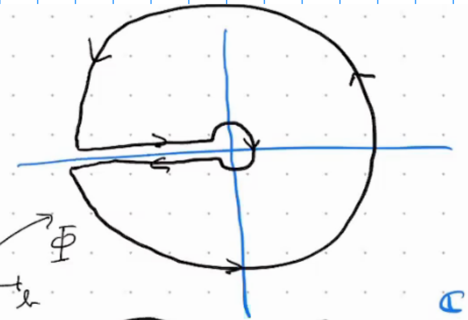
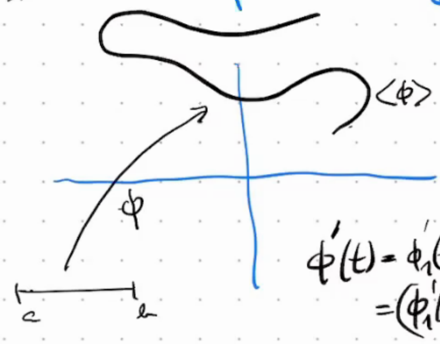
Křivkový integrál

- Nechť γ je regulární křivka (nebo konečný součet reg. křivek) a ϕ je její parametrizace (kladná pokud jde proti směru hodinových ručiček)
- $f: \mathbb{C} \to \mathbb{C}$ spojitá na množině obsahující γ



$$\boxed{\int_\gamma f(z)\,dz := \int_a^b f(\phi(t))\,\phi'(t)\,dt}$$



$$\phi'(t) = \phi_1'(t) + i\phi_2'(t) = (\phi_1'(t), \phi_2'(t))$$

VAROVÁNÍ: JDE O NOVÝ TYP KŘIVKOVÉHO INTEGRÁLU

Věta 15.4 (O vztahu $\int_\gamma f(z)\,dz$ s křivkovým integrálem 2. druhu)

Buď $f(z) = f_1(z) + i f_2(z)$. Pak

$$\int_\gamma f(z)\,dz = \int_\gamma (f_1, -f_2)\cdot \vec{ds} + i \int_\gamma (f_2, f_1)\cdot \vec{ds}$$

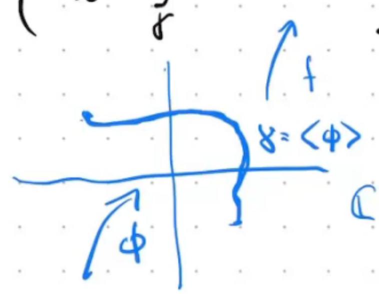
křivkový integrály 2. druhu

Tvrzení (odhad komplexního křivkového integrálu)

- Buď γ regulární křivka v $\mathbb{C}$ délky ℓ $\quad \left(\ell := \int_\gamma ds \right)$ křivk. integrál 1. druhu
- Buď $f: \mathbb{C} \to \mathbb{C}$

Pak

$$\left| \int_\gamma f(z)\,dz \right| \le \sup_{z\in\gamma} |f(z)|\,\ell$$



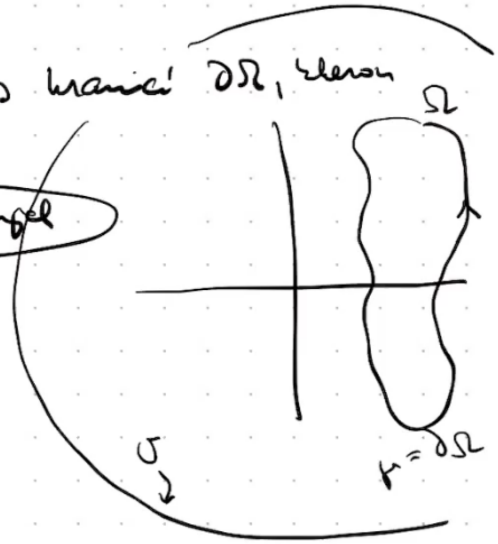
Tvrzeníčko

Je-li $g: (a,b) \to \mathbb{C} \Rightarrow \left| \int_a^b g\,dt \right| \le \int_a^b |g|\,dt$

