MAF II - ćwiczenia 6.5.2021

(P_1) $f(x) = \dfrac{e^{-a|x|}}{|x|^s}$ $\quad s\in\mathbb{R}$: $\dfrac{0<s<1}{a>0}$

w $\mathbb{R}^1$

Należy znaleźć $\mathcal{F}(f(x))(\xi)$.

- $\boxed{f\in L^1 \;?\quad \checkmark}$
- $f\in S$? ✗ ($e^{-a|x|}\notin S$? $\wedge$)
- $f\in L^2$ (Czade)

bierzemy $\xi>0$ (bo parzysta)

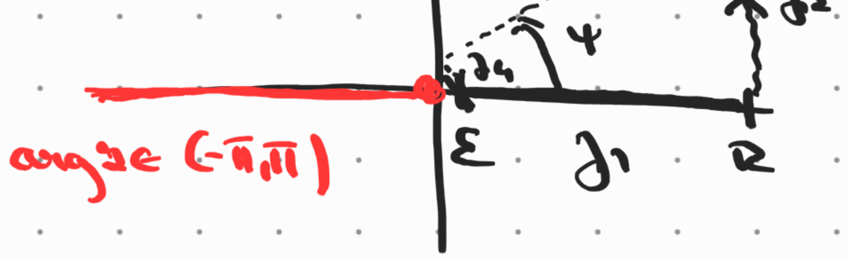
$$\mathcal{F}(f(x))(\xi) = \underline{\int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x\xi}\,dx} = \int_{-\infty}^{\infty} \underbrace{\frac{e^{-a|x|}}{|x|^s}}_{\text{parzysta!}} e^{-2\pi i x\xi}\,dx$$

$$= 2\int_0^\infty \frac{e^{-ax}}{x^s}\cos(2\pi x\xi)\,dx = 2\,\mathrm{Re}\underbrace{\int_0^\infty \frac{e^{-ax}}{x^s}\, e^{2\pi i x\xi}\,dx}_{I}$$

$$= 2\lim_{\substack{\varepsilon\to 0+\\ R\to\infty}} \int_\varepsilon^R \frac{e^{-ax}}{x^s}\, e^{2\pi i x\xi}\,dx$$

$$\underline{f(z)} \overset{\text{def}}{=} \frac{e^{-az}}{z^s}\, e^{2\pi i z\xi}$$

$\uparrow$ obrana wocnina !



zjednoduší se pro vhodné φ?

$$\gamma_1:\ z = t,\ t\in[\varepsilon,R]$$

$$\gamma_2:\ z = Re^{it},\ t\in[0,\varphi]$$

$$\gamma_3:\ z = e^{i\varphi}\cdot t,\ t\in[R,\varepsilon]$$

$$\gamma_4:\ z = \varepsilon e^{it},\ t\in[\varphi,0]$$

$$I_1 \xrightarrow[R\to\infty]{\varepsilon\to0+} J$$

$$I_2, I_4 \xrightarrow[R\to\infty]{\varepsilon\to0+} 0 \quad \underline{\text{snad}}$$

$$I_3 = -\int_\varepsilon^R \frac{e^{(2\pi i\xi - a)e^{i\varphi}\cdot t}}{\underbrace{(e^{i\varphi}\cdot t)^s}_{\overset{\shortparallel}{e^{i\varphi s}\cdot t^s}}}\, e^{i\varphi}\,dt =$$

$$= -\left(e^{i\varphi}\right)^{1-s}\int_\varepsilon^R \frac{e^{-(a-2\pi i\xi)e^{i\varphi}\cdot t}}{t^s}\,dt$$

co dál? $\boxed{e^{i\varphi} = \frac{a+2\pi i\xi}{\sqrt{a^2+4\pi^2\xi^2}}}$ $\quad \underline{a>0 \atop \xi>0}$

$$= -\left(e^{i\varphi}\right)^{1-s}\int_\varepsilon^R \frac{e^{-\sqrt{a^2+4\pi^2\xi^2}\,t}}{t^s}\,\underline{dt} \quad \Downarrow$$

