# 7. LINEAR OPERATORS IN HILBERT SPACES

Let $H$ be a vector space over $\mathbb{K}$. We say that $H$ is the space with scalar product (or the space with inner product) or pre-Hilbert space if there is a map $H \times H \to \mathbb{R}$ so that

(1) $(x,x)_H \geq 0 \quad \forall x \in H$ and equality holds if and only if $x=0$

(2) $(x+y,z) = (x,z)+(y,z)$ and $(\alpha x,z) = \alpha(x,z)$ $\quad \forall x,y,z \in H \quad \forall \alpha \in \mathbb{K}$

(3) $(x,y) = \overline{(y,x)}$

NOTE that
- $(x,y+z) = \overline{(y+z,x)} = \overline{(y,x)} + \overline{(z,x)} = (x,y) + (x,z)$
- $(x,\alpha z) = \overline{(\alpha z,x)} = \bar{\alpha}\overline{(z,x)} = \bar{\alpha}(x,z)$

Recall:
- $\|x\|_H := \sqrt{(x,x)_H}$ defines a norm on $H$
- if $H$ is complete $(H,\|\cdot\|_H)$, then $\boxed{H \text{ is called a Hilbert space}}$
- two basic inequalities: $\boxed{|(x,y)_H| \leq \|x\|_H \|y\|_H \quad \forall x,y \in H}$ Cauchy-Schwarz-Buniakowski

  $\boxed{\|x+y\|_H \leq \|x\|_H + \|y\|_H \quad \forall x,y \in H}$ Minkowski or $\Delta$-ineq.!
- nice identity called parallelogram identity (with respect to the rule in $\mathbb{R}^2$)

$$\boxed{\|x+y\|_H^2 + \|x-y\|_H^2 = 2\|x\|_H^2 + 2\|y\|_H^2}$$

Pf $\quad (x+y,x+y) + (x-y,x-y) = 2\|x\|_H^2 + (x,y) + (y,x) + 2\|y\|_H^2 - (x,y) - (y,x)$ ▨

Hilbert spaces „differ" from Banach spaces by the presence of scalar product. This additional structural property allows one: to generalize the concept of orthogonality from $\mathbb{R}^d$ to $H$

to characterize/describe well all bdd linear functionals on $H$, i.e. to characterize $H'$.

## 7.1 Ortogonality

- Given $S \subseteq H$, define $\boxed{\mathrm{span}(S) \equiv \left\{ \sum_{i=1}^{N} \alpha_i x_i \ , \alpha_i \in \mathbb{K}, x_i \in S, N \geq 1 \right\}}$

Then span(S) is a subspace, but <u>not</u> necessarily closed.

define $V := \overline{\mathrm{span}(S)}$ ... a space generated by $S$

If $V = H$, then one says that $S$ is total.
It means, if $S$ is total, then $\forall x \in H \ \exists x_n \in \mathrm{span}\, S$ so that $\|x_n - x\|_H \to 0$.

- We say that $x, y \in H$ are orthogonal $\overset{\text{df.}}{\equiv} (x,y)_H = 0$.

- Given $S \subseteq H$, define $S^\perp \equiv \{ y \in H ; (y,x) = 0 \ \ \forall x \in S \}$.

<u>NOTE</u> $S^\perp$ is always a closed subspace of $H$. Verify!

**Theorem 7.1** Let $H$ be Hilbert and $V \subset H$ a <u>closed</u> subspace of $H$.

Then

(i) $\boxed{H = V \oplus V^\perp}$ i.e. $\forall x \in H \ \exists!\ y \in V$ and $z \in V^\perp$ : $x = y + z$

(ii) $y$ is the unique point in $\boxed{V}$ having minimal distance from $x$; $y \equiv P_V(x)$
$z$ — " — in $\boxed{V^\perp}$ — " — $x$; $z = P_{V^\perp}(x)$

(iii) The perpendicular projections $x \mapsto P_V(x)$ and $x \mapsto P_{V^\perp}(x)$ are linear continuous with the norm $\leq 1$.
In fact, if $V \neq \{0\}$ then $\|P_V\|_{\mathcal{L}(H,H)} = \|P_{V^\perp}\|_{\mathcal{L}(H,H)} = 1$.