$\boxed{\text{Věta 15.5}}$ (Cauchy–Goursatova věta)

- Buď $\Omega \subset U \subset \mathbb{C}$ omezená, otevřená s hranicí $\partial\Omega$, kterou lze zapsat jako součet regulárních křivek (vhodně orientovaných)
- Nechť $f \in H(U)$

Pak $\displaystyle\int_{\partial\Omega} f(z)\,dz = 0$



$\boxed{\text{Def}}$ Primitivní fce F k f

Řekneme, že $F : \mathbb{C} \to \mathbb{C}$ je PF k $f : \mathbb{C} \to \mathbb{C}$ v $\Omega \subset \mathbb{C}$ otevř.

$\overset{\text{def}}{\iff}$ $F'(z) = f(z) \quad \forall z \in \Omega$.

$\boxed{\text{Věta 15.6}}$ Nechť $\Omega \subset \mathbb{C}$ otevřená, omezená a souvislá a $f : \Omega \to \mathbb{C}$ spojitá.

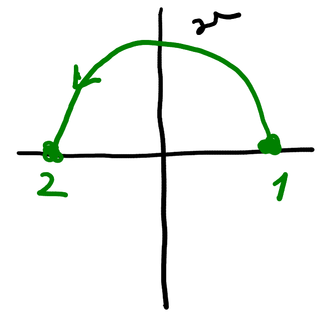
Pak násl. podmínky jsou ekvivalentní:

(i) f má v Ω primitivní funkci

(ii) Pro $\forall$ součet uzavř. regulárních křivek γ : $\displaystyle\int_\gamma f(z)\,dz = 0$

(iii) Komplexní křivkový integrál nezávisí na volbě křivky

- $\int_\gamma z\,dz$ $\quad\gamma$ půlkružnice $|z|=1$, $(1,0)\longrightarrow(-1,0)$, přes horní polorovinu

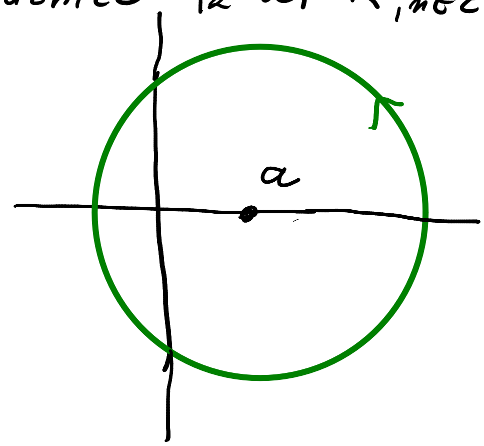


γ: $z=e^{i\varphi}$ $\quad \varphi\in[0,\pi]$ $\quad \swarrow \frac{d}{d\varphi}$

$$\int_\gamma z\,dz \underset{z\ \text{def}}{=} \int_0^\pi e^{i\varphi}(e^{i\varphi})'\,d\varphi =$$

$$= \int_0^\pi i\,e^{2i\varphi}\,d\varphi = \frac{1}{2}\left[e^{2i\varphi}\right]_0^\pi = e^{2\pi i}-e^0 = 0$$

- $\int_\gamma (z-a)^n\,dz$ $\quad\gamma$: kl. orientovaná kružnice $|z-a|=R$, $n\in\mathbb{Z}$, $a\in\mathbb{C}$



γ: $z=a+Re^{i\varphi}$ $\quad \varphi\in[0,2\pi)$

$$\int_\gamma (z-a)^n\,dz = \int_0^{2\pi}(Re^{i\varphi})^n(a+Re^{i\varphi})'\,d\varphi$$

$$= R^{n+1}\,i\int_0^{2\pi} e^{i\varphi(n+1)}\,d\varphi =$$

$$= \begin{cases} n=-1 & i\int_0^{2\pi} d\varphi = 2i\pi \\ n\neq -1 & R^{n+1}\,\frac{1}{(n+1)}\left[e^{i\varphi(n+1)}\right]_0^{2\pi} = 0 \end{cases}$$

