MORE ON LINEAR OPERATORS

Recall: Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be two normed spaces over the same $\mathbb{K}$.

We say

- $f: X \to Y$ is continuous $\overset{\text{df.}}{\equiv}$ $x_n \xrightarrow[n\to\infty]{} x$ in $X \Rightarrow f(x_n) \xrightarrow[n\to\infty]{} f(x)$ in Y

- $L : \mathrm{Dom}\, L \subset X \to Y$ is linear $\overset{\text{df.}}{\equiv}$ $\left.\begin{array}{l} L(x+y) = Lx + Ly \\ L(\alpha x) = \alpha L x \end{array}\right\}$ $\forall x, y \in \mathrm{Dom}\, L$, $\forall \alpha \in \mathbb{K}$

- Domain of L
- $\mathrm{Ker}\, L = \mathrm{Null}(L)$
- $\mathcal{R}(L) = \mathrm{Range}\, L = \mathrm{Im}\, L$

In finite-dimensional X, each linear map is continuous automatically (see also below). In infinite-dimensional X's, this is not anymore true. There is however a nice simple concept, namely that of boundedness, that characterizes the continuity of linear operators.

Def. Let $L : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ be linear, i.e. $\mathrm{Dom}\, L = X$. $(\dagger)$

We say

- L is bounded $\equiv$ $\|L\| := \sup\limits_{\|x\|_X \le 1} \|Lx\|_Y < +\infty$ (B)

Example: $Lx = ax$, $a \ne 0 \Rightarrow \|L\| = |a|$

We observe:

i) $L(0) = 0$ (take $\alpha = 0$ in the second condition of linearity)

ii) $\|Lx\|_Y \le \|L\| \, \|x\|_X$ $\forall x \in X$

iii) If $\exists M > 0$ s.t. $\boxed{\|Lx\|_Y \le M \|x\|_X \quad \forall x \in X}$ $(*)$, then L is bounded

Also, $\|L\| = \inf M$ for which $(*)$ holds

Theorem 1.1. For L satisfying $(\dagger)$: L is continuous (in X) $\Leftrightarrow$ L is bounded

Pf $\boxed{\Rightarrow}$ Since L is continuous in X, it is continuous at 0 (in particular).

For $\varepsilon = 1$ there is $\delta > 0$: $\|x\|_X \le \delta \Rightarrow \|L(x)\|_Y \le 1$.

Hence for $y \in X$, $\|y\|_X \le 1$: $\|\delta y\|_X \le \delta$ $\Rightarrow \|L(y)\|_Y \, \delta = \|L(\delta y)\|_Y \le 1$

$$\Rightarrow \|Ly\|_Y \le \frac{1}{\delta} \quad \text{for all } y : \|y\|_X \le 1$$

i.e. L is bounded.

$\boxed{\Leftarrow}$ If L is bounded, then

$$\|Lx_1 - Lx_2\|_Y \overset{\text{lin.}}{=} \|L(x_1-x_2)\|_Y \overset{\text{lin.}}{=} \left\| L\left(\frac{x_1-x_2}{\|x_1-x_2\|_X}\right) \|x_1-x_2\|_X \right\|_Y$$

$$\overset{\circledast}{\leq} \|L\| \, \|x_1-x_2\|_X$$ where $\|L\| < \infty$. This gives (Lipschitz) continuity. ▨

$\boxed{\text{NOTATION}}$ $\mathcal{L}(X,Y) := \{ L : X \to Y ;$ linear & bounded$\} =: \mathcal{B}(X,Y)$

$\Updownarrow$

linear & continuous

$$\mathcal{L}(X) := \mathcal{L}(X,X)$$

$$X' := \mathcal{L}(X,\mathbb{K}) =: X^*$$

$\boxed{\text{Theorem 1.2.}}$ • $\left(\mathcal{L}(X,Y), \|\cdot\|_{\mathcal{L}(X,Y)}\right)$ where $\|\cdot\|_{\mathcal{L}(X,Y)}$ is given in (B) is normed space (assuming X,Y are normed)

• If, in addition, Y is Banach, then $\left(\mathcal{L}(X,Y), \|\cdot\|_{\mathcal{L}(X,Y)}\right)$ is Banach

Consequently, $\left(X', \|\cdot\|_{X'}\right)$, where $\|L\|_{X'} := \sup_{\|x\|\leq 1} \|Lx\|_{\mathbb{K}}$ is always Banach.