**Proof** **Step 1** Let $x \in H$ be arbitrary, but fix.
Let $\alpha := d(x, V) = \inf_{y \in V} \|x - y\|_H$. Then $\exists\ y_n \in V$ so that $\lim_{n\to\infty} \|x - y_n\|_H = \alpha$. We shall show that $\{y_n\}$ is Cauchy.

If so and as $V$ is closed, $V$ is complete. Hence there is $y \in V$ : $y_n \to y$ in $H$ and $\|x - y\|_H = \alpha$.

We need to show that $y$ is a unique point satisfying $\|x - y\|_H = \alpha$.

**Step 2** The facts that $\boxed{\{y_n\}_{n=1}^{\infty} \text{ is Cauchy}}$ and $\boxed{y \text{ is unique}}$ are proved using very similar arguments.

Recalling $\|u + v\|_H^2 + \|u - v\|_H^2 = 2\|u\|_H^2 + 2\|v\|_H^2 \qquad \forall u, v \in H$

and taking $u = x-y_n$ and $v = x-y_m$, we get

$$\|y_n-y_m\|_H^2 = 2\|x-y_m\|_H^2 + 2\|x-y_n\|_H^2 - 4\left\|x-\frac{y_n+y_m}{2}\right\|_H^2$$

Since $y_n, y_m \in V$, then $\frac{y_n+y_m}{2}\in V$ and $\left\|x-\frac{y_n+y_m}{2}\right\|_H^2 \ge \alpha^2$

Hence

$$\limsup_{m,n\to\infty}\|y_n-y_m\|_H^2 \le 2\limsup_{m\to\infty}\|x-y_m\|_H^2 + 2\limsup_{n\to\infty}\|x-y_n\|_H^2 - 4\liminf_{n,m\to\infty}\left\|x-\frac{y_n+y_m}{2}\right\|_H^2$$

$$\le 2\alpha^2+2\alpha^2-4\alpha^2 = 0.$$

Similarly, if $y$ and $y'$ would be two minimizing points, then ($u := x-y$ $v = x-y'$)

$$\|y-y'\|_H^2 = 2\|x-y\|_H^2+2\|x-y'\|_H^2 - 4\Big\|x-\underbrace{\frac{y+y'}{2}}_{\in V}\Big\|_H^2$$

$$\le 4\alpha^2-4\alpha^2=0.$$

Hence $y=y'$. It means that the mapping ($P_V$) that gives to any $x\in H$ a point $y\in V$ (having the minimal distance) is well-defined.

$\boxed{\text{Step 3}}$ We show that $P_V(x)$ can be characterized as the unique $y\in V$: $\quad x-y\in V^\perp$.

$\boxed{\text{Existence}}$ For arbitrary $v\in V$ consider the mapping

$$\underset{\in\mathbb{R}}{t} \longmapsto \|x-(y+tv)\|_H^2 = \|x-y\|_H^2 + t^2\|v\|_H^2 + 2\operatorname{Re}(x-y, tv)$$

By Step 1 and 2, this mapping attains minimum at $t=0$.

Hence $\quad \operatorname{Re}(x-y,v)=0 \quad \forall\, v\in V$

Replacing $v$ by $-iv$, we conclude that

$\operatorname{Im}(x-y,v) = \operatorname{Re}(x-y,-iv)=0$. Hence $x-y\in V^\perp$

$\boxed{\text{Uniqueness}}$ If there are two (such) points $y,y'\in V$, then

$$\|y-y'\|_H^2 = (y-y',y-y') = (\underbrace{y-y'}_{\in V}, \underbrace{x-y'}_{\in V^\perp}) - (\underbrace{y-y'}_{\in V},\underbrace{x-y}_{\in V^\perp}) = 0.$$

$\boxed{\text{Step 4}}$ Properties of $P_V$ and $P_{V^\perp}$.

