6. Adjoint operators. Compact operators

We start this section by two observations of general interest.

Assertion 1 Let $L \in \mathcal{L}(X,Y)$. Then

$$\|L\|_{\mathcal{L}(X,Y)} := \sup_{\|x\|_X \le 1} \|Lx\|_Y \quad \text{equals to} \quad \sup_{\|x\|_X = 1} \|Lx\|_Y .$$

Pf Set $l := \sup_{\|x\|_X=1} \|Lx\|_Y$. Clearly $l \le \|L\|_{\mathcal{L}(X,Y)}$. We want to show that for any $\varepsilon > 0$: $\|L\|_{\mathcal{L}(X,Y)} \le l + \varepsilon \iff \|L\|_{\mathcal{L}(X,Y)} - \varepsilon \le l$.

However, from the definition of $\|L\|_{\mathcal{L}(X,Y)}$; for $\|L\|_{\mathcal{L}(X,Y)} - \varepsilon$ there is x_ε, $\|x_\varepsilon\| \le 1$, so that $\|L\|_{\mathcal{L}(X,Y)} - \varepsilon < \|Lx_\varepsilon\|_Y$

or $\underbrace{\|L\|_{\mathcal{L}(X,Y)} - \varepsilon} < \|L\tilde{x}_\varepsilon\|_Y \|x_\varepsilon\|_X \le \underbrace{\|L\tilde{x}_\varepsilon\|_Y}$ where $\|\tilde{x}_\varepsilon\|_X = 1$, $\tilde{x}_\varepsilon := \frac{x_\varepsilon}{\|x_\varepsilon\|_X}$.

This implies $\|L\|_{\mathcal{L}(X,Y)} - \varepsilon \le \sup_{\|\tilde{x}_\varepsilon\|_X = 1} \|L\tilde{x}_\varepsilon\|_Y = l$, and we are done. ☺

Assertion 2 (Dual description of the norm) Let $(X, \|\cdot\|_X)$.

Then

$$\boxed{\|z\|_X = \sup_{\substack{f \in X^* \\ \|f\|_{X^*} = 1}} |f(z)|}$$

Pf If $f \in X^*$ with $\|f\|_{X^*} = 1$, then $|f(z)| \le \|f\|_{X^*} \|z\|_X \le \|z\|_X$

Hence $\sup_{\substack{f \in X^* \\ \|f\|_{X^*}=1}} |f(z)| \le \|z\|_X$. To prove the opposite ineq. consider $z \ne 0$. Then by H-B theorem: $\exists f \in X^*$: $f(z) = \|z\|_X$ & $\|f\|_{X^*} = 1$

Hence $\|z\|_X = f(z) \le \sup_{\substack{f \in X^* \\ \|f\|_{X^*}=1}} |f(z)|$ and the proof is complete. ∎

ADJOINT OPERATORS

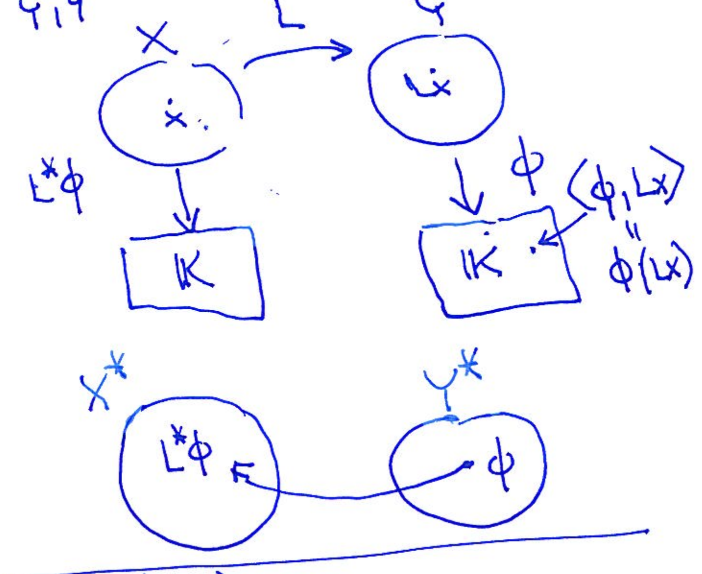
▷ Given $V \subset\subset X$, X being a Banach space, we define the orthogonal set $V^\perp$ to V via $\boxed{V^\perp := \{\phi \in X^* ;\ \langle \phi, x \rangle = 0 \text{ for all } x \in V\}}$