$\boxed{\text{Proof}}$ • $\mathcal{L}(X,Y)$ is a $\boxed{\text{vector space}}$ once we define, for $L_1, L_2 \in \mathcal{L}(X,Y)$: $(L_1+L_2)(x) := L_1x + L_2x$ and $(\alpha L_1)(x) := \alpha L_1(x)$

• $\boxed{\text{Verification of (N1)–(N3)}}$ (N1) If $L = 0$, then $Lx = 0$ for all $x \in X$, and then $\|Lx\|_Y = 0$ for all $x \in X$ and hence $\|L\| = 0$.

• On the other hand, if L is not zero operator, there is $x_0 \in X$ s.t. $Lx_0 \neq 0$. Then

$$\|L\| = \sup_{\substack{x \in X \\ \|x\| \leq 1}} \|Lx\|_Y \geq \left\| L\left(\frac{x_0}{\|x_0\|_X}\right) \right\|_Y = \frac{1}{\|x_0\|_X} \|Lx_0\|_Y > 0.$$

(N2) If $\alpha \in \mathbb{K}$, then $\|\alpha L\| = \sup_{\|x\|_X \leq 1} \|\alpha L x\|_Y = |\alpha| \sup_{\|x\|_X \leq 1} \|Lx\|_Y = |\alpha| \, \|L\|$

(N3) For any $x \in X$, $\|x\|_X \leq 1$:

$$\underset{\sim\sim\sim}{\|(L_1+L_2)(x)\|_Y} = \|L_1x + L_2x\|_Y \leq \|L_1x\|_Y + \|L_2x\|_Y$$

$$\leq \sup_{\|x\|\leq 1} \|L_1x\| + \sup_{\|x\|\leq 1} \|L_2x\|_Y = \underset{\sim\sim\sim}{\|L_1\| + \|L_2\|}$$

which implies

$$\underset{\sim\sim}{\|L_1+L_2\|} = \sup_{\|x\|\leq 1} \|(L_1+L_2)(x)\|_Y \leq \underset{\sim\sim\sim}{\|L_1\| + \|L_2\|}$$, and (N3) is proved.

- $\boxed{\mathcal{L}(X,Y) \text{ is Banach}}$ Assuming Y is Banach.

  Let $\{L_n\}_{n=1}^{\infty}$ be Cauchy in $\mathcal{L}(X,Y)$. We aim at finding $\boxed{L \in \mathcal{L}(X,Y)}$ (linear, bdd) so that $\boxed{\|L_n - L\|_{\mathcal{L}(X,Y)} \xrightarrow{n\to\infty} 0.}$

From ⊛ follows: $\|L_n x - L_m x\|_Y \le \underbrace{\|L_n - L_m\|_{\mathcal{L}(X,Y)}}_{\xrightarrow{n,m\to\infty} 0} \|x\| \quad \forall x \in X$

Due to the assumption $\{L_n x\}_{n=1}^{\infty}$ is Cauchy in Y (for any $x \in X$ fixed)

But Y is Banach, i.e. for each $x \in X$ there is an element in Y, let us denote it by Lx, so that

$$\|L_n x - Lx\|_Y \xrightarrow[n\to\infty]{} 0.$$

- Since L_n's are linear, L is linear as well.
- L is bounded. Indeed, since $\{L_n\}_{n=1}^{\infty}$ is Cauchy, for $\varepsilon = 1$ $\exists N \in \mathbb{N}$ so that for all $k > N$: $\|L_N - L_k\| < 1$

  Then for any $x \in X$; $\|x\|_X \le 1$:

  $$\|Lx\|_Y = \lim_{k\to\infty} \|L_k x\| \le \lim_{k\to\infty} \|L_k x - L_N x\|_Y + \|L_N x\|_Y \le 1 + \|L_N x\|$$

  which gives $\|L\| \le 1 + \|L_N\|$

- Finally, $\|Lx - L_N x\|_Y = \lim_{k\to\infty} \|L_k x - L_N x\| \overset{⊛}{\le} \lim_{k\to\infty} \|L_k - L_N\| < \varepsilon$

  for $x \in X$; $\|x\| \le 1$

∎

This is a consequence of the following standard inequality:

$$\big|\|L_k x\|_Y - \|Lx\|_Y\big| \le \|L_k x - Lx\|_Y \qquad (▲)$$

$\boxed{\text{Pf of (▲)}}$

$$\left.\begin{aligned} \|L_k x\|_Y &\le \|L_k x - Lx\|_Y + \|Lx\|_Y \\ \|Lx\| &\le \|L_k x - Lx\|_Y + \|L_k x\|_Y \end{aligned}\right\} \Rightarrow \pm\left(\|L_k x\|_Y - \|Lx\|_Y\right) \le \|L_k x - Lx\|_Y$$

∎

<u>Example</u> Let H be a (separable) Hilbert space with an orthonormal basis $\{\varphi_k\}_{k=1}^{\infty}$.

