $\forall\varphi\in C^\infty$ $\to$ $C^\infty_{0,div}$, $C^\infty_{n,div}$
- platí $(E)_{NS}$
- $\lim_{t\to0+}\|v(t,\cdot)-v_0\|_2=0$
- v je <u>nějak</u> spojité v čase v prostoru L^2

test

Def. slabé řešení $(P)_{NS}$

Řekneme, že v je slabé ř.š. $(P)_{NS}$

pokud

- $v \in L^\infty(0,T;L^2) \cap L^2(0,T;W^{1,2})$ → $W^{1,2}_{0,div}\,|_{v=0}$; → $W^{1,2}_{n,div}\,|_{v\cdot n=0}$
- WF platí $\forall \varphi \in C^\infty$ → $C^\infty_{0,div}$; → $C^\infty_{n,div}$
- platí $(E)_{NS}$
- $\lim_{t\to 0+} \|v(t,\cdot) - v_0\|_2 = 0$
- v je <u>nějak</u> spojité v čase v prostoru L^2

L4 Leray's theory for NSE (Leray–Hopf)

$v=(v_1,\dots,v_d)$, p

$$(P)\quad \begin{cases} \operatorname{div} v = 0 & \\ \dfrac{\partial v}{\partial t} + v_k \dfrac{\partial v}{\partial x_k} - \nu \Delta v = -\nabla p & \text{in } \underbrace{(0,T)\times\Omega}_{Q_T},\ \Omega\subset\mathbb{R}^d,\ d\geq 2 \\ v = 0 & \text{on } (0,T)\times\partial\Omega \\ v(0,\cdot) = v_0 & \text{in } \Omega \end{cases}$$

$W_0^{1,2} = \text{closure } \overline{C_0^\infty}^{\|\cdot\|_{1,2}}$ | $W^{1,2}_{0,div}$ | $L^2_{n,div} = \overline{C^\infty_{0,div}}^{\|\cdot\|_2}$

Definition (Let $v_0 \in L^2_{n,div}$) We say that v is weak solution to (P) if

- $v \in L^\infty(0,T;L^2_{n,div}(\Omega)) \cap L^2(0,T;W^{1,2}_{0,div}) \cap C([0,T];L^2_{weak}(\Omega))$
- $\dfrac{\partial v}{\partial t} \in L^\alpha(0,T;(W^{1,2}_{0,div})^*)$ for $\alpha > 1$, $\alpha = \frac{4}{3}$ if $d=3$, 2 if $d=2$
- $\left\langle \dfrac{\partial v}{\partial t}, \varphi \right\rangle_{(W^{1,2}_{0,div})^*} - (v\otimes v, \nabla\varphi) + \nu(\nabla v,\nabla\varphi) = 0 \quad \forall \varphi \in W^{1,2}_{0,div}$ for a.a. $t\in[0,T]$.
- $\frac{1}{2}\|v(t)\|_2^2 + \nu\int_0^t \|\nabla v\|_2^2\,ds = \frac{1}{2}\|v_0\|_2^2$
- $\lim_{t\to 0+} \|v(t,\cdot) - v_0\|_2^2 = 0$

Goal: to show existence of weak solution

Notes: In 2D:
- $\frac{\partial v}{\partial t} \in L^2(0,T;(W^{1,2}_{0,div})^*)$
- v is admissible test function in weak form
- weak sol. is unique

In $d\geq 3$: uniqueness open

- PDE theory of weak sol. - done in 2 steps
  1) stability of PDE or its weak formulation w.r.t. weakly converging subsequences
  and the convergence take place in function spaces in which we have apriori estimates
  2) complete proof starting from some approximations, suitable (close) showing the existence of solutions for these approximations and passing to the limit from approximations to original problem.

Ad 1) for 3D NSEs Let $\{v^\varepsilon\}_{\varepsilon>0}$ be weak solution of our problem (P).

In particular, (●) $\frac{1}{2}\|v^\varepsilon(t)\|_2^2 + \nu_* \int_0^t \|\nabla v^\varepsilon(t)\|_2^2 \leq \frac{1}{2}\|v_0^\varepsilon\|_2^2$ a.a. $\forall\, t \in [0,T]$

(*) $\left\langle \frac{\partial v^\varepsilon}{\partial t}, \varphi \right\rangle - (v^\varepsilon \otimes v^\varepsilon, \nabla\varphi) + \nu_*(\nabla v^\varepsilon, \nabla\varphi) = 0$ $\quad \forall \varphi \in W^{1,2}_{0,div}$

