cvičení MAF II 3.6.2021

KONVOLUČNÍ ROVNICE

na dnes • $y * g = \boxed{f}$ motivace

vstup, zadaná distribuce filtr, výstup

často fundamentální řešení:

$$\boxed{y_F} * g = \delta$$

PROČ fundamentální? :

$$(y_F * f) * g \overset{\text{komutativita?}}{=} (f * y_F) * g \overset{!}{=} f * (\underbrace{y_F * g}_{\delta}) \overset{?}{=} f$$

Připomenutí: konvoluce distribucí

Buď $T, G \in \mathcal{D}'(\mathbb{R}^n)$. Pokud $\exists$ jednozn.

$$\langle T*G, \varphi \rangle := \lim_{k\to\infty} \langle T(x) \otimes G(y), \eta_k(x,y)\,\varphi(x+y) \rangle$$

seřezání

$$= \langle T(x), \langle G(y), \varphi(x+y) \rangle \rangle$$

• δ je jednička vzhledem $*$: $T \in \mathcal{D}'$: $\boxed{T*\delta = \delta * T = T}$

• $\exists\, T*G \implies \exists\, G*T$ a $\boxed{G*T = T*G}$ (komut. o.k.)

• asociativita T, G, H $\quad (T*G)*H \overset{?}{=} T*(G*H)$

| bolwześ NE ! |

a) $(T_1 * D\delta_0) * T_H = \underline{T_0}$!

ale:

b) $T_1 * (\underline{D\delta_0 * T_H}) = T_1$

vlastnosti: $T * G \ \ldots \ \exists D^\alpha (T * G)$

$$\boxed{D^\alpha (T * G) = D^\alpha T * G = T * D^\alpha G}$$

a) $(T_1 * D\delta_0) * T_H = (\underbrace{DT_1}_{} * \delta) * T_H = (*)$

$DT_1 = T_0$

$\langle DT_1, \varphi \rangle = -\langle T_1, D\varphi \rangle = -\int_{-\infty}^{\infty} 1 \cdot \varphi' \, dx = -\varphi \Big]_{-\infty}^{+\infty} = 0$

$\langle D^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, D^\alpha \varphi \rangle$ $\quad\quad \langle T_0, \varphi \rangle \quad \forall \varphi \in \mathcal{D}$

$(*) = T_0 * T_H = \underline{T_0}$

$T_1 * (D\delta_0 * T_H) = T_1 * (\delta_0 * \underbrace{DT_H}_{\delta_0}) = T_1 * \delta_0 = \underline{T_1}$

δ_0

alternativně:

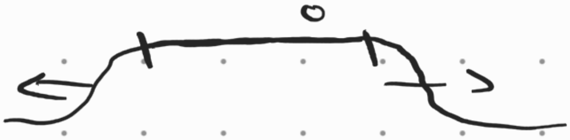
$\langle T_1 * D\delta, \varphi \rangle = \lim \langle T_1(x) \otimes D\delta(y), \eta_k^{(x,y)} \varphi(x+y) \rangle$

$$= \lim_{k\to\infty} \langle T_1(x), \langle D\delta(y), \eta_k(x,y)\,\varphi(x+y)\rangle\rangle$$

$$= \lim_{k\to\infty} \langle T_1(x), -\langle \delta(y), \frac{\partial \eta_k}{\partial y}\,\varphi(x+y) + \eta_k^{(x,y)}\,\varphi'(x+y)\rangle\rangle$$

$$= \lim_{k\to\infty} \langle T_1(x), -\left(\frac{\partial \eta_k}{\partial y}(x,0)\,\varphi(x) + \eta_k(x,0)\,\varphi'(x)\right)\rangle$$

$$= -\lim_{k\to\infty} \int_{-\infty}^{\infty} \underbrace{\frac{\partial \eta_k}{\partial y}(x,0)}_{\to 0}\,\varphi(x) + \underbrace{\eta_k(x,0)}_{\to 1}\,\varphi'(x)\,dx$$



$$= -\int_{-\infty}^{\infty} \varphi'(x)\,dx = 0 = \langle T_0, \varphi\rangle$$

∎

F.T. konvoluce • $T, G \in S'(\mathbb{R})$ a G má kompaktní nosič

$$\mathcal{F}(T * G) = \mathcal{F}(T)\cdot \underbrace{\mathcal{F}(G)}_{g}$$

$g \in \Theta_M$ (pomalu rost.)