- If $y=P_V(x)$, $y'=P_V(x')$, then for $\alpha,\alpha'\in\mathbb{K}$, $\alpha y+\alpha' y'\in V$ and $\alpha x+\alpha' x' - \alpha y - \alpha' y' \in V^\perp$

  Hence, by Step 3, $P_V(\alpha x+\alpha' x') = \alpha y+\alpha' y'$. Hence $P_V$ is linear. As $P_{V^\perp} = I-P_V$, $P_{V^\perp}$ is linear as well.

- $$\|x\|_H^2 = \|\underbrace{x-P_V(x)}_{V^\perp} + \underbrace{P_V(x)}_{V}\|_H^2 = \|x-P_V(x)\|_H^2+\|P_V(x)\|_H^2$$ (Pythagoras's theorem)

$\Rightarrow \sup_{\|x\|_H=1}\|P_V(x)\|_H \le 1$, $\sup_{\|x\|_H=1}\|(I-P_V)(x)\|_H\le 1$.

If $V\ne\{0\}$, then $P_V(x)=x$ for $x\in V$.

## $\boxed{7.2}$ Linear functionals on a Hilbert space. Riesz representation theorem.

$\boxed{\text{Theorem 7.2}}$ Let $H$ be a Hilbert space. Then it holds:

(1) For every $x \in H$: $y \mapsto (y,x)_H \in H^*$, i.e.

$\phi^x := [\, y \mapsto (y,x)_H ]$ is linear continuous map of $H$ into $\mathbb{K}$.

$\boxed{\text{Mapping: } \overset{\in H}{x} \mapsto \overset{\in H^*}{\phi^x} \text{ is isometry.}}$

(2) For every $\phi \in H^*$: $\exists!\, a \in H$ $\langle \phi, x \rangle_{H^*,H} = \phi(x) = (a,x)_H$ for all $x \in H$.

$\boxed{\text{Proof}}$ $\boxed{\text{Ad (1)}}$ Simple. Do it yourself.

$\boxed{\text{Ad (2)}}$ Let $\phi \in H^*$ be given. ▶ If $\phi(y) = 0$ for all $y \in H$, the the conclusion holds with $a = 0$.

▶ If $\phi$ is nontrivial, then $\mathrm{Ker}\,\phi$ is closed subspace of $H$ that is proper, i.e. $\mathrm{Ker}\,\phi \neq V$. Then $\exists\, b \in [\mathrm{Ker}\,\phi]^\perp$ that can be normalized; $\|b\|_H = 1$. Since for any fixed $x \in H$:

$$\phi\big(b\phi(x) - x\phi(b)\big) = \phi(b)\phi(x) - \phi(x)\phi(b) = 0,$$

the vector $b\phi(x) - x\phi(b) \in \mathrm{Ker}\,\phi$ and is then ortogonal to $b$, which implies that

$$0 = b \cdot b\phi(x) - b \cdot x\phi(b) \Rightarrow \phi(x) = \overline{\phi(b)}\, b \cdot x.$$

Hence, setting $a = \overline{\phi(b)}\, b$, we are done with the existence part of Theorem.

▶ Regarding uniqueness, assume that there are two points $a_1, a_2 \in H$ such that $\phi(x) = (a_i, x)$ $\forall x \in H$. Then $(a_1 - a_2, x) = 0$ $\forall x \in H$, which however implies that $a_1 = a_2$.

For the sake of completness, we add a proof of the part (1).

$\boxed{\text{Ad (1)}}$ On one hand, we have: $|\phi^x(y)| \leq \|y\|_H \|x\|_H \Rightarrow \boxed{\|\phi^x\|_{H^*} \leq \|x\|_H}$.

Note that linearity of $\phi^x$ is trivial. Hence $\phi^x \in \mathcal{L}(H;\mathbb{K}) = H^*$.