$v^\varepsilon(0,\cdot) = v_0^\varepsilon$ in Ω

$\|v_0^\varepsilon - v_0\|_2 \to 0$, $\varepsilon \to 0$

$\{v^\varepsilon\}$ is bdd in $L^\infty(0,T;L^2) \cap L^2(0,T;W^{1,2}_{0,div})$

since $\sup_{t\in[0,T]} \|v^\varepsilon(t)\|_2^2 \leq \|v_0\|_2^2$

$\nu_* \int_0^T \|\nabla v^\varepsilon\|_2^2 \leq \|v_0\|_2^2$

there is $v \in L^\infty(0,T;L^2) \cap L^2(0,T;W_0^{1,2})$

$v^\varepsilon \to v$ *-weakly $L^\infty(0,T;L^2)$

weakly $L^2(0,T;W^{1,2}_{0,div})$

Q: Is v weak solution to (P)

Steps

1) $\left\{\frac{\partial v^\varepsilon}{\partial t}\right\}$ is bounded in $L^{4/3}(0,T,(W^{1,2}_{0,div})^*)$

2) $v^\varepsilon \to v$ strongly in $L^2(0,T;L^2(\Omega))$ which suffices to take the limit in (*).

3) $v \in C_{weak}([0,T], L^2_{n,div})$

4) v fulfills the energy ineq. (●)

5) $\lim \|v(t) - v_0\|_2^2$

$\boxed{\text{Ad 1)}}$ From weak formulation, for smooth function in time, we conclude

$$\int_0^T \left\langle \frac{\partial v^\varepsilon}{\partial t}, \psi \right\rangle = \underbrace{\int_0^T (v^\varepsilon \otimes v^\varepsilon, \nabla\psi)} - \nu_* \int_0^T (\underbrace{\nabla v^\varepsilon}_{\in L^2}, \nabla\psi)$$

$\forall\, \psi \in C^\infty(0,T; W^{1,2}_{0,\mathrm{div}})$ L^2

$$\boxed{\left\| \frac{\partial v^\varepsilon}{\partial t} \right\|_{X^*} := \sup_{\|\psi\|_X \le 1} \left| \left\langle \frac{\partial v^\varepsilon}{\partial t}, \psi \right\rangle_{X^*, X} \right|}$$

$$\le \nu_* \underbrace{\left(\int_0^T \|\nabla v^\varepsilon\|_2^2\right)^{1/2}}_{< \frac{1}{2}\|v_0\|_2^2} \underbrace{\left(\int_0^T \|\nabla\psi\|_2^2\right)^{1/2}}_{\le 1}$$

$$\int_0^T \underbrace{\|v^\varepsilon\|_4^2}\, \|\nabla\psi\|_2 \le \int_0^T \|\nabla v^\varepsilon\|_2^2\, \|\nabla\psi\|_2$$

$W^{1,2} \hookrightarrow L^6$ Imbedding, Poincaré

$\boxed{d=3}$

$\boxed{d=4}$ Homework: $\boxed{\text{① Do derivative of } \frac{\partial v^\varepsilon}{\partial t} \text{ if } d=4}$ ③

② Show that $v^\varepsilon \to v$ strongly in L^2

Better via interpolation: by Hölder ineq.

$$\|z\|_q \le \|z\|_{p_1}^{\lambda} \|z\|_{p_2}^{1-\lambda} \qquad 1 \le p_1 \le q \le p_2 \le +\infty$$

$$\lambda \in [0,1] \qquad \boxed{\frac{1}{q} = \frac{\lambda}{p_1} + \frac{1-\lambda}{p_2}}$$

$$\boxed{\|v^\varepsilon\|_4 \le \|v^\varepsilon\|_2^{1/4} \|v^\varepsilon\|_6^{3/4}}$$

$$\frac{1}{4} = \frac{\lambda}{2} + \frac{1-\lambda}{6}$$

$$3 = 6\lambda + 2 - 2\lambda$$

$$\lambda = 1/4 \qquad 1-\lambda = \frac{3}{4}$$

$$\int_0^T \|v^\varepsilon\|_4^2 \|\nabla\psi\|_2 \le \int_0^T \|v^\varepsilon\|_2^{1/2} \|v^\varepsilon\|_6^{3/2} \|\nabla\psi\|_2 \le$$

Hölder in time

$$\le C \sup_t \|v^\varepsilon(t)\|_2^{1/2} \int_0^T \|\nabla v^\varepsilon\|_2^{3/2} \|\nabla\psi\|_2 \le$$

$$\delta = \frac{4}{3} \qquad \delta' = 4$$

$$\le \underbrace{C(\|v_0\|_2)^{1/2}} \Big(\underbrace{\int_0^T \|\nabla v^\varepsilon\|_2^2}_{\le \|v_0\|_2^2}\Big)^{3/4} \Big(\underbrace{\int_0^T \|\nabla\psi\|_2^4}_{\le 1}\Big)^{1/4} < +\infty$$