• $T \in S'$, $\eta \in S$

$$\mathcal{F}(T * G_\eta) = \mathcal{F}(T) \cdot \mathcal{F}(G_\eta)$$

$$T_\eta \in S$$

z pohledu počítání (v kontextu konvol. rovnic)

počítáme formálně a pak ... zpětně ověřujeme korektnost

... zkouška

Př Řešme $y * \underline{T_{e^{-\alpha|x|}}} = \delta$ v S', $\alpha > 0$

postup: aplikuji formálně F.T.

$$\mathcal{F}(y) \cdot \underbrace{\mathcal{F}(T_{e^{-\alpha|x|}})} = 1$$

$$\Big\| \quad (e^{-\alpha|x|} \in L^1$$

$$T_{\mathcal{F}(e^{-\alpha|x|})} = T_{\frac{2\alpha}{4\pi^2\xi^2+\alpha^2}}$$ (bylo na přednášce)

$$\underline{\mathcal{F}(y)} \cdot \underline{\frac{2\alpha}{4\pi^2\xi^2+\alpha^2}} = 1$$

$$\boxed{\mathcal{F}(y) = \frac{4\pi^2\xi^2}{2\alpha} + \frac{\alpha}{2}}$$

$$\left\langle \mathcal{F}(y), \frac{2\alpha}{4\pi^2\xi^2+\alpha^2}, \varphi \right\rangle = \langle T_1, \varphi \rangle \quad \forall \varphi \in S$$

$$\|$$

$$\left\langle \mathcal{F}(y), \underbrace{\frac{2\alpha}{4\pi^2\xi^2+\alpha^2} \cdot \varphi}_{\psi} \right\rangle = \langle T_1, \varphi \rangle$$

$$\overset{\|}{(\ast)}$$

$$(\ast) = \left\langle T_1, \frac{4\pi^2\xi^2+\alpha^2}{2\alpha} \cdot \overbrace{\frac{2\alpha}{4\pi^2\xi^2+\alpha^2}\varphi}^{\psi} \right\rangle \quad \forall \varphi \in S \quad \forall \psi$$

$$\psi$$

$$= \left\langle T_1 \cdot \frac{4\pi^2\xi^2+\alpha^2}{2\alpha}, \psi \right\rangle$$

$$\langle \mathcal{F}(y), \psi \rangle = \left\langle T_1 \cdot \frac{4\pi^2\xi^2+\alpha^2}{2\alpha}, \psi \right\rangle \quad \forall \psi \in S$$

$\llcorner$

služal $\quad y = \mathcal{F}^{-1}(\mathcal{F}(y)) =$

$$= \mathcal{F}^{-1}\left(\frac{2\pi^2}{\alpha}\xi^2\right) + \mathcal{F}^{-1}\left(\frac{\alpha}{2}\right) =$$

$$= \frac{2\pi^2}{\alpha}\mathcal{F}^{-1}(\xi^2) + \frac{\alpha}{2}\mathcal{F}^{-1}(1)$$

$$\boxed{\mathcal{F}^{-1}(\xi^n) = \left(\frac{1}{2\pi i}\right)^n D^n \delta}$$

$$= \frac{2\pi^2}{\alpha} \frac{\delta''}{-4\pi^2} + \frac{\alpha}{2}\delta$$

$$= -\frac{1}{2\alpha}\delta'' + \frac{\alpha}{2}\delta$$

$\in S'$ s kompaktním nosičem

... můžeme projít postup výpočtu s nalezenou y a ověřit korektnost

... zkouška:

počítejme $\left(\frac{\alpha}{2}\delta - \frac{1}{2\alpha}\delta''\right) * T_{e^{-\alpha|x|}} =$

$$= \frac{\alpha}{2}\underbrace{\delta * T_{e^{-\alpha|x|}}}_{= T_{e^{-\alpha|x|}}} - \frac{1}{2\alpha}\underbrace{\delta'' * T_{e^{-\alpha|x|}}}_{= \delta' * D T_{e^{-\alpha|x|}}} =$$

$$= \delta * D^2 T_{e^{-\alpha|x|}}$$

$$= D^2 T_{e^{-\alpha|x|}}$$

$$= \frac{\alpha}{2} T_{e^{-\alpha|x|}} - \frac{1}{2\alpha} \boxed{D^2 T_{e^{-\alpha|x|}}} = (\ast\ast)$$