On the other hand, for $\frac{x}{\|x\|}$, which is at the unit sphere in $H$, we have

$$\phi^x\left(\frac{x}{\|x\|}\right) = \frac{x \cdot x}{\|x\|_H} = \|x\|_H,$$ which leads to $\boxed{\|x\|_H \leq \|\phi^x\|_{H^*}}$.

Hence $\|x\|_H = \|\phi^x\|_{H^*}$

$\boxed{\text{Theorem 7.3}}$ (Dual to H or $\boxed{H^* = H}$) Let $H$ be a Hilbert space. A mapping that maps $\phi^{\in H^*}$ to $a^{\in H}$ is one-to-one isometry of $H^*$ onto $H$ and denoting this mapping $\Psi$ we have

$$\Psi(\phi_1 + \phi_2) = \Psi(\phi_1) + \Psi(\phi_2) \quad \text{and} \quad \Psi(\alpha\phi) = \bar{\alpha}\Psi(\phi)$$
$$\forall \phi_1, \phi_2 \in H^* \quad \forall \alpha \in \mathbb{K}.$$

If $\mathbb{K} = \mathbb{R}$, then $\Psi$ is bijection isometry of $H^*$ onto $H$.

(Pf) It follows from the previous theorem that $\Psi : \phi^{\in H^*} \mapsto a^{\in H}$ is bijection and isometry.

If $\phi_1, \phi_2 \in H^*$ and $a_1 = \Psi(\phi_1)$, $a_2 = \Psi(\phi_2)$ then for all $x \in H$: $(\phi_1 + \phi_2)(x) = (a_1, x) + (a_2, x) = (a_1 + a_2, x)$

As $\Psi$ is one-to-one
$$\Psi(\phi_1 + \phi_2) = a_1 + a_2 = \Psi(\phi_1) + \Psi(\phi_2).$$
▨

$\boxed{\text{Theorem 7.4}}$ (Reflexivity of Hilbert spaces)

Every Hilbert space is reflexive

(Pf) Pick any $\Phi \in H^{**}$. Our goal is to find $x \in H$ so that

$$J_x = \Phi \qquad \text{where } J : H \to H^{**} \text{ is canonical embedding.}$$
$$\Updownarrow$$
$$\boxed{\langle J_x, \varphi \rangle = \Phi(\varphi) \quad \forall \varphi \in H^*}$$

Consider, for all $h \in H$, mapping $\varphi_h : x \mapsto (x, h)$ $(x \in H)$

Then $\varphi_h \in H^*$. Furthermore,
$$h \mapsto \overline{\Phi(\varphi_h)} \quad \text{maps } H \to \mathbb{K}$$
and it is bounded and linear functional.

By Riesz representation theorem $\exists!\, x \in H$:
$$\overline{\Phi(\varphi_h)} = (h, x)_H \quad \forall h \in H.$$

To conclude: for $\varphi \in H^*$ we find $h \in H$ so that $\varphi = \varphi_h$ then
$$\Phi(\varphi) = \Phi(\varphi_h) = \overline{(h, x)} = (x, h) = \varphi_h(x) = \varphi(x) = J_x(\varphi).$$
▨

## Well-posedness for positive definite operators. Lax-Milgram lemma.

A basic problem of LA: $\boxed{\text{to solve } Ax = b \,,\; A \in \mathbb{R}^{n\times n},\; b \in \mathbb{R}^n}$.

If $Ax \cdot x > 0 \;\; \forall x \neq 0$, then LA says: $\exists!\, x \in \mathbb{R}^n$ solving the problem.

Similar result holds in infinite-dimensional (Hilbert) spaces.

$\boxed{\text{Def}}$ $A : H \to H$ is strictly positive definite:

$\boxed{H \text{ over } \mathbb{R}}$ $\quad \exists \beta > 0 : \quad (Au, u) \geq \beta \|u\|_H^2 \quad \forall u \in H.$ (*)

$\boxed{\text{Theorem 7.5}}$ $H$ ... Hilbert over $\mathbb{R}$. Let $A \in \mathcal{L}(H;H)$ fulfills (*).

