Ad 4) v^ε

(E_ε) $\quad \frac{1}{2}\|v^\varepsilon(t,\cdot)\|_2^2 + \nu_* \int_0^t \|\nabla v^\varepsilon\|_2^2 \leq \frac{1}{2}\|v_0^\varepsilon\|_2^2$

$\nabla v^\varepsilon \to \nabla v$

$\varepsilon \to 0$

- $\overline{\lim\limits_{\varepsilon\to 0}}\,(a_\varepsilon + b_\varepsilon) \geq \overline{\lim}\, a_\varepsilon + \underline{\lim}\, b_\varepsilon$
- $v_0^\varepsilon \to v_0$ in L^2
- $v^\varepsilon \to v$ in $L^2(0,T;L^2(\Omega)) \Rightarrow \forall t \in [0,T) \setminus E$ $\quad$ zero measure

$v^\varepsilon(t,\cdot) \to v(t,\cdot)$ in $L^2(\Omega)$

$\overline{\lim\limits_{\varepsilon\to 0}}\, E_\varepsilon$

(E) $\quad \frac{1}{2}\|v(t,\cdot)\|_2^2 + \nu_* \int_0^t \|\nabla v\|_2^2 \leq \frac{1}{2}\|v_0\|_2^2$

X reflexive Banach.

$x_n \xrightarrow{w} x$ in $X \Rightarrow \|x\|_X \leq \underline{\lim}\, \|x_n\|_X$

Ad 5) by 3) a 4)

$$\|v(t,\cdot) - v_0\|_2^2 = (v(t,\cdot) - v_0, v(t,\cdot) - v_0) =$$

$$\|v(t,\cdot)\|_2^2 - 2(v(t), v_0) + \|v_0\|_2^2$$

$$\underset{2.(E)}{\leq} 2\|v_0\|_2^2 - 2\,(v(t), v_0) \xrightarrow{t\to 0+} 0$$

(v_0, v_0)

$\boxtimes$

Questions 1) Where is the pressure?
2) Suitable weak sol.
3) Regularity, weak-strong uniqueness
4) Approximations.

Remark One can show for L-H solution.: $\forall$ a.a. $t_0 \in [0,T]$ including $t_0 = 0$, $\forall\, t_1 \in (0,T)$:

$$\frac{1}{2}\|v(t_1,\cdot)\|_2^2 + \nu_* \int_{t_0}^{t_1} \|\nabla v\|_2^2 \le \frac{1}{2}\|v(t_0,\cdot)\|_2^2$$

Ad 1) Heuristic approach

$$\operatorname{div}\Big(\partial_t v - \underset{\substack{\parallel \\ v_k \frac{\partial v}{\partial x_k}}}{\operatorname{div}(v\otimes v)} - \nu_* \Delta v = -\nabla p\Big)$$

$$-\Delta p = \begin{cases} \operatorname{div}\left(v_k \frac{\partial v}{\partial x_k}\right) \\ \operatorname{div}\operatorname{div}(v\otimes v) \end{cases}$$

$$-\cancel{\operatorname{div}}(\cancel{\nabla} p) = \cancel{\operatorname{div}}\,\cancel{\operatorname{div}}(v\otimes v)$$

$$\Rightarrow \boxed{p \sim |v|^2}$$

$$v \in L^\infty(0,T;L^2) \cap L^2(W^{1,2}) \hookrightarrow L^s(0,T;L^q(\Omega))$$

$$2 \le q \le 6, \quad 2 \le s \le +\infty \qquad s = q = \frac{10}{3}$$

$$v \in L^{10/3}((0,T)\times\Omega) \Rightarrow |v|^2 \in L^{5/3}$$

Conjecture: $p \in L^{5/3}((0,T)\times\Omega)$

Proof of the conjecture will follow and will be based on the results for evolutionary Stokes system

Lemma (Global linear estimates for ev. Stokes s. or maximal $L^s(L^q)$-regularity)

Let $\Omega \in C^{2,\mu}$, $\Omega \subset \mathbb{R}^3$, $\mu \in (0,1)$, $s, q \in (1,+\infty)$, $0 \leq T \leq +\infty$.

Then, for any $u_0 \in W^{1,2}_{0,\mathrm{div}}$ a $f \in L^s(0,T; L^q(\Omega)^3)$,

$\exists!$ (u,p) solving

- $\partial_t u - \Delta u + \nabla p = f$, $\mathrm{div}\, u = 0$, $u|_{(0,T)\times\partial\Omega} = 0$, $u(0,\cdot) = u_0$ in Ω

and satisfying

$$u \in L^s(0,T'; W^{2,q}(\Omega)) \qquad \text{for } \forall T' \leq T$$

and

$$\|\partial_t u, \nabla^2 u, \nabla p\|_{L^s(0,T;L^q)} \leq C_0 \left(\|u_0\|_{1,2} + c_1 \|f\|_{L^s(0,T;L^q)}\right)$$

Theory:

- Solonnikov (1977) $s = q$ but $C_0 = C(T)$
- Sohr, von Wahl (1986) $s \neq q$
- Giga, Sohr (1991) $s \neq q$

Parabolic problems:

- Ladyzhenskaya, Solonnikov, Ural'tseva 1968 $q = s$
- von Wahl 1982 $q \neq s$
- Krylov 1997
- Amann; Lunardi.