Similarly, for $W \subset\subset X^*$, we define $\boxed{W^\perp = \{x \in X ;\ \langle \phi, x\rangle = 0\ \forall \phi \in W\}}$

▷ Let X, Y be Banach, $L \in \mathcal{L}(X, Y)$.
Then for any $\phi \in Y^*$, the composed mapping $\phi \circ L$ maps X to $\mathbb{K}$, i.e. $\phi \circ L \in X^*$. The map that assings to $\phi \in Y^*$ the functional $\phi \circ L \in X^*$ is called adjoint (or dual) map to L. It holds, by its definition,

(*) $$\langle L^* \phi, x \rangle_{X^*, X} = \langle \phi, Lx \rangle_{Y^*, Y} \quad \text{for all } x \in X,$$

see Fig. 1.



Sometimes: $\phi \sim y^*$
Then (*) takes the form
$$\langle L^* y^*, x \rangle_{X^*, X} = \langle y^*, Lx \rangle_{Y^*, Y}$$

Theorem 6.1 (Properties of adjoint operators)
Let $L \in \mathcal{L}(X, Y)$ and $L^* : Y^* \to X^*$ be its adjoint. Then
(i) $L^* \in \mathcal{L}(Y^*, X^*)$ and $\|L\|_{\mathcal{L}(X,Y)} = \|L^*\|_{\mathcal{L}(Y^*, X^*)}$
(ii) $\operatorname{Ker} L = [\operatorname{Range} L^*]^\perp = [\operatorname{Im} L^*]^\perp$ and $\operatorname{Ker} L^* = [\operatorname{Range} L]^\perp = [\operatorname{Im} L]^\perp$

Pf $\boxed{\text{Ad (i)}}$

$$\|L^*\|_{\mathcal{L}(Y^*, X^*)} = \sup\{\|L^*\phi\|_{X^*} ;\ \|\phi\|_{Y^*} = 1\} = \sup\{|\langle L^*\phi, x\rangle| ;\ \|x\|_X = 1 \ \&\ \|\phi\|_{Y^*} = 1\}$$
$$= \sup\{\langle \phi, Lx\rangle ;\ \|x\|_X = 1, \|\phi\|_{Y^*} = 1\}$$
$$= \sup\{\|Lx\|_Y ;\ \|x\|_X = 1\} = \|L\|_{\mathcal{L}(X,Y)}.$$

definition of L^*

Assertion 2

Ad (ii) $\boxed{\operatorname{Ker} L = [\operatorname{Im} L^*]^\perp}$

Pf. $\boxed{x \in \operatorname{Ker} L} \iff Lx = 0 \iff \langle \phi, Lx \rangle = 0 \quad \forall \phi \in Y^*$
$\iff \langle L^*\phi, x \rangle = 0$
$\iff x \in [\operatorname{Im} L^*]^\perp$ ∎

$\boxed{\operatorname{Ker} L^* = (\operatorname{Im} L)^\perp}$

Pf. $\phi \in \operatorname{Ker} L^* \iff L^*\phi = 0$
$\iff \langle L^*\phi, x \rangle = 0 \quad \forall x \in X$
$\iff \langle \phi, Lx \rangle = 0 \quad \forall x \in X$
$\iff \phi \in (\operatorname{Im} L)^\perp$ ∎

COMPACT OPERATORS

Def. Let X, Y be Banach spaces. We say that $L \in \mathcal{L}(X,Y)$ is compact

$\overset{df.}{\equiv}$ $\forall \{x_n\}_{n=1}^\infty \subset X$ bdd $\exists \{x_{n_k}\}_{k=1}^\infty \subset \{x_n\}_{n=1}^\infty : Lx_{n_k} \to Lx$ in X
$(\exists\, Lx \in Y)$

$\iff \forall\, \mathcal{U} \subset X$ bdd is $\overline{L(\mathcal{U})}$ compact (or $L(\mathcal{U})$ is precompact)

Every bdd sequence contains a subsequence so that its image by L is converging in Y. ~~in X~~

Q: What operators are compact?

