Recall
$$\begin{cases} \frac{\partial u^\varepsilon}{\partial t} - \operatorname{div} \vec{q}^\varepsilon = g & \text{in } (0,T)\times\Omega =: Q \\ \nabla u^\varepsilon = f(q^\varepsilon) + \varepsilon q^\varepsilon \\ u(0,\cdot) = u_0 \end{cases} \qquad f(q^\varepsilon) := \frac{q^\varepsilon}{(1+|q^\varepsilon|^2)^{1/2}}$$

It remains to show that $\boxed{q^\varepsilon \longrightarrow q \text{ a.e. in } Q}$

Recall
$$\nabla u^\varepsilon \longrightarrow \nabla u \quad \text{in } L^2(0,T; L^2)$$
$$\Downarrow$$
$$\nabla u^\varepsilon \longrightarrow \nabla u \quad \text{a.e. in } Q \qquad \text{(modulo subsequences)}$$

By Egorov theorem: $\forall\, \varepsilon>0\ \exists\, Q_\varepsilon \subset Q$

$$\nabla u^\varepsilon \rightrightarrows \nabla u \quad \text{in } Q_\varepsilon \quad \text{and} \quad \underbrace{|Q\setminus Q_\varepsilon|}_{\text{Lebesgue measure}} < \varepsilon$$
$$\Updownarrow$$
$$\boxed{\nabla u^\varepsilon \longrightarrow \nabla u \quad \text{in } L^\infty(Q_\varepsilon)}$$

Recall $\{\vec{q}^\varepsilon\}_\varepsilon$ bdd in $L^1(Q)$

Chacon's biting lemma Let $\{g^n\}_{n=1}^\infty$ be bdd in $L^1(\Omega)$, $\Omega \subset \mathbb{R}^m$. Then $\exists\, g \in L^1(\Omega)$, a subsequence, still denoted $\{g^n\}$, and non-increasing sequence of $\{E_k\}_{k=1}^\infty$ such that $|E_k| \searrow 0$ as $k\to\infty$, s.t.

$$g^n \rightharpoonup g \quad \text{weakly in } L^1(\Omega\setminus E_k) \quad \text{for all } k\in\mathbb{N}.$$

Egorov theorem Let f_k be such that $f_k \to f$ a.e. in Ω, where $|\Omega| < +\infty$. Then for $\varepsilon>0$ $\exists\, E\subset\Omega$ measurable s.t.

- $|E| < \varepsilon$
- $f_k \rightrightarrows f$ in $\Omega\setminus E$.

$f_k := \nabla u^k$

Let $\delta>0$ be arbitrary then for Q_δ (by Egorov and Cheers)

$$\nabla u^\varepsilon \to \nabla u \quad \text{in } L^\infty(Q_\delta) \text{ strongly}$$
$$q^\varepsilon \to q \quad \text{in } L^1(Q_\delta) \text{ weakly}$$

Recall that $f(q)$ is strictly monotone: $f(q^\varepsilon)=\frac{q^\varepsilon}{(1+|q^\varepsilon q^\varepsilon|)^{1/2}}$

$$0 \le \int_{Q_\delta} \underbrace{(f(q^\varepsilon)-f(q))\cdot(q^\varepsilon-q)}_{\geq 0}$$

$$= \int_{Q_\delta} (\nabla u^\varepsilon - \varepsilon q^\varepsilon)\cdot(q^\varepsilon-q) - \underbrace{\int_{Q_\delta} \overset{L^\infty}{f(q)}\cdot \underline{(q^\varepsilon-q)}}_{\xrightarrow[\sigma(\varepsilon)]{} 0}$$

$$= \int_{Q_\delta} \underbrace{(\nabla u^\varepsilon-\nabla u)}_{\to 0 \; L^\infty}\cdot \underbrace{q^\varepsilon}_{L^1} + \int_{Q_\delta}\nabla u\cdot \underset{\searrow \nabla u\cdot q}{q^\varepsilon}$$

$$- \varepsilon \underbrace{\int |q^\varepsilon|^2}_{\le 0} - \int \underset{\searrow \nabla u\cdot q}{\nabla u^\varepsilon\cdot q} + \underset{\sqrt{\varepsilon}\,\sqrt{\varepsilon}}{\varepsilon}\int \underset{L^2}{q^\varepsilon}\cdot q \quad + \sigma(\varepsilon)$$

$$\Rightarrow \quad 0 \le \lim_{\varepsilon\to 0}\int_{Q_\delta} (f(q^\varepsilon)-f(q))\cdot(q^\varepsilon-q) \le 0$$

$$\Rightarrow \quad \lim_{\varepsilon\to 0}\int_{Q_\delta} \underbrace{(f(q^\varepsilon)-f(q))\cdot(q^\varepsilon-q)}_{\geq 0 \; =: b^\varepsilon} = 0$$

$$\Leftrightarrow \quad \|b_\varepsilon\|_{1,Q_\delta} \to 0$$

$\mathbb{R}^d$

$\lim_{\varepsilon\to 0}(f(q^\varepsilon)-f(q))\cdot(q^\varepsilon-q)=0$ for a.a. $(t,x)\in Q_\delta$