Application to NS

$$\frac{\partial v}{\partial t} - \Delta v + \nabla p = v_2 \frac{\partial v}{\partial x_2} = f \quad \text{NS}$$

$\boxed{\Omega \in C^{2,\mu},\ v_0 \in W^{1,2}_{0,\mathrm{div}}}$

$\nabla p \in L^{5/4}(Q_T)$ ⟵ previous Lemma ($10/3$, 2, $5/4$)

$$\frac{1}{\alpha} = \frac{1}{2} + \frac{3}{10} = \frac{8}{10} = \frac{4}{5}$$

$$v_2 \frac{\partial v}{\partial x_2} \in L^{5/4}((0,T)\times\Omega)$$

$$f \in L^s(L^q)$$

$$s = q = 5/4$$

Taking p s.t. $\int_\Omega p\,dx = 0$

Poincaré:

$$p \in L^{5/4}(0,T; L^{\beta}(\Omega))$$

$$\beta: \quad W^{1,5/4} \hookrightarrow L^{\beta}(\Omega)$$

How to modify this scheme to get $p \in L^{5/3}((0,T)\times\Omega)$

Aim: $\nabla p \in L^{5/3}(0,T; L^{\gamma}(\Omega))$ $\quad W^{1,\gamma} \hookrightarrow L^{5/3}$ check! $\gamma = \frac{15}{14}$

Objective: Does $v_2 \frac{\partial v}{\partial x_2} \in L^{5/3}(0,T; L^{15/14}(\Omega))$?

Proof of $\boxed{}$:

$$\|(v\cdot\nabla)v\|_{\frac{15}{14}}^{\frac{15}{14}} = \int_\Omega |v|^{\frac{15}{14}} \underbrace{|\nabla v|^{\frac{15}{14}}}_{} dx \le$$

Hölder: $\delta = \frac{28}{15}$, $\delta' = \frac{28}{13}$

$$\int_0^T \|(v\cdot\nabla)v\|_{\frac{15}{14}}^{5/3} \le C \int_0^T \|\nabla v\|_2^2$$

$$\le \|\nabla v\|_2^{\frac{15}{14}} \|v\|_{\frac{30}{13}}^{\frac{15}{14}}$$

$$\frac{1}{5}\cdot\frac{15}{14}$$

$$\le \|\nabla v\|_2^{\frac{15}{14}} \underbrace{\|v\|_2^{(1-\lambda)\frac{15}{14}}}_{\le C} \underbrace{\|v\|_6}_{\text{imbedding}} \le$$

$$\le C \|\nabla v\|_2^{\frac{15}{14}+\frac{3}{14}} \qquad \frac{15}{14}+\frac{3}{14} = \frac{18}{14}$$

$$\|z\|_{\frac{30}{13}} \le \|z\|_2^{1-\lambda} \|z\|_6^{\lambda}$$

$$\frac{13}{30} = \frac{1-\lambda}{2} + \frac{\lambda}{6} \Rightarrow 13 = 15 - 15\lambda + 5\lambda = 15 - 10\lambda$$

$$\lambda = \frac{1}{5}$$

$\int_0$

Ad 2) Suitable weak solution adds to L-H w.s.

so-called local energy ineq.

$$\frac{1}{2}\frac{\partial}{\partial t}(|v|^2) + \operatorname{div}\left(\frac{|v|^2}{2}v\right) - \mu_* \operatorname{div}(\nabla v \cdot v) + \mu_*|\nabla v|^2 + \operatorname{div}(pv) = 0$$

$$\Big|\ \psi \in \mathcal{D}(Q_T)$$

$$2\int_\Omega dx$$

$$\int_\Omega \frac{\partial}{\partial t}|v|^2 \psi - \int_\Omega \frac{|v|^2}{2} v \cdot \nabla\psi \, dx + \mu_* \int_\Omega \nabla|v|^2 \cdot \nabla\psi$$

$$+2\mu_* \int_\Omega |\nabla v|^2 \psi - 2\int_\Omega p\vec{v} \cdot \nabla\psi \, dx = 0$$

$$\int_0^t$$

$$\int_\Omega |v(t,\cdot)|^2 \psi + 2\mu_* \int_0^t \int_\Omega \psi |\nabla v|^2 dx \leq$$

$$= \int_\Omega |v|^2 \left(\frac{\partial\psi}{\partial t} + v \cdot \nabla\psi - \mu_* \Delta\psi\right) + 2\int p\vec{v} \cdot \nabla\psi$$

$$\forall \psi \in \mathcal{D}(Q_T)^\Omega$$

Why it is useful?

In 3D, one ~~cannot~~ use v as the test function in weak formulation!

It replaces this deficiency.

It is used to estimate size of possible ~~singular~~ points where singularity of the NS eq. can occur.

Theorem (Leray 1933) • $\Omega \subset \mathbb{R}^3$ bdd smooth or $\mathbb{R}^3$

$T \in (0, +\infty)$ • Let v be Leray-Hopf w.s. in $(0,T) \times \Omega$ satisfying strong energy inequality (t_0, t_1)

Let $\Sigma := \{ t \in (0,T] ;\ v$ is singular at (x_0, t) for some $x_0 \in \Omega \}$

Then $H^{1/2}(\Sigma) = 0$ — $1/2$ Hausdorff measure.

Books on NSE

- Constantin, Foias : NSE, Chicago Press, 1988
- Temam : NSE : analysis & numerics, 2001
  5th edition, AMS
- Tai-Peng Tsai : Lectures on NSEs
  AMS, 2017.

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