Theorem 6.2 (First two simple, but important cases) Let X, Y be Banach.

(1) Let $L \in \mathcal{L}(X,Y)$ and $\dim Y < +\infty$. Then L is compact.

(2) Let $L_n : X \to Y$ be compact for each $n \geq 1$ and $\lim_{n\to\infty} \|L_n - L\|_{\mathcal{L}(X,Y)} = 0$. Then L is compact.

Proof Ad (1) Due to linearity and equivalent definition of compactness:
$L \in \mathcal{L}(X,Y)$ is compact $\iff \overline{L(B_1(0))}$ is compact. However, if $\operatorname{Im} L$ is finitedimensional and L is bounded, then the closure $\overline{L(B_1(0))}$ is a closed bdd set of a finite-dim. space, hence compact. ∎

Ad (2) As Y is complete, then the statement $\boxed{\overline{L(B_1(0))} \text{ is compact}}$ is equivalent to $L(B_1(0))$ is precompact, i.e. $\forall \varepsilon > 0$ the set $L(B_1(0))$ can be covered by a finite balls of radius ε

Let $\varepsilon>0$ be given. As $\|L_n - L\|_{\mathcal{L}(X,Y)} \to 0$ for $n\to\infty$, there is k : $\|L-L_k\| < \frac{\varepsilon}{2}$. Since L_k is compact, there are

$y_1,\dots,y_N \in Y$: $\quad L_k(B_1(0)) \subset \bigcup_{i=1}^{N} B_{\frac{\varepsilon}{2}}(y_i) \quad (☆)$

Finally, if $\|x\| \le 1$ (i.e. $x\in \overline{B_1(0)}$), then $\|Lx - L_k x\|_Y \le \|L-L_k\| \|x\|_X \le \frac{\varepsilon}{2}$

- By (☆) : $\exists\, y_i$ ($i\in\{1,\dots,N\}$) : $\|L_k x - y_i\|_Y < \frac{\varepsilon}{2}$.

By Δ-ineq: $\|Lx - y_i\|_Y \le \|Lx - L_k x\|_Y + \|L_k x - y_i\|_Y < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$.

This proves that the L-image of $B_1(0)$ is covered by N (i.e. finite) number of balls of radius ε. ▨

There are two important criteria of compactness: one concerns subsets of $C(K)$, the other concerns subsets of $L^p(\mathbb{R}^d)$.

▷ $\boxed{\text{Arzelà–Ascoli theorem}}$ Let $K\subset\mathbb{R}^d$ be compact and $\boxed{\mathcal{F}\subset C(K)}$

($\mathcal{F}$ is a subset in the space of continuous functions; it can be subset of $C(K,Y) = \{f: K\to Y,\ f \text{ continuous}\}$, where Y is finite-dim.)

Then

$\boxed{\mathcal{F} \text{ is precompact in } C(K)}$ or $C(K,Y)$ $\iff$ $\boxed{\mathcal{F} \text{ is bdd and (uniformly) equicontinuous : it means}}$

(i) $\sup_{f\in\mathcal{F}} \sup_{x\in K} |f(x)| < \infty$

(ii) $\sup_{f\in\mathcal{F}} |f(x)-f(y)| \to 0$ whenever $|x-y|\to 0$

(Pf) See course Math. analysis for physicist I. ▨

▷ $\boxed{\text{Theorem by Riesz}}$ (or Kolmogorov criterium of compactness in $L^p(\Omega)$)