Then $\forall f \in H \;\exists!\, u \in H \;(u := A^{-1}f)$ so that $Au = f$.

The inverse $A^{-1} \in \mathcal{L}(H,H)$ satisfies $\|A^{-1}\|_{\mathcal{L}(H,H)} \leq \frac{1}{\beta}$.

Pf. Goal: to show that (*) implies $A$ is onto and one-to-one.

$\boxed{\text{Step 1}}$ (*) implies that

$$\beta \|u\|_H^2 = (Au, u) \leq \|Au\|_H \|u\|_H \Rightarrow \boxed{\beta \|u\|_H \leq \|Au\|_H \quad \forall u \in H} \quad (**)$$

Hence if $Au = 0$, then $u = 0$ and $A$ is one-to-one.

$\boxed{\text{Step 2}}$ $\boxed{\text{Range } A \text{ is closed}}$ Consider $v_n \in \text{Range } A$, $v_n \to v$ in $H$.

We look for $u \in H$: $Au = v$. However, by assumption $v_n = Au_n$ for some $u_n \in H$. Hence

$$\|u_n - u_m\| \leq \frac{1}{\beta} \|Au_n - Au_m\| = \frac{1}{\beta} \|v_n - v_m\| \to 0$$

Hence $\{u_n\}_{n\in\mathbb{N}}$ is Cauchy and $\exists u \in H$: $u_n \to u$ in $H$

But, by Heine def. of continuity, $Au_n \to Au$ in $H$

Hence $Au = v$.

$\boxed{\text{Step 3}}$ $\boxed{\text{Range } A = H}$. If not, as Range $A$ is closed, $\exists\, w \neq 0$ such that $w \in [\text{Range}]^\perp$ (see Theorem 7.1). But then

$$\beta \|w\|_H^2 \leq (Aw, w) = 0 \;, \text{ which gives } \lightning \text{ with } w \neq 0.$$

$\boxed{\text{Step 4}}$ As $A \in \mathcal{L}(H,H)$ is bijection, $Au = f$ has ! solution for any $f \in H$, denoted $A^{-1}f$. Then by (**)

$$\beta \|A^{-1}f\|_H = \beta \|u\|_H \leq \|Au\|_H = \|f\|_H \,,$$

which implies that $\|A^{-1}f\|_H \leq \frac{1}{\beta}$.

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**Theorem 7.6** Let $H$ be a Hilbert space and $\mathbb{K}=\mathbb{R}$. Assume that $B : H \times H \to \mathbb{R}$ satisfies

LINEARITA or SESQUILINEARITY
$$\begin{cases} B(\alpha u_1 + \beta u_2, v) = \alpha B(u_1, v) + \beta B(u_2, v) & \forall u_1, u_2, v \in H \quad \forall \alpha_1, \alpha_2 \in \mathbb{R} \\ B(u, \gamma v_1 + \delta v_2) = \gamma B(u, v_1) + \delta B(u, v_2) & \forall u, v_1, v_2 \in H \quad \forall \beta_1, \beta_2 \in \mathbb{R} \end{cases}$$

BOUNDEDNESS $\exists\, C>0$: $\quad |B(u,v)| \leq C \|u\|_H \|v\|_H \quad \forall u,v \in H$

H-coercivity $\exists\, \beta>0 \quad B(u,u) \geq \beta \|u\|_H^2 \quad \forall u \in H.$

Then, for every $f \in H$, $\exists!\, u \in H$:
$$B(u,\varphi) = (f,\varphi) \quad \forall \varphi \in H$$

Moreover,
$$\|u\|_H \leq \frac{\|f\|}{\beta}$$

(Pf) For every $u \in H$: $\varphi \longmapsto B(u,\varphi) \in H^*$. Hence, by Riesz representation theorem 7.2, $\exists!$ vector in $H$, we call it $Au$ so that
$$B(u,\varphi) = (Au, \varphi)_H \quad \forall \varphi \in H.$$
We will show that $A$ is bounded, strictly positive definite linear operator.