2π periodická

- $I = \int_\gamma z e^z \, dz$ $\quad \gamma$ oblouk kubické paraboly $\gamma(t) = t + i t^3$ $\quad 0 \le t \le 1$

$z e^z \in H(\mathbb{C}) \rightarrow$ výpočet nezávisí na cestě, jen na koncových bodech

ale umíme vypočítat $F(z)$

$$\int z e^z \, dz = \begin{vmatrix} u' = e^z & u = e^z \\ v = z & v' = 1 \end{vmatrix} =$$

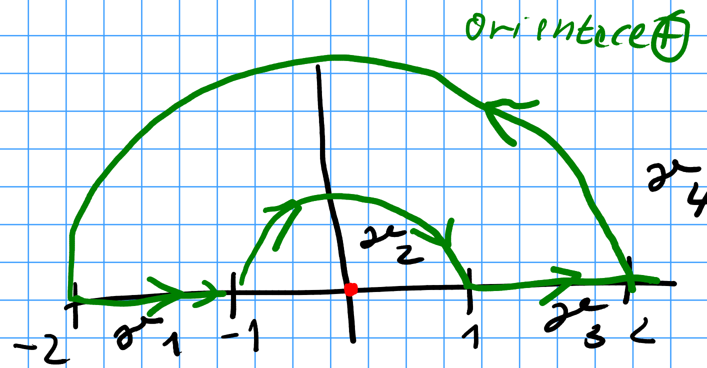
$$= z e^z - \int e^z \, dz = (z-1) e^z + C$$

$$\rightarrow F(b) - F(a) = i e^{1+i} + 1 = e\,(i \cos 1 - \sin 1) + 1$$

$a: \quad 0$

$b: \quad 1+i$

$$\int_{\gamma} \frac{z}{\bar z}\, dz$$



$\gamma_1: \quad z = t \qquad t \in [-2,-1]$

$\gamma_3: \quad z = t \qquad t \in [1,2]$

$\gamma_2: \quad z = e^{i\varphi} \qquad \varphi \in [\pi, 0]$

$\gamma_4 \quad z = 2e^{i\varphi} \qquad \varphi \in [0,\pi]$

$$f(z) = \frac{z}{\bar z} = \quad \left.\begin{matrix} ① \\ ③ \end{matrix}\right\} \frac{z}{\bar z} = 1 \qquad ② \ \frac{z}{\bar z} = \frac{e^{i\varphi}}{e^{-i\varphi}} = e^{2i\varphi} \qquad ④ \ \frac{z}{\bar z} = \frac{2e^{i\varphi}}{2e^{-i\varphi}} = e^{2i\varphi}$$

$$① \int_{\gamma_1} \frac{z}{\bar z}\, dz \underset{\text{def}}{\overset{\uparrow}{=}} \int_{-2}^{-1} 1\, dt = [t]_{-2}^{-1} = -1-(-2) = 1$$

$$③ \int_{\gamma_3} f(z)\, dz = \int_1^2 1\, dt = [t]_1^2 = 1$$

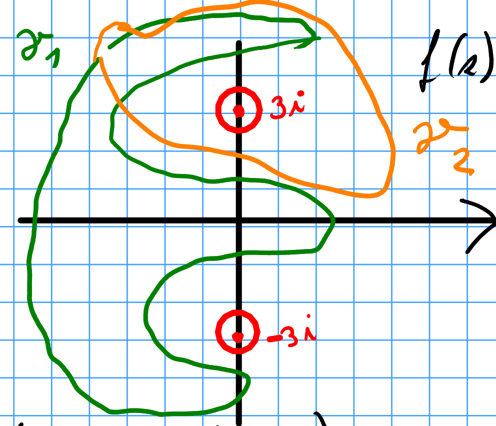
$$② \int_{\gamma_2} f(z)\, dz = \int_{\pi}^{0} e^{2i\varphi}\, i\, e^{i\varphi}\, d\varphi = i \int_{\pi}^{0} e^{3i\varphi}\, d\varphi =$$