$\varphi \in (0, \frac{\pi}{2})$

$$= \Big| \tilde{d} = \sqrt{a^2+4\pi^2\xi^2}\, d$$

$$= -\left(\frac{a+2\pi i\xi}{\sqrt{a^2+4\pi^2\xi^2}}\right)^{1-s} \int_{\tilde{\varepsilon}}^{\tilde{R}} \frac{e^{-\tilde{d}}}{\tilde{d}^{s}}\, d\tilde{d} \cdot \left(\sqrt{a^2+4\pi^2\xi^2}\right)^{-1+s}$$

$$\shortparallel$$

$$\left(\frac{1}{\sqrt{a^2+4\pi^2\xi^2}}\right)^{1-s}$$

$$= \textcircled{-} \left(\frac{a+2\pi i\xi}{\underline{a^2+4\pi^2\xi^2}}\right)^{\boxed{1-s}} \underbrace{\int_{\tilde{\varepsilon}}^{\tilde{R}} \frac{e^{-\tilde{d}}\, d\tilde{d}}{\tilde{d}^{\Delta}}}_{=\Gamma(1-\Delta)} = (*)$$

für $\tilde{\varepsilon} = \varepsilon \cdot \sqrt{a^2+4\pi^2\xi^2}$

$\tilde{R} = R \cdot$ —"—

$$\lim_{\substack{\tilde{\varepsilon} \to 0 \\ \tilde{R} \to \infty}} \int_{\tilde{\varepsilon}}^{\tilde{R}} \frac{e^{-\tilde{d}}\, d\tilde{d}}{\tilde{d}^{\Delta}}$$

- $\Gamma(y) = \int_0^\infty x^{y-1} e^{-x}\, dx$

$$\Gamma(1-s) = \int_0^\infty \underline{x^{-s}}\, e^{-x}\, dx$$

$$(*) \xrightarrow[R \to \infty]{\varepsilon \to 0+} -\frac{\Gamma(1-s)}{(a-2\pi i\xi)^{1-s}}$$

$$\Phi = \int f(x)dx = \Big(\boxed{\overline{T}} + T + \quad T \quad T$$

$\int_{\Gamma_\psi}$ I_2 I_3 $+ I_4$

$$\Big\downarrow \begin{matrix}\varepsilon \to 0 \\ R \to \infty\end{matrix}$$

$$-\frac{\Gamma(1-s)}{(a-2\pi i \xi)^{s-1}}$$

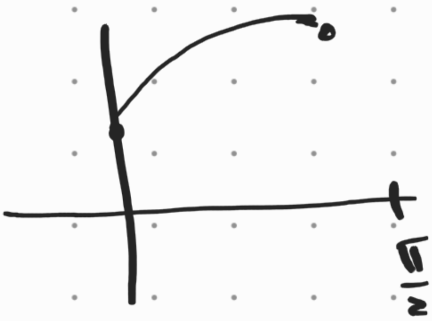
$$I_2, I_4 \xrightarrow[R\to\infty]{\varepsilon\to 0} 0$$

$$|I_2| = \left| \int_0^\psi \frac{e^{(2\pi i\xi - a)Re^{i\alpha}}}{\underset{= R^s e^{is\alpha}}{(Re^{i\alpha})^s}} \, i\underline{R}e^{i\alpha} \, d\alpha \right| \leq$$

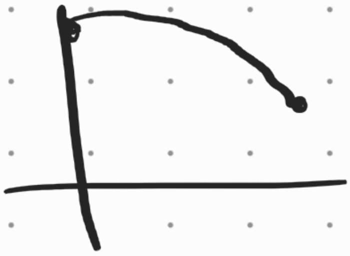
$$\leq \int_0^\psi |\cdots| \, d\alpha \leq \psi R^{1-s} \max_\alpha \left| e^{(2\pi i\xi - a)Re^{i\alpha}} \right|$$

$$\left| e^{R(2\pi i\xi - a)(\cos\alpha + i\sin\alpha)} \right| =$$

$$= e^{-R\underbrace{(a\cos\alpha + 2\pi\xi\sin\alpha)}_{h(\alpha) \text{ konkávní fce}}} \qquad \alpha \in (0, \tfrac{\pi}{2})$$



nebo



$$a\cos\alpha + 2\pi\xi\sin\alpha \geq \min(h(0), h(\psi)) > 0$$

$$\leq \frac{\pi}{2} R^{1-s} \underline{e^{-R h_m}} \xrightarrow{R \to \infty} 0$$

$$|I_4| \leq \frac{\pi}{2} \varepsilon^{1-s} \xrightarrow{\varepsilon \to 0+} 0$$

$$\boxed{\mathcal{F}(f) = 2\,\mathrm{Re}\, \frac{\Gamma(1-s)}{(a-2\pi i \xi)^{1-s}}}$$

