Cvičení Matematika pro fyziky 1 19. října 2020
19.10.2020 17:20–18:50

Posloupnosti a řady funkcí

Bodová a stejnoměrná konvergence

Posloupnost čísel

$$\varphi_n \to A \overset{def}{\iff} (\forall \varepsilon > 0) \exists n_0 (\forall n \geq n_0) \, |\varphi_n - A| < \varepsilon$$

Posloupnost funkcí

Definice Řekneme, že f_n konvergují k f bodově v M, a píšeme $\boxed{f_n \to f \text{ v } M}$, právě když

$$(\forall z \in M)(\forall \varepsilon > 0)(\exists n_0 = n_0(\varepsilon, z))(\forall n \geq n_0) \, |f_n(z) - f(z)| < \varepsilon$$

Definice Řekneme, že f_n konvergují k f stejnoměrně v M, píšeme $f_n \rightrightarrows f$ v M pro $n \to \infty$, pokud

$$(\forall \varepsilon > 0)(\exists N_0 = N_0(\varepsilon))(\forall x \in M)(\forall n \geq n_0) \, |f_n(x) - f(x)| < \varepsilon.$$

Příklad 1 Najděme obor bodové konvergence a limitu posloupnosti

$$f_n(x) = e^x \frac{\sin x \sin 2x \cdots \sin nx}{\sqrt{n}} \qquad x \in \mathbb{R}$$

- bodová konv. $\longrightarrow$ zvol $x \in \mathbb{R}$ pevné libovolné

$$0 \leq |f_n(x)| = |\cdots| \leq \frac{e^x}{\sqrt{n}} |\sin x| |\sin 2x| \cdots |\sin nx|$$

$$\leq \frac{e^x}{\sqrt{n}} \xrightarrow{n \to \infty} 0_+ \quad \forall x \in \mathbb{R}$$

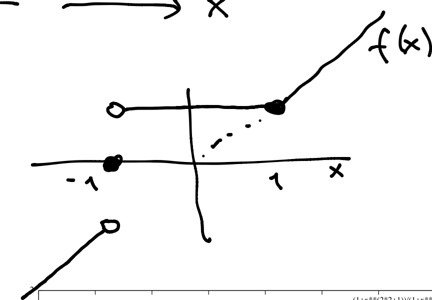
$$\lim_{n \to \infty} f_n(x) = 0 \quad \forall x \in \mathbb{R} \qquad \underline{f_n \to 0 \text{ na } (-\infty, \infty)}$$

Příklad 2 Najděme obor bodové konvergence a limitu posloupnosti

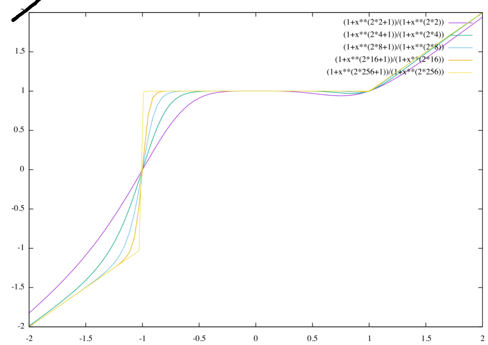
$$f_n(x) = \frac{1 + x^{2n+1}}{1 + x^{2n}} \qquad x \in \mathbb{R}$$

$$|x| < 1 \Rightarrow x^{2n+1}, x^{2n} \xrightarrow{n \to \infty} 0 \Rightarrow \lim f_n(x) = \lim \frac{1 + x^{2n+1}}{1 + x^{2n}} = 1$$

$$\begin{cases} |x|>1 \Rightarrow f_n(x) = \dfrac{x^{2n+1}\left(\frac{1}{x^{2n+1}}+1\right)}{x^{2n}\left(\underbrace{\frac{1}{x^{2n}}}_{\to 0}+1\right)} \xrightarrow{n\to\infty} x \\ |x|=1: \quad x=+1 \quad f_n(x)\to 1 \\ \qquad\qquad\; x=-1 \quad f_n(x)\to 0 \end{cases}$$