$$= \frac{i}{3i} \left[e^{3i\varphi}\right]_{\pi}^{0} = \frac{1}{3}\left[1-(-1)\right] = \frac{2}{3}$$

$$④ \int_{\gamma_4} f(z)\, dz = \int_0^{\pi} e^{2i\varphi}\, 2i\, e^{i\varphi}\, d\varphi = -\frac{4}{3}$$

$$1 + 1 + \frac{2}{3} - \frac{4}{3} = \underline{\underline{\frac{4}{3}}}$$

- $$\int_\gamma \frac{dz}{z^2+9}$$ γ uzavretá



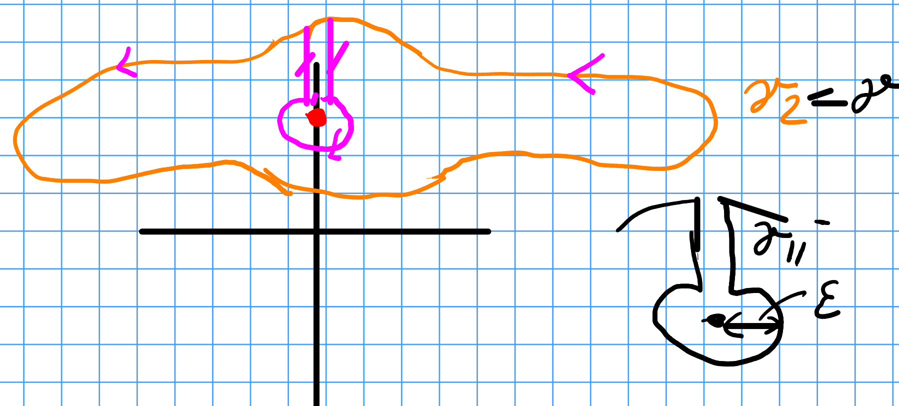
$f(z) \in H(\mathbb{C} \setminus \{\pm 3i\})$

$$f(z) = \frac{1}{z^2+9} = \frac{1}{(z+3i)(z-3i)}$$

$$= \frac{A}{z-3i} + \frac{B}{z+3i} = \frac{1}{6i}\left(\underbrace{\frac{1}{z-3i}}_{f_1} - \underbrace{\frac{1}{z+3i}}_{f_2}\right)$$

$$\int_{\gamma_1} f(z)\,dz = 0$$

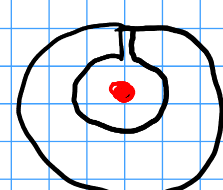
$$\int_{\gamma_2} f(z)\,dz = ?$$



$$\Gamma = \gamma + \gamma_{11}^- + \gamma_{\varepsilon}^- + \gamma_{11}^+$$

$$\int_\Gamma f(z) = 0 = \int_\gamma f(z)\,dz + \underbrace{\int_{\gamma_{11}^-} f(z)\,dz + \int_{\gamma_{11}^+} f(z)\,dz}_{0} + \int_{\gamma_\varepsilon^-} f(z)\,dz$$

$$\int_\gamma f(z)\,dz = -\int_{\gamma_\varepsilon \ominus} f(z)\,dz = \int_{\gamma_\varepsilon} f(z)\,dz$$