F.T. radiálních funkcí

- víme F.T. zachovává paritu (v $\mathbb{R}^1$)
- platí více:

Lemma: (o zachování radiální symetrie pro F.T.)

Nechť $f \in S(\mathbb{R}^d)$ je radiálně symetrická (tj. $f(x) = g(|x|)$ pro $g : \mathbb{R} \to \mathbb{C}\,(\mathbb{R})$)

potom i $\mathcal{F}(f)$ je radiálně symetrická.

Speciálně: v $\boxed{\mathbb{R}^3}$!!!

$$\mathcal{F}(f(x))(\xi) = \frac{2}{\boxed{|\xi|}} \int_0^\infty \underline{\overset{f(|x|)}{\overset{\shortparallel}{g}}(\rho)}\,\rho \sin(2\pi\rho\boxed{|\xi|})\,d\rho$$

$$\shortparallel$$

$$\int_{\mathbb{R}^3} f(x) e^{-2\pi i \boxed{(x,\xi)}}\,dx$$

Použití: najděme F.T. $f(x) = \frac{1}{4\pi^2|x|^2+k^2}$

motivace (Helmholtzův operátor |

$\Delta + k^2$ $k \in \mathbb{R}$

- máme $-\Delta + k^2$ v $\mathbb{R}^3$
- hledejme tzv. fundamentální řešení

$$(-\Delta + k^2)u_0 = \delta \qquad /\mathcal{F}$$

formálně: aplikuj F.T. a využij $\mathcal{F}(\delta) = 1$

$$\mathcal{F}(-\Delta u) = ? = 4\pi^2|\xi|^2\mathcal{F}(u)$$

$$\mathcal{F}\left(\frac{\partial}{\partial x_j} u\right) = 2\pi i \xi_j \mathcal{F}(u)$$

$$\Delta = \sum_j \frac{\partial^2}{\partial x_j^2}$$

$$(4\pi^2|\xi|^2 + \underline{\varepsilon^2}) \cdot \mathcal{F}(u_0) = 1$$

$$\boxed{\underline{\mathcal{F}(u_0)} = \frac{1}{4\pi^2|\xi|^2 + \underline{\varepsilon^2}}}$$

⊥

výpočet $\mathcal{F}\left(\underbrace{\frac{1}{4\pi^2|x|^2+\varepsilon^2}}_{\text{radiální fce}}\right) = ?$ v $\mathbb{R}^3$

$$f(x) = \frac{1}{4\pi^2|x|^2+\underline{\varepsilon^2}}$$

- $f(x) \in S(\mathbb{R}^3)$? ✗
- $f(x) \in L^1(\mathbb{R}^3)$?

$$\int_{\mathbb{R}^3} |f(x)|\, dx = 4\pi^2 \int_0^\infty \underbrace{\frac{1}{4\pi^2|x|^2+\varepsilon^2}\, \underline{|x|^2}\, d|x|}_{\notin L^1(\mathbb{R}^3)}$$

- $\underline{f(x) \in L^2(\mathbb{R}^3)}$

$$\int_{\mathbb{R}^3} |f(x)|^2 \, dx = 4\pi^2 \int_0^\infty \frac{1}{(4\pi^2|x|^2+\varepsilon^2)^2} |x|^2 \, d|x|$$

$$< \infty$$

<u>Úvaha</u> : $\mathcal{F}(f) = \lim_{R\to\infty} \mathcal{F}(f \cdot \underline{\chi_{B_R(0)}})$

··· ve smyslu L1

kombinace důkazu + vzorečku pro F.T.