$\underline{f_n \to f}$



Příklad 3 Vyšetřeme kovergenci $f_n(x) = x^n - x^{n+1}$ na $[0,1]$

Věta 1 KRITÉRIUM STEJNOMĚRNÉ KONVERGENCE

Buď $f_n: \mathbb{C}\to\mathbb{C}$. Platí

$$f_n \rightrightarrows f \text{ v } M \text{ pro } n\to\infty \iff \sigma_n := \sup_{z\in M} |f_n(z)-f(z)| \to 0 \text{ pro } n\to\infty.$$

Bodová

$$\left.\begin{aligned} f_n(x) &= x^n - x^{n+1} = x^n(1-x) \\ f_n(0) &= 0 \;\forall n \\ f_n(1) &= 0 \;\forall n \\ x\in(0,1):\quad 0 < f_n(x) &= x^n(1-x) \xrightarrow{n\to\infty} 0 \end{aligned}\right\} \quad f_n \to 0 \text{ na } [0,1]$$

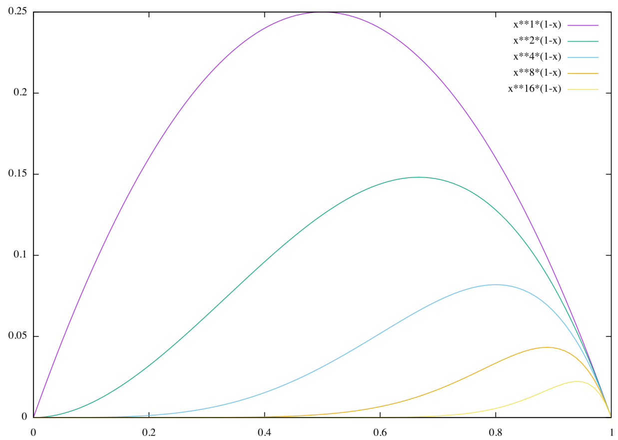
Stejnoměrná?

$$\sigma_n := \sup_{x\in[0,1]} |f_n(x) - f(x)| = \max_{x\in[0,1]} x^n(1-x)$$

$$f_n'(x) = nx^{n-1} - (n+1)x^n = x^{n-1}[n-(n+1)x] \stackrel{!}{=} 0 \iff \begin{cases} x=0 \\ x_{max} = \frac{n}{n+1} \in (0,1) \end{cases}$$

$$\sigma_n = f_n(x_{max}) = \left(\frac{n}{n+1}\right)^n\left(1-\frac{n}{n+1}\right) = \underbrace{\frac{1}{\left(1+\frac{1}{n}\right)^n}}_{\to e}\underbrace{\frac{1}{n+1}}_{\to 0} \xrightarrow{n\to\infty} 0 \checkmark$$

$\underline{f_n \rightrightarrows 0}$ na $[0,1]$



Příklad 4 Vyšetřeme kovergenci

$$f_n(x) = x^n - x^{2n} \quad \text{na } [0,1]$$

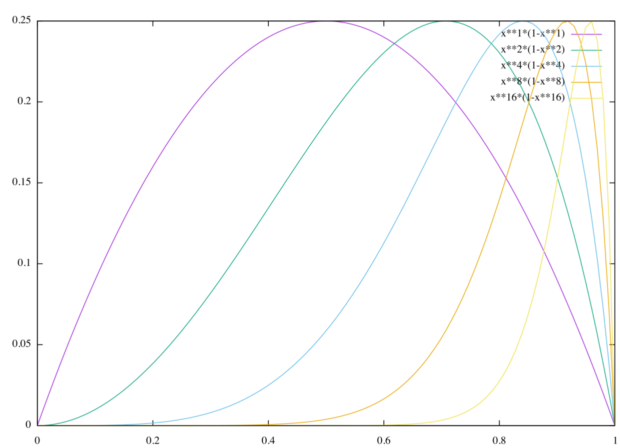
Bodová: $f_n(x) = x^n(1-x^n) \xrightarrow{n\to\infty} 0 \qquad f_n \to 0$