možnosť 1

$$f(z) = \frac{1}{z^2+9} = \frac{1}{(z+3i)(z-3i)} =$$

$$= \frac{1}{6i}\left(\frac{1}{z-3i} - \frac{1}{z+3i}\right)$$

$$6i \int_{\gamma_\varepsilon} f(z)\,dz = \int_{\gamma_\varepsilon} f_1(z)\,dz = \int_{\gamma_\varepsilon} \frac{1}{z-3i}\,dz = (*)$$

$$\gamma_\varepsilon : \quad 3i + \varepsilon e^{i\varphi} \quad \varphi \in [0, 2\pi)$$

$$(*) = \int_0^{2\pi} \frac{1}{\varepsilon e^{i\varphi}}\, i\varepsilon e^{i\varphi}\, d\varphi = i \int_0^{2\pi} d\varphi = 2\pi i$$

$$\int_{\gamma} f(z)\,dz = \int_{\gamma_\varepsilon} f(z)\,dz = \frac{2\pi i}{6i} = \underline{\frac{\pi}{3}}$$

<u>možnost 2</u>

$$f(z) = \frac{1}{z^2+9} = \frac{1}{(z+3i)\underbrace{(z-3i)}_{(3i+\varepsilon e^{i\varphi})}} =$$

$$\int_{\gamma_\varepsilon} f(z)\,dz = \int_0^{2\pi} \frac{1}{\varepsilon e^{i\varphi}(6i+\varepsilon e^{i\varphi})}\, \varepsilon i e^{i\varphi}\, d\varphi$$

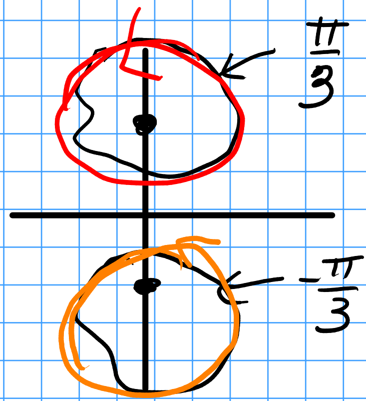
$$i \int_0^{2\pi} \frac{1}{\underset{\nearrow}{6i + \varepsilon e^{i\varphi}}}\, d\varphi$$

$$\lim_{\varepsilon \to 0^+} i \int_0^{2\pi} \frac{1}{6i + \varepsilon e^{i\varphi}}\, d\varphi \overset{\text{Lebesgue}}{=} i \int_0^{2\pi} \lim_{\varepsilon \to 0^+} \frac{1}{6i + \varepsilon e^{i\varphi}}\, d\varphi =$$

$$= \frac{i}{6i} \int_0^{2\pi} d\varphi = \frac{\pi}{3}$$

$$\left| \frac{1}{6i + \varepsilon e^{i\varphi}} \cdot \frac{\varepsilon e^{-i\varphi} - 6i}{\varepsilon e^{-i\varphi} - 6i} \right| \leq \frac{6+\varepsilon}{36 - 12\varepsilon + \varepsilon^2} \leq \frac{6+\varepsilon}{36 - 12\varepsilon}$$