radiální fce :

$$\mathcal{F}(f) = \lim_{R\to\infty} \frac{2}{|\xi|} \int_0^R f(s) s \sin(\underline{2\pi s|\xi|}) ds$$

$$= \lim_{R\to\infty} \frac{2}{|\xi|} \int_0^R \frac{s}{4\pi^2 s^2+\varepsilon^2} \sin(\underline{2\pi|\xi|s}) ds$$

$$= \left| \, \alpha = 2\pi|\xi|s \right.$$

$$\tilde{R}$$

$$= \lim_{R\to\infty} \frac{2}{|\xi|} \int_0 \frac{\left(\frac{d}{2\pi|\xi|}\right) \sin d}{\left(\frac{d}{|\xi|}\right)^2 + \varepsilon^2} \frac{dd}{2\pi|\xi|}$$

$$= \lim_{\tilde{R}\to\infty} \frac{1}{2\pi^2|\xi|} \underbrace{\int_0^{\tilde{R}} \frac{d \sin d}{d^2+\varepsilon^2|\xi|^2} \, dd}_{? \; = J}$$

Sei $\tilde{R} = R \cdot 2\pi|\xi|$

$$J = \frac{1}{2} \int_{-\tilde{R}}^{\tilde{R}} \frac{d \sin d}{d^2+\varepsilon^2|\xi|^2} \, dd =$$

$$= \frac{1}{2} \operatorname{Im} \int_{-\tilde{R}}^{\tilde{R}} \frac{d}{d^2+\varepsilon^2|\xi|^2} e^{id} \, dd$$

$$f(z) = \boxed{\frac{z}{z^2+\varepsilon^2|\xi|^2}} \, e^{iz} \; : \mathbb{C} \to \mathbb{C}$$

$\searrow$ Jordan $\Delta \alpha = 1$
$\Rightarrow$ bound

$-i\ell|\xi|$

Pólus $f(z)$ $\quad z = \pm i\ell|\xi|$ $\quad$ ($\ell > 0$ BÚNO)

$$I_1 \xrightarrow{R\to\infty} 2J$$

$$I_2 \xrightarrow{R\to\infty} 0$$ (Jordan $\alpha = 1$, $\max|\tilde{f}| \sim \frac{1}{R}$)

Jordan + res. tétel:

$$G(x) = \frac{1}{4\pi^2|\xi|} \operatorname{Im}\left[2\pi i \operatorname{Res}_{i\ell|\xi|} \frac{z e^{iz}}{z^2 + \ell^2|\xi|^2}\right]$$

$$= \frac{1}{4\pi^2|\xi|} \operatorname{Im}\left[2\pi i \left.\frac{\cancel{z} e^{iz}}{2\cancel{z}}\right|_{z = i\ell|\xi|}\right]$$

$$= \frac{1}{4\pi^2} \operatorname{Im}\left[\cancel{2}\pi i \, e^{-\ell|\xi|}\right] =$$

$4\pi^2|\xi| \quad \Big[\quad \neq \quad \Big]$

$$= \frac{1}{4\pi|\xi|} e^{-\varepsilon|\xi|}$$ radialna f-cja $|\xi|$

$$\Longrightarrow u_0 \quad \left((-\Delta+\underline{\varepsilon^2})u_0 = \delta \text{ na } \mathbb{R}^3 \right)$$

$$\Longrightarrow u_0(|x|) = \frac{1}{4\pi|x|} e^{-\varepsilon|x|}$$

☒

Pozn $$\mathcal{F}^{-1}\Big(\underbrace{\frac{1}{4\pi^2|\xi|^2+\varepsilon^2}}_{} \Big)(x) =$$

- realna
- radialna

$$= \mathcal{F}\Big(\frac{1}{4\pi|\xi|^2+\varepsilon^2} \Big)(x) =$$

$$= \frac{2}{\quad} \int^{R} \frac{1}{\quad} \, |\xi| \sin(2\pi|\xi||x|$$

$$|x| \qquad {}_{0} \qquad 4\pi|\xi|^2+\varepsilon^2 \qquad \xi \qquad d|\xi|$$

$$= \frac{2}{|x|} \int_0^R \frac{1}{4\pi|\xi|^2+\varepsilon^2} |\xi| \sin(2\pi|\xi||x|) \, d|\xi|$$