Souvislost klasické a distributivní derivace

<u>věta</u>: nechť $n \in \mathbb{N}$ a $f \in \tilde{C}^n(\mathbb{R}\setminus\{0\})$.

$$D^n T_f = \underbrace{T_{D^n f}}_{\text{klasická derivace } f \text{ tam, kde existuje}} + \sum_{k=0}^{n-1} A_{n-1-k}\, \delta^{(k)}$$

kde $A_k \overset{\text{def}}{=} f^{(k)}(0+) - f^{(k)}(0-)$

$\llcorner$

speciálně pro $f \in \tilde{C}^1(\mathbb{R}\setminus\{0\})$

$$DT_f = T_{Df} + [f(0+) - f(0-)]\delta$$

$\llcorner$

chci $\underline{D^2 T_{e^{-\alpha|x|}}} = D\Big(T_{De^{-\alpha|x|}} + \cancel{0\delta}\Big)$

$$De^{-\alpha|x|} = \underline{-\alpha \operatorname{sgn} x\, e^{-\alpha|x|}}$$

$$= T_{(-\alpha\,\mathrm{sgn}\,x)^2 e^{-\alpha|x|}} + (-2\alpha)\delta$$

$\quad\ \ \shortparallel$
$\quad\ \ \alpha^2$

<u>POZOR</u> : POZOR na operace funkcí

$$D\,T_{(\mathrm{sgn}\,x)^2} \cancel{=} \underline{2(2\delta)\cdot \mathrm{sgn}\,x} \quad ↯$$

$\shortparallel \qquad\qquad\qquad$ ↓ není definované $\nu\,\mathcal{D}'$

T_1

$\shortparallel$

T_0

$$\ldots| = \cancel{\frac{\alpha}{2} T_{e^{-\alpha|x|}}} - \frac{1}{2\alpha}\left(\cancel{T_{\alpha^2 e^{-\alpha|x|}}} - 2\alpha\delta\right)$$

$$= \delta \quad ↯$$

Př $\qquad y * \underline{\sin\alpha|x|} = \underline{\delta} \qquad \alpha > 0$

$/\mathcal{F}$

$$\mathcal{F}(y)\cdot \mathcal{F}(\sin\alpha|x|) = 1$$

$\qquad\qquad\qquad\quad ↯_1$

$$\underline{\sin\alpha|x|} = \underline{\underline{\mathrm{sgn}\,x}}\cdot \underline{\underline{\sin\alpha x}}$$

$$\mathcal{F}(\sin\alpha|x|) = \mathcal{F}(\mathrm{sgn}\,x) * \mathcal{F}(\sin\alpha x)$$

$\qquad\qquad\qquad\qquad \shortparallel \qquad\qquad\qquad\qquad\qquad \shortparallel$

$$\frac{1}{i\pi}\ \mathrm{v.p.}\frac{1}{\xi} \qquad\qquad \mathcal{F}\left(\frac{e^{i\alpha x} - e^{-i\alpha x}}{2i}\right) = \frac{1}{2i}\left(\delta_{\frac{\alpha}{2\pi}}^{\xi} - \delta_{\frac{-\alpha}{2\pi}}^{\xi}\right)$$

$$= \frac{-1}{2\pi}\, \underline{v.p.\frac{1}{\xi}} * \left(\delta_{\frac{\alpha}{2\pi}} - \delta_{-\frac{\alpha}{2\pi}}\right) =$$

$$\left[\; T * \delta_a = \underline{T_{-a}} \quad (\text{sami}) \;\right]$$

$$\boxed{= \frac{1}{-2\pi}\left(v.p.\, \underline{\frac{1}{\xi - \frac{\alpha}{2\pi}}} - v.p.\, \underline{\frac{1}{\xi + \frac{\alpha}{2\pi}}} \right)} \; G$$

$$\left\langle v.p.\frac{1}{\xi - \frac{\alpha}{2\pi}}, \varphi \right\rangle \overset{def}{=} \lim_{\varepsilon \to 0+} \int_{\left|\xi - \frac{\alpha}{2\pi}\right| > \varepsilon} \frac{\varphi(\xi)}{\xi - \frac{\alpha}{2\pi}}\, d\xi$$