Stejnoměrně? $\sigma_n = \sup_{x\in[0,1]} |f_n(x) - \underbrace{f(x)}_{0}| = \max_{x\in[0,1]} f_n(x)$

$$f_n'(x) = n x^{n-1} - 2n x^{2n-1} = n x^{n-1}(1-2x^n) \overset{!}{=} 0 \quad \begin{cases} x = 0 \\ x = \frac{1}{\sqrt[n]{2}} \in (0,1) \end{cases}$$

$$\sigma_n = f_n\left(\frac{1}{\sqrt[n]{2}}\right) = x^n(1-x^n)\Big|_{x=\frac{1}{\sqrt[n]{2}}} = \frac{1}{2}\left(1-\frac{1}{2}\right) = \frac{1}{4} \xrightarrow{n\to\infty} \frac{1}{4} \neq 0$$

$f_n \not\rightrightarrows 0$ na $[0,1]$



Co kdybych vyšetřoval $f_n \overset{?}{\rightrightarrows}$ na $[0,1-\delta]$ $\delta > 0$

→ Pro $n > n_0$, kde n_0 je takové, že $1-\delta > \frac{1}{\sqrt[n_0]{2}}$

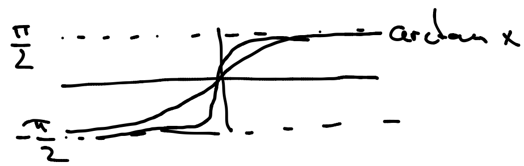
f nabývá maxima v $x = 1-\delta$

$$\Rightarrow \sigma_n^{[0,1-\delta]} = f_n(1-\delta) = (1-\delta)^n\left(1-(1-\delta)^n\right) \xrightarrow{n\to\infty} 0 \checkmark$$

$$f_n(x) \rightrightarrows 0 \quad \text{na} \quad [0, 1-\delta]$$

Příklad 5 Vyšetřeme kovergenci

$$f_n(x) = \arctan(nx) \quad \text{na} \quad (0,\infty)$$



Bodová

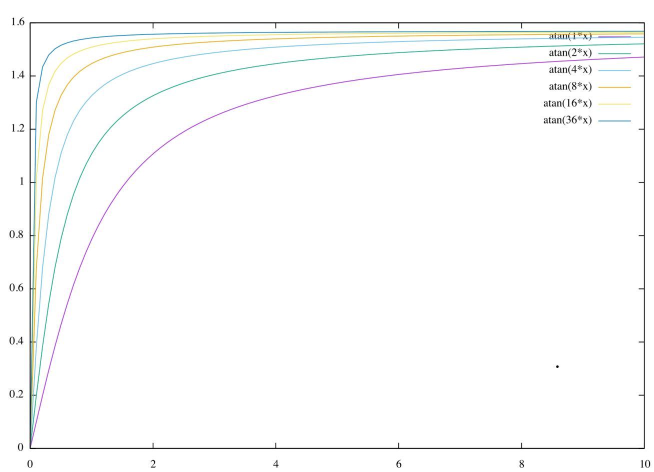
$$\forall x \in (0,\infty) \ \lim_{n\to\infty} \arctan(nx) = \frac{\pi}{2} \qquad f_n \to \frac{\pi}{2} \ \text{na} \ (0,\infty)$$

Stejnoměrná ?

$$\sigma_n = \sup_{x\in(0,\infty)} |f_n(x) - f(x)| = \sup_{x\in(0,\infty)} \left|\arctan(nx) - \frac{\pi}{2}\right| = \boxed{\sup_{x\in(0,\infty)} \left(\frac{\pi}{2} - \arctan(nx)\right)}$$

$$= \sup\left(\frac{\pi}{2}\right) - \inf_{x\in(0,\infty)} \arctan(nx) = \frac{\pi}{2} - 0 = \frac{\pi}{2} \xrightarrow{n\to\infty} \frac{\pi}{2} \neq 0$$

$$\arctan(nx) \not\rightrightarrows \frac{\pi}{2} \quad \text{na} \quad (0,\infty)$$