$$\int_{\gamma_\varepsilon} f(z)\,dz = \frac{\pi}{3}$$



$k\,\frac{\pi}{3}$

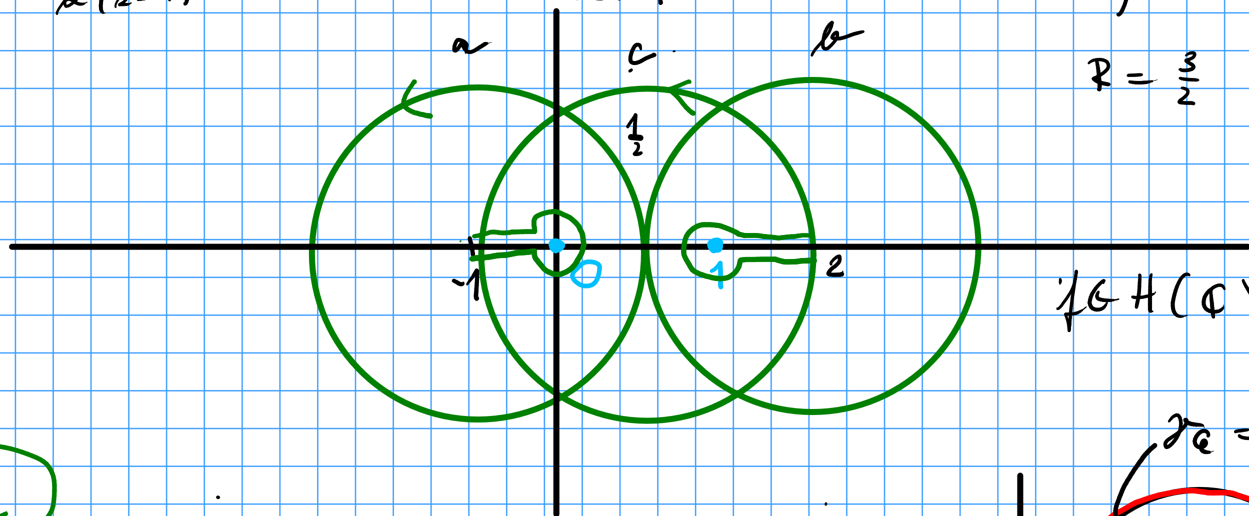
$k = \ell - m$

počet

počet

• $\frac{1}{2\pi i}\int \underbrace{\frac{e^z}{z(z-1)^3}}_{f(z)} dz$

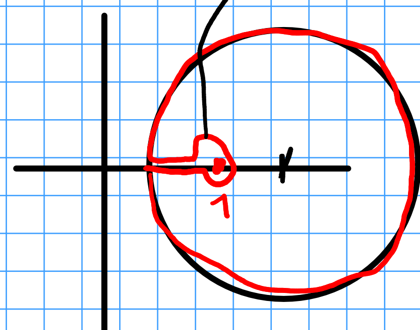
$$f(z) = \frac{e^z}{z(z-1)^3} = e^z\left(\overset{f_0}{-\frac{1}{z}} + \overset{f_{11}}{\frac{1}{z-1}} - \overset{f_{12}}{\frac{1}{(z-1)^2}} + \overset{f_{13}}{\frac{1}{(z-1)^3}}\right)$$



$R = \frac{3}{2}$

$f \in H(\mathbb{C} \setminus \{1,0\})$

$\gamma_\varepsilon = 1 + \varepsilon e^{i\varphi}$



γ_a

γ_b $\quad z = 2 + R e^{i\varphi}$

$$\int_{\gamma_b} f(z)\,dz = \int_{\gamma_\varepsilon} f(z)\,dz =$$

$$= \int_{\gamma_\varepsilon} f_{11}\,dz + \int f_{12}\,dz + \int f_{13}\,dz$$

$$\int_{\gamma_\varepsilon} \frac{e^z}{(z-1)^n}\,dz = \left| 1 + \varepsilon e^{i\varphi} \right|$$

$$= \int_0^{2\pi} \frac{e^{1+\varepsilon e^{i\varphi}}}{(\varepsilon e^{i\varphi})^n}\,\varepsilon i e^{i\varphi}\,d\varphi = ie\int_0^{2\pi} \frac{e^{\varepsilon e^{i\varphi}}}{(\varepsilon e^{i\varphi})^{n-1}}\,d\varphi$$

$$= ie \int_0^{2\pi} \frac{\sum_{m=0}^{\infty} \frac{(\varepsilon e^{i\varphi})^m}{m!}}{(\varepsilon e^{i\varphi})^{n-1}} d\varphi = ie \int_0^{2\pi} \frac{1}{(n-1)!} + \sum_{\substack{m=0 \\ m \neq n-1}}^{\infty} \frac{[\varepsilon e^{i\varphi}]^{m-n+1}}{m!} d\varphi$$

$$m = n-1$$

$$m-n+1 \neq 0 \qquad e^{i\varphi k} \to \left[\frac{e^{i\varphi k}}{ik}\right]_0^{2\pi}$$