$$\langle G, \varphi \rangle = \lim_{\varepsilon \to 0+} \int_{-\infty}^{-\frac{\alpha}{2\pi} - \varepsilon} \Big(\underbrace{\frac{1}{\xi - \frac{\alpha}{2\pi}} - \frac{1}{\xi + \frac{\alpha}{2\pi}}}_{} \Big) \varphi(\xi)\, d\xi$$

$$+ \int_{-\frac{\alpha}{2\pi} + \varepsilon}^{\frac{\alpha}{2\pi} - \varepsilon} \text{— " —}\ \varphi(\xi)\, d\xi$$

$$+ \int_{\frac{\alpha}{2\pi} + \varepsilon}^{\infty} \text{— " —}\ \varphi(\xi)\, d\xi$$

$$\overset{def}{=} 4\pi\alpha\ v.p. \int \frac{\varphi(\xi)}{\quad}$$ (ve singular

$$\ldots \frac{\xi}{4\pi^2\xi^2 - \alpha^2} \qquad \text{výjde}\,)$$

$$= \left\langle \underline{4\pi\alpha \text{ v.p.}} \frac{1}{4\pi^2\xi^2 - \alpha^2}, \varphi \right\rangle$$

jediný kandidát

$$\mathcal{F}(y) = \frac{1}{4\pi\alpha} \underline{\left(4\pi^2\xi^2 - \alpha^2\right)} (-2\pi)$$

$$= -\frac{2\pi^2}{\alpha}\underline{\xi^2} + \frac{\alpha}{2} 1$$

$$y = -\frac{2\pi^2}{\alpha} \underbrace{\mathcal{F}^{-1}(\xi^2)}_{\frac{\delta''}{(2\pi i)^2}} + \frac{\alpha}{2}\underbrace{\mathcal{F}^{-1}(1)}_{\delta} =$$

$$= \frac{1}{2\alpha}\delta'' + \frac{\alpha}{2}\delta \qquad \in \mathcal{S}' \text{ s komp. nosičem}$$

zkouška: (sami) ...

- Buď f omezená spojitá fce na $\mathbb{R}$

$0 < \lambda < 1$, $a > 0$

Najděte $y \in \mathcal{S}'$:

• $\lambda y(x-a) + \frac{1}{\lambda} y(x+a) = f(x)$ в S'

$$y * \left(\lambda \delta_a + \frac{1}{\lambda}\delta_{-a}\right) = f \quad \text{в } S'$$

F.T.

$$\mathcal{F}(y)\cdot\left(\lambda \underbrace{\mathcal{F}(\delta_a)}_{= e^{-2\pi i a\xi}} + \frac{1}{\lambda}\underbrace{\mathcal{F}(\delta_{-a})}_{= e^{2\pi i a \xi}}\right) = \mathcal{F}(f)$$

$$\mathcal{F}(y)\left(z + \frac{1}{z}\right) = \mathcal{F}(f)$$

(~~Rdle~~ $z \overset{def}{=} \lambda e^{-2\pi i \xi a}$

$$\frac{z^2+1}{z}$$

$$\mathcal{F}(y) = \mathcal{F}(f)\cdot\frac{z}{z^2+1} = \mathcal{F}(f)\, z\, \frac{1}{1-(-z^2)} =$$

$$|z| = \lambda < 1$$

$$= \mathcal{F}(f)\, z \sum^{\infty} (-1)^n (z)^{2n}$$

$$n=0$$

$$= \mathcal{F}(g) \cdot \sum_{n=0}^{\infty} (-1)^n \lambda^{2n+1} \underline{e^{-2\pi i a \xi (2n+1)}}$$

$$y = \underbrace{\mathcal{F}^{-1}(\mathcal{F}(g))}_{=\, g} * \sum_{n=0}^{\infty} (-1)^n \lambda^{2n+1} \underbrace{\mathcal{F}^{-1}\left(e^{-2\pi i a (2n+1) \xi}\right)}_{=\, \delta_{(2n+1)a}}$$

$$= g * \sum_{n=0}^{\infty} (-1)^n \lambda^{2n+1} \underline{\delta_{(2n+1)a}}$$

$$= \sum_{n=0}^{\infty} (-1)^n \underbrace{\lambda^{2n+1}}_{<1} \underbrace{g(x-(2n+1)a)}_{\text{omezená}} \qquad K$$

<u>ověřte sami zkouškou</u> (explicitním výpočtem)