Uvážuji $x \in [\delta, \infty)$ $\delta > 0$

$$\sigma_n^{[\delta,\infty)} = \frac{\pi}{2} - \arctan(n\delta) \xrightarrow{n\to\infty} 0 \checkmark$$

$$\underline{\arctan(nx) \rightrightarrows \frac{\pi}{2} \ \text{na} \ [\delta,\infty)}$$
$$\delta > 0$$

Příklad 6 Vyšetřeme kovergenci.

$$f_n(x) = \left(1 + x^{4n}\right)^{\frac{2}{n}} \quad \text{na} \ \mathbb{R}$$

Bodová $\ulcorner$ $x \in (-1,1)$ $\lim_{n\to\infty} f_n(x) = 1$ $)$

$$\left[\begin{array}{lll} x = \pm 1 & \lim\limits_{n\to\infty} f_n(x) = 1 \\ |x| > 1 & \lim\limits_{n\to\infty} f_n(x) = x^8 \end{array}\right\} \quad f(x) = \begin{cases} 1 & \text{pro } x \in [-1,1] \\ x^8 & \text{pro } |x| > 1 \end{cases}$$

$f_n \to f$

Stejnoměrně? $\to$ $f_n(x)$ sudé $\to$ vyšetřuji na $[0,\infty)$

$$\sigma_n = \sup_{x\in[0,\infty)} \big|\underbrace{f_n(x) - f(x)}_{g_n(x)}\big|$$

$- \; [0,1]$

$$g_n(x) = \left(1+x^{4n}\right)^{\frac{2}{n}} - 1$$

$$g_n(0) = 0$$

$$g_n(1) = 2^{2/n} - 1$$

$$g_n'(x) = \frac{2}{n}\left(1+x^{4n}\right)^{\frac{2}{n}-1} 4n\, x^{4n-1} > 0 \quad \forall x \in (0,1)$$

$$\sigma_n = \sup_{x\in[0,1]} |g_n(x)| = 2^{2/n} - 1 \xrightarrow{n\to\infty} 0 \checkmark$$

$- \; (1,\infty)$

$$g_n(x) = \left(1+x^{4n}\right)^{\frac{2}{n}} - x^8$$

krajní body

$$\begin{cases} g_n(1) = 2^{2/n} - 1 \\ \lim\limits_{x\to\infty} g_n(x) = \lim\limits_{x\to\infty} x^8\left(\frac{1}{x^{4n}} + 1\right)^{\frac{2}{n}} - x^8 \end{cases}$$

$$= \lim_{x\to\infty} x^8\left(\cancel{1} + \frac{2}{n}\frac{1}{x^{4n}} + o\left(\frac{1}{x^{4n}}\right)\right) \cancel{- x^8}$$

$$= \lim_{x\to\infty} \frac{2}{n} x^{8-4n} + o\left(x^{8-4n}\right) = 0 \checkmark$$

uvnitř? $\to$

$$g_n'(x) = \frac{2}{\cancel{n}}\left(1+x^{4n}\right)^{\frac{2}{n}-1} 4\cancel{n}\, x^{4n-1} - 8x^7$$

$$= 8\,\frac{\left(1+x^{4n}\right)^{\frac{2}{n}} x^{4n-1} - x^7\left(1+x^{4n}\right)}{1+x^{4n}} \overset{?}{=} 0$$

$$\left(1+x^{4n}\right)^{\frac{2}{n}} x^{4n-1} \overset{?}{=} x^7\left(1+x^{4n}\right)$$

$$x^{4n-8} \overset{?}{=} \left(1+x^{4n}\right)^{\frac{n-2}{n}}$$

$$x^{4n^2-8n} \neq \left(1+x^{4n}\right)^{n-2}$$

$<$ $\quad g_n'(x) < 0$ na $[1,\infty)$

na $(1,\infty)$ najväčšia $g_n(x)$ sup. $g_n(1) = 2^{2/n} - 1 \xrightarrow{n\to\infty} 0$ ✓

$\sigma_n \xrightarrow{n\to\infty} 0$ & sudosť $f_n(x)$ $\Rightarrow$ $f_n(x) \rightrightarrows f(x)$ na $\mathbb{R}$
$[0,\infty)$