Then for $f\in H$; $\boxed{\phi \mapsto (f,\phi)_H \in H'}$ (i.e. it is a bounded linear functional)

and for $f,g\in H$

$$\boxed{\xi \longmapsto (f,\xi)_H\, g \in \mathcal{L}(H)}$$

Pf. • $(f,\alpha\phi_1+\phi_2) = \bar{\alpha}(f,\phi_1)+(f,\phi_2) \Rightarrow$ linearity

• $|(f,\phi)|_{\mathbb{K}} \le \|f\|_H\|\phi\|_H \Rightarrow \sup\limits_{\|\phi\|_H\le 1} |(f,\phi)|_{\mathbb{K}} \le \|f\|_H$ $\Rightarrow$ boundedness.

Similarly, $\xi\mapsto (f,\xi)_H\, g$ is <u>linear</u>, due to linearity of the scalar product and

$$\|(f,\xi)g\|_H \le |(f,\xi)|_{\mathbb{K}}\, \|g\|_H \le \|f\|_H\|g\|_H\|\xi\|_H$$

implying $\sup\limits_{\substack{\xi\in H\\ \|\xi\|_H\le 1}}\|(f,\xi)g\|_H \le \|f\|_H\|g\|_H < \infty.$ ☒

<u>Examples.</u> (1) Každá $\boxed{n\times m \text{ matice } A=(a_{ij})}$ určuje <u>omezený</u> <u>lineární</u> operátor $L: \mathbb{R}^m\to\mathbb{R}^n$ definovaný vztahem $\boxed{x\longmapsto y:=Ax}$

resp. $(x_1,\dots,x_m)\longmapsto(y_1,\dots,y_n)$ $\quad y_i = A_{ij}x_j = \sum_{j=1}^{m} A_{ij}x_j$

(2) Diagonální operátory na prostoru posloupností (sequences)
Diagonal operators on a space of sequences.

Let $1\le p\le\infty$; $X=l_p$; define $L: l_p\to l_p$ as

$$\boxed{L(\{x_n\}_{n=1}^{\infty}) = L((x_1,x_2,\dots,\dots)) = (\lambda_1x_1,\lambda_2x_2,\dots,\lambda_nx_n,\dots)}$$

kde $(\lambda_1,\dots,\lambda_n,\dots)=\{\lambda_i\}_{i=1}^{\infty}$ jsou dané čísla lze L zobrazit jako ∞-matici

Vzhledem k bázi $(\{e_1\},\dots,\{e_n\},\dots)$

$$\begin{pmatrix} \lambda_1 & & & & 0 \\ & \lambda_2 & & & \\ & & \ddots & & \\ 0 & & & \lambda_n & \\ & & & & \ddots \end{pmatrix}$$

Two cases:

(1) If $\{\lambda_k\}_{k=1}^{\infty}$ is bounded, the L is bounded lin. operator. Its norm $\|L\| := \sup\limits_{k\in\mathbb{N}}|\lambda_k|$

(2) If $\{\lambda_k\}_{k=1}^{\infty}$ is unbounded, the L is not bounded $\mathrm{dom}(L) = \{x\in l_p;\ Lx\in l_p\}$

(3) Operator of differentiation Consider $I = (0,\pi)$, $X = BC(I)$

bounded & continuous functions on I

with $\|f\| := \sup_{x \in (0,\pi)} |f(x)|$

Define $Lf = f'$. Then L is linear.

But L is not bounded, Consider $f_k(x) = \sin kx$. Then $f_k'(x) = k \cos kx$ and consequently

$$\|f_k\| = 1, \quad \|f_k'\| = k = \|Lf\| \quad \text{for all } k \geq 1.$$

and. $\text{Dom } L = \{ f \in BC(I) ; f' \in BC(I)\} \subsetneqq BC(I)$.

(4) $\boxed{\text{Shift operator on } L^p(\mathbb{R}^d)}$ Let $1 \le p \le +\infty$. Fix $a \in \mathbb{R}^d$.

For $f \in L^p(\mathbb{R}^d)$, define $(L_a f)(x) \doteq f(x-a)$. Clearly $\|L_a f\|_p = \|f\|_p$. Therefore $L_a : L^p \to L^p$ is a bounded linear operator with $\|L_a\| = 1$. Note that L_a is surjective (onto) and injective (one-to-one) operator.