Let $1\le p<+\infty$ and $\mathcal{F}\subset L^p(\mathbb{R}^d)$. {or $\mathcal{F}\subset L^p(\mathbb{R}^d;Y)$ with $\dim Y<+\infty$}

Then

$\boxed{\mathcal{F} \text{ is precompact in } L^p(\mathbb{R}^d)}$ $\iff$

(1) $\sup_{f\in\mathcal{F}} \|f\|_{L^p(\mathbb{R}^d)} < +\infty$

(2) $\sup_{f\in\mathcal{F}} \|f(\cdot+h) - f\|_{L^p(\mathbb{R}^d)} \to 0$ as $|h|\to 0$

(3) $\sup_{f\in\mathcal{F}} \|f\|_{L^p(\mathbb{R}^d\setminus B_R(0))} \to 0$ as $R\to+\infty$

- boundedness
- generalization of uniform continuity
- control of behavior at "∞"!

(Pf) omitted: based on approximation of L^p-functions by smooth functions and on Arzelà–Ascoli theorem. ▨

$\boxed{\text{Theorem 6.3}}$ (Adjoint operator of a compact operator)

Let X, Y be Banach and $L \in \mathcal{L}(X,Y)$. Then

$$\boxed{L \text{ is compact}} \iff \boxed{L^* \text{ is compact}}$$

Pf ▸ We prove only the following $\boxed{\Rightarrow}$. From definition, we want to find, for arbitrary (now fixed) $\boxed{\{\phi_n\}_{n=1}^\infty \subset Y^* \text{ with } \|\phi_n\|_{Y^*} \le 1}$ a subsequence $\{\phi_{n_q}\}_{q=1}^\infty$ so that $L^*\phi_{n_q}$ is converging strongly in X^*.

▸ As L is compact, we know that $\boxed{E := \overline{L(B_1(0))}} \subset Y$ is compact.

▸ For any $n \in \mathbb{N}$, $\phi_n \in Y^*$ maps Y into $\mathbb{K}$. Consider $\phi_n|_E \in C(E)$. We will show that $\phi_n|_E$ fulfills the assumptions of Arzelà-Ascoli theorem. Indeed:

- $|\phi_n|_E(y) - \phi_n|_{E'}(y)| \le \|\phi_n\| \, \|y-y'\|_Y \le \|y-y'\|_Y$
- $|\phi_n|_E(y)| \le \|\phi_n^*\|_{Y^*} \|y\|_Y \le \underset{\substack{\cap \\ E}}{\|y\|_Y} = \|\underset{\substack{\uparrow \\ x \in B_1(0)}}{Lx}\|_Y \le \|L\|_{\mathcal{L}(X,Y)}$ $\quad \|x\| \le 1$

Thus, by Arzelà-Ascoli theorem, $\exists \{\phi_{n_j}|_E\}$ and $f \in C(E)$:

$$\phi_{n_j}|_E \to f \text{ in } C(E)$$

▸ Next,
$$\|L^*\phi_{n_j} - L^*\phi_{n_q}\|_{X^*} = \sup_{\|x\|=1} |\langle L^*\phi_{n_j} - L^*\phi_{n_q}, x\rangle|$$
$$= \sup_{\|x\|=1} |\langle \phi_{n_j} - \phi_{n_q}, Lx\rangle|$$
$$= \sup_{\|x\| \le 1} |\phi_{n_j}(Lx) - \phi_{n_q}(Lx)| \longrightarrow 0$$

implying that $\{L^*\phi_{n_j}\}_{j=1}^\infty$ is a Cauchy sequence.

Another important application of Arzelà-Ascoli theorem is formulated in the following theorem

$\boxed{\text{Theorem 6.4}}$ Let $K : [a,b] \times [a,b] \to \mathbb{R}$ be continuous, i.e. $K \in C([a,b]^2)$.

Then
$$Lf(x) := \int_a^b K(x,y) f(y)\, dy$$
is compact linear operator from $C([a,b])$ into $C([a,b])$

Pf Verify the assumptions of Arzelà-Ascoli theorem by means of Cantor's theorem on equicontinuity of continuous fcts.

▨