$\Rightarrow \exists$ not relabelled subsq. $\boxed{b_\varepsilon \to 0}$ a.e. in Q_δ

$$\boxed{\text{Since } f \text{ is strictly monotone} \Rightarrow \quad q^\varepsilon \to q \text{ a.e. in } Q_\delta}$$

Since $\delta>0$ was arbitrary, it implies

$$\underline{q^\varepsilon \to q \quad \text{a.e. in } Q.}$$

G. Dal Maso, F. Murat (1996)

Lemma • Let $(X, |\cdot|)$ be finite-dimensional Hilbert ($\mathbb{R}^N$)

▶ Let • $\{\beta_k\}$ be monotone $-$ $(\beta_k(z)-\beta_k(u))\cdot(z-u)\geq 0$ $\forall k\in\mathbb{N}$

• $\beta_k \rightrightarrows \beta$ for all $K\subset X$ compact

▶ Let • β is strictly monotone $-$ $(\beta(z)-\beta(u))\cdot(z-u)>0$ for $z\neq u$

▶ Let $\{z_k\}\subset X$ and $z\in X$ fulfil

$$\lim_{k\to\infty} \underbrace{(\beta_k(z_k)-\beta_k(z))}_{(f(q^\varepsilon)\ -\ f(q))}\cdot\underbrace{(z_k-z)}_{(q^\varepsilon-q)} = 0$$

$\Rightarrow (\beta(z)-\beta(\xi))\cdot(z-\xi)=0 \Rightarrow z=\xi$

$\forall\delta>0\ \exists k_0\ \forall k\geq k_0$

Then $z_k \longrightarrow z$ in X

Proof By contradiction. If z_k does not converge to z, then $\exists\,\delta>0$ and a subsequence, still denoted z_k, $|z_k-z|\geq\delta$ for $\forall k$.

Set $t_k := \dfrac{\delta}{|z_k-z|}$ and $\xi_k := t_k z_k + (1-t_k)z$. Pal

$t_k\in(0,1]$

$|\xi_k - z| = \dfrac{\delta}{|z_k-z|}|z_k-z| = \delta \Rightarrow \{\xi_k\}$ is bdd $\Rightarrow |\xi-z|=\delta$

$\Rightarrow \exists$ subsequence, still denoted ξ_k, a $\xi\in X$: $\xi_k\longrightarrow\xi$ in X

From monotonicity of β_k $\quad 1\geq t_k\geq 0$

$$(\beta_k(\xi_k)-\beta_k(z))\cdot(z_k-z) \geq (\beta_k(\xi_k)-\beta_k(z))\cdot(\xi_k-z) \geq 0 \quad \text{mon. of } \beta_k$$

$\xi_k - z = t_k(z_k-z)$

Also

$$(\beta_k(z_k)-\beta_k(\xi_k))\cdot(z_k-z) \geq (\beta_k(z_k)-\beta_k(\xi_k))\cdot(z_k-\xi_k)\geq 0$$

$z_k-\xi_k = (1-t_k)(z_k-z)$

$$(\beta_k(z_k)-\beta_k(z))\cdot(z_k-z) = \pm\beta_k(\xi_k) \geq 0 + \boxed{(\beta_k(\xi_k)-\beta_k(z))\cdot(z_k-z)} \geq 0$$

$\Rightarrow$

$$\Rightarrow \boxed{\lim_{k\to\infty} \left(\beta_k(\xi_k) - \beta_k(z)\right)\cdot(\xi_k - z) = 0}$$

$$\Rightarrow \left(\beta(\xi) - \beta(z)\right)\cdot(\xi - z) = 0 \Rightarrow \xi = z \;\; ↯$$

as $|\xi - z| = \delta$ ∎

SUMMARY Apparently we looked for a very minor modification of heat equation

$$\frac{\partial u}{\partial t} - \mathrm{div}\,\vec{q} = g \qquad \vec{q} \sim \nabla u \text{ in nice nonlinear way}$$

But to perform the large-data global in-time $\exists!$ of weak solution we needed:

- monotone operator theory (in the case where is sudden loss of info)

  strict monotonicity essential
- regularization of ∞-Laplacean
- higher differentiability techniques in space, in time

  but it gives regularity only for $a \in \left(0, \frac{2}{d+1}\right)$

  $\Downarrow$

  almost everywhere conv. for $q^\varepsilon \to q$
- with help of
  - Egorov
  - Chacon
  - Fatou $\Rightarrow q \in L^1(Q)$

~~$\int |q| \le \liminf_{\varepsilon\to 0} \int |q^\varepsilon| < +\infty$~~

- renormalization $\left(T_k(|q^\varepsilon|)\right)$ $\begin{matrix} k \to \infty \\ \varepsilon \to 0+ \end{matrix}$

Cauchy-Schwarz ineq. working with $|\cdot|\, 1_{A(q^\varepsilon)}$ ∎