(5) $\boxed{\text{Shift operator on } \ell^p}$ Let $1 \le p \le +\infty$. Define the operators

$$L_+ (x_1, \dots, x_n, \dots) \stackrel{df.}{=} (0, x_1, x_2, \dots)$$
$$L_- (x_1, \dots) \stackrel{df.}{=} (x_2, x_3, \dots)$$

L_+, L_- are linear operators with $\|L_+\| = \|L_-\| = 1$

However L_+ is one-to-one, but not onto

L_- is not one-to-one, but is onto

(6) $\boxed{\text{Multiplication operator}}$ Let $\Omega \subset \mathbb{R}^d$, $g : \Omega \to \mathbb{R}$ be bounded measurable function*). For any $1 \le p \le \infty$, consider on $L_p(\Omega)$ the operator

$$M_g : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d) \qquad (M_g f)(x) \stackrel{def}{=} g(x) f(x).$$

*) $g \in L^\infty(\Omega)$ Then M_g is linear and bounded (hence continuous) with the norm

$$\|M_g\| \stackrel{df.}{=} \sup_{\|f\|_p \le 1} \|gf\|_{L^p} \leqq \|g\|_{L^\infty}$$

(7) $\boxed{\text{Integral operator}}$ Let $a < b$, consider $X = C([a,b])$

and $(Lf)(x) \stackrel{df.}{=} \int_a^x f(y)\, dy$

Then $L : X \to X$ is a bounded lin. operator. Indeed

$$|(Lf)(x)| \le \left| \int_a^x f(y)\,dy \right| \le \int_a^x |f(y)|\,dy \le (b-a) \max_{x \in \langle a,b\rangle} |f(x)| \Rightarrow \|Lf\| \le (b-a)$$

Finite-dimensional spaces

Def. Let X be a vector space. Then two norms $\|\cdot\|_{\blacksquare}, \|\cdot\|_{\bullet}$ are equivalent $\overset{df.}{=}$ $\exists\, C>0$ $\frac{1}{C}\|x\|_{\blacksquare} \le \|x\|_{\bullet} \le C\|x\|_{\blacksquare}$ $\forall x \in X$.

- Equivalent norms give the same Cauchy sequences, the same topology.
- In infinite-dim spaces, there can be many non-equivalent norms.

In finite-dimensional spaces all norms are equivalent.

Theorem Let X be a normed vector space over $\mathbb{K}$, $\dim X < +\infty$.
Let $\mathcal{B} = \{x_1, \dots, x_N\}$ be a basis of X.

Then
(i) X is complete, and hence Banach space
(ii) Defining, for any $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{K}^N$; $L\alpha \overset{df.}{=} \alpha_1 x_1 + \dots + \alpha_N x_N$
then $L: \mathbb{K}^N \to X$ is bijective (surjective + injective) and bounded.
Moreover, $L^{-1}: X \to \mathbb{K}^N$ is also bdd.

Proof [1] As $\{x_1, \dots, x_N\}$ is basis in X, we see that L is bijective.
Hence L^{-1} is well defined.

[2] $$\|L\alpha\|_X = \Big\|\sum_{i=1}^N \alpha_i x_i\Big\|_X \le \sum_{i=1}^N \|\alpha_i x_i\|_X = \sum_{i=1}^N |\alpha_i|\,\|x_i\|_X$$
$$\le |\alpha|_{\mathbb{K}^N} \sum_{i=1}^N \|x_i\|_X$$
which implies $\|L\| < +\infty$. Hence L is $\underbrace{\text{bounded}}_{\text{continuous}}$ linear operator.

[3] To prove that L^{-1} is bounded, assume, on contrary, that
$\exists \{y_n\}_{n=1}^\infty \subset X$; $\|y_n\|_X \le 1$ $\forall n$ and $\|L^{-1}y_n\|_{\mathbb{K}^N} \to \infty$

Consider $$\beta_n := \frac{L^{-1}y_n}{\|L^{-1}y_n\|_{\mathbb{K}^N}} \in \mathbb{K}^N$$

Then $\|\beta_n\|_{\mathbb{K}^N} = 1$ and $L\beta_n \to 0$ as $n \to \infty$. Since $\{\beta_n\}_{n=1}^\infty$ is bdd in $\mathbb{K}^N$, it admits a convergent subsequence: i.e. $\beta_{n_k} \to \beta \in \mathbb{K}^N$

Clearly
$$\left.\begin{aligned} \|\beta\| &= \lim_{k\to\infty} \|\beta_{n_k}\| = 1 \\ L\beta &= \lim L\beta_{n_k} = \lim \frac{\overset{\text{bdd}}{y_{n_k}}}{\underbrace{\|L^{-1}y_{n_k}\|}_{\to\infty}} = 0 \end{aligned}\right\}$$ which contradicts to the fact $L\beta = 0 \Leftrightarrow \beta = 0$.