**Linearity** $(A(u+v), \varphi) = B(u+v, \varphi) \underset{\text{lin. of } B}{=} B(u,\varphi) + B(v,\varphi)$

**Boundedness**
$$\|Au\|_H = \sup_{\|\varphi\|_H=1} |(Au,\varphi)| \leq \sup_{\|\varphi\|_H=1} |B(u,\varphi)| \underset{\text{boundedness of } B}{\leq} C\|u\|_H$$

which implies
$$\|A\|_{\mathcal{L}(H,H)} \leq C$$

**Strict positive definiteness** follows from
$$(Au,u) = B(u,u) \geq \beta \|u\|_H^2$$

Hence, by Theorem 7.5, we conclude that $Au=f$ has unique solution $u = A^{-1}f$ satisfying $\|u\|_H \leq \frac{\|f\|_H}{\beta}$.

■ Weak convergence in the Hilbert spaces takes, due to Riesz representation theorem, slightly simpler form. Recall

$x_n \to x$ weakly in $H \iff \phi(x_n) \to \phi(x) \quad \forall \phi \in H^*$

By Riesz representation theorem we can associate with any $\phi$ unique $a_\phi \in H$ so that $\phi(x) = \langle \phi, x \rangle = (a_\phi, x)_H$ for all $x \in X$

We can then say

$$\boxed{x_n \to x \text{ weakly in } H \iff (y, x_n)_H \to (y, x) \text{ for all } y \in H}$$

Uniqueness of the limit can be re-checked again: If $x_n \to x$ weakly in $H$ and $x_n \to \tilde{x}$ weakly in $H$, then

$$0 = \lim_{n\to\infty} (x-\tilde{x}, x_n)_H - \lim (x-\tilde{x}, x_n)_H = (x-\tilde{x}, x) - (x-\tilde{x}, \tilde{x})$$
$$= (x-\tilde{x}, x-\tilde{x}) = \|x-\tilde{x}\|_H^2$$ □

$\boxed{\text{Theorem 7.8}}$ Let $x_n \to x$ weakly in $H$ and $L \in \mathcal{L}(H,H)$ be compact. Then $\lim_{n\to\infty} \|Lx_n - Lx\|_H = 0$ (strong convergence).

Pf Since $x_n \to x$ weakly in $H$, $\{x_n\}_{n\geq1}^\infty$ is bdd. As $L$ is compact $\{Lx_n\}_{n\geq 1}^\infty$ contains subsequence that is converging, i.e. there is $y \in H$: $\|Lx_{n_k} - y\|_H \to 0$ as $k \to \infty$.

It remains to show that $Lx = y$. However, using the adjoint operator, we have

$$(v, Lx_n - Lx) = (L^* v, x_n - x) \xrightarrow[n\to\infty]{} 0 \quad \forall v \in H$$

Hence $Lx_n \to Lx$ weakly in $H$, but also $Lx_{n_k} \to y$ weakly in $H$. From the uniqueness of the weak limit $Lx = y$.

To check that the whole sequence $Lx_n$ converges to $Lx$, assume, on the contrary, that there is a subsequence $\{x_{n_\ell}\} \subset \{x_n\}$ so that $L(x_{n_\ell} - x)$ does not converge to zero, i.e. $\exists\, \varepsilon_0 > 0$ so that $\|L(x_{n_\ell} - x)\|_H \geq \varepsilon_0 > 0$. But as $x_{n_\ell} - x$ is bdd, there is a subsequence (denoted for simplicity again $x_{n_\ell}$) so that as $L$ is compact $Lx_{n_\ell} \to z$ as $\ell \to \infty$. However, from the first part of the proof $z = Lx$, which contradicts to

$$\|Lx_{n_\ell} - Lx\|_H \geq \varepsilon_0 > 0 \text{ for all } \ell \in \mathbb{N}.$$ □