$\left\{ \frac{\partial v^\varepsilon}{\partial t} \right\}_\varepsilon$ bdd in $L^{4/3}(0,T;(W^{1,2}_{0,div})^*)$

$\boxed{\text{Ad 2)}}$

$$X_0 \hookrightarrow\hookrightarrow X_1 \hookrightarrow X_2$$

$$W^{1,2}_{0,div} \hookrightarrow\hookrightarrow L^2_{n,div} = (L^2_{n,div})^* \hookrightarrow (W^{1,2}_{0,div})^*$$

$p \geq 1$

$$\left\{ u \in L^2(0,T;W^{1,2}_{0,div});\ \frac{\partial u}{\partial t} \in L^p(0,T;(W^{1,2}_{0,div})^*) \right\} \hookrightarrow\hookrightarrow L^2(0,T;L^2_{n,div})$$

Having $v^\varepsilon \rightharpoonup v$ weakly in $L^2(0,T;W^{1,2}_{0,div})$

$$\frac{\partial v^\varepsilon}{\partial t} \rightharpoonup \frac{\partial v}{\partial t} \quad \text{— " —} \quad L^{4/3}(0,T;(W^{1,2}_{0,div})^*)$$

$\Longrightarrow$ $v^\varepsilon \to v$ STRONGLY in $L^2(0,T;L^2_{n,div})$

How part 3 — Show that if $d=3,4$

$$\int_0^T \int_\Omega v^\varepsilon \otimes v^\varepsilon : \nabla\psi \longrightarrow \int_0^T \int_\Omega v \otimes v : \nabla \psi$$

$\boxed{\text{Ad 3)}}$ $v \in C_{weak}([0,T;L^2_{n,div})$

Take weak form for v^ε write in the following way:
and integrate per partes w.r.t. time

$\psi \in C^\infty(0,T;W^{1,2}_{n,div})$, $\psi(T,\cdot)=0$

$$-\int_0^T \left(v^\varepsilon, \frac{\partial\psi}{\partial t}\right) - \int_0^T (v^\varepsilon \otimes v^\varepsilon, \nabla\psi) + \nu_* \int_0^T (\nabla v^\varepsilon, \nabla\psi) = (v_0^\varepsilon, \psi(0,\cdot))$$

$\varepsilon \to 0$ $\downarrow$

$$-\int_0^T \left(v, \frac{\partial\psi}{\partial t}\right) - \int_0^T (v \otimes v, \nabla\psi) + \nu_* \int_0^T (\nabla v, \nabla\psi) = (v_0, \psi(0,\cdot))$$

$$\lim_{t \to t_0} \underbrace{(v(t,\cdot) - v(t_0,\cdot), \varphi)}_{} = 0 \quad \forall \varphi \in L^2_{n,div}$$

$$\stackrel{def}{=} \quad v \in C_{weak}([0,T], L^2_{n,div})$$

v satisfies W.F. for $\psi \in L^4(0,T; W^{1,2}_{0,\mathrm{div}})$

Taking $\psi(t,x) = \varphi(x)\chi_{[t_0,t]}$ $\qquad \underline{\varphi \in W^{1,2}_{0,\mathrm{div}}}$

$$t > t_0 \qquad \int_0^T \left\langle \frac{\partial v}{\partial t}, \psi(t) \right\rangle dt = \int_{t_0}^{t} \left\langle \frac{\partial v}{\partial t}, \varphi \right\rangle dt = \big(v(t,\cdot) - v(t_0,\cdot), \varphi\big)$$

$\|$ weak formulation

$$\underbrace{-\nu \int_{t_0}^{t} (\nabla v, \nabla \varphi)} + \int_0^t \left(\underline{v \otimes v}_2, \underline{\nabla \varphi}_2 \right)$$

$$|v^\varepsilon|^4$$

$$\leq \int_{t_0}^{t} \|\nabla v\|_2 \cdot \|\nabla \varphi\|_2 \, 1 \leq \|\nabla\varphi\|_2 \underbrace{\left(\int_{t_0}^{t} \|\nabla v\|_2^2 \right)^{1/2}}_{< C} |t - t_0|^{1/2} \xrightarrow[t \to t_0]{} 0$$

$$\Rightarrow \lim_{t \to t_0} \big(\underbrace{v(t) - v(t_0,\cdot)}_{L^\infty(0,T;}, \varphi\big)_2 = 0 \qquad \underline{\forall \varphi \in W^{1,2}_{0,\mathrm{div}}}$$

$\forall \varphi \in L^2_{n,\mathrm{div}}$ by density