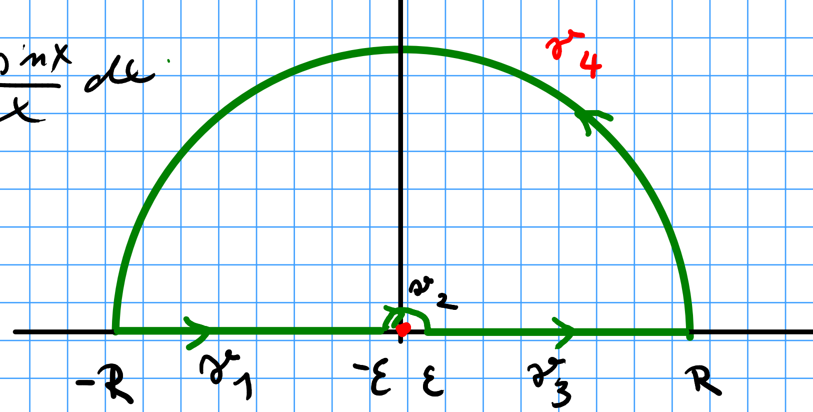
$$= ie \int_0^{2\pi} \frac{1}{(n-1)!} d\varphi = \frac{2\pi i e}{(n-1)!}$$

$$\int f_{11}\, dz + \int f_{12}\, dz + \int f_{13}\, dz = \left(\left(\frac{1}{0!} - \frac{1}{1!}\right) + \frac{1}{2!}\right) 2\pi i e =$$

$$= \pi i e \xrightarrow{\frac{1}{2\pi i}} \frac{e}{2}$$

$$\int_{\gamma_\delta} f(z)\, dz = \frac{e}{2}$$

• $I = \int_0^{\infty} \frac{\sin x}{x}\, dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx$



$f(z) = \frac{e^{iz}}{z}$

$\gamma_1 + \gamma_3$

• $\int_{\gamma_1} f(z)\,dz + \int_{\gamma_3} f(z)\,dz$ $\quad z = t \quad t \in [-R, -\varepsilon]$ γ_1

$t \in [\varepsilon, R]$ γ_3

$= \int_{-R}^{-\varepsilon} \frac{e^{it}}{t}\,dt + \int_{\varepsilon}^{R} \frac{e^{it}}{t}\,dt \xrightarrow[\varepsilon \to 0^+]{R \to \infty} \int_{-\infty}^{\infty} \frac{e^{it}}{t}\,dt$

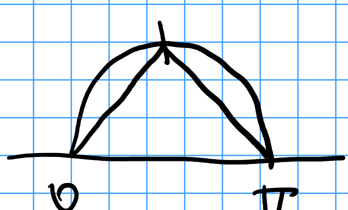
$\mathrm{Im} \int_{-\infty}^{\infty} \frac{e^{it}}{t}\,dt = I$

• γ_2

$0 \le \left| i \int_0^{\pi} \frac{e^{i(Re^{i\varphi})}}{Re^{i\varphi}} Re^{i\varphi}\, d\varphi \right| = \left| \int_0^{\pi} e^{iRe^{i\varphi}}\, d\varphi \right|$

$= \left| \int_0^{\pi} e^{iR(\cos\varphi + i\sin\varphi)}\, d\varphi \right| \le \int_0^{\pi} \left| e^{iR\cos\varphi - R\sin\varphi} \right| d\varphi$

$\le \int_0^{\pi} e^{-R\sin\varphi}\, d\varphi = 2\int_0^{\frac{\pi}{2}} e^{-R\sin\varphi}\, d\varphi \le 2\int_0^{\frac{\pi}{2}} e^{-R\frac{2}{\pi}\varphi}\, d\varphi$