Hence L^{-1} is continuous.

[4] To prove that X is complete, let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in X. Then $L^{-1}x_n$ defines a Cauchy seq. in $\mathbb{K}^N$, which converges to some $\beta \in \mathbb{K}^N$ (as $\mathbb{K}^N$ is complete); i.e. $x_n \to L\beta \in X$.

YES

$\boxed{\text{Corollary}}$ In a finite-dimensional space, all <u>norms are equivalent</u>

<u>Proof</u>: Let $\|\cdot\|_{\square}$ and $\|\cdot\|_{\circ}$ be two norms in X. Let $\mathcal{B} = \{x_1, \dots, x_N\}$ be a basis in X and $L : \mathbb{K}^N \to X$ is defined as above, then L, L^{-1} are bounded linear operators. Hence there are $C', C'' > 0$:

$$\frac{1}{C'} \|L^{-1}x\|_{\mathbb{K}^N} \leq \|x\|_{\square} \leq C' \|L^{-1}x\|_{\mathbb{K}^N}$$

$$\frac{1}{C''} \|L^{-1}x\|_{\mathbb{K}^N} \leq \|x\|_{\circ} \leq C'' \|L^{-1}x\|_{\mathbb{K}^N}.$$

for all $x \in X$.

Hence $\|\cdot\|_{\square}$ and $\|\cdot\|_{\circ}$ are equivalent. ▨

Bolzano-Weierstrass theorem states that

(COMPACTNESS PROPERTY) $\boxed{\text{each bounded sequence contain converging subsequence}}$

This property holds in all finite-dimensional spaces
fail in all $\boxed{\text{INFINITE}}$ —— " ——

$\boxed{\text{Theorem}}$ <u>Locally compact normed spaces are finite-dimensional.</u>

Let X be a normed space. The following holds:

$$X \text{ is finite-dimensional} \iff \overline{B_1(0)} \text{ is compact.}$$

<u>Proof.</u> $\boxed{\Rightarrow}$ Since X is finite-dimensional, there is, by previous theorem a <u>linear</u> homeomorphismus $L : \mathbb{K}^N \to X$ with bounded inverse.
Since $\overline{B_1(0)}$ is closed and bounded, $L^{-1}(\overline{B_1(0)})$ is also closed a bdd. By Bolzano-Weierstrass: $L^{-1}(\overline{B_1(0)})$ is compact.
As L is continuous and $\overline{B_1(0)}$ ~~is~~ a continuous image of a compact set, $\overline{B_1(0)}$ is compact.

(3)

$\boxed{\Leftarrow}$ By assumption, $B_1(0)$ is compact and can be covered by a finite number of balls $B_{1/2}(p_i)$, where $p_i \in \overline{B_1(0)}$.

Set $V = \text{span}\{p_1, \dots, p_n\}$. It is a subspace of X that is finite-dimensional, $\dim V \le n$. By preceding theorem V is complete, i.e. V is closed.

The goal is to show that $V = X$. (Then X is finite-dimensional.) If not, we could find $x \in X \setminus V$. Let $\varrho \overset{\text{def}}{=} d(x, V) = \inf_{y \in V} \|y - x\|_X$. Then $\varrho > 0$, as V is closed. Furthermore, there is an $v \in V$

(*) $$\varrho < \|x - v\|_X \le \frac{3}{2}\varrho$$

Considering
$$z = \frac{x-v}{\|x-v\|} \in \overline{B_1(0)},$$
by the above construction there is a point $p_i \in \overline{B_1(0)}$ s.t. $\|z - p_i\| < \frac{1}{2}$.

Then
$$x = v + z\|x-v\|_X = \underbrace{v + \|x-v\| p_i}_{\in V} + \|x-v\|(z-p_i)$$

But $v + \|x-v\|p_i \in V$, which implies
$$\|x-v\|\|(z-p_i)\| \ge d(x, V) \ge \varrho$$
which gives
$$\|x-v\| \ge 2\varrho$$
which contradicts to (*).

Hence $X = V$, the proof is complete. $\square$