$= \left[2e^{-R\frac{2}{\pi}\varphi} \left(-\frac{\pi}{2R}\right) \right]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{R}\left(1 - e^{-R}\right) \xrightarrow[R \to \infty]{} 0$

$$\int_{\gamma_2 = \gamma_{\varepsilon}^-} f(z)\,dz = \int_{\pi}^{0} \frac{e^{i(\varepsilon e^{i\varphi})}}{\varepsilon e^{i\varphi}} \left(i\varepsilon e^{i\varphi}\right) d\varphi$$

$$= -i\int_0^{\pi} e^{i(\varepsilon e^{i\varphi})}\, d\varphi$$

$$\lim_{\varepsilon \to 0^+} -i\int_0^{\pi} e^{i(\varepsilon e^{i\varphi})}\, d\varphi \overset{\text{Lebesgue?}}{=} -i\int_0^{\pi} d\varphi = -i\pi$$

$$\left| e^{i\varepsilon e^{i\varphi}} \right| = \left| e^{i\varepsilon(\cos\varphi + i\sin\varphi)} \right| =$$

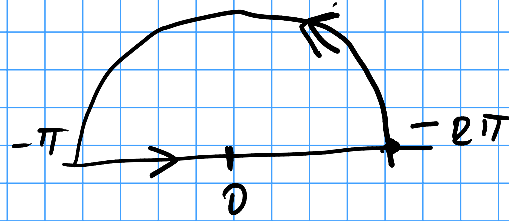
$$= \left| \underbrace{e^{i\varepsilon\cos\varphi}} e^{-\varepsilon\sin\varphi} \right| \le e^{-\varepsilon}$$

$$\underbrace{\int_{\gamma_1+\gamma_3} f(z)\,dz} + \underbrace{\int_{\gamma_{\varepsilon}^-} f(z)\,dz} + \int_{\gamma_4} f(z)\,dz = 0$$

$$\int_{-\infty}^{\infty} \frac{e^{it}}{t}\,dt \quad - i\pi \quad + 0 = 0.$$

$$\underbrace{\mathrm{Im}\int_{-\infty}^{\infty} \frac{e^{it}}{t}\,dt}_{2I} = -\mathrm{Im}(-i\pi) = \pi = 2I \to \boxed{I = \frac{\pi}{2}}$$

• $\displaystyle\int \frac{dz}{\sqrt{z}}$



$$\sqrt{z} = z^{\frac{1}{2}} = e^{\frac{1}{2}(\ln|z| + \operatorname{Arg} z)} \qquad z^{\alpha} \equiv e^{\alpha(\ln|z| + i \operatorname{Arg} z)}$$

• volme $\varphi \in (-2\pi, 0]$

$z = 1 \qquad \sqrt{1} = e^{\frac{1}{2}(\ln 1 + i\, 0)} = 1$

$$\int_\gamma \frac{dz}{\sqrt{z}} = \int_{-2\pi}^{-\pi} \frac{i e^{i\varphi}}{e^{i\frac{\varphi}{2}}}\, d\varphi = i \int_{-2\pi}^{-\pi} e^{i\frac{\varphi}{2}}\, d\varphi = \frac{2}{i}\, i \left[e^{i\frac{\varphi}{2}} \right]_{-2\pi}^{-\pi}$$

$$= 2\left(e^{-i\frac{\pi}{2}} - e^{-i\pi}\right) = 2(-i+1)$$

• volme $\varphi \in (0, 2\pi]$

$$\sqrt{1} = e^{\frac{1}{2}(\underbrace{\ln 1}_{0} + i 2\pi)} = -1$$

$$\int_\gamma \frac{dz}{\sqrt{z}} = \int_0^{\pi} i\, e^{i\frac{\varphi}{2}}\, d\varphi = 2\left[e^{i\frac{\varphi}{2}}\right]_0^{\pi} = 2(i-1)$$